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A closed-form solution for a two-server heterogeneous retrial queue with threshold policy

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Abstract. In this paper, we reconsider a two-server heterogeneous retrial queue with threshold policy. However, the computation time with the existing method is prohibitively large for certain values of the threshold parameter. Applying the spectral expansion method, we derive a closed-form expression for the eigenvalues and eigenvectors matrix that are needed to determine the steady-state distribution of a quasi-birth-death process describing the queue. As a result, the computation time for the performance measures does not depend on the threshold parameter.

Keywords. Retrial queue; two servers; threshold; spectral expansion; closed-form.

1. Introduction

Queueing theory has been applied to analyze the performance of telecommunication systems, modern information and communication technology (ICT) systems, production-inventory systems and manufacturing systems for long time [1–5]. Retrial queues where blocked customers may re-request for service after a certain timeout [1, 6–20] form a specific research topic in the queueing theory. It is worth mentioning that the impatient behaviour of customers can be modelled by retrial queues. Furthermore, a retrial mechanism can be also applied to control the access of resources in a certain system [21].

[22] considered a two-server heterogeneous retrial queue with threshold policy. They modelled the system as a quasi-birth-and-death (QBD) process with threshold dependent block-tridiagonal infinitesimal matrix and applied the general theory of matrix-geometric solutions. Thus, the computation of the rate matrix R (the minimal non-negative solution to the matrix equation) is based on the iteration algorithm. However, their analysis procedure (see [22]) has limited applicability because the computational time significantly depends on the value of a threshold. To enhance the applicability of the two-server heterogeneous retrial queue with threshold policy for the analysis of practical systems such as ICT systems and manufacturing systems, we derive a closed-form solution for the steady-state probabilities. Therefore, the computational time needed to obtain the performance measures does not depend on the threshold, which is demonstrated by numerical results.

The rest of the paper is organized as follows. In section 2, we describe a model. We present a mathematical derivation for the closed form solution in section 3. Numerical results are presented in section 4.

2. A two-server heterogeneous retrial queue with threshold policy

In this paper, we consider a retrial queue with two servers. The service time of a customer follows an exponential distribution with rate μ_1 if the customer is served by the fast server and with rate μ_2 if the customer is served by the slow server. Note that $\mu_1 > \mu_2$. Customers arrive according to a Poisson process with rate λ. A customer (either arriving or retrial) gets service from the system if

- either the fast server is idle, or

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- the fast server is busy, the slow server is idle and the number of customers in the orbit is above threshold q_2.

There is an orbit where customers are stored for the case when customers could not obtain service from the servers. According to the First Come First Serve (FCFS) principle, only the customer at the head of the orbit can retry for accessing servers. If the customer at the head of the orbit could not get service, he/she will return to the head of the orbit. The times between retrials (if the current retried customer is moved to the orbit when the orbit is not empty) follow the exponential distribution with parameter γ.

Let $J(t)$ denote the number of customers in the orbit at time t, and $I(t)$ describe the state of the servers as follows:

$$I(t) = \begin{cases} 0 & \text{if two servers are idle} \\ 1 & \text{the fast server is busy and the slow server is idle} \\ 2 & \text{the fast server is idle and the slow server is busy} \\ 3 & \text{if two servers are busy} \end{cases}$$

The entire system is described by Continuous Time Markov Chain (CTMC) $Y = \{I(t), J(t)\}$ with state space $\{(i,j): 0 \le i \le 3,\ j \ge 0\}$. We denote the steady state probabilities by $\pi_{i,j} = \lim_{t\to\infty} \Pr(I(t) = i, J(t) = j)$, and introduce $\pi_j = (\pi_{0,j}, \pi_{1,j}, \pi_{2,j}, \pi_{3,j})$.

The evolution of CTMC Y is driven by the following transitions.

(a) $A_j(i,k)$ denotes a transition rate from state (i, j) to state (k, j) $(0 \le i,k \le 3; j = 0,1,\ldots)$, which is caused by either the arrival of customers or the departure of customers after service. Matrix A_j is defined as the matrix with elements $A_j(i,k)$.

