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ON SIMULTANEOUS PELL EQUATIONS AND RELATED THUE EQUATIONS

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Abstract. In this paper, we prove that the simultaneous Pell equations

$$x^2-(m^2-1)y^2=1,\ z^2-(n^2-1)y^2=1$$

have only positive integer solution $(x,y,z)=(m,1,n)$ if $m<n\le m+m^{\varepsilon}, 0<\varepsilon<1$ and $m\ge 202304^{\frac{1}{1-\varepsilon}}$. Using a computational reduction method we can omit the lower bound for m when $m<n\le m^{\frac{1}{5}}$. Moreover, we apply our main result to a family of Thue equations in two parameters studied by Jadrijević [?]-[?].

1. Introduction

Let a and b be distinct non-square positive integers. There is a long history of the simultaneous Pell equations

(1) $$x^2-ay^2=1, z^2-by^2=1.$$

This system of equations plays a crucial role in the classical theory of the figurate numbers and the modern approach of congruent number problem as well, see [?] and [?], respectively. By the work of Thue [?] and Siegel [?], the number of positive integer solutions (x,y,z) of (??) is finite. The first absolute upper bound for the number of solutions to (??) was given by Schlickewei [?], later Masser and Rickert [?] improved his exponential bound to 16. In 1998, combining with simultaneous diophantine approximation and the theory of linear forms in logarithms, Bennett [?] showed that there are at most 3 positive integer solutions (x,y,z) to the system of equations (??). Yuan [?] and [?] proved that for sufficiently large a and b, the number of solution is at most two. The best known bound is due to Bennett, Cipu, Mignotte and Okazaki [?]. In fact, they proved that Yuan's result is true for every value of a and b.

As it was pointed out by Bennett (see [?]), if the system of equations (??) has positive integer solutions, then it is equivalent to

(2) $$x^2-(m^2-1)y^2=1, z^2-(n^2-1)y^2=1, \quad 1<m<n.$$

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Bennett [?] proved that there are exactly two positive integer solutions of (**??**) when

$$n = \frac{\left(m+\sqrt{m^2-1}\right)^{2l} - \left(m-\sqrt{m^2-1}\right)^{2l}}{4\sqrt{m^2-1}}, \tag{3}$$

with $l \geq 2$ and $m \geq 2\cdot 10^7\sqrt{l}\log^2 l$. Yuan [?] made the following conjecture.

Conjecture 1. *Apart from when n has the form in (**??**), there is at most one positive integer solution (x, y, z) to (**??**).*

Le [?] showed that the conjecture is true if m and n are sufficiently large and have sufficiently large common divisor. In this paper, we will verify the conjecture when the difference of n and m is small. More precisely, we prove the following result.

Theorem 1. *If $m < n \leq m + m^{\varepsilon}$, for some real number $\varepsilon \in (0, 1)$, then the simultaneous Pell equations (**??**) have only the positive integer solution $(x, y, z) = (m, 1, n)$ for $m > 202304^{\frac{1}{1-\varepsilon}}$.*

When the constant ε is small, we have

Theorem 2. *If $m < n \leq m + m^{\frac{1}{5}}$, then the simultaneous Pell equations (**??**) possess the unique positive integer solution $(x, y, z) = (m, 1, n)$.*

For the organization of this paper, we will use the gap principle to prove some lemmas in Section **??**. By the means of a linear form in two logarithms, we prove Theorem **??** in Section **??**. We use the reduction method of Baker-Davenport [?] is in Section **??** to solve the system of Pell equations (**??**) for the small values of m, n, therefore we complete the proof of Theorem **??**. In the last section, we apply our main result to a family of Thue equations in two parameters studied by Jadrijević [?]-[?].

