DE GRUYTER Forum Math. 2015; aop

Research Article

Giovanni Falcone and Ágota Figula*

The action of a compact Lie group on nilpotent Lie algebras of type $\{n, 2\}$

Abstract: We classify finite-dimensional real nilpotent Lie algebras with 2-dimensional central commutator ideals admitting a Lie group of automorphisms isomorphic to $SO_2(\mathbb{R})$. This is the first step to extend the class of nilpotent Lie algebras $\mathfrak{h}$ of type $\{n, 2\}$ to solvable Lie algebras in which $\mathfrak{h}$ has codimension one.

Keywords: Nilpotent Lie algebras, pair of alternating forms

MSC 2010: 17B05, 17B30, 15A63

DOI: 10.1515/forum-2014-0170
Received September 30, 2014; revised April 10, 2015

Communicated by: Karl Strambach

1 Introduction

The most simple (non-abelian) Lie algebras are the $(2n + 1)$-dimensional Heisenberg Lie algebras, defined on a vector space $\mathfrak{h} = V \oplus \langle x \rangle$ by a non-degenerate alternating form F on the $2n$-dimensional subspace V, putting $[u, v] = F(u, v)x$ for any $u, v \in V$.

According to the literature beginning with Vergne [9], metabelian Lie algebras $\mathfrak{h} = V \oplus \langle x, y \rangle$ of dimension $(n + 2)$ defined by a pair of alternating forms F_1, F_2 on the n-dimensional vector space V, putting $[u, v] = F_1(u, v)x + F_2(u, v)y$ for any $u, v \in V$, are called nilpotent Lie algebras of type $\{n, 2\}$, where the *type* $\{p_1, \dots, p_c\}$ of a nilpotent Lie algebra $\mathfrak{g}$ with descending central series $\mathfrak{g}^{(i)} = [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$ is defined by the integers

$$p_i = \dim \frac{\mathfrak{g}^{(i-1)}}{\mathfrak{g}^{(i)}}.$$

Nilpotent (real or complex) Lie algebras of type $\{n, 2\}$ have been classified firstly by Gauger [4], applying the canonical reduction of the pair F_1, F_2. We mention that also nilpotent Lie algebras of type $\{n, 1, 1\}$ can be explicitly described (cf. [1]). According to results of Belitskii, Lipyanski, and Sergeichuk [3], this line of investigation cannot be carried further. A possible way of broadening these families of Lie algebras appears therefore to be that of considering their derivations.

In this paper, we want to study derivations of a real nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$, whereas derivations of a nilpotent Lie algebra of type $\{n, 1, 1\}$ are being considered in another paper [2].

As a first question, we ask whether $\mathfrak{h}$ admits a compact Lie algebra of derivations, that is, the Lie algebra of a compact Lie group. This question is interesting for the study of the isometry groups of homogeneous nilmanifolds. Considering an invariant inner product on $\mathfrak{h}$, the compact Lie algebra of derivations of $\mathfrak{h}$ belongs to the Lie algebra of the isometry group of a simply connected nilmanifold associated to $\mathfrak{h}$ (cf. [10], p. 337). Since a non-commutative simple compact Lie algebra cannot have a 2-dimensional representation, a non-commutative simple compact Lie algebra of derivations of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ must induce the null map on the 2-dimensional commutator ideal $\mathfrak{h}'$.

Giovanni Falcone: Dipartimento di Matematica e Informatica, via Archirafi 34, 90123 Palermo, Italy,
e-mail: giovanni.falcone@unipa.it
***Corresponding author: Ágota Figula:** Institute of Mathematics, University of Debrecen, 4010 Debrecen, P.O. Box 12, Hungary,
e-mail: figula@science.unideb.hu

Up to isomorphisms, the smallest example of a nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ is the 5-dimensional Lie algebra of type $\{3, 2\}$ given by

$$[u_1, u_2] = x, \quad [u_1, u_3] = y$$

(cf. [5, p. 646] for the Lie algebra $L_{5,8}$ and [7, p. 162] for the Lie algebra L_5^1). This Lie algebra does not have a non-commutative compact Lie algebra of derivations, since its derivations inducing the null map on $\mathfrak{h}'$ are defined with respect to the basis $\{u_1, u_2, u_3, x, y\}$ by the matrices

$$\left(\begin{array}{c|cc|cc} a & 0 & 0 & 0 & 0 \\ \hline b & -a & 0 & 0 & 0 \\ c & 0 & -a & 0 & 0 \\ \hline d_1 & d_2 & d_3 & 0 & 0 \\ d_4 & d_5 & d_6 & 0 & 0 \end{array}\right).$$

But its group of automorphisms contains the group $\mathrm{SO}_2(\mathbb{R})$, acting as in Theorem 4.1, that is, as the group of automorphisms of the form $\exp(0 \oplus it \oplus it)$ for $t \in \mathbb{R}$.

In general, the structure of a Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$ is not particularly rigid and the following example shows that, as soon as $n = 4$, the algebra of derivations contains compact simple subalgebras.

Example 1.1. Let $\mathcal{B} = \{u_1, u_2, u_3, u_4, x, y\}$ be a basis of the 6-dimensional Lie algebra $\mathfrak{h}$ of type $\{4, 2\}$ defined by

$$[u_1, u_3] = x, \quad [u_1, u_4] = -y, \quad [u_2, u_3] = y, \quad [u_2, u_4] = x$$

(cf. [5, p. 647] for the Lie algebra $L_{6,22}$ $(\epsilon = -1)$ and [7, p. 168] for the Lie algebra n. 5 $(\gamma = -1)$). A direct computation shows that, with respect to the basis $\mathcal{B}$, the derivations of $\mathfrak{h}$ inducing the null map on $\mathfrak{h}'$ are represented by matrices of the form

$$\left(\begin{array}{cccc|cc} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ -a_2 & a_1 & a_4 & -a_3 & 0 & 0 \\ -b_2 & c_2 & -a_1 & a_2 & 0 & 0 \\ c_2 & b_2 & -a_2 & -a_1 & 0 & 0 \\ \hline d_1 & d_2 & d_3 & d_4 & 0 & 0 \\ d_5 & d_6 & d_7 & d_8 & 0 & 0 \end{array}\right).$$

With $a_1 = 0$, $c_2 = -a_4$, $b_2 = a_3$ and all the entries d_i equal to zero, we get an algebra isomorphic to the compact real form $\mathfrak{su}_2$. This Lie algebra $\mathfrak{h}$ is isomorphic to the Lie algebra of the complex Heisenberg group

$$N = \left\{ \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{C} \right\},$$

which is an interesting example of nilmanifolds. Namely, the group N is the smallest H-type group with 2-dimensional centre (cf. [8, Section 5]).