(b) $B_j(i,k)$ represents one step upward transition rate from state (i, j) to state $(k, j+1)$ $(0 \le i,k \le 3; j = 0,1,\ldots)$, which is due to the arrival of a customer when it could not obtain service. In the similar way, matrix B_j (B) is defined with elements $B_j(i,k)$.

(c) $C_j(i,k)$ is the transition rate from state (i, j) to state $(k, j-1)$ $(0 \le i,k \le c; j = 1,\ldots)$, which is due to the successful retrial of a request from the orbit. Matrix C_j $(\forall j \ge 1)$ is defined with elements $C_j(i,k)$.

Based on the operation rule, we obtain

$$A_j = \begin{bmatrix} 0 & \lambda & 0 & 0 \\ \mu_1 & 0 & 0 & 0 \\ \mu_2 & 0 & 0 & \lambda \\ 0 & \mu_2 & \mu_1 & 0 \end{bmatrix} \quad \text{for} \quad 0 \le j < q_2;$$

$$A_j = \begin{bmatrix} 0 & \lambda & 0 & 0 \\ \mu_1 & 0 & 0 & \lambda \\ \mu_2 & 0 & 0 & \lambda \\ 0 & \mu_2 & \mu_1 & 0 \end{bmatrix} \quad \text{for} \quad j \ge q_2;$$

$$B_j = A_{01} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \quad \text{for} \quad 0 \le j \le q_2 - 1;$$

$$B_j = A_{02} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \quad \text{for} \quad j > q_2 - 1;$$

$$C_j = A_{21} = \begin{bmatrix} 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for} \quad 1 \le j \le q_2;$$

$$C_j = A_{22} = \begin{bmatrix} 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{for} \quad j > q_2. \tag{1}$$

Let us introduce $A_{00} = A_0 - D^{A_0} - D^{B_0}$, $A_{10} = A_1 - D^{A_1} - D^{A_{01}} - D^{A_{21}}$, $A_{11} = A_{q_2} - D^{A_{q_2}} - D^{A_{02}} - D^{A_{21}}$ and $A_{12} = A_{q_2} - D^{A_{q_2}} - D^{A_{22}} - D^{A_{02}}$, where D^Z $(Z = A_j, B_j, C_j)$ is a diagonal matrix whose diagonal element is the sum of all elements in the corresponding row of Z.

By equating the flow out and in each state, the balance equations can be expressed as below.

- In band 1, we have the balance equations

$$\pi_{i-1}A_{01} + \pi_i A_{10} + \pi_{i+1}A_{21} = 0, \quad 1 \le i \le q_2 - 1. \tag{2}$$

- The balance equations of band 2 is expressed as follows.

$$\pi_{i-1}A_{02} + \pi_i A_{12} + \pi_{i+1}A_{22} = 0, \quad i \ge q_2 + 1. \tag{3}$$

The boundary balance equations are

$$\pi_0 A_{00} + \pi_1 A_{21} = 0, \tag{4}$$

$$\pi_{q_2-1}A_{01} + \pi_{q_2}A_{11} + \pi_{q_2+1}A_{22} = 0. \tag{5}$$

3. A closed-form solution

Following [23], we obtain the expression for π_i, $0 \le i \le q_2$, from Eq. (2)

$$\pi_i = \sum_{k=1}^{4} a_{1,k}\psi_{1,k}x_{1,k}^i + \sum_{k=1}^{4} b_{1,k}\phi_{1,k}y_{1,k}^{q_2-1-i} \quad \forall i = 0,\ldots,q_2-1, \tag{6}$$

where $a_{1,k}$'s and $b_{1,k}$'s are the coefficients to be determined, and $(x_{1,k},\ \psi_{1,k})$ $(k = 1,\ 2,\ 3,\ 4)$ are the eigenvalue, left-eigenvector solution pairs of the matrix equations

$$\psi_{1,}\left[A_{01}+A_{10}x_{1,}+A_{21}x_{1,}^{2}\right]=0. \tag{7}$$

and $(y_{1,k},\ \phi_{1,k})$ (k = 1, 2, 3, 4) are the eigenvalue, left-eigenvector solution pairs of the matrix equations

