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PROBLEMS AND CONJECTURES AROUND SHIFT RADIX SYSTEMS

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Abstract. Some basic open problems and conjectures concerning shift radix systems are listed and their relations to well-known concepts and questions are outlined.

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1. Introduction and basic definitions

In 2005 Akiyama et al. [2] introduced the notion of a shift radix system and pointed out connections of this simple dynamical system to well-known number systems such as *beta-numeration* and *canonical number systems*. Let us first recall the definitions (here for $y \in \mathbb{R}$ we denote by $\lfloor y \rfloor$ the largest $n \in \mathbb{Z}$ with $n \le y$).

Definition 1.1. *Let $d \in \mathbb{N}$ and $\mathbf{r} = (r_0, \dots, r_{d-1}) \in \mathbb{R}^d$.*

(i) *The mapping $\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d$ given by*

$$\tau_{\mathbf{r}}(\mathbf{z}) = (z_1, \dots, z_{d-1}, -\lfloor \mathbf{rz} \rfloor)^t \qquad (\mathbf{z} = (z_0, \dots, z_{d-1})^t \in \mathbb{Z}^d)$$

is called a shift radix system *(SRS for short), where we set $\mathbf{rz} := r_0 z_0 + \cdots + r_{d-1} z_{d-1}$.*

(ii) *We say that $\tau_{\mathbf{r}}$ has the* finiteness property *if for each $\mathbf{z} \in \mathbb{Z}^d$ there is $k \in \mathbb{N}$ such that the k-fold iterate of the application of $\tau_{\mathbf{r}}$ to $\mathbf{z}$ satisfies $\tau_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}$.*

This definition agrees with the one in [11], but the SRS in [2] coincide with our SRS with finiteness property. Our definition is equivalent to the property that $\tau_{\mathbf{r}}(z_0, \dots, z_{d-1}) = (z_1, \dots, z_{d-1}, z_d)^t$, where z_d is the unique integral solution of the linear inequality

$$0 \le r_0 z_0 + \cdots + r_{d-1} z_{d-1} + z_d < 1\,.$$

Therefore, we can write the mapping $\tau_{\mathbf{r}}$ as the sum of a linear function and a small error term. More explicitly, we have

$$\tau_{\mathbf{r}}(\mathbf{z}) = R_{\mathbf{r}}\,\mathbf{z} + \mathbf{v}(\mathbf{z}) \qquad (\mathbf{z} \in \mathbb{Z}^d),$$

where we put $\mathbf{v}(\mathbf{z}) := (0, \dots, 0, \mathbf{rz} - \lfloor \mathbf{rz} \rfloor)^t$ (in particular, $||\mathbf{v}(\mathbf{z})||_\infty < 1$) and $R_{\mathbf{r}}$ is a companion matrix of the polynomial

$$\chi_{\mathbf{r}}(X) := X^d + r_{d-1}X^{d-1} + \cdots + r_1 X + r_0 \qquad (\mathbf{r} = (r_0, \dots, r_{d-1}) \in \mathbb{R}^d).$$

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Since $\chi_{\mathbf{r}}$ agrees with the characteristic polynomial of the linear recurrence

$$z_n = -r_{d-1}z_{n-1} - \cdots - r_0 z_{n-d},$$

SRS is viewed as an *almost linear recurrence.*

SRS and their relation to beta-numeration seem to have appeared for the first time in Hollander's PhD thesis [15] in 1996; two years earlier Vivaldi [26] had studied similar mappings in his detailed investigation of discretized rotations (see also Reeve-Black and Vivaldi [22]). The dynamical aspects of SRS are described in a broader context by Barat et al. [9].

The aim of the present paper is to provide a concise list of open problems and conjectures concerning SRS thereby extending work of the first [1] and the third authors [21, Section 6]. The reader is referred to [17] for detailed background information, illustrations and algorithms.

2. Problems and conjectures

The following sets play a central role in the study of various aspects of SRS.