This and the fact that any maximal compact subgroup of a connected real solvable Lie group is a torus, allow one, in our opinion, to study the action of a torus on nilpotent Lie algebras of type $\{n, 2\}$.

The simple structure of a nilpotent Lie algebra of type $\{n, 2\}$ admitting a 1-dimensional compact group T of automorphisms is hidden by three obstacles that can be removed by a clear notation. The first is the representation of $2h \times 2k$ real matrices as $h \times k$ matrices with coefficients in the algebra of split-quaternions. The second is the reduction to canonical form of a pair of skew-Hermitian forms. The third is the reduction to the *T-indecomposable* case. In Section 2, we summarize some known facts, fix the notation and derive the relations (2.7), that are basic for the classification. Thereafter, in Section 3, we consider the case where T induces the identity on the commutator ideal. It turns out that the classification depends on the reduction to canonical form of pairs of skew-Hermitian forms (cf. Theorem 3.1). In Section 4, we consider the case where T operates effectively on the commutator ideal. In this case, the classification is more rich and we consider four cases. The classification is always carried out by using parameters that are linked to the dimension of the eigenspaces of T and by a certain arbitrariness in the reduction to the echelon form of the blocks of the matrices describing the Lie algebra $\mathfrak{h}$. In one case (cf. Theorem 4.3), a class is possibly given by the real form of an arbitrary skew-symmetric complex matrix. Our classes give remarkable examples for nilmanifolds M such that the group of isometries of M contains the compact group $SO_2(\mathbb{R})$. In Theorem 3.1 and Remark 3.3, we describe the H-type groups with 2-dimensional centre. They are precisely the complex Heisenberg groups (cf. [8, Section 5]).

2 Notation

2.1 Split-quaternions and matrix notation

We denote by 0 any $n \times m$ (real or complex) zero matrix, by I_m the (real or complex) m-dimensional identity matrix, and by $\widetilde{I}_{n\times m}$ an $n \times m$ matrix having rank m, obtained by the identity matrix I_m by inserting $n - m$ zero rows (without specifying, however, which ones). By

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we denote the real matrix corresponding to the imaginary unit, by A' the transpose of A, and by $A^\dagger$ the conjugate transpose of A. Finally, by $A \oplus B$ we denote the diagonal block matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

and by $(\oplus A)$ we denote the diagonal block matrix $A \oplus \cdots \oplus A$.

Throughout the paper, we represent the Clifford algebra of split-quaternions as the set

$$\mathbb{H}_- = \{z_1 + z_2\omega : z_i \in \mathbb{C}, \omega z = \bar{z}\omega, \omega^2 = 1\}.$$

We recall that, through the usual identification of the complex number $z = a + ib$ with the real matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

and of the reflection

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with the split-quaternion ω, one obtains an isomorphism

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(\frac{a+d}{2} + i\frac{b-c}{2}\right) + \left(\frac{b+c}{2} + i\frac{a-d}{2}\right)\omega$$

of the algebra of real 2×2 matrices with the algebra $\mathbb{H}_-$ and, more generally, of the space of $\mathbb{R}^{2n\times 2m}$ matrices with the space of $\mathbb{H}_-^{n\times m}$ matrices.

In more detail, with the identification of the split-quaternion matrix ωI_n with the $2n \times 2n$ reflection $\Omega_{2n} = \Omega \oplus \cdots \oplus \Omega$, any matrix $A \in \mathbb{R}^{2n\times 2m}$ can be written in a unique way as $A = A_1 + A_2\Omega_{2m}$, where A_1 and A_2 are *real forms* of complex matrices $\widehat{A}_1 = (z_{ij}), \widehat{A}_2 = (u_{ij}) \in \mathbb{C}^{n\times m}$ such that for $\overline{\widehat{A}}_1 = (\bar{z}_{ij})$ and $\overline{\widehat{A}}_2 = (\bar{u}_{ij})$, one has $\omega I_n \widehat{A}_i = \overline{\widehat{A}}_i\, \omega I_m$ for $i = 1, 2$.

2.2 Canonical form of a pair of skew-Hermitian forms

Since a bilinear form F is skew-Hermitian precisely when iF is Hermitian, the problem of the simultaneous reduction to canonical form of a pair of skew-Hermitian forms reduces to the one concerning a pair of Hermitian forms, which has a long history that was finally presented in [6], which is our reference for what follows.

A pair (H_1, H_2) of Hermitian matrices can be reduced, by simultaneous congruence $H_i \mapsto A^\dagger H_i A$, into the direct sum of diagonal blocks which have one of the four types (other than the null pair)

$$\begin{cases} \text{(i)} & (\pm F_\epsilon, \pm G_\epsilon), \\ \text{(ii)} & (\pm(\alpha F_\epsilon + G_\epsilon), \pm F_\epsilon), \\ \text{(iii)} & \left(G_{2\epsilon+1}, \begin{pmatrix} 0 & 0 & F_\epsilon \\ 0 & 0 & 0 \\ F_\epsilon & 0 & 0 \end{pmatrix} \right), \\ \text{(iv)} & \left(\begin{pmatrix} 0 & \beta F_\epsilon + G_\epsilon \\ \bar\beta F_\epsilon + G_\epsilon & 0 \end{pmatrix}, \begin{pmatrix} 0 & F_\epsilon \\ F_\epsilon & 0 \end{pmatrix} \right), \end{cases} \tag{2.1}$$

where F_ϵ is the matrix that maps $(x_1, x_2, \dots, x_{\epsilon-1}, x_\epsilon)$ onto $(x_\epsilon, x_{\epsilon-1}, \dots, x_2, x_1)$, G_ϵ is the matrix that maps $(x_1, x_2, \dots, x_{\epsilon-1}, x_\epsilon)$ onto $(x_{\epsilon-1}, x_{\epsilon-2}, \dots, x_1, 0)$, α is a real number and β is a complex non-real number. With the only exception of changing β with $\bar\beta$, different values of the parameters α and β, and of the sign $\pm$, correspond to pairs that are not congruent.