$$\phi_{1,}\left[A_{21}+A_{10}y_{1,}+A_{01}y_{1,}^{2}\right]=0. \tag{8}$$

This means, we have to determine the appropriate eigenvalues and eigenvectors of the characteristic matrix polynomial

$$Q_1(x_1)=A_{01}+x_1A_{10}+x_1{}^2A_{21}$$
$$=\begin{bmatrix} x_1(-\gamma-\lambda) & x_1^2\gamma+x_1\lambda & 0 & 0\\ x_1\mu_1 & \lambda+x_1(-\lambda-\mu_1) & 0 & 0\\ x_1\mu_2 & 0 & x_1(-\gamma-\lambda-\mu_2) & x_1^2\gamma+x_1\lambda\\ 0 & x_1\mu_2 & x_1\mu_1 & \lambda+x_1(-\lambda-\mu_1-\mu_2)\end{bmatrix}. \tag{9}$$

We can get eigenvalues $x_{1,}$ by solving $Det[Q_1(x_1)]=0$ as

$$x_{1,1}=0, x_{1,2}=0,$$
$$x_{1,3}=\frac{\omega-\sqrt{4\gamma\mu_1(-\gamma\lambda-\lambda^2-\lambda\mu_2)+\omega^2}}{2\gamma\mu_1}, \tag{10}$$

$$x_{1,4}=\frac{\lambda(\gamma+\lambda)}{\gamma\mu_1}, x_{1,5}=1,$$
$$x_{1,6}=\frac{\omega+\sqrt{4\gamma\mu_1(-\gamma\lambda-\lambda^2-\lambda\mu_2)+\omega^2}}{2\gamma\mu_1}, \tag{11}$$

where $\omega=\gamma\lambda+\lambda^2+\gamma\mu_1+\gamma\mu_2+2\lambda\mu_2+\mu_1\mu_2+\mu_2^2$.

The eigenvectors are obtained as follows $\psi_{1,1}=[1,0,0,0]$, $\psi_{1,2}=[0,0,1,0]$, $\psi_{1,3}=[-1,-\frac{\gamma+\lambda+\mu_2}{\mu_1},1,\frac{\gamma+\lambda+\mu_2}{\mu_1}]$, $\psi_{1,4}=[1,\frac{\gamma+\lambda}{\mu_1},0,0]$.

To obtain eigenvalues $y_{1,}$ we solve $Det[Q_2(y_1)]=0$, where

$$Q_2(y_1)=A_{21}+y_1A_{10}+y_1{}^2A_{01}$$
$$=\begin{bmatrix} y_1(-\gamma-\lambda) & \gamma+y_1\lambda & 0 & 0\\ y_1\mu_1 & y_1^2\lambda+y_1(-\lambda-\mu_1) & 0 & 0\\ y_1\mu_2 & 0 & y_1(-\gamma-\lambda-\mu_2) & \gamma+y_1\lambda\\ 0 & y_1\mu_2 & y_1\mu_1 & y_1^2\lambda+y_1(-\lambda-\mu_1-\mu_2)\end{bmatrix}.$$

The results are

$$y_{1,1}=0, y_{1,2}=0, y_{1,3}=\frac{\gamma\mu_1}{\lambda(\gamma+\lambda)},$$
$$y_{1,4}=1, y_{1,5}=\frac{\omega-\sqrt{4\gamma\mu_1(-\gamma\lambda-\lambda^2-\lambda\mu_2)+\omega^2}}{2(\gamma\lambda+\lambda^2+\lambda\mu_2)},$$
$$y_{1,6}=\frac{\omega+\sqrt{4\gamma\mu_1(-\gamma\lambda-\lambda^2-\lambda\mu_2)+\omega^2}}{2(\gamma\lambda+\lambda^2+\lambda\mu_2)},$$

and the eigenvectors are $\phi_{1,1}=[0,1,0,0]$, $\phi_{1,2}=[0,0,0,1]$, $\phi_{1,3}=[1,-\frac{\gamma+\lambda+\mu_2}{\mu_1},1,\frac{\gamma+\lambda+\mu_2}{\mu_1}]$, $\phi_{1,4}=[1,\frac{\gamma+\lambda}{\mu_1},0,0]$.

