Nullity distributions associated with Chern connection

Nabil L. Youssef[1] **and S. G. Elgendi**[2,3]

[1]Department of Mathematics, Faculty of Science,
Cairo University, Giza, Egypt

[2]Department of Mathematics, Faculty of Science,
Benha University, Benha, Egypt

[3]Institute of Mathematics, University of Debrecen,
Debrecen, Hungary

E-mails: nlyoussef@sci.cu.edu.eg, nlyoussef2003@yahoo.fr
salah.ali@fsci.bu.edu.eg, salahelgendi@yahoo.com

Abstract. The nullity distributions of the two curvature tensors $\overset{*}{R}$ and $\overset{*}{P}$ of the Chern connection of a Finsler manifold are investigated. The completeness of the nullity foliation associated with the nullity distribution $\mathcal{N}_{R^*}$ is proved. Two counterexamples are given: the first shows that $\mathcal{N}_{R^*}$ does not coincide with the kernel distribution of $\overset{*}{R}$; the second illustrates that $\mathcal{N}_{P^*}$ is not completely integrable. We give a simple class of a non-Berwaldian Landsberg spaces with singularities.

Keywords: Klein-Grifone formalism, Chern connection, nullity distribution, kernel distribution, nullity foliation, autoparallel submanifold.

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1. Introduction

Adopting the pullback approach to Finsler geometry, the nullity distribution has been investigated, for example, in [1, 2, 14]. In 2011, Bidabad and Refie-Rad [3] studied a more general distribution called k-nullity distribution. On the other hand, in 1982, Youssef [12, 13] studied the nullity distributions of the curvature tensors of Barthel and Berwald connections, adopting the Klein-Grifone approach to Finsler geometry. Moreover, Youssef et al. [18] studied the nullity distributions associated to the Cartan connection.

In their paper [17], the present authors investigated the existence and uniqueness of the Chern connection and studied the properties of its curvature tensors following the Klein-Grifone approach. In this paper, we investigate the nullity distributions associated with the Chern connection. We prove the integrability and the autoparallel property of the nullity distribution $\mathcal{N}_{R^*}$ of the Chern h-curvature $\overset{*}{R}$. Moreover, we prove the completeness of the nullity foliation associated with $\mathcal{N}_{R^*}$. We give two interesting counterexamples. The first shows that the nullity distribution $\mathcal{N}_{R^*}$ does not coincide with the kernel distribution of $\overset{*}{R}$ ($\mathcal{N}_{R^*}$ is a proper sub-distribution of Ker_{R^*}). The second shows that $\mathcal{N}_{P^*}$ is not completely integrable. As a by-product, this allows us to give a simple class of non-Berwaldian Landsberg spaces with singularities.

2. Notation and Preliminaries

In this section we present a brief account of the basic concepts of Klein-Grifone's theory of Finsler manifolds. For details, we refer to references [6, 7, 8, 13]. We begin with some notational conventions.

Throughout, M is a smooth manifold of finite dimension n. The $\mathbb{R}$-algebra of smooth real-valued functions on M is denoted by $C^\infty(M)$; $\mathfrak{X}(M)$ stands for the $C^\infty(M)$-module of vector fields on M. The tangent bundle of M is $\pi_M : TM \longrightarrow M$, the subbundle of nonzero tangent vectors to M is $\pi : \mathcal{T}M \longrightarrow M$. The vertical subbundle of TTM is denoted by $V(TM)$. The pull-back of TM over π is $P : \pi^{-1}(TM) \longrightarrow \mathcal{T}M$. If $X \in \mathfrak{X}(M)$, i_X and $\mathcal{L}_X$ denote the interior product by X and the Lie derivative with respect to X, respectively. The differential of $f \in C^\infty(M)$ is df. A vector ℓ-form on M is a skew-symmetric $C^\infty(M)$-linear map $L : (\mathfrak{X}(M))^\ell \longrightarrow \mathfrak{X}(M)$. Every vector ℓ-form L defines two graded derivations i_L and d_L of the Grassman algebra of M such that

$$i_L f = 0, \quad i_L df = df \circ L \quad (f \in C^\infty(M)),$$

$$d_L := [i_L, d] = i_L \circ d - (-1)^{\ell-1} d i_L.$$

We have the following short exact sequence of vector bundle morphisms:

$$0 \longrightarrow \mathcal{T}M \times_M TM \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \mathcal{T}M \times_M TM \longrightarrow 0.$$

