

ANALYSIS OF A PARTIALLY LOADED PARABOLOID OF REVOLUTION AND POSSIBLE APPROXIMATIONS OF ITS SOLUTION

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In this paper a partially loaded paraboloid of revolution is investigated by the theory of bent shallow shells. After establishing the sixth order linear system of equations it is concluded that the unknown coefficients cannot be expressed, cannot be given as practical formulas, thus useful approximate solutions are constructed for practical usage. The applicability limit of the column reaction approximated by concentrated load is investigated and the generated error of membrane action for this approximation is determined.

Keywords: punching, reinforced concrete slabs, membrane action, approximate solutions

1. Introduction

Reinforced concrete slabs on columns were developed by Turner and Maillart at the beginning of the 20th century [1, 2]. In the early years large mushroom-shaped column capitals were used for the slab-column connections to facilitate the concentrated effects of the column reactions. In the 1950s flat slabs without capitals started to become dominant, where the slab failure is the punching shear failure around the column.

The design codes typically give empirical expressions for the punching shear resistance of flat slabs, which are based on experimental investigations. A similar pattern is observed in the critical shear crack width theory too, where the semi-empirical failure criterion is a function of the width of the critical crack [3].

However, in another approach, it can be verified that the membrane forces have significant effect around the column for calculating the punching shear resistance. Hence the load bearing around the column can be investigated more adequately on the basis of the theory of bent shallow shells than that of thin plates. On the basis of the shear resistance of the concrete compression zone and the theory of bent shallow shells, a simple mechanical model and an expression for the punching shear capacity can be given. The calculated results of this model were in good agreement with the test results of flat slabs without punching shear reinforcement [4].

In discussions on the bending of a flat slab it is mostly assumed that the column reactions can be approximated by concentrated loads, however, the column reactions are distributed uniformly over the areas of the cross-section of the columns. Moreover, the experimental investigations often include the diameter of circular column as a variable, thus it raises the question, what the applicability limit of the column reaction approximated by concentrated load is.

Consequently, the error caused by this approximation in the calculated values of the membrane action has to be investigated.

2. Analysis of the bent shallow shell

The punching shear resistance of a flat slab supported by columns of a square mesh is investigated by considering a representative slab element surrounding a column. The theory of thin elastic plates shows that, in the case of small values of c/L , where c is the radius of a circular column and L is the axis-to-axis spacing of the columns, the bending moments in the radial direction practically form a zero circle of radius $r = 0.22L$. Thus the plate around the column and inside this circle can be approximated as a circular plate simply supported along the circle $r = 0.22L$ [5]. The load bearing around the column can be investigated more adequately on the basis of the theory of bent shallow shells, than that of thin plates [4]. The assumed shell around the column and its geometry is shown in Fig. 1.

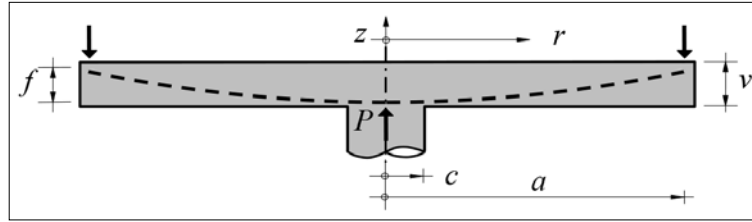


Fig. 1. Geometry of shell around the column head

Let the shape function of the flat shell in an r, ϑ, z cylindrical coordinate system

$$z = (1/2)\alpha_0 r^2, \quad \text{in which} \quad \alpha_0 = 2f/a^2, \quad (1)$$

where a is the boundary radius, and f is the depth of the shell. The load and the supports of this paraboloid are assumed axisymmetric. The material is assumed to be homogenous and isotropic with elastic constants E and ν .

For analyzing the bent shallow shell, the method of the generator function can be applied. The application of this method is shown in detail according to [6]. In the case of a shell carrying a concentrated load (see in Fig. 1) the application of this same method is shown according to [4].