2.3 *T*-indecomposable nilpotent Lie algebra of type $\{n, 2\}$

Let $\mathfrak{h}$ be a nilpotent Lie algebra of type $\{n, 2\}$, that is, with a 2-dimensional commutator ideal $\mathfrak{h}'$ coinciding with the centre $\mathfrak{z}$. Let T be a group of automorphisms of $\mathfrak{h}$ isomorphic to $SO_2(\mathbb{R})$ and $\mathfrak{t}$ the corresponding compact algebra of derivations of $\mathfrak{h}$. We let $n = 2m$ if n is even and $n = 2m + 1$ if n is odd, with $n \geq 3$. By the complete reducibility of T, we find a basis $\{e_1, \dots, e_n, x, y\}$ of $\mathfrak{h}$ such that $\{x, y\}$ is a basis of $\mathfrak{h}' = \mathfrak{z}$ and such that $\mathfrak{t}$ operates on $\mathfrak{h}$ as the algebra of matrices $\partial(t)$ with parameter $t \in \mathbb{R}$, defined as

$$\partial(t) := \begin{cases} (\alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) \oplus \beta t \cdot J & \text{for } n = 2m, \\ (0 \oplus \alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) \oplus \beta t \cdot J & \text{for } n = 2m + 1, \end{cases} \tag{2.2}$$

where $\beta t \cdot J$ is the 2×2 matrix operating on $\mathfrak{h}' = \langle x, y \rangle$. Notice that, up to rescaling the parameter t, we can assume that either $\beta = 0$ or $\beta = 1$. Moreover, up to interchanging the basis vector of each T-invariant plane in $\mathfrak{h}$, we can assume that α_i is non-negative for all $i = 1, \dots, m$ and, up to interchanging the ordering of the planes in the basis, we can assume that $\alpha_i \leq \alpha_{i+1}$ for all $i = 1, \dots, m - 1$.

If $\mathfrak{h}$ contains two proper ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ which are invariant under T such that $[\mathfrak{i}_1, \mathfrak{i}_2] = 0$ and $\mathfrak{i}_1 \cap \mathfrak{i}_2 = \mathfrak{h}'$, we say that $\mathfrak{h}$ is *T-decomposable* into the direct sum of $\mathfrak{i}_1$ and $\mathfrak{i}_2$ with amalgamated centre and we restrict our interest on T-indecomposable Lie algebras $\mathfrak{h}$ of type $\{n, 2\}$. Namely, if $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are two T-indecomposable nilpotent Lie algebras of type $\{n, 2\}$ such that the action of the group T on the centre $\mathfrak{z}_1$ of $\mathfrak{h}_1$ and on $\mathfrak{z}_2$ of $\mathfrak{h}_2$ coincides, then the direct sum of $\mathfrak{h}_1$ and $\mathfrak{h}_2$ with amalgamated centre is a T-decomposable nilpotent Lie algebra of type $\{n, 2\}$ and any T-decomposable nilpotent Lie algebra of type $\{n, 2\}$ is obtained in this way.

We define the pair of alternating matrices $(T_1 = (a_{ij}), T_2 = (b_{ij}))$ by putting

$$[e_i, e_j] = a_{ij} x + b_{ij} y.$$

Clearly, the Lie algebra $\mathfrak{h}$ is T-decomposable if and only if the matrices T_1 and T_2 can be put into the same diagonal block form and T leaves invariant the subspaces corresponding to the blocks.

Writing

$$\partial_0(t) := \begin{cases} (\alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) & \text{for } n \text{ even}, \\ (0 \oplus \alpha_1 t \cdot J \oplus \cdots \oplus \alpha_m t \cdot J) & \text{for } n \text{ odd}, \end{cases}$$

for $t \in \mathbb{R}$, since $\mathfrak{t}$ operates as an algebra of derivations of $\mathfrak{h}$, that is,

$$[e_i, e_j]^{\partial(t)} = [e_i^{\partial(t)}, e_j] + [e_i, e_j^{\partial(t)}],$$

for a generator of $\mathfrak{t}$, e.g., for $t = 1$, we get

$$\beta T_2 = \partial_0(1)' T_1 + T_1 \partial_0(1), \quad -\beta T_1 = \partial_0(1)' T_2 + T_2 \partial_0(1). \tag{2.3}$$

Hence, we find

$$\beta^2 T_1 = -\partial_0(1)'^2 T_1 - 2\partial_0(1)' T_1 \partial_0(1) - T_1 \partial_0(1)^2, \quad \beta^2 T_2 = -\partial_0(1)'^2 T_2 - 2\partial_0(1)' T_2 \partial_0(1) - T_2 \partial_0(1)^2. \tag{2.4}$$

We arrange the matrices T_1 and T_2 into 2×2 blocks A_{hk} and B_{hk} with $h, k = 1, \dots, m$ (in the case where $n = 2m + 1$, we denote the 1×2 blocks of the first row with A_{0k} and B_{0k} and we put $\alpha_0 = 0$). Then, (2.3) is equivalent to

$$\beta B_{hk} = -\alpha_h J A_{hk} + \alpha_k A_{hk} J, \quad -\beta A_{hk} = -\alpha_h J B_{hk} + \alpha_k B_{hk} J \tag{2.5}$$

and (2.4) is equivalent to

$$-\beta^2 \cdot A_{hk} = -(\alpha_h^2 + \alpha_k^2) A_{hk} + 2\alpha_h \alpha_k J' A_{hk} J, \quad -\beta^2 \cdot B_{hk} = -(\alpha_h^2 + \alpha_k^2) B_{hk} + 2\alpha_h \alpha_k J' B_{hk} J. \tag{2.6}$$

Notice that since $\alpha_0 = 0$, the above equations still hold, with a slight abuse of notation, in the case where $h = 0$.

Considering T_1 and T_2 as split-quaternion matrices, we write

$$A_{hk} = z_1 + z_2 \omega, \quad B_{hk} = z_3 + z_4 \omega$$

for suitable complex numbers z_1, z_2, z_3, z_4. Then, the equations (2.6) give

$$(\alpha_h - \alpha_k)^2 z_1 = \beta^2 z_1, \quad (\alpha_h - \alpha_k)^2 z_3 = \beta^2 z_3, \quad (\alpha_h + \alpha_k)^2 z_2 = \beta^2 z_2, \quad (\alpha_h + \alpha_k)^2 z_4 = \beta^2 z_4. \tag{2.7}$$

Remark 2.1. Notice that if h is a given index such that for any $k \neq h$ all the blocks A_{hk} and B_{hk} are zero, then $\mathfrak{h}$ is T-decomposable. Moreover, from equations (2.5) it follows that in the case where $\beta \neq 0$, A_{hk} is zero if and only if B_{hk} is zero.

3 The case where $\beta = 0$

This is the case when, in particular, T is contained in a non-commutative compact group of automorphisms of $\mathfrak{h}$ (see Remark 3.4).