Here $\rho := (\pi_{\mathcal{T}M}, \pi_*)$, and γ is defined by $\gamma(u, v) := j_u(v)$, where j_u is the canonical isomorphism from $T_{\pi_M(v)}M$ onto $T_u(T_{\pi_M(v)}M)$. Then, $J := \gamma \circ \rho$ is a vector 1-form on TM called the vertical endomorphism. The Liouville vector field on TM is the vector field defined by $C := \gamma \circ \overline{\eta}$, $\overline{\eta}(u) = (u, u)$, $u \in TM$.

A differential form ω (resp. a vector form L) on TM is semi-basic if $i_{JX}\omega = 0$ (resp. $i_{JX}L = 0$ and $JL = 0$), for all $X \in \mathfrak{X}(TM)$. A vector 1-form G on TM is called a *Grifone connection* if it is smooth on $\mathcal{T}M$, continuous on TM and satisfies $JG = J$, $GJ = -J$. The vertical and horizontal projectors v and h associated to G are defined by

$$v := \frac{1}{2}(I - G) \text{ and } h := \frac{1}{2}(I + G).$$

The almost complex structure determined by G is the vector 1-form $\mathbf{F}$ characterized by $\mathbf{F}J = h$ and $\mathbf{F}h = -J$.

A Grifone connection G induces the direct sum decomposition

$$TTM = V(TM) \oplus H(TM), \quad H(TM) := Im(h).$$

The subbundle $H(TM)$ is called the G-horizontal subbundle of TTM, the module of its smooth sections will be denoted by $\mathfrak{X}^h(\mathcal{T}M)$.

A Grifone connection G is homogeneous if $[C,G]=0$. The torsion and the curvature of G are the vector 2-forms $t := \frac{1}{2}[J,G]$ and $\mathfrak{R} := -\frac{1}{2}[h,h]$, respectively. Note that in the last three equalities the brackets mean Frölicher-Nijenhuis bracket [5].

A function $E : TM \longrightarrow \mathbb{R}$ is called a *Finslerian energy* function if it is of class C^1 on TM and C^∞ on $\mathcal{T}M$; $E(u) > 0$ if $u \in \mathcal{T}M$ and $E(0)=0$; $C \cdot E = 2E$, i.e., E is 2^+-homogeneous; the fundamental 2-form $\Omega := dd_J E$ has maximal rank. A *Finsler manifold* is a manifold together with a Finslerian energy. If (M,E) a Finsler manifold, then

(i) there exists a unique spray S for M such that $i_S\Omega = -dE$;

(ii) there exists a unique homogeneous Grifone connection on TM with vanishing torsion, namely $G=[J,S]$, such that $d_h E = 0$ ('G is conservative').

We say that S is the *canonical spray* and G is the *canonical connection* or *Barthel connection* of (M,E).

If (M,E) is a Finsler manifold, then the map $\overline{g}$ given by

$$\overline{g}(JX,JY) := \Omega(JX,Y); \quad X,Y \in \mathfrak{X}(\mathcal{T}M)$$

is a metric tensor on $V(TM)$. It can be extended to a metric tensor g on $T(TM)$ by

$$g(X,Y) := \overline{g}(JX,JY) + \overline{g}(vX,vY) = \Omega(X,\mathbf{F}Y). \tag{2.1}$$