Investigations show analogy of the bent shallow shell and the circular plate on elastic foundation (see in Fig. 2), where the intensity of the reaction of the subgrade is given by the curvature of the middle surface, and the fictitious Winkler-type foundation is $C = \alpha_0^2 Et$, where t is the thickness of the shell.

Expression (2) is the general solution of displacement of a paraboloid of revolution by the method of the generator function [4]. And expression (3) is the general solution of a circular plate on elastic foundation according to [7],

$$w = \alpha_0 L^{-2} [4c_2 + 4c_4(1 + \ln x) - c_5 \text{bei}(x) + c_6 \text{ber}(x) + c_7 \text{ker}(x) - c_8 \text{kei}(x)], \quad (2)$$

$$w = (q/C) + c_1 \text{ber}(x) + c_2 \text{bei}(x) + c_3 \text{kei}(x) + c_4 \text{ker}(x), \quad (3)$$

where $\text{ber}(x)$, $\text{bei}(x)$, $\text{ker}(x)$ and $\text{kei}(x)$ are the zero order Thomson functions, $x = r/L$ is a dimensionless radial coordinate, and

$$L = [K/(\alpha_0^2 Et)]^{1/4} \quad (4)$$

is the characteristic length, in which K is the flexural stiffness of the plate.

Solutions (2) and (3) can be clearly corresponded in their structure as well as in their calculated constants.

3. Analysis of a partially loaded paraboloid of revolution

To investigate the load bearing near the column head, let us consider the case in which the column reaction is distributed uniformly over some areas corresponding to the cross-section of the column (see in Fig. 3). Assuming that b is the radius of a circular column $q = P/\pi b^2$.

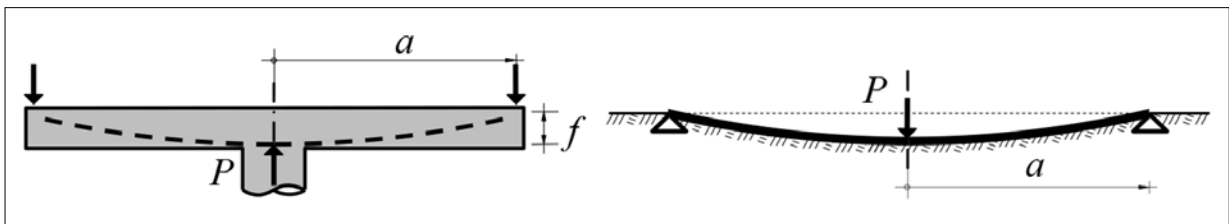


Fig. 2. Bent shallow shell and its analogy as circular plate on elastic foundation

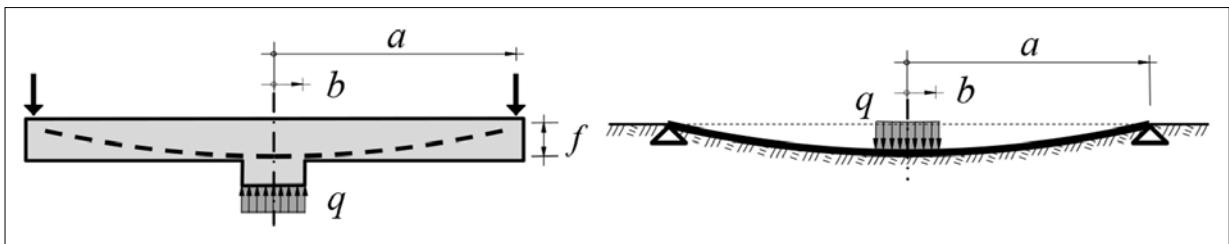


Fig. 3. Partially loaded paraboloid of revolution and circular plate on elastic foundation

Using expression (3) the displacement with respect to the inner and the outer portion of the plate is represented in the following form:

$$w_{x \leq \beta} = w_{\text{inner}} = (q/C) + c_1 \text{ber}(x) + c_2 \text{bei}(x) + c_3 \text{kei}(x) + c_4 \text{ker}(x), \quad (5)$$

$$w_{\beta \leq x \leq \alpha} = w_{\text{outer}} = c_5 \text{ber}(x) + c_6 \text{bei}(x) + c_7 \text{kei}(x) + c_8 \text{ker}(x), \quad (6)$$

where $\beta = b/L$ and $\alpha = a/L$ are dimensionless radial coordinates.