2. Gap principles

We will rewrite the equations into an equality of two Lucas numbers. We have $y = U_j = W_k$, i.e.

$$y = U_j = \frac{\alpha^j - \alpha^{-j}}{2\sqrt{a}} = \frac{\beta^k - \beta^{-k}}{2\sqrt{b}} = W_k, \tag{4}$$

where j, k are odd positive integers, $\alpha = m + \sqrt{m^2-1}$, $\beta = n + \sqrt{n^2-1}$ are the fundamental solutions to the equations $x^2 - ay^2 = 1$ and $z^2 - by^2 = 1$ respectively, with $a = m^2 - 1$ and $b = n^2 - 1$.

In this section, we will consider the following linear form in logarithms

$$\Lambda = j\log\alpha - k\log\beta + \log\left(\sqrt{\frac{b}{a}}\right). \tag{5}$$

Lemma 1. *If $j, k > 1$, then $j > k$.*

Proof. Since $U_j = \frac{\alpha^j - \alpha^{-j}}{2\sqrt{a}}$, by mathematical induction one can show that $\alpha^{j-1} < U_j < (2m)^{j-1}$, for $j > 1$. Similarly, we have $\beta^{k-1} < W_k < (2n)^{k-1}$, for $k > 1$. From $U_j = W_k$, we get $\beta^{k-1} < (2m)^{j-1}$. Since $n \geq m+1$, we have $\beta = n + \sqrt{n^2-1} \geq 2n - 1 \geq 2m+1$. This implies $(2m+1)^{k-1} < (2m)^{j-1}$ and then $k < j$. □

Lemma 2. *If equation (**??**) holds with $k \geq 1$, then $0 < \Lambda < \frac{\alpha^2}{\alpha^2-1} \cdot \alpha^{-2j}$.*

Proof. One can verify that $\alpha^j < \beta^k$. Indeed, if $\alpha^j > \beta^k$, then $\alpha^j - \alpha^{-j} > \beta^k - \beta^{-k}$. Since $m < n$, then we have $\frac{\alpha^j - \alpha^{-j}}{\sqrt{m^2-1}} > \frac{\beta^k - \beta^{-k}}{\sqrt{n^2-1}}$. This contradicts (**??**). Therefore, we get

$$0 < \Lambda = \log\left(\frac{1-\beta^{-2k}}{1-\alpha^{-2j}}\right) < -\log(1-\alpha^{-2j}) < \frac{\alpha^2}{\alpha^2-1} \cdot \alpha^{-2j}.$$

□

Lemma 3. *If $k > 1$, then $(j-1)\log\alpha < (k-1)\log\beta$.*

Proof. The condition $k > 1$ and Lemma **??** give $j > 1$. By Lemma **??**, we have $\Lambda < \frac{1}{(\alpha^2-1)\alpha^2}$. Since

$$\begin{aligned}\alpha\sqrt{b} - \beta\sqrt{a} &= (m+\sqrt{a})\sqrt{b} - (n+\sqrt{b})\sqrt{a} = m\sqrt{b} - n\sqrt{a} \\ &= \tfrac{m^2(n^2-1)-n^2(m^2-1)}{m\sqrt{b}+n\sqrt{a}} = \tfrac{n^2-m^2}{m\sqrt{b}+n\sqrt{a}} \geq \tfrac{(m+1)^2-m^2}{m\sqrt{b}+n\sqrt{a}} > \tfrac{2m+1}{2m\sqrt{b}} > \tfrac{1}{\sqrt{b}},\end{aligned}$$

we obtain

$$\begin{aligned}(j-1)\log\alpha - (k-1)\log\beta &= \Lambda + \log\left(\tfrac{\beta\sqrt{a}}{\alpha\sqrt{b}}\right) < \Lambda + \tfrac{\beta\sqrt{a}-\alpha\sqrt{b}}{\alpha\sqrt{b}} < \Lambda - \tfrac{1}{\alpha b} \\ &< \tfrac{1}{(\alpha^2-1)\alpha^2} - \tfrac{1}{\alpha b} = \tfrac{b-(\alpha^2-1)\alpha}{(\alpha^2-1)\alpha^2 b} < 0.\end{aligned}$$

The last inequality is easy to get as $\alpha = m + \sqrt{m^2-1} > 2\sqrt{m^2-1}$ and $n < 2m$. □

Let $\Delta = j - k$. Then we have.