Theorem 3.1. *With the above notation, let $\mathfrak{h}$ be a T-indecomposable Lie algebra of type $\{n, 2\}$ and let $\beta = 0$. Then, n is even and the pair (T_1, T_2) is the real form of a pair of complex skew-Hermitian matrices $(\widehat{T_1} = iH_1, \widehat{T_2} = iH_2)$, where, up to a change of basis, (H_1, H_2) is one of the four pairs in* (2.1). *The group T operates, with respect to the chosen basis, as the group of automorphisms* $\exp(\partial(t))$, *where*

$$\partial(t) = t \cdot ((\oplus \alpha_1 J)) \oplus 0.$$

Proof. From equations (2.5) we deduce that if $\alpha_h = 0 \neq \alpha_k$, then A_{hk} and B_{hk} are zero. As $\mathfrak{h}$ is T-indecomposable, it follows from Remark 2.1 that α_h is positive for any $h = 1, \dots, m$ and $n = 2m$ is even. Write $A_{hk} = z_1 + z_2\omega$, $B_{hk} = z_3 + z_4\omega$ for suitable complex numbers z_1, z_2, z_3, z_4. Then, equations (2.7) give

$$(\alpha_h - \alpha_k)^2 z_1 = 0, \quad (\alpha_h - \alpha_k)^2 z_3 = 0, \quad (\alpha_h + \alpha_k)^2 z_2 = 0, \quad (\alpha_h + \alpha_k)^2 z_4 = 0.$$

The latter forces $z_2 = 0 = z_4$, that is, A_{hk} and B_{hk} are the real form of two complex numbers. Moreover, from the former we obtain that either A_{hk} and B_{hk} are zero or $\alpha_h = \alpha_k$. As $\mathfrak{h}$ is T-indecomposable, we exclude the first case, hence we have that $\partial(t) = t \cdot ((\oplus \alpha J)) \oplus 0$. Since T_1 and T_2 are the real form of two complex $m \times m$ skew-Hermitian matrices $\widehat{T_1}$ and $\widehat{T_2}$ and T operates on them as the complex scalar matrix $\alpha i I_m$, up to a suitable change of basis in the m-dimensional complex space, which leaves T invariant, we can assume that $(\widehat{T_1}, \widehat{T_2})$ is in the canonical form given in the claim. These are T-indecomposable over the real numbers, since the only T-invariant real planes are $\Pi_1 = \langle e_1, e_2 \rangle, \dots, \Pi_{n/2} = \langle e_{n-1}, e_n \rangle$. □

Remark 3.2. Up to rescaling the parameter t, we can assume that $\alpha_1 = 1$ in the above theorem, but we prefer not to, because in the case where $\mathfrak{h}$ is T-decomposable, different values of α_k can occur. For the same reason, we do not simplify the case where $(\widehat{T_1}, \widehat{T_2}) = (\pm(\alpha i F_\epsilon + iG_\epsilon), \pm iF_\epsilon)$, which by the change of basis $\{x' = x, y' = \alpha x + y\}$ in $\mathfrak{h}'$ would transform in $(\pm iG_\epsilon, \pm iF_\epsilon)$.

Remark 3.3. For $n = 4$ and $(\widehat{T_1}, \widehat{T_2})$ of type (iv) in (2.1) with $\epsilon = 1$ and $\beta = -i$, we obtain

$$(\widehat{T_1}, \widehat{T_2}) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\right),$$

which corresponds to Example 1.1.

Pairs obtained by the direct sum of t copies of this pair correspond to the $(2t + 1)$-dimensional complex Heisenberg groups, which are special nilmanifolds, the H-type groups with 2-dimensional centre (cf. [8, Section 5]). These are the only nilpotent real Lie algebras of type $\{n, 2\}$ which are complex Lie algebras (of type $\{n/2, 1\}$).

Remark 3.4. Let $\mathfrak{k}$ be a simple compact algebra of derivations of the nilpotent Lie algebra $\mathfrak{h}$ of type $\{n, 2\}$. Since $\mathfrak{k}$ cannot have a 2-dimensional representation, it induces on the 2-dimensional commutator subalgebra $\mathfrak{h}'$ the null map. Any element in $\mathfrak{k}$ generates a 1-dimensional compact subalgebra of derivations of $\mathfrak{h}$, thus $\mathfrak{h}$ has the structure given in Theorem 3.1 and its algebra of derivations can be directly computed.

4 The cases where $\beta \neq 0$

Up to rescaling the parameter t, if $\beta \neq 0$, then we can assume that $\beta = 1$. From now on, we need to distinguish the cases where the smallest coefficient α_h is zero or, respectively, smaller, equal or greater than 1/2. The arguments are more or less the ones we give in the following theorem.

Theorem 4.1. *With the notation given in (2.2), if $\beta = 1$ and if the smallest coefficient α_h is zero, then, with respect to a suitable basis of $\mathfrak{h}$, the group T operates as the group of automorphisms $\exp(\partial(t))$, where*

$$\partial(t) = t \cdot ((\oplus 0) \oplus (\oplus J) \oplus (\oplus 2J) \oplus \cdots \oplus (\oplus lJ)) \oplus t \cdot J \tag{4.1}$$

with the diagonal blocks $(\oplus kJ)$ of dimension $d_k \times d_k$ (with d_k even for $k > 0$) and the T-indecomposable Lie algebra $\mathfrak{h}$ is described by the pair (T_1, T_2), where

$$T_1 = \left(\begin{array}{c|c|c|c|c|c} 0 & W_0 & & & & \\ \hline -W_0' & 0 & W_1 & & & \\ \hline & -W_1' & 0 & W_2 & & \\ \hline & & -W_2' & \ddots & \ddots & \\ \hline & & & \ddots & 0 & W_l \\ \hline & & & & -W_l' & 0 \end{array}\right) \tag{4.2}$$

and T_2 has the same shape with the blocks $W_k(\oplus J)$ instead of W_k. The blocks W_k have dimension $d_k \times d_{k+1}$ and the following hold.

(i) *The block W_0 can be written into the echelon form*

$$W_0 = \left(\begin{array}{c|c|cc|c} 0 & \oplus L_1 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{2s} & 0 & \Omega_{2s} \\ 0 & 0 & 0 & I_{2t} & 0 \end{array}\right),$$

where $L_1 = (1, 0)$ and Ω_{2s} is the real form of the split-quaternion matrix ωI_s, $s \geq 0$, $t \geq 0$, such that the first zero columns as well as the blocks L_1 can be missing and $W_0 \neq 0$ has no zero rows.