From the expression of displacement, for the slope, the radial bending moment and the radial shear force, the following expressions can be given as

$$\begin{aligned} \varphi &= -\frac{\partial w}{\partial r}, \\ M_r &= -K \left[\frac{\partial^2 w}{\partial r^2} + \nu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \vartheta^2} \right) \right], \\ Q_r &= -K \frac{\partial}{\partial r} (\Delta w). \end{aligned} \quad (7)$$

For the displacement at point $x = 0$ ($r = 0$) becomes infinitely large, since $\text{ker}(0) = \infty$, the constant c_4 must be zero. Since there is no concentrated load at the point $x = 0$, the value of the shear force is $(Q_{r,\text{inner}})_{r=0} = 0$, thus $c_3 = 0$ can be obtained. The remaining constants can be calculated from the continuity conditions along the circle of radius b (8) and from the boundary conditions (9) as follows:

$$\begin{aligned} (w_{\text{inner}})_{r=b} &= (w_{\text{outer}})_{r=b}, & (\varphi_{\text{inner}})_{r=b} &= (\varphi_{\text{outer}})_{r=b}, \\ (M_{r,\text{inner}})_{r=b} &= (M_{r,\text{outer}})_{r=b}, & (Q_{r,\text{inner}})_{r=b} &= (Q_{r,\text{outer}})_{r=b}, \end{aligned} \quad (8)$$

$$(w_{\text{outer}})_{r=a} = 0, \quad (M_{r,\text{outer}})_{r=a} = 0. \quad (9)$$

Using this conditions $\mathbf{Ax} = \mathbf{b}$ a sixth order linear system of equations is obtained, in which the elements of \mathbf{x} are the unknown coefficients c_1, c_2, c_5, c_6, c_7 , and c_8 , while the elements of the coefficient matrix \mathbf{A} are:

$$\begin{aligned} a_{11} &= \text{ber}(\beta), & a_{12} &= \text{bei}(\beta), & a_{13} &= -\text{ber}(\beta), \\ a_{14} &= -\text{bei}(\beta), & a_{15} &= -\text{kei}(\beta), & a_{16} &= -\text{ker}(\beta), \end{aligned}$$

$$\begin{aligned} a_{21} &= \text{ber}'(\beta), & a_{22} &= \text{bei}'(\beta), & a_{23} &= -\text{ber}'(\beta), \\ a_{24} &= -\text{bei}'(\beta), & a_{25} &= -\text{kei}'(\beta), & a_{26} &= -\text{ker}'(\beta), \end{aligned}$$

$$\begin{aligned} a_{31} &= -\text{bei}(\beta) - [(1-\nu)/\beta] \text{ber}'(\beta), \\ a_{32} &= \text{ber}(\beta) - [(1-\nu)/\beta] \text{bei}'(\beta), \\ a_{33} &= \text{bei}(\beta) + [(1-\nu)/\beta] \text{ber}'(\beta), \\ a_{34} &= -\text{ber}(\beta) + [(1-\nu)/\beta] \text{bei}'(\beta), \\ a_{35} &= -\text{ker}(\beta) + [(1-\nu)/\beta] \text{kei}'(\beta), \\ a_{36} &= \text{kei}(\beta) + [(1-\nu)/\beta] \text{ker}'(\beta), \end{aligned}$$