Lemma 4. *If $k > 1$, $m \geq 8$, then $k - 1 \geq 0.99\Delta m^{1-\varepsilon}\log\alpha$.*

Proof. By Lemma **??**, we have $\frac{j-1}{k-1} < \frac{\log\beta}{\log\alpha}$. This implies

$$\begin{aligned}\frac{\Delta}{k-1} &< \frac{\log\beta}{\log\alpha} - 1 = \frac{\log(\beta/\alpha)}{\log\alpha} = \frac{\log(1+(\beta-\alpha)/\alpha)}{\log\alpha} < \frac{\beta-\alpha}{\alpha\log\alpha} \\ &= \frac{n+\sqrt{b}-m-\sqrt{a}}{\alpha\log\alpha} = \frac{n-m+\frac{b-a}{\sqrt{b}+\sqrt{a}}}{\alpha\log\alpha} = \frac{(n-m)\left(1+\frac{n+m}{\sqrt{b}+\sqrt{a}}\right)}{(m+\sqrt{a})\log\alpha} \\ &< \frac{m^\varepsilon\left(1+\frac{m}{\sqrt{m^2-1}}\right)}{1.99m\log\alpha}.\end{aligned}$$

Therefore, we have

$$k-1 > \Delta\frac{1.99m^{1-\varepsilon}}{1+\frac{m}{\sqrt{m^2-1}}}\log\alpha > 0.99\Delta m^{1-\varepsilon}\log\alpha.$$

□

3. Linear forms in two logarithms

Now we recall the following result due to Laurent (see [?], Corollary 2, page 328) on linear forms in two logarithms. For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X-\gamma^{(j)}\right)$, we denote by

$$h(\gamma)=\frac{1}{d}\left(\log|a|+\sum_{j=1}^{d}\log\max\left(1,\left|\gamma^{(j)}\right|\right)\right)$$

its absolute logarithmic height.

Lemma 5. *Let* α_1 *and* α_2 *be multiplicatively independent, each of* $\alpha_1,\alpha_2,\log\alpha_1$ *and* $\log\alpha_2$ *is real and positive.* b_1 *and* $b_2\in\mathbb{Z}^+$ *and*

$$\Lambda=b_2\log\alpha_2-b_1\log\alpha_1.$$

Let $D:=[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}]$, *for* $i=1,2$ *let*

$$\log A_i\geq\max\left\{h(\alpha_i),\frac{|\log\alpha_i|}{D},\frac{1}{D}\right\}$$

and

$$b'\geq\frac{b_1}{D\,\log A_2}+\frac{b_2}{D\,\log A_1}.$$

If $|\Lambda|\neq 0$, *then we have*

$$\log|\Lambda|\geq-17.9\cdot D^4\left(\max\left\{\log b'+0.38,\frac{30}{D},1\right\}\right)^2\log A_1\log A_2.$$

In order to apply Lemma **??**, we rewrite Λ, (see (**??**)), into the form

$$\Lambda=\log\left(\alpha^{\Delta}\cdot\sqrt{\frac{b}{a}}\right)-k\log\left(\frac{\beta}{\alpha}\right). \tag{6}$$

Hence, we take

$$D=4,\quad b_2=1,\quad b_1=k,\quad \alpha_2=\alpha^{\Delta}\cdot\sqrt{\frac{b}{a}},\quad \alpha_1=\frac{\beta}{\alpha}.$$

If α_1 and α_2 are multiplicatively dependent, then we have $\alpha_1^p=\alpha_2^q$ with nonzero integers p and q. Without loss of generality, we suppose that $p>0$. This implies $a^q\beta^{2p}=b^q\alpha^{2p+2\Delta q}$. Define that $\alpha^j=V_j+U_j\sqrt{a}$ and $\beta^k=T_k+W_k\sqrt{b}$. Combining with the fact V_j,U_j,T_k and W_k are both integers and equality

$$a^q(T_{2p}+W_{2p}\sqrt{b})=b^q(V_{2p+2\Delta q}+U_{2p+2\Delta q}\sqrt{a})$$

we get

$$a^qT_{2p}=b^qV_{2p+2\Delta q},\quad a^qW_{2p}\sqrt{b}=b^qU_{2p+2\Delta q}\sqrt{a}.$$