Now we recall three famous covariant derivative operators on a Finsler manifold, called also 'connections'. They are the *Berwald connection* $\overset{\circ}{D}$, the *Cartan connection* D and the *Chern connection* $\overset{*}{D}$, given by

$$\overset{\circ}{D}_{JX}JY = J[JX,Y], \quad \overset{\circ}{D}_{hX}JY = v[hX,JY], \quad \overset{\circ}{D}\mathbf{F}=0; \tag{2.2}$$

$$D_{JX}JY = \overset{\circ}{D}_{JX}JY + \mathcal{C}(X,Y), \quad D_{hX}JY = \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X,Y), \quad D\mathbf{F}=0; \tag{2.3}$$

$$\overset{*}{D}_{JX}JY = J[JX,Y], \quad \overset{*}{D}_{hX}JY = v[hX,JY] + \mathcal{C}'(X,Y), \quad \overset{*}{D}\mathbf{F}=0, \tag{2.4}$$

$(X,Y \in \mathfrak{X}(\mathcal{T}M))$. In the formulas (2.3) and (2.4) $\mathcal{C}$ is the *Cartan tensor*, $\mathcal{C}'$ is the *Landsberg tensor* of (M,E). For their definition, see [7], p. 329. The tensors $\mathcal{C}$ and $\mathcal{C}'$ are symmetric, semi-basic and for arbitrary semispray S on TM, we have

$$\mathcal{C}(X,S) = \mathcal{C}'(X,S) = 0. \tag{2.5}$$

Let $\overset{*}{R}$ and $\overset{*}{P}$ be the h-curvature and the hv-curvature of $\overset{*}{D}$, respectively. We list some important identities from [17], which will be needed in the sequel. Below X, Y, Z, W are vector fields, S is a semispray on $\mathcal{T}M$.

$$[hX,hY] = h(\overset{*}{D}_{hX}Y - \overset{*}{D}_{hY}X) - \mathfrak{R}(X,Y); \tag{2.6}$$

$$\overset{*}{R}(X,Y)Z = R(X,Y)Z - \mathcal{C}(\mathbf{F}\mathfrak{R}(X,Y),Z), \tag{2.7}$$

where R is the h-curvature of D;

$$\overset{*}{P}(X,Y)Z = \overset{\circ}{P}(X,Y)Z - (\overset{*}{D}_{JY}\mathcal{C}')(X,Z), \tag{2.8}$$

where $\overset{\circ}{P}$ is the hv-curvature of $\overset{\circ}{D}$;

$$\overset{*}{R}(X,Y)S = \mathfrak{R}(X,Y); \tag{2.9}$$

$$\overset{*}{P}(X,Y)S = \overset{*}{P}(S,Y)X = \mathcal{C}'(X,Y), \ \overset{*}{P}(X,S)Z = 0; \tag{2.10}$$

$$\mathfrak{S}_{X,Y,Z}\,\{\overset{*}{R}(X,Y)Z\} = 0; \tag{2.11}$$

$$\mathfrak{S}_{X,Y,Z}\,\{(\overset{*}{D}_{hX}\,\mathfrak{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{\,\mathcal{C}'(\mathbf{F}\mathfrak{R}(X,Y),Z)\}; \tag{2.12}$$

$$\mathfrak{S}_{X,Y,Z}\{\,(\overset{*}{D}_{hX}\overset{*}{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{\overset{*}{P}(X,\mathbf{F}\mathfrak{R}(Y,Z))\}; \tag{2.13}$$

$$(\overset{*}{D}_{hX}\overset{*}{P})(Y,Z) - (\overset{*}{D}_{hY}\overset{*}{P})(X,Z) + (\overset{*}{D}_{JZ}\overset{*}{R})(X,Y) = \overset{*}{P}(X,\mathbf{F}\mathcal{C}'(Y,Z)) - \overset{*}{P}(Y,\mathbf{F}\mathcal{C}'(X,Z)); \tag{2.14}$$

If $\mathfrak{R} = 0$, then

$$\overset{*}{R}(X,Y,Z,W) = \overset{*}{R}(Z,W,X,Y), \tag{2.15}$$

where $\overset{*}{R}(X,Y,Z,W) := g(\overset{*}{R}(X,Y)Z, JW)$.