(ii) *For any $k > 0$, the block W_k is the real form of a complex matrix $\widehat{W}_k$ that can be reduced to the* almost echelon *form $\widehat{W}_k = (0 \,|\, \widetilde{I}_{r_k})$ (and, in particular, $\widehat{W}_l = \widetilde{I}_{r_l}$), where $\widetilde{I}_{r_k}$ is obtained by the complex identity matrix by possibly adding zero rows.*

Moreover, the blocks W_k are such that there do not exist two successive columns of T_1 of indices $2j-1, 2j$ which are both zero.

Proof. Recall that we have chosen a basis such that $0 \le \alpha_i \le \alpha_{i+1}$ and that, by equations (2.5), we get

$$A_{hk} = 0 \text{ if and only if } B_{hk} = 0.$$

If the coefficient α_h is zero, then equations (2.6) become

$$-A_{hk} = -\alpha_k^2 A_{hk}, \quad -B_{hk} = -\alpha_k^2 B_{hk}$$

and we see that if $\alpha_k \neq 1$, then $A_{hk} = B_{hk} = 0$. As $\mathfrak{h}$ is T-indecomposable, by Remark 2.1 we find that the smallest non-zero coefficients must be equal to 1. For the same reason, the next possible coefficients must be equal to 2 and so on, that is, the derivations must be of the form given in (4.1).

Moreover, equations (2.7) for $A_{hk} = z_1 + z_2\omega$, which with $\beta = 1$ become

$$(\alpha_h - \alpha_k)^2 z_1 = z_1, \quad (\alpha_h + \alpha_k)^2 z_2 = z_2,$$

show that $A_{hk} \neq 0$ only if $|\alpha_h - \alpha_k| = 1$ and that A_{hk} is the real form of a non-zero complex number as soon as $0 < \alpha_h = \alpha_k - 1$. Thus, T_1 must be of the form given in (4.2) and the blocks W_i are the real form of complex matrices $\widehat{W}_i$ for $i > 0$. By the first of equations (2.5), which for $\beta = 1$ gives

$$B_{hk} = -\alpha_h \cdot JA_{hk} + \alpha_k \cdot A_{hk}J,$$

the block of T_2 corresponding to $\alpha_h = 0$ and $\alpha_k = 1$ is equal to $W_0(\oplus J)$ and the same holds for the blocks corresponding to $0 < \alpha_h = \alpha_k - 1$ because in these cases A_{hk} commutes with J, hence they are of the form $W_i(\oplus J)$ also for $i > 0$.

Consider a basis change block diagonal matrix of the form

$$X = (X_0 \oplus X_1 \oplus \cdots \oplus X_{l+1}) \oplus I_2,$$

where the blocks X_i are $d_i \times d_i$ matrices that are the real form of complex matrices $\widehat{X}_i$ for $i > 0$, and notice that it leaves the derivations invariant and changes the block W_i of T_1 into the block $X_i' W_i X_{i+1}$ (and the corresponding block of T_2, accordingly).

In order to reduce the blocks W_i into the form given in the claim, we now perform the following algorithm.

(i) Starting from the last block W_l and working upward one by one, by left multiplication $X_i' W_i$ with a suitable matrix X_i, we can assume that the blocks W_i are reduced to *lower* echelon form, that is, such that the pivot of the h-th row is on the *right* of the pivot of the k-th row for $h < k$. Moreover, we annihilate the entries also below any pivot (as well as above it). The matrices X_0 and W_0 are not necessarily the real forms of complex matrices for $i = 0$. On the contrary, they are for $i > 0$.

(ii) Let $i > 0$ and proceed downward. In order to reduce to zero all the row entries which are on the right of any pivot, we operate on the real form W_i of the complex matrix $\widehat{W}_i$ by adding to a given (complex) column a linear combination of the *previous* (complex) columns of W_i, that is, by the multiplication

$$\begin{pmatrix} W_i \\ \hline 0 \\ \hline -W_{i+1}' \end{pmatrix} X_{i+1}$$

with a suitable matrix, that we still indicate as X_{i+1}, which is the real form of an upper triangular complex matrix with any diagonal (complex) entry equal to I and this operation did not change the lower echelon form of W_{i+1}. Notice, in fact, that the transpose of a lower echelon matrix is still a lower echelon matrix. Thus, we can assume that the columns of all blocks W_i are either zero or vectors from the canonical basis (taken in the reverse order).

(iii) We have to distinguish the case where $i = 0$ because W_0 is not necessarily the real form of a complex matrix but X_1 is such, hence the real matrix X_1 operates on *pairs* of columns with indices $2j-1, 2j$. As W_0 is a lower echelon matrix with zeros above and below any pivot, considering the pivot of a row and its position with respect to the pivot of the previous row, we have the three cases

$$\left(\begin{array}{c|cc|c} \cdots & 0 & 1 & \cdots \\ \cdots & 1 & 0 & \cdots \end{array}\right), \quad \left(\begin{array}{c|cc|c} \cdots & 0 & 0 & \cdots \\ \cdots & 1 & a & \cdots \end{array}\right), \quad \left(\begin{array}{c|cc|c} \cdots & 0 & 0 & \cdots \\ \cdots & 0 & 1 & \cdots \end{array}\right).$$

As any row is virtually the second row of a two-row real form of a single complex row, in the last two cases the second row can be reduced to a vector e_{2i+1} of the real canonical basis by multiplying on the right with the real form X_1 of an upper triangular complex matrix $\widehat{X}_1$ which has the identity in any (complex) diagonal entry but the one corresponding to the pivot, which has to be

$$\frac{1}{1+a^2}\begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{4.3}$$

respectively. In order to show that we can assume that also the entries below $(1, 0)$ are zero, we consider, for instance, the minimal case of the matrix

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & a \\ 1 & 0 & 0 & b \end{array}\right).$$

The following multiplications, first on the left

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -b & 1 & 0 \\ a & 0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & a \\ 1 & 0 & 0 & b \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & -b & a \\ 1 & 0 & a & b \end{array}\right)$$

with a real matrix and second on the right

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & -b & a \\ 1 & 0 & a & b \end{array}\right)\left(\begin{array}{cc|cc} 1 & 0 & -a & -b \\ 0 & 1 & b & -a \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

with the real form of a complex matrix, show therefore that the block $L_1 = (1, 0)$ has only zero blocks on left and right, and above and below.