$$\begin{aligned} a_{41} &= \text{bei}'(\beta), & a_{42} &= -\text{ber}'(\beta), & a_{43} &= -\text{bei}'(\beta), \\ a_{44} &= \text{ber}'(\beta), & a_{45} &= \text{ker}'(\beta), & a_{46} &= -\text{kei}'(\beta), \end{aligned}$$

$$\begin{aligned} a_{51} &= 0, & a_{52} &= 0, & a_{53} &= \text{ber}(\alpha), \\ a_{54} &= \text{bei}(\alpha), & a_{55} &= \text{kei}(\alpha), & a_{56} &= \text{ker}(\alpha), \end{aligned}$$

$$\begin{aligned} a_{61} &= 0, & a_{62} &= 0, \\ a_{63} &= -\text{bei}(\alpha) - [(1-\nu)/\alpha] \text{ber}'(\alpha), \\ a_{64} &= \text{ber}(\alpha) - [(1-\nu)/\alpha] \text{bei}'(\alpha), \\ a_{65} &= \text{ker}(\alpha) - [(1-\nu)/\alpha] \text{kei}'(\alpha), \\ a_{66} &= -\text{kei}(\alpha) - [(1-\nu)/\alpha] \text{ker}'(\alpha) \end{aligned}$$

and the elements of \mathbf{b} are

$$b_1 = -q/C, \quad b_2 = b_3 = b_4 = b_5 = b_6 = 0.$$

For the special functions Wolfram Mathematica Code is used to solve the system of equations and to determine the unknown coefficients. However, due to the structure of matrix \mathbf{A} , and due to their sizes, the unknown coefficients cannot be expressed, cannot be given as practical formulas.

The obtained results in the case of $f = 0$ as a partially loaded circular plate and in the case of $f \neq 0$ as a circular plate on elastic foundation were verified. The verified cases are shown in Fig. 4.

Taking advantage of the tools offered by Wolfram Mathematica the membrane effect was investigated. This effect is an important part of the load bearing around a column. Thus the membrane action can be determined as

$$\mu_{r=0} = p_{\text{membrane}}/p = 1 - (w/w_0)_{r=0}, \quad (10)$$

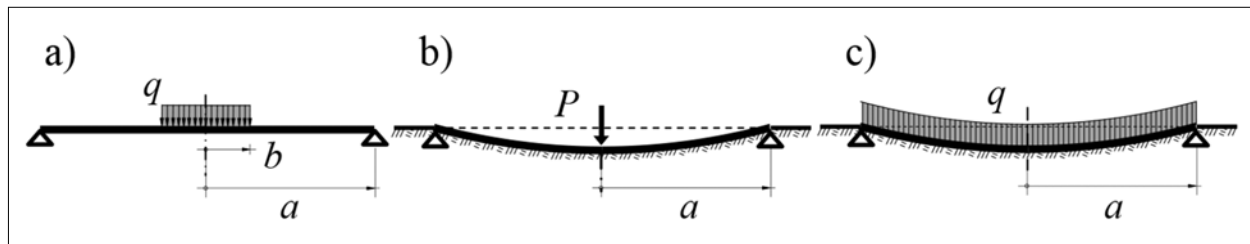


Fig. 4. Verified cases of the obtained results

where p is the total load, and p_{membrane} is the part of p equilibrated by membrane forces. In expression (10) w is the displacement calculated by the theory of bent shallow shells, and w_0 is the displacement calculated by means of the theory based on small deflections according to [7] as follows:

$$(w_0)_{\rho \leq \beta_0} = \frac{pa^4}{64K} \left\{ \left[4 - 5\beta_0^2 + 4(2 + \beta_0^2) \ln \beta_0 \right] \beta_0^2 + 2c_0(1 - \rho^2) / (1 + \nu) + \rho^4 \right\}, \quad (11)$$

$$(w_0)_{\rho \geq \beta_0} = \frac{pa^4 \beta_0^2}{32K} \left\{ \left[2(3 + \nu) - (1 - \nu)\beta_0^2 \right] (1 + \nu)^{-1} \times (1 - \rho^2) + 2 \ln \rho (2\rho^2 + \beta_0^2) \right\}, \quad (12)$$

where $c_0 = [4 - (1 - \nu)\beta_0^2 - 4(1 + \nu) \ln \beta_0] \beta_0^2$, and $\rho = r/a$ and $\beta_0 = b/a$ are relative radial coordinates.