It follows that

$$a^{2q}=a^{2q}(T_{2p}^2-bW_{2p}^2)=b^{2q}(V_{2p+2\Delta q}^2-aU_{2p+2\Delta q}^2)=b^{2q}.$$

It results $a = b$, which is a contradiction. So α_1 and α_2 are multiplicatively independent. It is easy to see that $h\left(\alpha^\Delta\right) = \frac{1}{2}\Delta \log \alpha$ and

$$h\left(\sqrt{\frac{b}{a}}\right) = \frac{1}{2}\log b = \frac{1}{2}\log(n^2-1) < \frac{1}{2}\log((2m-1)^2-1) < \log(2m-1) < \log \alpha.$$

Thus we have

$$h(\alpha_2) = h\left(\alpha^\Delta \cdot \sqrt{\frac{b}{a}}\right) \leq h\left(\alpha^\Delta\right) + h\left(\sqrt{\frac{b}{a}}\right) < \frac{1}{2}(\Delta+2)\log\alpha =: \log A_2.$$

Moreover, $\gamma = \frac{n+\sqrt{n^2-1}}{m+\sqrt{m^2-1}}$ is a root of

$$X^4 - 4mnX^3 + (4m^2+4n^2-2)X^2 - 4mnX + 1.$$

The absolute values of its conjugates greater than 1 are β/α and $\alpha\beta$. Hence

$$\log A_1 := h(\gamma) = \frac{1}{4}\left(\log(\beta/\alpha) + \log(\alpha\beta)\right) = \frac{1}{2}\log\beta.$$

We assume from now and on that $m^{1-\varepsilon} \geq 10^5$. We have $\beta = n + \sqrt{n^2-1} > 2m \geq 2m^{1-\varepsilon} \geq 2 \cdot 10^5$. This implies $\frac{|b_2|}{Dh_1} = \frac{1}{2\log\beta} < 0.041$. This leads to

$$b' = \frac{k}{2(\Delta+2)\log\alpha} + 0.041. \tag{7}$$

Notice that $\Delta = j - k$ is a positive integer. From Lemma **??**, we have

$$\frac{k-1}{2(\Delta+2)\log\alpha} > \frac{0.99\Delta m^{1-\varepsilon}\log\alpha}{2(\Delta+2)\log\alpha} \geq 0.165m^{1-\varepsilon}.$$

This implies $\log b' + 0.38 \geq 10 > 30/D$. Therefore, by Lemma **??** we obtain

$$\log|\Lambda| \geq -17.9 \cdot 4^4 \left(\log b' + 0.38\right)^2 \cdot \frac{1}{2}\log\beta \cdot \frac{1}{2}(\Delta+2)\log\alpha. \tag{8}$$

On the other hand, from Lemma **??**, we get

$$\log|\Lambda| < \log\left(\frac{\alpha^2}{\alpha^2-1}\right) - 2j\log\alpha < \frac{1}{\alpha^2-1} - 2j\log\alpha. \tag{9}$$

Combining (**??**) and (**??**), we have

$$\frac{j}{\log\beta} < \frac{1}{2(\alpha^2-1)\log\alpha\log\beta} + 17.9 \cdot 32\left(\log b' + 0.38\right)^2(\Delta+2).$$

It is easy to show that $j\log\alpha > (k-1)\log\beta$, then above inequality imply

$$\frac{k}{2(\Delta+2)\log\alpha} - 0.014 < \frac{k-1}{2(\Delta+2)\log\alpha} < 0.001 + 17.9 \cdot 16\left(\log b' + 0.38\right)^2.$$

It follows that

$$b' < 0.055 + 286.4\left(\log b' + 0.38\right)^2. \tag{10}$$

We calculate that $b' < 33380$. Therefore, by (**??**) we get

$$k < 66760(\Delta+2)\log\alpha.$$

Combining this and $k-1 \geq 0.99\Delta m^{1-\varepsilon}\log\alpha$ (see Lemma **??**), we obtain

$$m^{1-\varepsilon} < \frac{66760}{0.99}\cdot\frac{\Delta+2}{\Delta} \leq \frac{66760}{0.99}\cdot 3 < 202304.$$

Thus, we have $m < 202304^{\frac{1}{1-\varepsilon}}$. This completes the proof of Theorem **??**.