Following [24], the probability π_i, $i\geq q_2$, is given by

$$\pi_i=\sum_{k=1}^{4}a_{2,k}\psi_{2,k}x_{2,k}^{i-q_2},\quad \forall i\geq q_2, \tag{12}$$

where $(x_{2,k},\psi_{2,k})\,(k=1,2,3,4)$ are the eigenvalue, left-eigenvector solution pairs of the matrix equations

$$\psi_{2,}\left[A_{02}+A_{12}x_{2,}+A_{22}x_{2,}^{2}\right]=0. \tag{13}$$

Note that $|x_{2,k}|<1$ for $k=1,\ 2,\ 3,\ 4$. This means, we have to determine the appropriate eigenvalues and eigenvectors of the characteristic matrix polynomial

$$Q(x) = A_{02} + xA_{12} + x^2A_{22} = \begin{bmatrix} -(\gamma+\lambda)x & x(\lambda+\gamma x) & 0 & 0 \\ \mu_1 x & -(\gamma+\lambda+\mu_1)x & 0 & x(\lambda+\gamma x) \\ \mu_2 x & 0 & -(\gamma+\lambda+\mu_2)x & x(\lambda+\gamma x) \\ 0 & \mu_2 x & \mu_1 x & \lambda-(\lambda+\mu_1+\mu_2)x \end{bmatrix}.$$

It yields

$$\begin{aligned} Det[Q(x)] = {} & x^3(x-1)\Big[\lambda((\gamma+\lambda)^2+\gamma\mu_1)(\gamma+\lambda+\mu_2) \\ & -\gamma\big(\mu_1(\gamma^2+2\lambda^2+\gamma(3\lambda+\mu_1)) \\ & +\Big((\gamma+\lambda)^2+2(\gamma+2\lambda)\mu_1+\mu_1^2\Big)\mu_2 \\ & +(\gamma+\lambda+\mu_1)\mu_2^2)x+\gamma^2\mu_1(\mu_1-\mu_2)x^2\Big] \\ = {} & x^3(x-1)\left[\omega_0-\omega_1 x+\omega_2 x^2\right], \end{aligned} \tag{14}$$

where

$$\begin{aligned} \omega_0 &= \lambda\Big((\gamma+\lambda)^2+\gamma\mu_1\Big)(\gamma+\lambda+\mu_2), \\ \omega_1 &= \gamma\big(\mu_1(\gamma^2+2\lambda^2+\gamma(3\lambda+\mu_1)) \\ &\quad +\Big((\gamma+\lambda)^2+2(\gamma+2\lambda)\mu_1+\mu_1^2\Big)\mu_2 \\ &\quad +\big(\gamma+\lambda+\mu_1\big)\mu_2^2\big), \\ \omega_2 &= \gamma^2\mu_1(\mu_1-\mu_2). \end{aligned}$$

As a consequence, $Det[Q(x)]$ has three zero roots ($x_{2,1}=x_{2,2}=x_{2,3}=0$), one root equal to 1. In addition,

- $Det[Q(x)]$ has one root $x^*_{2,4} = \frac{\gamma^2\lambda+2\gamma\lambda^2+\lambda^3+\gamma\lambda\mu_2}{\gamma\mu_2(2\gamma+3\lambda+2\mu_2)}$ if $\mu_1=\mu_2$.
- $Det[Q(x)]$ has two roots $x_{2,4} = \frac{\omega_1-\sqrt{\omega_1^2-4\omega_0\omega_2}}{2\omega_2}$ and $x_{2,5} = \frac{\omega_1+\sqrt{\omega_1^2-4\omega_0\omega_2}}{2\omega_2}$ if $\mu_1\neq\mu_2$.

Note that the eigenvalues of Eq. (13) are the roots of $Det[Q(x)]$. Following the same argument as in [8], if the QBD process is ergodic, $Q(x)$ should have four eigenvalues inside the unit circle. As a consequence, $|x^*_{2,4}|<1$ (for $\mu_1=\mu_2$) and $|x_{2,4}|<1$ (for $\mu_1\neq\mu_2$). In what follows, we also use $x_{2,4}$ to refer to $x^*_{2,4}$ when $\mu_1=\mu_2$.