The performed numerical calculations show that the column reaction approximated by concentrated load causes an error in the membrane action, which is less than 5% for the maximum ratio of $\beta_0 = 0.395$.

However, if the membrane action is calculated for the punching resistance according to [4], this error should be kept below 2%, from which the ratio $\beta_0 = 0.195$ can be calculated, because, in fact, the error of membrane action has no linear effect on the punching resistance. The error of punching resistance δ_V can be expressed in the following form:

$$\delta_V = \frac{(1 + \delta_\mu)(\mu - 1)}{(1 + \delta_\mu)\mu - 1} - 1, \quad (13)$$

where δ_μ is the error of the membrane action. Let us assume that the value of membrane action is $\mu = 0.600$ and its error is 5% ($\delta_\mu = 0.05$). In this case, from expression (13), $\delta_V = 0.135$ can be obtained, thus the generated error is 13.5%.

4. Approximation of the load bearing for a partially loaded paraboloid of revolution

4.1. Approximation by combination with two circular plates on elastic foundation

On the basis of the available literature [7], an approximate solution can be given with the combination of the cases of Fig. 4b and 4c. The general form of this combination is:

$$w \cong w_{\text{appr.}(2)} = [1 - F(\beta_0)]w_1 + F(\beta_0)w_2, \quad (14)$$

where $F(\beta_0)$ is a function of the ratio of the loaded area $\beta_0 = b/a$, w_1 and w_2 are the functions of displacement according to (3) taking into account the cases of Fig. 4b and 4c. If $F(\beta_0)$ is chosen as a linear function, expression (14) becomes

$$w \cong w_{\text{appr.}(1)} = (1 - \beta_0)w_1 + \beta_0w_2 = \left(1 - \frac{b}{a}\right)w_1 + \frac{b}{a}w_2. \quad (15)$$

The deviation of the approximate function at the point $r = 0$ versus the ratio b/a , which is also the error of the membrane action, is shown in Fig. 5. In this figure the relative depth of the shell is denoted f/t .

In cases, when $f/t = 0.00 \div 1.00$, the curves are located between the curves given in Fig. 5. As shown in Fig. 5, the error of the approximation for $b/a = 0.50$ is significant, it approaches 40%. The figure also shows that the approximate expression must contain the ratio f/t , beside the ratio b/a . Thus the approximate function of displacement at the point $r = 0$ can be expressed in the following form:

$$w_{\text{appr.}(2)} = [1 - F(\beta_0, f/t)]w_1 + F(\beta_0, f/t)w_2, \quad (16)$$

where $F(\beta_0, f/t) = F_1(\beta_0) + (f/t)F_2(\beta_0)$. In expression (16) F_1 and F_2 are both polynomials, choosing them as fourth-degree expressions, take the form

$$F_i = b_{i1} + b_{i2}\beta_0 + b_{i3}\beta_0^2 + b_{i4}\beta_0^3 + b_{i5}\beta_0^4. \quad (17)$$

Based on the deviation of the approximate function at the point $r = 0$ the following constants of (17) can be determined

$$\begin{aligned} b_{11} &= 0.0000, & b_{12} &= 0.0990, & b_{13} &= 1.7213, \\ & & b_{14} &= -1.1322, & b_{15} &= 0.3119, \\ b_{21} &= 0.0000, & b_{22} &= 0.1460, & b_{23} &= 0.6136, \\ & & b_{24} &= -1.3209, & b_{25} &= 0.5667. \end{aligned}$$

With these values of the constants the deviation of $w_{\text{appr.}(2)}$ at the point $r = 0$ versus the ratio b/a is shown in Fig. 6.