3. Nullity distribution of the Chern h-curvature

In this section, we investigate the nullity distribution of the Chern connection. It should be noted that the nullity distributions of the Barthel, Berwald and Cartan connections have already been studied in [12, 13, 18], respectively. First, we study the nullity distribution of the h-curvature tensor.

Definition 3.1. Let $\overset{*}{R}$ be the h-curvature tensor of the Chern connection. The nullity space of $\overset{*}{R}$ at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by

$$\mathcal{N}_{R^*}(z) := \{v \in H_z(TM) |\, \overset{*}{R}_z(v,w) = 0, \text{ for all } w \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_{R^*}(z)$, denoted by $\mu_{R^*}(z)$, is the nullity index of $\overset{*}{R}$ at z. If the nullity index μ_{R^*} is constant, then the map $\mathcal{N}_{R^*} : z \mapsto \mathcal{N}_{R^*}(z)$ defines a distribution $\mathcal{N}_{R^*}$ of rank μ_{R^*}, called the nullity distribution of $\overset{*}{R}$. Any smooth section in the nullity distribution $\mathcal{N}_{R^*}$ is called a nullity vector field. We denote by $\Gamma(\mathcal{N}_{R^*})$ the $C^\infty(TM)$-module of the nullity vector fields. We shall assume that $\mu_{R^*} \neq 0$ and $\mu_{R^*} \neq n$.

Let $\mathcal{N}_{R^*}(x) := \pi_*(\mathcal{N}_{R^*}(z))$ if $\pi(z) = x$. Then $\mathcal{N}_{R^*}(x)$ isomorphic to $\mathcal{N}_{R^*}(z)$ via the isomorphism $\pi_* \lceil_{H_z(TM)}$.

Definition 3.2. The kernel of $\overset{*}{R}$ at the point $z \in TM$ is defined by

$$Ker_{R^*}(z) := \{u \in H_z(TM) |\, \overset{*}{R}_z(v,w)u = 0, \text{ for all } v,w \in H_z(TM)\}.$$

We have $Ker_{R^*}(x) = \pi_*(Ker_{R^*}(z))$; $x = \pi(z)$.

Proposition 3.3. *The nullity distribution $\mathcal{N}_{R^*}$ has the following properties:*

(1) $\mathcal{N}_{R^*} \neq \phi$ *and* $\mathrm{Ker}_{R^*} \neq \phi$.

(2) $\mathcal{N}_{R^*} \subseteq \mathcal{N}_{\mathfrak{R}}$, *where $\mathcal{N}_{\mathfrak{R}}$ is the nullity distribution of the curvature $\mathfrak{R}$ of the Barthel connection.*

(3) $\mathcal{N}_{R^*} \subseteq \mathrm{Ker}_{R^*}$.

(4) *If the canonical spray S belongs to $\Gamma(\mathcal{N}_{R^*})$, then $\mathfrak{R} = 0$.*

(5) *If $X \in \Gamma(\mathcal{N}_{R^*})$, then $[C, X] \in \Gamma(\mathcal{N}_{R^*})$ and, consequently, $[C, X] \in \Gamma(\mathcal{N}_{\mathfrak{R}})$.*

Proof. (2) Let X be a nullity vector field. Using (2.9), we have

$$\begin{aligned} X \in \Gamma(\mathcal{N}_{R^*}) &\Longrightarrow \overset{*}{R}(X,Y)Z = 0 \quad \text{for all } Y, Z \in \mathfrak{X}(TM) \\ &\Longrightarrow \overset{*}{R}(X,Y)S = 0 \quad \text{for all } Y \in \mathfrak{X}(TM) \\ &\Longrightarrow \mathfrak{R}(X,Y) = 0 \quad \text{for all } Y \in \mathfrak{X}(TM) \\ &\Longrightarrow X \in \Gamma(\mathcal{N}_{\mathfrak{R}}). \end{aligned}$$

(3) Let $Z \in \Gamma(\mathcal{N}_{R^*})$, then, by (2.11), we have $\mathfrak{S}_{X,Y,Z}\{\overset{*}{R}(X,Y)Z\} = 0$. Since $\overset{*}{R}(Y,Z)X = \overset{*}{R}(Z,X)Y = 0$, then the result follows.