It is easy to check that independent left-eigenvectors corresponding to three null-eigenvalues are $\psi_{2,1}=[1,0,0,0]$, $\psi_{2,2}=[0,1,0,0]$,...,$\psi_{2,3}=[0,0,1,0]$.

Let $\psi_{2,4}=[\psi_{2,4,1},\psi_{2,4,2},\psi_{2,4,3},1]$ be the eigenvector corresponding to $x_{2,4}$. Utilizing $\psi_{2,4}Q(x_{2,4})=0$, we get

$$\psi_{2,4,1} = \frac{2\gamma\mu_1\mu_2+2\lambda\mu_1\mu_2+\mu_1^2\mu_2+\mu_1\mu_2^2}{(\gamma+\lambda+\mu_2)(\gamma^2+2\gamma\lambda+\lambda^2+\gamma\mu_1-\gamma\mu_1 x_{2,4})}, \tag{15}$$

$$\psi_{2,4,2} = -\frac{-\gamma^2\mu_2-2\gamma\lambda\mu_2-\lambda^2\mu_2-\lambda\mu_1\mu_2-\gamma\mu_2^2-\lambda\mu_2^2-\gamma\mu_1\mu_2 x_{2,4}}{(\gamma+\lambda+\mu_2)(\gamma^2+2\gamma\lambda+\lambda^2+\gamma\mu_1-\gamma\mu_1 x_{2,4})}, \tag{16}$$

$$\psi_{2,4,3} = \frac{\mu_1}{\gamma+\lambda+\mu_2}. \tag{17}$$

Besides (2), (3), (4) and (5) the normalization equation can be used to determine the coefficients:

$$\sum_{i=0}^{\infty}\pi_i e = \sum_{i=0}^{q_2-1}\pi_i e + \sum_{i=q_2}^{\infty}\pi_i e = 1 \tag{18}$$

Since

$$\begin{aligned} \sum_{i=0}^{q_2-1}\pi_i e &= \sum_{k=1}^{4} a_{1,k}\Psi_{1,k}\frac{-1+x_{1,k}^{q_2}}{-1+x_{1,k}}e \\ &\quad + \sum_{k=1}^{4} b_{1,k}\Phi_{1,k}\frac{-1+y_{1,k}^{q_2}}{-1+y_{1,k}}e \\ \sum_{i=q_2}^{\infty}\pi_i e &= \sum_{k=1}^{4} a_{2,k}\Psi_{2,k}\frac{1}{1-x_{2,k}}e, \end{aligned}$$

the solution for coefficients are

$$\begin{aligned} & a_{1,1}=G_{11}a_{1,3}, a_{1,2}=\frac{1}{\lambda+\mu_2}\gamma(1-\tau_6 y_{1,3}\mu_1 G_{12})a_{1,3}, \\ & a_{1,4}=G_{14}a_{1,3}, b_{1,1}=0, b_{1,2}=0, b_{1,3}=GB_{13}a_{1,3}, \\ & b_{1,4}=GB_{14}a_{1,3}, a_{2,1}=G_{21}a_{1,3}, a_{2,2}=G_{22}a_{1,3}, a_{2,3}=0, \\ & a_{2,4}=G_{24}a_{1,3}, \\ & a_{1,3}=1/\Bigg(G_{11}+\frac{\gamma(1-G_{12}\mu_1\tau_6 y_{1,3})}{\lambda+\mu_2}+G_{14}\tau_{13}\tau_{14}+G_{14}\tau_{13} \\ & \qquad +G_{21}+G_{22}+\frac{G_{24}(-\tau_1+\tau_2-1)}{x_{2,4}-1}+GB_{14}(\tau_{14}+1)\Bigg), \end{aligned} \tag{19}$$