The error of the approximate function at the point $r = 0$ for $f/t = 1.00$ is 0.27%, and the maximum error in case of $f/t = 0.50$ is 1.66%. Due to the error of the membrane action, considering that the decrease of the relative depth of the shell causes decrease in the membrane action, the generat-

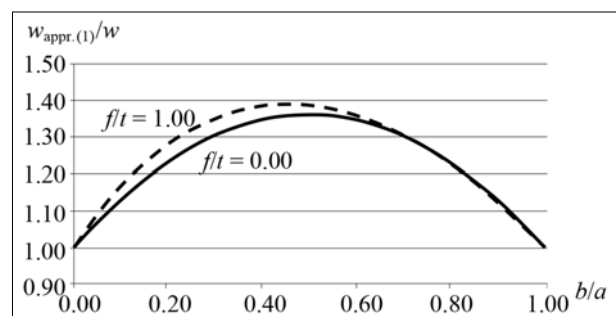


Fig. 5. Deviation of the approximate function at the point $r = 0$

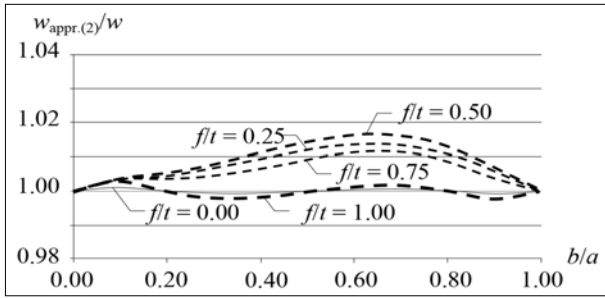


Fig. 6. Deviation of $w_{appr.(2)}$ at the point $r = 0$

ed error of punching resistance, in all cases remains under 2.50%.

Taking into account that the approximate function $w_{appr.(2)}$ at the point $r = 0$ is constructed, in cases of $r \neq 0$ the deviation of this function is larger than given above. In order to reduce this deviation, a function, called shape function has to be chosen, which does not change the value of $w_{appr.(2)}$ at the point $r = 0$. The shape function is constructed as a quadratic, and the corresponding polynomial F_3 is chosen in the form of (17). Thus

$$w_{appr.(3)} = w_{appr.(2)}(1 + w_{appr.(shape)}), \quad (18)$$

$$w_{appr.(shape)} = F_3(\beta_0) \left(0.45 + 0.55 \frac{f}{t} \right) \left[\left(\frac{x}{\alpha} \right)^2 - 2 \frac{x}{\alpha} \right], \quad (19)$$

in which the constants for F_3 are

$$b_{31} = 0.000, \quad b_{32} = -0.100, \quad b_{33} = -0.648, \\ b_{34} = 0.277, \quad b_{35} = 0.471.$$

The results of expression (18) with various ratio of b/a for $f/t = 1.00$, which is the maximum deviation of (18), is shown in Fig. 7.

As it is shown in Fig. 7 the maximum error of $w_{appr.(3)}$ for $b/a = 0.40$ is 4%.

4.2. Approximation by combination with two circular plates on elastic foundation

To get an approximate solution advantage can be taken of the fact that the system of differential equations for bent shallow shells can be derived from the general equations of large deflections of plates [8]. Thus simple approximate solutions of large deflections of plates can also be used. In the followings, based on the above, a simple approximation method can be applied, which consists of a combination of the known solutions given by the theory of small deflections and the membrane theory [5].