We are left with the first case

$$\left(\begin{array}{c|cc|c} \cdots & 0 & 1 & \cdots \\ \cdots & 1 & 0 & \cdots \end{array}\right),$$

which is the case where the two rows can be seen as a row with entries in the algebra $\mathbb{H}_-$ of split-quaternions and with pivot ω. By multiplying on the right with the real form of an upper triangular complex matrix which has the identity in any (complex) diagonal entry, we reduce each non-zero entry $z_1 + z_2\omega$ to z_1 and, by another multiplication, we can assume that only the most leftward entry z_1 is non-zero in the row. Finally, we reduce it to 1 by multiplying on the right with the real form of an upper triangular complex matrix which has z_1^{-1} in the suitable (complex) diagonal entry.

Thus, also in the case $i = 0$, we can assume that the columns of the block W_0 are either zero or vectors from the canonical basis. Moreover, this operation did not change the lower echelon form of W_1, however some row has been multiplied by a complex scalar at point (4.3) and in the above reduction of z_1 to 1. These rows can be reduced again to vectors of the canonical basis by multiplying on the right with a suitable complex diagonal matrix and so on with the successive blocks W_i.

(iv) Starting again from the last block and working upward, we change now the lower echelon blocks into upper echelon blocks by multiplying on the left with the suitable permutation matrix. We still indicate the blocks by W_k.

(v) Starting now from the first block W_0, by multiplying on the left by a real permutation matrix Q_0' and on the right by a complex permutation matrix Q_1, in the most general case we reduce it to the form

$$Q_0' W_0 Q_1 = \left(\begin{array}{c|c|cc|c} 0 & \oplus L_1 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{2s} & 0 & \Omega_{2s} \\ 0 & 0 & 0 & I_{2t} & 0 \end{array}\right),$$

where Ω_{2s} is the real form of the split-quaternion matrix ωI_s, $s \geq 0$, $t \geq 0$, and the first zero columns as well as the blocks L_1 can be missing (notice that W_0 cannot have zero rows). The second block is now $Q_1' W_1$ and its (complex) columns are vectors from the canonical basis, in the order permuted by the multiplication by Q_1'. The zero rows are not necessarily at the bottom now and we cannot move them to the bottom without permuting the columns of W_0. On the contrary, by multiplying on the right by a complex permutation matrix Q_2, we can permute the columns and reduce W_1 to the *almost echelon* form $\widehat{W}_1 = (0 \,|\, \widetilde{I}_{r_1})$, where $\widetilde{I}_{r_1}$ is obtained by the complex identity matrix by adding zero rows. Going down, we cannot move the rows of $Q_2' W_2$ without permuting the columns of W_1, but we can permute the columns and reduce $\widehat{W}_2$ to the form $\widehat{W}_2 = (0 \,|\, \widetilde{I}_{r_2})$, where $\widetilde{I}_{r_2}$ is obtained by the complex identity matrix by adding zero rows.

The claim follows after repeating the same argument till the last block $\widehat{W}_l$, which, in particular, will be reduced to the form $\widetilde{I}_{r_l}$ because it cannot have zero (complex) columns. □

Theorem 4.2. *With the notation given in* (2.2), *if* $\beta = 1$ *and if the smallest coefficient* α_h *is greater than* $1/2$, *then, with respect to a suitable basis of* $\mathfrak{h}$, *the group* T *operates as the group of automorphisms* $\exp(\partial(t))$, *where*

$$\partial(t) = t \cdot ((\oplus \alpha J) \oplus (\oplus(\alpha+1)J) \oplus \cdots \oplus (\oplus(\alpha+l)J)) \oplus t \cdot J,$$

with the diagonal blocks $(\oplus(\alpha+k-1)J)$ *of dimension* $d_k \times d_k$ *(with* d_k *even), and the* T*-indecomposable Lie algebra* $\mathfrak{h}$ *is described by the pair* (T_1, T_2), *where*

$$T_1 = \left(\begin{array}{c|c|c|c|c|c} 0 & W_1 & & & & \\ \hline -W_1' & 0 & W_2 & & & \\ \hline & -W_2' & 0 & W_3 & & \\ \hline & & -W_3' & \ddots & \ddots & \\ \hline & & & \ddots & 0 & W_l \\ \hline & & & & -W_l' & 0 \end{array}\right)$$

and T_2 *has the same shape with the blocks* $W_k(\oplus J)$ *instead of* W_k. *The block* W_k *has dimension* $d_k \times d_{k+1}$ *and is the real form of a complex matrix* $\widehat{W}_k$ *that can be reduced to the* almost echelon *form* $\widehat{W}_k = (0 \,|\, \widetilde{I}_{r_k})$ *(and, in particular,* $\widehat{W}_1 = (0 \,|\, I_{r_1})$ *and* $\widehat{W}_l = \widetilde{I}_{r_l}$*), where* $\widetilde{I}_{r_k}$ *is obtained by the complex identity matrix by adding zero rows and such that no two successive columns of* T_1 *of indices* $2j-1, 2j$ *are both zero.*

Proof. Let the smallest coefficient α_h be greater than $1/2$ and let $\alpha_k \neq \alpha_h$. By equations (2.7), if $1 - \alpha_h \neq \alpha_k \neq 1 + \alpha_h$, then $A_{hk} = B_{hk} = 0$. Since $\mathfrak{h}$ is T-indecomposable, we have that the closest coefficients are $\alpha_k = 1 - \alpha_h$ or $\alpha_k = 1 + \alpha_h$. But, since $\alpha_h > 1/2$, we have that $1 - \alpha_h < \alpha_h$, thus $\alpha_k = 1 - \alpha_h$ would be a contradiction to the minimality of α_h. Therefore, in this case, the coefficients are α_h, $1 + \alpha_h$, $2 + \alpha_h$ and so on. The claim follows using the same arguments as in Theorem 4.1. In this case, also the first block W_1 is the real form of a complex matrix $\widehat{W}_1$. □

The following case is, somehow, exceptional. In fact, in addition to the various choices of $\widetilde{I}_r$, here it is possible that the real form of an arbitrary skew-symmetric complex matrix gives a class of T-indecomposable Lie algebras $\mathfrak{h}$.