where

$$
\begin{aligned}
x_{1,3} &= 1/(2\gamma\mu_1)(\gamma\lambda + \lambda^2 + \gamma\mu_1 + \gamma\mu_2 + 2\lambda\mu_2 + \mu_1\mu_2 + \mu_2^2 \\
&\quad - \sqrt{-4\gamma\lambda\mu_1(\gamma+\lambda+\mu_2) + (\lambda^2 + 2\lambda\mu_2 + \mu_2(\mu_1+\mu_2) + \gamma(\lambda+\mu_1+\mu_2))^2}),\\
y_{1,3} &= 1/(2\lambda(\gamma+\lambda+\mu_2))(\gamma\lambda + \lambda^2 + \gamma\mu_1 + \gamma\mu_2 + 2\lambda\mu_2 + \mu_1\mu_2 + \mu_2^2 \\
&\quad - \sqrt{-4\gamma\lambda\mu_1(\gamma+\lambda+\mu_2) + (\lambda^2 + 2\lambda\mu_2 + \mu_2(\mu_1+\mu_2) + \gamma(\lambda+\mu_1+\mu_2))^2}),\\
x_{2,4} &= 1/\left(2\gamma^2\mu_1(\mu_1-\mu_2)\right)\left(\gamma^3\mu_1 + 3\gamma^2\lambda\mu_1 + 2\gamma\lambda^2\mu_1 + \gamma^2\mu_1^2 + \gamma^3\mu_2 + 2\gamma^2\lambda\mu_2\right.\\
&\quad + \gamma\lambda^2\mu_2 + 2\gamma^2\mu_1\mu_2 + 4\gamma\lambda\mu_1\mu_2 + \gamma\mu_1^2\mu_2 + \gamma^2\mu_2^2 + \gamma\lambda\mu_2^2 + \gamma\mu_1\mu_2^2\\
&\quad - \left(\gamma^2\left(-4\lambda\mu_1\left(\gamma^2+\lambda^2+\gamma(2\lambda+\mu_1)\right)(\mu_1-\mu_2)(\gamma+\lambda+\mu_2) + \left(\gamma^2(\mu_1+\mu_2)\right.\right.\right.\\
&\quad + \mu_1\mu_2(\mu_1+\mu_2) + \lambda^2(2\mu_1+\mu_2) + \lambda\mu_2(4\mu_1+\mu_2)\\
&\quad \left.\left.\left.\left. + \gamma\left((\mu_1+\mu_2)^2 + \lambda(3\mu_1+2\mu_2)\right)\right)^2\right)\right)^{0.5}\right),\\
\tau_1 &= ((-1+x_{2,4})\lambda + x_{2,4}(\mu_1+\mu_2))/(x_{2,4}(x_{2,4}\gamma+\lambda)),\\
\tau_2 &= (\mu_2 + (\gamma+\lambda+\mu_1)(\mu_1/(\gamma+\lambda+\mu_2) - \tau_1))/(x_{2,4}\gamma+\lambda),\\
\tau_3 &= \left(\lambda^{q_2-1}(\gamma+\lambda)^{q_2}\right)/\left(\gamma^{q_2-1}\mu_1^{q_2}\right);\\
\tau_4 &= x_{1,3}^{q_2-1}(\gamma+\lambda+\mu_2),\\
\tau_5 &= \left(\lambda^{q_2-1}(\gamma+\lambda)^{q_2-1}\right)/\left(\gamma^{q_2-1}\mu_1^{q_2-1}\right),\\
\tau_6 &= y_{1,3}^{q_2-2},\\
\tau_7 &= \mu_1/(\gamma+\lambda+\mu_2),\\
\tau_8 &= x_{1,3}^{q_2-2},\\
\tau_9 &= \mu_1^{q_2-2},\\
\tau_{10} &= \left(\lambda^{q_2-2}(\gamma+\lambda)^{q_2-1}\right)/\gamma^{q_2-2},\\
\tau_{11} &= (\lambda^{q_2}(\gamma+\lambda)^{q_2})/\left(\gamma^{q_2}\mu_1^{q_2}\right),\\
\tau_{12} &= (\gamma+\lambda+\mu_1)/\mu_1,\\
\tau_{13} &= (-1+\tau_{11})/(-1+(\lambda(\gamma+\lambda))/(\gamma\mu_1)),\\
\tau_{14} &= (\gamma+\lambda)/\mu_1,
\end{aligned}
$$

and G_{11}, G_{12}, G_{14}, G_{21}, GB_{22}, G_{24}, GB_{13} and GB_{14} are expressed as long equations in the e-companion [25].