The deflection at the center $w_{0,plate}$ of a circular plate by the theory of small deflections, according to [7], is given by the following expression:

$$w_{0,plate} = (w_0)_{r=0} = \frac{qa^2b^2}{64K(1+\nu)} \left[4(3+\nu) - (7+3\nu)\beta_0^2 + 4(1+\nu)\beta_0^2 \ln \beta_0 \right]. \quad (20)$$

Considering the circular plate as a circular membrane, the deflection at the center is

$$w_{0,membrane} = (1 - \ln \beta_0^2) \left[\frac{3(1-\nu)qa^2b^3}{8Et(4a-3b)} \right]^{1/3}. \quad (21)$$

Substituting $b = a$ in expression (21) the known solution for the deflection of a uniformly loaded circular membrane [9] can be obtained, which is

$$w_{0,membrane} = [\{ 3(1-\nu)qa^4 \} / (8Et)]^{1/3}. \quad (22)$$

Having expressions (20) and (21) for the deflections, can be obtained

$$q_{plate} = w_{0,plate} [64K(1+\nu)] / (a^2b^2C_{plate}), \quad (23)$$

where $C_{plate} = 4(3+\nu) - (7+3\nu)\beta_0^2 + 4(1+\nu)\beta_0^2 \ln \beta_0$ and

$$q_{membrane} = w_{0,membrane}^3 \times \frac{8Et(4a-3b)}{3(1-\nu)qa^2b^3(1-\ln \beta_0^2)^3}. \quad (24)$$

The deflection w_0 is obtained from the equation $q = q_{plate} + q_{membrane}$ which gives

$$w_0 = \frac{qa^2b^2C_{plate}}{64K(1+\nu)} \left[1 + B \frac{w_0^2}{t^2} \right]^{-1}, \quad (25)$$

where

$$B = (1/2)[C_{plate}(4a-3b)] / [b(1-\ln \beta_0^2)^3]. \quad (26)$$

The expression (25) can be used to determine the deflection by method of successive approximations. Observing expression (25) it can be concluded that the

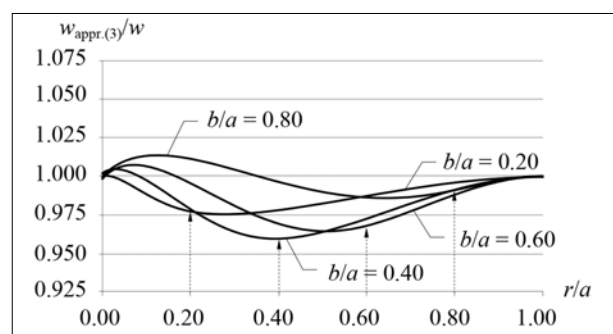


Fig. 7. Deviation of $w_{appr.(3)}$ at the point $r = 0$

last factor on the right-hand side is the plate action. In the case when the deflection is small in comparison with the thickness of the plate, this factor hardly differs from one, thus the membrane action can be neglected.

Taking advantage of the relationship between the differential equation system of the bent shallow shells and the general equations of the large deflections of plates, by replacing w_0 with f in divisor of formula (25), we get an approximate solution for partially loaded paraboloid of revolution, as follows

$$w_{\text{appr.}(4)} = \frac{q a^2 b^2 C_{\text{plate}}}{64K(1+\nu)} \left[1 + B \frac{f^2}{t^2} \right]^{-1} \quad (27)$$

The deviation of $w_{\text{appr.}(4)}$ at the point $r = 0$ versus the ratio b/a is shown in Fig. 8. It is evident that in case of $f = 0$ this approximate solution gives the solution of a partially loaded circular plate by the theory of small deflections.

As it is shown in Fig. 8, the error of the approximate function $w_{\text{appr.}(4)}$, in cases of $b/a \geq 0.025$, is less than 14%.

It should be noted that, in cases of $f/t \neq 0.00$ for very small values of the ratio b/a , B becomes infinitely large and expression (27) tends to zero. This means that a concentrated load cannot be balanced only by membrane forces.