Definition 2.1. *For $d \in \mathbb{N}$ set*

$$\mathcal{D}_d := \left\{ \mathbf{r} \in \mathbb{R}^d \;:\; \forall \mathbf{z} \in \mathbb{Z}^d \,\exists k \in \mathbb{N}\,\exists \ell \in \mathbb{N}_{>0} : \tau_{\mathbf{r}}^k(\mathbf{z}) = \tau_{\mathbf{r}}^{k+\ell}(\mathbf{z}) \right\} \quad \textit{and}$$
$$\mathcal{D}_d^{(0)} := \left\{ \mathbf{r} \in \mathbb{R}^d \;:\; \tau_{\mathbf{r}} \textit{ is an SRS with finiteness property} \right\}.$$

Our fundamental open problem can roughly be described in the following way:

Problem 2.2. *Give a complete description of $\mathcal{D}_d$ and $\mathcal{D}_d^{(0)}$ for each $d \geq 2$.*

In the sequel we break up Problem 2.2 into several subproblems and conjectures. Computational experiments suggest the following (cf. [21]):

Problem 2.3. *Prove that $\mathbf{r} \in \mathcal{D}_d^{(0)} \cap \mathbb{Q}^d$ cannot be verified in polynomial time. Is it true that this problem does not belong to the NP complexity class?*

SRS are closely related to number systems:

- For an algebraic integer $\beta > 1$ the restriction of the *beta-transformation* T_β to $\mathbb{Z}[\beta]$ is conjugate to an SRS associated to a parameter defined by β.
- Akiyama et al. [5] investigated number systems with rational bases and established relations of these number systems to Mahler's $\frac{3}{2}$-problem (cf. [20]). These number systems can also be regarded as special cases of SRS (see Steiner and Thuswaldner [25]) and there seem to be relations between the $\frac{3}{2}$-problem and the length of SRS tiles (see below) associated with $\tau_{-2/3}$.
- The *backward division mapping* used to define *canonical number systems*[1] is conjugate to $\tau_{\mathbf{r}}$ for certain parameters $\mathbf{r}$; thus, characterizing all bases of canonical number systems is a special case of describing certain vectors $\mathbf{r} \in \mathbb{Q}^d$ giving rise to SRS with finiteness property (cf. Akiyama et al. [2]).

[1]In [16] the term "complete base" was coined.

Problem 2.4. *Characterize all parameters* $\mathbf{r}$ *for which the digits of the underlying number systems gives a language of a sofic shift (see* [18] *for the definition).*

The *Schur-Cohn region* $\mathcal{E}_d$ (see Schur [24]) is the set of all vectors $\mathbf{r} \in \mathbb{R}^d$ which define a contractive polynomial $\chi_{\mathbf{r}}$.

Conjecture 2.5. *We have* $\mathcal{D}_d^{(0)} \subset \mathcal{E}_d$.

This conjecture has only been settled for $d \in \{1, 2, 3\}$ (see [4, 14]).

Problem 2.6. *What is the measure of* $\mathcal{D}_d^{(0)}$*? What can we say about the topology of* $\mathcal{D}_d^{(0)}$*? What is the Hausdorff dimension of the boundary of* $\mathcal{D}_d^{(0)}$*?*

These problems have only been settled for $d = 1$.

Results on the topology of $\mathcal{D}_2^{(0)}$ are given by Weitzer [27], in particular, he showed that $\mathcal{D}_2^{(0)}$ is neither connected nor simply connected, and explicitly exhibited "holes" and components. It would be interesting to prove the following conjecture.

Conjecture 2.7. *The fundamental group of* $\mathcal{D}_3^{(0)}$ *is non-trivial, i.e., it has a handle.*

The following task was put forward and commented in [21].

Problem 2.8. *Given* $\mathbf{r} \in \mathcal{E}_d \cap \mathbb{Q}^d$*, is* $\mathbf{r} \in \mathcal{D}_d$ *algorithmically decidable?*

Periodicity of the orbits on $\partial\mathcal{D}_d$ is an open problem even in dimension 2. For instance, the special case $(1, \lambda) \in \partial\mathcal{D}_2$ can be expressed in the following simple arithmetical form:

Conjecture 2.9. *Let* $\lambda \in \mathbb{R}$ *be such that* $-2 < \lambda < 2$*. Further, let* $a_0, a_1 \in \mathbb{Z}$ *and define*

$$a_{n+1} := -a_{n-1} - \lfloor \lambda a_n \rfloor \qquad (n \geq 1).$$

Then the sequence $a_0, a_1, a_2, \ldots$ *is periodic.*

Some partial results have been obtained in [3] (see also [19, 6]). A weaker conjecture is that for any λ there exist infinitely many non-symmetric periodic orbits.