4. Proof of Theorem ??

The Diophantine approximation algorithm called the Baker-Davenport reduction method, is used in many papers. The following lemma is a slight modification of the original version of Baker-Davenport reduction method. (See [**?**, Lemma 5a]).

Lemma 6. *Let M be a positive integer and let δ, μ, A and B be real numbers with $A > 0$ and $B > 1$. Assume that p/q be the convergent of the continued fraction expansion of δ such that $q > 6M$ and let*

$$\eta = \|\mu q\| - M\cdot\|\delta q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < j\delta - k + \mu < AB^{-j}$$

in integers j and k with

$$\frac{\log(Aq/\eta)}{\log B} \leq j \leq M.$$

For any fixed $\varepsilon \in (0,1)$, we only need to check that these cases are determined by $2 \leq m \leq 202304^{\frac{1}{1-\varepsilon}}$. Since n satisfies $m < n < m+m^{\varepsilon}$, then there are about

$$N(\varepsilon) = \sum_{m=2}^{202304^{1/(1-\varepsilon)}}\sum_{n=m+1}^{m+m^{\varepsilon}} 1 \leq \int_1^{202304^{1/(1-\varepsilon)}}\int_t^{t+t^{\varepsilon}} dv dt$$

$$= \int_1^{202304^{1/(1-\varepsilon)}} t^{\varepsilon} dt = \frac{t^{1+\varepsilon}}{1+\varepsilon}\Big|_1^{202304^{1/(1-\varepsilon)}} \leq \frac{1}{1+\varepsilon}202304^{\frac{1+\varepsilon}{1-\varepsilon}}$$

pairs (m,n).

When we choose $\varepsilon = 1/5$, then $m < 4290478$ and $N(\varepsilon) < 7.6\cdot 10^7$. This shows (see equation (11) of [**?**]) that

$$\frac{j}{\log(ej)} < 4.26\times 10^{13}\log^2\beta.$$

In our case, we have $j < 10^{18}$. By Lemma **??** we may apply Lemma **??** with

$$\delta = \frac{\log\alpha}{\log\beta},\quad \mu = \frac{\log\left(\sqrt{\frac{b}{a}}\right)}{\log\beta},\quad A = \frac{\alpha^2}{(\alpha^2-1)\log\beta},\quad B = \alpha^2$$

and $M = 10^{18}$.

The program was developed in PARI/GP running with 200 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the condition. We checked in the ranges $2 \le m \le 4290478$ and $m+1 \le n < m + m^{1/5}$.

All the computations were done in about 18 hours. The use of the second convergent was needed in 6147425 cases (5.57%), the third convergent was used in 986330 cases (0.26%), etc., the 11th was needed only in $(m, n) = (1219283, 1219292)$. In all cases we obtained $j \le 18$. By Lemma **??**, if $m \ge 6$, then we have

$$0.99m^{4/5} \log(m + \sqrt{m^2-1}) < k-1 < j-1 \le 17.$$

This implies $m \le 8$. In this range, n has to be $m+1$ with the inequalities $m < n < m + m^{1/5}$. One can refer to Theorem 4 of [?], there is only the solution $j = k = 1$. Thus, in any case there is no positive integer solution with $jk > 1$. This completes the proof of Theorem **??**.