(4) This is an immediate consequence of (2.9).

(5) Let $X \in \Gamma(\mathcal{N}_{R^*})$. Since $\overset{*}{D}_C \overset{*}{R} = 0$ [17] , we get $(\overset{*}{D}_C \overset{*}{R})(X,Y) = 0$, which leads to $\overset{*}{R}(\overset{*}{D}_C X, Y) = 0$. Using (2.4), we have $\overset{*}{R}([C,X],Y) = 0$. By the homogeneity of h, $[C,h] = 0$, from which $[C,hX] = h[C,X]$. That is, $[C,hX]$ is horizontal. Hence, $[C,X] \in \Gamma(\mathcal{N}_{R^*})$. Consequently, by (2), $[C,X] \in \Gamma(\mathcal{N}_{\mathfrak{R}})$. □

It is important to note that the reverse inclusion in the property (3) of Proposition 3.3 is not true; that is, $\mathrm{Ker}_{R^*} \not\subset \mathcal{N}_{R^*}$. This is shown by the next example in which the calculations are performed by using [15].

Example 3.4. Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 | x^2 > 0\}$ and $U = \{(x^1, ..., x^4; y^1, ..., y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : y^1 \neq 0, y^2 \neq 0\} \subset TM$. Define F on U by

$$F(x,y) := \left((x^2)^2 (y^1)^4 + (y^2)^4 + (y^3)^4 + (y^4)^4\right)^{1/4}.$$

According to [16], the nullity distribution of the Cartan h-curvature R of (M,F) is

$$\mathcal{N}_R = \{sh_3 + th_4 \in \mathfrak{X}^h(\mathcal{T}M) | \, s,t \in \mathbb{R}\} \tag{3.1}$$

and the kernel distribution $\ker_R$ of R is

$$\ker_R = \left\{ s\left(\frac{y^1}{y^2}h_1 + h_2 + \frac{x^2(y^1)^4 + (y^2)^4 + 2(y^3)^4 + 2(y^4)^4}{y^2(y^4)^3}h_4\right) \right. \\ \left. + t\left(h_3 - \frac{(y^3)^3}{(y^4)^3}h_4\right) \in \mathfrak{X}^h(\mathcal{T}M) | s, t \in \mathbb{R} \right\}, \tag{3.2}$$

where $h_i := \frac{\partial}{\partial x^i} - N_i^m \frac{\partial}{\partial y^m}$ form a basis of $\mathfrak{X}^h(\mathcal{T}M)$.

Now, by the NF-package [15], we can perform the following calculations.

Chern h-curvature $\overset{*}{R}$:

```
> show(Rchern[i, -h, -j, -k]);
```

$Rchern^{x1}_{x1x1x2} = \frac{1}{18}\frac{4y2^4 + x2^2 y1^4}{x2^2 y1 y2^3}$ $Rchern^{x1}_{x2x1x2} = -\frac{1}{9}\frac{4y2^4 + x2^2 y1^4}{x2^2 y2^4}$

$Rchern^{x2}_{x1x1x2} = \frac{1}{9}\frac{4y1 2y2^4 + x2^2 y1^6}{y2^6}$ $Rchern^{x2}_{x2x1x2} = -\frac{1}{18}\frac{4y1^3 y2^4 + x2^2 y1^7}{y2^7}$

Nullity distribution of $\overset{*}{R}$:

```
> definetensor(RchernW[h, -i, -k] = Rchern[h, -i, -j, -k]*W[j]);
> show(RchernW[h, -i, -k]);
```

$RchernW^{x1}_{x2x2} = -\frac{1}{9}\frac{(4y2^4 + x2^2 y1^4)W^{x1}}{x2^2 y2^4}$ $RchernW^{x2}_{x1x1} = -\frac{1}{9}\frac{(4y1^2 y2^4 + x2^2 y1^6)W^{x2}}{y2^6}$