Theorem 4.3. *With the notation given in* (2.2), *if* $\beta = 1$ *and if the smallest coefficient* α_h *is equal to* $1/2$, *then, with respect to a suitable basis of* $\mathfrak{h}$, *the group* T *operates as the group of automorphisms* $\exp(\partial(t))$, *where*

$$\partial(t) = t \cdot \left(\left(\oplus \frac{1}{2} J\right) \oplus \left(\oplus \frac{3}{2} J\right) \oplus \cdots \oplus \left(\oplus \frac{2l+1}{2} J\right)\right) \oplus t \cdot J.$$

If we denote by $d_k \times d_k$ (d_k even) the dimension of the block

$$\left(\oplus \frac{2k-1}{2} J\right),$$

then the T-indecomposable Lie algebra $\mathfrak{h}$ is described by the pair (T_1, T_2) with

$$T_1 = \left(\begin{array}{c|c|c|c|c|c} \Omega_{d_1} H & W_1 & & & & \\ \hline -W_1' & 0 & W_2 & & & \\ \hline & -W_2' & 0 & W_3 & & \\ \hline & & -W_3' & \ddots & \ddots & \\ \hline & & & \ddots & 0 & W_l \\ \hline & & & & -W_l' & 0 \end{array}\right), \tag{4.4}$$

where Ω_{d_1} is the real form of the split-quaternion matrix $\omega I_{d_1/2}$ and T_2 has the same shape with the blocks $\Omega_{d_1} H(\oplus J)$ and $W_k(\oplus J)$ instead of $\Omega_{d_1} H$ and W_k such that the following hold.

(i) *H and W_1 are the real form of the complex matrices*

$$\widehat{H} = \left(\begin{array}{c|cc} \widehat{H}_0 & 0 & 0 \\ \hline 0 & 0 & I_r \\ 0 & -I_r & 0 \end{array}\right), \quad \widehat{W}_1 = \left(\begin{array}{c|c|c} 0 & 0 & I_s \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

respectively, with $s \geq 0$, $r \geq 0$, $d_1 = 2(s + 2r)$ and $\widehat{H}_0$ is a $s \times s$ skew-symmetric complex matrix.

(ii) *For any $k > 1$, the block W_k of dimension $d_k \times d_{k+1}$ is the real form of a complex matrix $\widehat{W}_k$ that can be reduced to the* almost echelon *form $\widehat{W}_k = (0 \,|\, \widetilde{I}_{r_k})$ (and, in particular, $\widehat{W}_l = \widetilde{I}_{r_l}$), where $\widetilde{I}_{r_k}$ is obtained by the complex identity matrix by adding zero rows.*

Moreover, no two successive columns of T_1 of indices $2j-1$, $2j$ are both zero.

Proof. If $\alpha_h = 1/2$, then the assumption $\alpha_k = 1 - \alpha_h$ leads to a contradiction to $\alpha_k \neq \alpha_h$. Thus, also in this case, the distinct coefficients are α_h, $1 + \alpha_h$, $2 + \alpha_h$ and so on. By equations (2.7), we see, however, that if $\alpha_k = \alpha_h$, then $A_{hk} = z_2 \omega$, hence the $d_1 \times d_1$ block corresponding to the values of α_k equal to $1/2$ is the real form of a split-quaternion matrix $\omega \widehat{H}$. Notice that whereas the real form of $\widehat{H}$ is skew-symmetric if and only if $\widehat{H}$ is skew-Hermitian, the real form of $\omega \widehat{H}$ is skew-symmetric if and only if $\widehat{H}$ is skew-symmetric. This forces T_1 and T_2 to be of the form given in (4.4).

With the same argument of Theorem 4.1, we reduce W_i to the almost echelon form $(0 \,|\, \widehat{I}_{r_i})$ and, in particular, $W_l = \widehat{I}_{r_l}$ and, since we can operate on the first row by multiplication on the left with a further matrix, we reduce W_1 to the real form of

$$\widehat{W}_1 = \left(\begin{array}{c|c} 0 & I_s \\ \hline 0 & 0 \end{array}\right).$$

A basis change matrix of the form $X_1 \oplus I_{d_2} \oplus \cdots \oplus I_{d_l}$ transforms the blocks W_1 and $\Omega_{d_1} H$ into $X_1' W_1$ and $X_1' \Omega_{d_1} H X_1$, respectively.

In order to leave invariant the echelon form of W_1, we have to take

$$\widehat{X}_1 = \left(\begin{array}{c|c} I_s & 0 \\ \hline C & D \end{array}\right).$$

Notice now that $X_1' \Omega_{d_1} H X_1$ is the real form of

$$\widehat{X}_1^{\dagger} \omega I_{d_1/2} \widehat{H} \widehat{X}_1 = \omega I_{d_1/2} \widehat{X}_1' \widehat{H} \widehat{X}_1.$$

If we write

$$\widehat{H} = \left(\begin{array}{c|c} H_0 & H_1 \\ \hline -H_1' & H_2 \end{array}\right)$$

with H_0 of dimension $s \times s$, we see that the congruence $\widehat{X}_1' \widehat{H} \widehat{X}_1$ changes H_2 into $D' H_2 D$ and H_1 into $(H_1 + C' H_2) D$.

Thus we can reduce, firstly, H_2 to the canonical form

$$\left(\begin{array}{c|c} 0 & I_r \\ \hline -I_r & 0 \end{array}\right)$$

because H_2 has to be non-degenerate or T_1 would have a zero (complex) row. Finally, since H_2 is non-degenerate, we reduce H_1 to zero, taking $C' = -H_1 H_2^{-1}$. □

Remark 4.4. In the case where

$$\partial(t) = \left(\oplus \frac{1}{2} t \cdot J\right) \oplus t \cdot J,$$

we find that $(T_1, T_2) = (\oplus I_2, \oplus J)^{\nabla}$.

In the following last theorem we will change the ordering of the coefficients α_h defining the derivations $\partial(t)$.

Theorem 4.5. *With the notation given in* (2.2), *if* $\beta = 1$ *and if the smallest coefficient* α_h *is smaller than* $1/2$, *then, with respect to a suitable basis of* $\mathfrak{h}$, *the group* T *operates, in the most general case, as the group of automorphisms* $\exp(\partial(t))$, *where* $\partial(t) = \partial_1(t) \oplus \partial_2(t) \oplus t \cdot J$ *with*

$$\partial_1(t) = t \cdot ((\oplus(l_1 - \alpha)J) \oplus \cdots \oplus (\oplus(2 - \alpha)J) \oplus (\oplus(1 - \alpha)J)), \quad \partial_2(t) = t \cdot ((\oplus \alpha J) \oplus (\oplus(\alpha + 1)J) \oplus \cdots \oplus (\oplus(\alpha + l_2)J))$$

and the T*-indecomposable Lie algebra* $\mathfrak{h}$ *is described by the pair* (T_1, T_2) *with*

$$T_1 = \left(\begin{array}{c|c|c|c|c|c|c|c}
0 & V_{l_1-1} & & & & & & \\ \hline
-V'_{l_1-1} & \ddots & \ddots & & & & & \\ \hline
 & \ddots & 0 & V_1 & & & & \\ \hline
 & & -V'_1 & 0 & \Omega_{d_1} S & & & \\ \hline
 & & & -S'\Omega_{d_1} & 0 & W_1 & & \\ \hline
 & & & & -W'_1 & 0 & \ddots & \\ \hline
 & & & & & \ddots & \ddots & W_{l_2} \\ \hline
 & & & & & & -W'_{l_2} & 0
\end{array}\right), \tag{4.5}$$