Using the more adequate solution for partially loaded paraboloid of revolution, expression (27) can be improved in the following form

$$w_{\text{appr.}(5)} = w_{\text{appr.}(6)} [1 + (f/t) F_{\text{corr.}}(\beta_0)], \quad (28)$$

where $F_{\text{corr.}}(\beta_0)$ is a correction function. This correction function is chosen as a linear function in the following form

$$F_{\text{corr.}}(\beta_0) = -0.20 + 0.35 \beta_0 \quad (29)$$

or as a sixth degree polynomial

$$F_{\text{corr.}}(\beta_0) = -2.903 \beta_0 + 20.25 \beta_0^2 - 61.77 \beta_0^3 + 97.22 \beta_0^4 - 75.39 \beta_0^5 + 22.73 \beta_0^6 \quad (30)$$

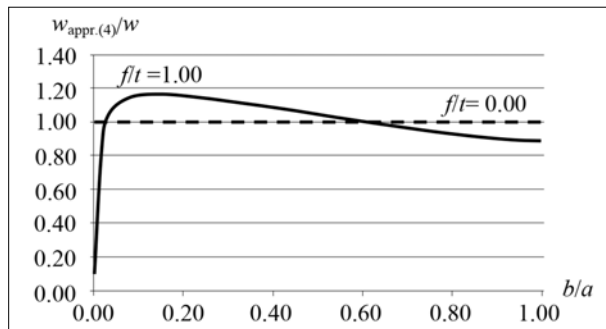


Fig. 8. Deviation of $w_{\text{appr.}(4)}$ at the point $r = 0$

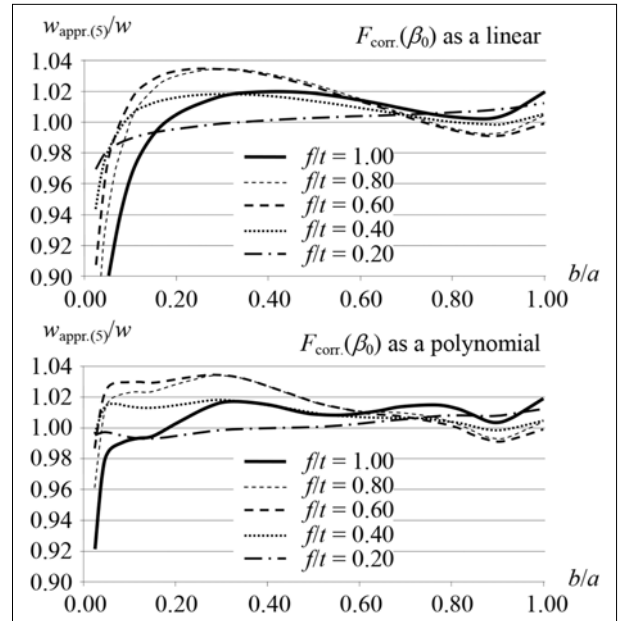


Fig. 9. Deviation of $w_{\text{appr.}(5)}$ using the correction functions

The deviation of the approximate function $w_{\text{appr.}(5)}$ at the point $r = 0$, using the correction function as a linear function and as a sixth degree polynomial, is shown in Fig. 9.

As it is shown in Fig. 9, the difference between the corrected functions only for small values of the ratio b/a is significant. Thus, in cases of $b/a \geq 0.150$ the application of a linear function and in cases of $0.050 \leq b/a < 0.150$ the application of a sixth degree polynomial can be proposed for the correction function.

5. Conclusions

The final results of the investigation show that the column reaction approximated by concentrated load causes an error in the membrane action, which is less than 2% and the generated error of the punching shear resistance is less than 5% for the maximum ratio of $b/a = 0.195$. Therefore it is verified that for the ratios used in practice, also in the case when the membrane action is calculated, the column reaction can be approximated by concentrated load.

Two useful approximate solutions are determined for calculating the membrane action. From these the first approximate solution, which is based on the combination of known solutions for two circular plates on elastic foundation, can be used for any ratio b/a , its maximum error is 1.66%. And the second approximate solution, which is based on the combination of the solutions given by the theory of small deflections and the membrane theory, can be used for the ratios $b/a \geq 0.050$, and its maximum error is 3.42%.

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