Putting $RchernW^{x1}_{x2x2} = 0$ and $RchernW^{x2}_{x1x1} = 0$, then we have a system of algebraic equations. The NF-package yields the following solution: $W^1 = W^2 = 0, W^3 = s, W^4 = t$, where $s, t \in \mathbb{R}$. Then, the nullity distribution is

$$\mathcal{N}_{R^*} = \{sh_3 + th_4 \in \mathfrak{X}^h(\mathcal{T}M) |\, s, t \in \mathbb{R}\}. \tag{3.3}$$

Kernel distribution of $\overset{*}{R}$:

```
> definetensor(RchernZ[h, -j, -k] = Rchern[h, -i, -j, -k]*Z[i]);
> show(RchernZ[h, -j, -k]);
```

$$RchernZ^{x1}_{x1x2} = \frac{1}{18}\frac{(4y2^4 + x2^2 y1^4)Z^{x1}}{x2^2 y1 y2^3} - \frac{1}{9}\frac{(4y2^4 + x2^2 y1^4)Z^{x2}}{x2^2 y2^4}$$

Putting $RchernZ^{x1}_{x1x2} = 0$, we get $Z^1 = \frac{2y1}{y2}r, Z^2 = r, Z^3 = s, Z^4 = t$; $r, s, t \in \mathbb{R}$. Then, the kernel distribution Ker_{R^*} is

$$\mathrm{Ker}_{R^*} = \left\{ r\left(\frac{2y1}{y2}h_1 + h_2\right) + sh_3 + th_4 \in \mathfrak{X}^h(\mathcal{T}M) |\, r, s, t \in \mathbb{R} \right\}. \tag{3.4}$$

Equations (3.3) and (3.4) show that Ker_{R^*} can not be a sub-distribution of $\mathcal{N}_{R^*}$.

Theorem 3.5. *The nullity distribution $\mathcal{N}_{R^*}$ of the Chern h-curvature and the nullity distribution $\mathcal{N}_R$ of the Cartan h-curvature coincide.*

Proof. Let $X \in \Gamma(\mathcal{N}_{R^*})$. Then, by (2.7) and Proposition 3.3 (2), $X \in \Gamma(\mathcal{N}_R)$. Hence $\mathcal{N}_{R^*}$ is a subset of $\mathcal{N}_R$. Conversely, let $X \in \Gamma(\mathcal{N}_R)$. Then, by (2.7) and by $\mathcal{N}_R \subset \mathcal{N}_{\mathfrak{R}}$ [18], we get $X \in \Gamma(\mathcal{N}_{R^*})$, whence, $\mathcal{N}_R \subset \mathcal{N}_{R^*}$. □

Remark 3.6. The above example shows that $\mathcal{N}_{R^*} \subset \mathrm{Ker}_{R^*}$ and the reverse inclusion is false by (3.3), (3.4). It also shows that although $\mathcal{N}_{R^*} = \mathcal{N}_R$ (see (3.1) and (3.3)), $\mathrm{Ker}_{R^*} \neq \mathrm{Ker}_R$ by (3.2), (3.4). In view of the above theorem, the reverse inclusion in (2) of Proposition 3.3 is not true either: $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_R = \mathcal{N}_{R^*}$ [16].

Definition 3.7. The conullity space of the h-curvature tensor at z, denoted by ${\mathcal{N}_{R^*}}^{\perp}(z)$, is the orthogonal complement of $\mathcal{N}_{R^*}$ in $H_z(TM)$, where the orthogonality is taken with respect to the metric g defined by (2.1).

Proposition 3.8. *For each point $z \in TM$, either $\mu_{R^*}(z) = n$ or $\mu_{R^*}(z) \leq n-2$. Consequently,* $\dim \mathrm{Ker}_{R^*} > n-2$.