where the blocks V_k, W_k *and* S *are the real form of complex matrices* $\widehat{V}_k$, $\widehat{W}_k$ *and* $\widehat{S}$ *that can be reduced to the* almost echelon *form* $\widehat{V}_k = (0 \,|\, \widetilde{I}_{r_i})$, $\widehat{W}_k = (0 \,|\, \widetilde{I}_{s_i})$ *and* $\widehat{S} = (0 \,|\, \widetilde{I}_s)$ *(and, in particular,* $\widehat{V}_{l_1-1} = (0 \,|\, I_{r_{l_1-1}})$ *and* $\widehat{W}_{l_2} = \widetilde{I}_{r_{l_2}}$*), where* $\widetilde{I}_{r_k}$ *is obtained by the complex identity matrix by adding zero rows,* Ω_{d_1} *is the real form of the split-quaternion matrix* $\omega I_{d_1/2}$ *and such that no two successive columns of* T_1 *of indices* $2j - 1, 2j$ *are both zero. The matrix* T_2 *has the same shape as* T_1 *with the blocks* $\Omega_{d_1} S(\oplus J)$, $V_k(\oplus J)$ *and* $W_k(\oplus J)$ *instead of* $\Omega_{d_1} S$, V_k *and* W_k.

Proof. At last, let the smallest coefficient α_h be smaller than $1/2$, thus the closest possible coefficients are $\alpha_k = 1 - \alpha_h$ and $\alpha_k = 1 + \alpha_h$. The latter gives, as above, the coefficients α_h, $1 + \alpha_h$, $2 + \alpha_h$ and so on. The former gives moreover the coefficients $1 - \alpha_h$, $2 - \alpha_h$ and so on, and no other, since $(1 - \alpha_h) - 1$ is negative and $1 - (1 - \alpha_h)$ is again α_h. Thus, in this case, the coefficients are

$$l_1 - \alpha, \ \ldots, \ 2 - \alpha, \ 1 - \alpha, \ \alpha, \ 1 + \alpha, \ 2 + \alpha, \ \ldots, \ l_2 + \alpha.$$

Notice that no coefficient of the form $a - \alpha_h$ can be equal to $b + \alpha_h$ because $\alpha_h < 1/2$. It follows that

(i) if $\alpha_h = a + \alpha$ and $\alpha_k = b + \alpha$, then $\alpha_h + \alpha_k$ is not an integer and, by equations (2.7), we see that A_{hk} is non-zero only if $|b - a| = 1$ and that A_{hk} is the real form of a complex number;

(ii) if $\alpha_h = a + \alpha$ and $\alpha_k = b - \alpha$, then $|\alpha_h - \alpha_k|$ is not an integer and, by equations (2.7), we see that A_{hk} is non-zero only if $a + b = 1$, that is, $\alpha_h = \alpha$ and $\alpha_k = 1 - \alpha$, and that A_{hk} is the real form of a split-quaternion ωz_2;

(iii) if $\alpha_h = a - \alpha$ and $\alpha_k = b - \alpha$, then $\alpha_h + \alpha_k$ is not an integer and, by equations (2.7), we see that again A_{hk} is non-zero only if $|b - a| = 1$ and that A_{hk} is the real form of a complex number.

Thus, T_1 is of the form given in (4.5). With the same arguments as in Theorem 4.1, we can reduce the blocks V_i, W_i and S to the form given in the claim. □

Remark 4.6. In Theorem 4.5, it can happen that $\partial(t) = (\partial_1(t) \oplus (\oplus \alpha t \cdot J)) \oplus t \cdot J$ and

$$T_1 = \left(\begin{array}{c|c|c|c|c} 0 & V_{l_1-1} & & & \\ \hline -V'_{l_1-1} & \ddots & \ddots & & \\ \hline & \ddots & 0 & V_1 & \\ \hline & & -V'_1 & 0 & \Omega_{d_1} S \\ \hline & & & -S'\Omega_{d_1} & 0 \end{array}\right)$$

or that $\partial(t) = \partial_2(t) \oplus t \cdot J$ and

$$T_1 = \left(\begin{array}{c|c|c|c} 0 & W_1 & & \\ \hline -W'_1 & 0 & \ddots & \\ \hline & \ddots & \ddots & W_{l_2} \\ \hline & & -W'_{l_2} & 0 \end{array}\right).$$

Funding: This research was supported by the Università di Palermo (2012-ATE-0446) and by the European Union's Seventh Framework Programme (FP7/2007–2013) under grant agreements no. 317721 and no. 318202.

References

[1] C. Bartolone, A. Di Bartolo and G. Falcone, Nilpotent Lie algebras with 2-dimensional commutator ideals, *Linear Algebra Appl.* **434** (2011), 650–656.
[2] C. Bartolone, A. Di Bartolo and G. Falcone, Solvable extensions of nilpotent complex Lie algebras of type $\{2n, 1, 1\}$, preprint (2015), available from giovanni.falcone@unipa.it.
[3] G. Belitskii, R. Lipyanski and V. V. Sergeichuk, Problems of classifying associative or Lie algebras and triples of symmetric or skew-symmetric matrices are wild, *Linear Algebra Appl.* **407** (2005), 249–262.
[4] M. A. Gauger, On the classification of metabelian Lie algebras, *Trans. Amer. Math. Soc.* **179** (1973), 293–329.
[5] W. A. de Graaf, Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2, *J. Algebra* **309** (2007), 640–653.
[6] P. Lancaster and L. Rodman, Canonical forms for Hermitian matrix pairs under strict equivalence and congruence, *SIAM Rev.* **47** (2005), 407–443.
[7] V. V. Morozov, Classification of nilpotent Lie algebras of sixth order (in Russian), *Izv. Vyssh. Uchebn. Zaved. Mat.* **4** (1958), 161–171.
[8] H. Tamaru and H. Yoshida, Lie groups locally isomorphic to generalized Heisenberg groups, *Proc. Amer. Math. Soc.* **136** (2008), 3247–3254.
[9] M. Vergne, Cohomologie des algébres de Lie nilpotentes. Application á l'étude de la variété des algebres de Lie nilpotentes, *Bull. Soc. Math. France* **98** (1970), 81–116.
[10] E. N. Wilson, Isometry groups on homogeneous nilmanifolds, *Geom. Dedicata* **12** (1982), 337–346.