4. Numerical results

4.1 *Comparing computational methods*

We compare the execution times of a method presented in [22] with our closed-form solution with parameters $\lambda = 2.2$, $\gamma = 16.5$, $\mu_1 = 2.6$ and $\mu_2 = 0.3$. Mathematica scripts were written and were executed in a machine with Intel Xeon E5410 2.33 GHz processor to produce results.

The execution times vs q_2 are depicted in figure 1. The execution times of a closed-form solution are independent

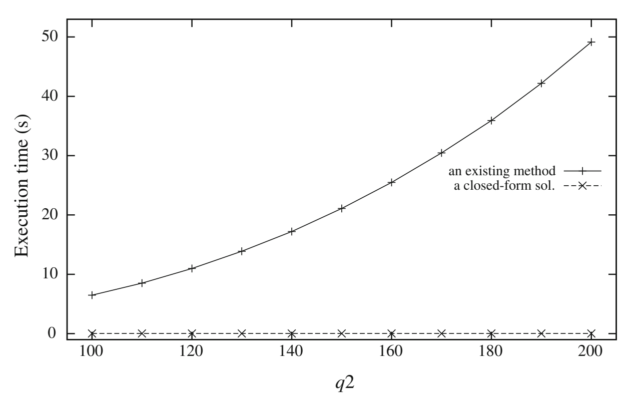

Figure 1. Comparison of computation methods.

from q_2 (see figure 2), while the execution times of a method by [22] rapidly increase. We get results for $q_2 = 4 \times 10^4$ when no result can be obtained with a method presented in [22].

Figure 3 shows the average time spent by customers in the system for the q_2 interval from 10 to 700 when the first server is (60,80,100,120)-fold faster than the second server. It can be observed that the average system time of customers can be minimized by the appropriate choice of q_2. It is also observed from table 1 that $r = \mu_1/\mu_2$ has a small impact on the choice of q_2.

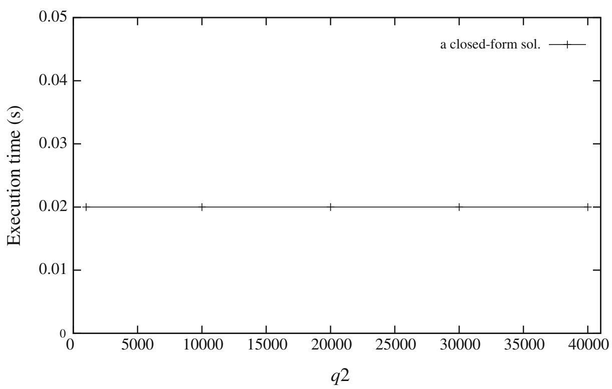

Figure 2. The execution times of a new method.

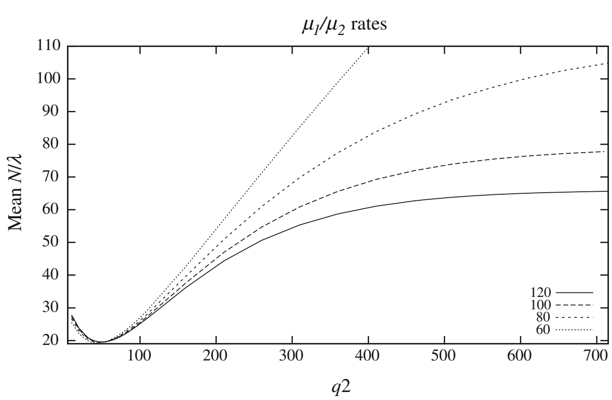

Figure 3. The average system time vs $r = \mu_1/\mu_2$.

Table 1. q_2 value when the average system time has the lowest value for $r = \mu_1/\mu_2$.

$r = \mu_1/\mu_2$	q_2	The average system time
60	44	19.259
80	46	19.4357
100	48	19.5188
120	50	19.5629

5. Conclusions

We have provided closed-form equations for the steady state probabilities and the performance measures of a two-server retrial queue with the threshold policy. Numerical results clearly demonstrate the advantage of the new method over the existing method.

The operation mode considered in this paper can be used to model a practical situation related to the application of two physical servers to provide IT service. The investigation will be our future work.

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