5. An application to a family of Thue equations with two parameters

We consider the following two-parametric family of Thue equations

$$X^4 - 4mnX^3Y + (4m^2 + 4n^2 - 2)X^2Y^2 - 4mnXY^3 + Y^4 = 1, \quad 1 < m < n, \tag{11}$$

in unknown integers X and Y. As a corollary of Theorems **??** and **??**, we have:

Corollary 1. *If $m < n \le m + m^{\varepsilon}$, for some real number $\varepsilon \in (0, 1)$, then the Thue equation (**??**) has only integer solutions $(X, Y) = (0, \pm 1), (\pm 1, 0)$, for $m > 202304^{\frac{1}{1-\varepsilon}}$. Furthermore, for $m < n \le m + m^{\frac{1}{5}}$, the Thue equation (**??**) has only integer solutions $(X, Y) = (0, \pm 1), (\pm 1, 0)$.*

Proof. Let (X, Y) be an arbitrary, but fixed solution. On setting

$$x = mX^2 - 2nXY + mY^2, \;\; y = |X^2 - Y^2|, \;\; z = nX^2 - 2mXY + nY^2, \tag{12}$$

we have the system of equations (**??**), so Theorems **??** and **??** complete the proof since $1 = y = |X^2 - Y^2|$ implies that $(X, Y) = (\pm 1, 0), (0, \pm 1)$. □

In 2005, Jadrijević [?]-[?] studied the two-parametric family of Thue equations

$$x^4 - 2mnx^3y + 2(m^2 - n^2 + 1)x^2y^2 + 2mnxy^3 + y^4 = 1$$

and transformed it into the system of Pell equations

$$V^2 - (m^2+2)U^2 = -2, \quad Z^2 - (n^2-2)U^2 = 2. \tag{13}$$

She proved that for every $0.5 < \varepsilon \le 1$, there exists an effectively computable constant $C(\varepsilon)$ such that if $m \ne 0$, $\max\{|m|, |n|\} \ge C(\varepsilon)$ and $\gcd(m, n) \ge \max\{|m|^{\varepsilon}, |n|^{\varepsilon}\}$, then the system of Pell equations (**??**) has only the trivial solutions $(V, Z, U) = (\pm m, \pm n, 1)$, and gave some particular values of $C(\varepsilon)$. Further, she proved that this family of Thue equations has no solutions if $\gcd(xy, mn) = 1$ and $xy \ne 0$.

Using a little more work and a method similar to that above, one can obtain the following result.

Theorem 3. *If $|m - n| \le \max\{m, n\}^{\varepsilon}, \varepsilon \in (0, 1)$, then the system of Pell equations (**??**) has only integer solution $(V, Z, U) = (\pm m, \pm n, 1)$, for $\max\{m, n\} \ge C(\varepsilon)$, where $C(\varepsilon)$ is an effectively computable constant depending on ε.*

The outline of proof of Theorem **??** is the following: Transform equations (**??**) into the intersections of two Lucas sequences such as

$$y = H_j = \frac{\nu^j - \bar{\nu}^j}{\sqrt{2(m^2+2)}} = K_k = \frac{\delta^k - \bar{\delta}^k}{\sqrt{2(n^2-2)}}, \tag{14}$$

where j, k are odd positive integers and

$$\nu = \frac{m + \sqrt{m^2+2}}{\sqrt{2}}, \ \bar{\nu} = \frac{m - \sqrt{m^2+2}}{\sqrt{2}}, \ \delta = \frac{n + \sqrt{n^2-2}}{\sqrt{2}}, \ \bar{\delta} = \frac{n - \sqrt{n^2-2}}{\sqrt{2}}.$$

Notice that the shapes of these two Lucas numbers are a little different. In order to deal with equation (**??**), we need to consider two cases: $m < n < m + m^{\varepsilon'}$ and $n < m < n + m^{\varepsilon'}$. In the case $n < m$, one can change the order of j and k (also, α and β) and follow the lines of Lemmas **??**-**??**. For the part of the proof related to the linear form in two logarithms, one can easily see that $\frac{\nu}{\delta}$ is a root of

$$1 + 2nmX + (2m^2 - 2n^2 + 2)X^2 - 2nmX^3 + X^4.$$

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