A remarkable conjecture of Schmidt [23] can be reformulated more generally in terms of the ultimate periodicity of $\tau_{\mathbf{r}}$, where $\mathbf{r}$ belongs to the hypersurface

$$E_d^{(\mathbb{C})} := \{\mathbf{r} \in \partial\mathcal{E}_d \ : \ R_{\mathbf{r}} \text{ has a non-real eigenvalue of modulus } 1\}.$$

Problem 2.10. *For which* $\mathbf{r} \in E_d^{(\mathbb{C})}$ *is each orbit of* $\tau_{\mathbf{r}}$ *ultimately periodic, in particular the orbit of* $(1, 0, \ldots, 0)^t$ *?*

For particular orbits addressed in Problem 2.10 we suspect the following more explicit result.

Conjecture 2.11. *Let* $\mathbf{r} \in E_d^{(\mathbb{C})}$ *be such that* $\chi_{\mathbf{r}}$ *is irreducible. Let* s *be the number of pairs of complex conjugate roots* $(\alpha, \bar{\alpha})$ *of* $\chi_{\mathbf{r}}$ *with* $|\alpha| = 1$*. Then every orbit of* $(1, 0, \ldots, 0)^t$ *under* $\tau_{\mathbf{r}}$ *is periodic if* $s \in \{1, 2\}$*, and there exist* $\mathbf{r}$ *such that the orbit of* $(1, 0, \ldots, 0)^t$ *is aperiodic if* $s \geq 3$ *(see also* [12], [13]*).*

SRS also admit a geometric theory, in particular, it is possible to define so-called *SRS tiles* (see Berthé et al. [11]). In the sequel we assume $r_0 \neq 0$.

Definition 2.12. *Let* $\mathbf{r} = (r_0, \ldots, r_{d-1}) \in \mathcal{E}_d$ *and* $\mathbf{z} \in \mathbb{Z}^d$ *be given. The set*

$$\mathcal{T}_{\mathbf{r}}(\mathbf{z}) = \operatorname*{Lim}_{n\to\infty} R_{\mathbf{r}}^n \, \tau_{\mathbf{r}}^{-n}(\{\mathbf{z}\})$$

is called the SRS tile associated with $\mathbf{r}$*; here the limit is taken with respect to the Hausdorff metric.*

A tiling is a collection of compact sets covering $\mathbb{R}^d$ with zero measure overlaps (see [25]). SRS tiles are conjectured to induce tilings of their representation spaces. A special case of this conjecture implies the *Pisot substitution conjectures* (see e.g. Arnoux and Ito [7], Baker-Barge-Kwapisz [8], Barge [10]) for Pisot beta substitutions. We present two challenging tasks.

Conjecture 2.13. *Let* $\mathbf{r} \in \mathcal{E}_d$*. Then* $\{\mathcal{T}_{\mathbf{r}}(\mathbf{z}) \,:\, \mathbf{z} \in \mathbb{Z}^d\}$ *is a tiling of* $\mathbb{R}^d$*.*

Problem 2.14. *Let* $\mathbf{r} \in \mathcal{E}_d$ *and* $\mathbf{z} \in \mathbb{Z}^d$*. Give criteria for* $\mathcal{T}_{\mathbf{r}}(\mathbf{z})$ *being the closure of its interior.*

Finally, we state a problem related to the connectivity of the so-called central SRS tiles $\mathcal{T}_{\mathbf{r}}(\mathbf{0})$.

Problem 2.15. *Describe the Mandelbrot sets*

$$\{\mathbf{r} \in \mathcal{E}_d \,:\, \mathcal{T}_{\mathbf{r}}(\mathbf{0}) \text{ is connected}\}$$

for $d \geq 2$*.*

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