Proof. If $\mu_{R^*}(z) \neq n$, then there is a non-zero horizontal vector $v \notin \mathcal{N}_{R^*}(z)$. It follows that there is a vector $w \in H_z(TM)$ such that $\overset{*}{R}_z(w,v) \neq 0$ and so $\overset{*}{R}_z(v,w) \neq 0$. Then $v, w \notin \mathcal{N}_{R^*}(z)$ and hence $v, w \in {\mathcal{N}_{R^*}}^{\perp}(z)$. By the antisymmetry of $\overset{*}{R}$, the vectors v and w are independent. Thus, $\dim {\mathcal{N}_{R^*}}^{\perp}(z) \geq 2$. Consequently, $\mu_{R^*}(z) \leq n-2$. □

Proposition 3.9. *If $\mathfrak{R} = 0$, then* $\mathrm{Im}(\overset{*}{R}) = (J\mathcal{N}_{R^*})^{\perp}$. Consequently, $\mathrm{rank}\,(\overset{*}{R}) = n - \mu_{R^*}$.

Proof. For all $X \in \Gamma(\mathcal{N}_{R^*})$ and $Y, Z, W \in \mathfrak{X}^h(\mathcal{T}M)$, we have

$$\begin{aligned} g(\overset{*}{R}(Y,Z)W, JX) &= \overset{*}{R}(Y,Z,W,X) \\ &= \overset{*}{R}(W,X,Y,Z) \ \text{(by (2.15))} \\ &= -\overset{*}{R}(X,W,Y,Z) \\ &= -g(\overset{*}{R}(X,W)Y, JZ) \\ &= 0 \ \text{(since } X \text{ is a nullity vector field)}, \end{aligned}$$

as wanted. □

As a direct consequence of Theorem 3.5 and the fact that $\mathcal{N}_R$ is completely integrable [18], we have the following result.

Corollary 3.10. *Let μ_{R^*} be constant on an open subset U of TM. The nullity distribution $z \mapsto \mathcal{N}_{R^*}(z)$ is completely integrable on U.*

According to the Frobenius theorem, there exists a foliation of M by μ_{R^*}-dimensional maximal connected submanifolds as leaves, such that the nullity space at a point $x \in M$ is the tangent space to the leaf at x. We call the foliation induced by the nullity distribution $\mathcal{N}_{R^*}$ the nullity foliation and denote it again by $\mathcal{N}_{R^*}$. So, by Corollary 3.10, we have the following result.

Theorem 3.11. *The leaves of the nullity foliations $\mathcal{N}_{R^*}$ and $\mathcal{N}_{\mathfrak{R}}$ are auto-parallel submanifolds with respect to the Chern connection.*

Proof. The fact that $\mathcal{N}_{R^*}$ is auto-parallel with respect to Chern connection can be proved in a similar manner as the analogous result in [18].

On the other hand, the integrability of the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ of the curvature of Barthel connection has been proved in [12]. We show that if $X, Y \in \Gamma(\mathcal{N}_{\mathfrak{R}})$, then $\overset{*}{D}_X Y \in \Gamma(\mathcal{N}_{\mathfrak{R}})$. By (2.12), we have

$$\mathfrak{S}_{X,Y,Z}\{(\overset{*}{D}_X \mathfrak{R})(Y,Z)\} = \mathfrak{S}_{X,Y,Z}\{\mathcal{C}'(Z, \mathbf{F}\mathfrak{R}(X,Y))\}.$$

Since $X, Y \in \Gamma(\mathcal{N}_{\mathfrak{R}})$, $\mathfrak{S}_{X,Y,Z}\{(\overset{*}{D}_X \mathfrak{R})(Y,Z)\} = 0$. Consequently, $\mathfrak{R}(\overset{*}{D}_X Y, Z) = 0$ for every vector field $Z \in \mathfrak{X}(TM)$ and $\overset{*}{D}_X Y \in \Gamma(\mathcal{N}_{\mathfrak{R}})$. □

Due to the torsion-freeness of the Levi-Civita connection in Riemannian geometry, the two concepts 'autoparallel submanifold' and 'totally geodesic submanifold' coincide [9]. This is not true in Finsler geometry. However, every auto-parallel submanifold is totally geodesic [4]. So, we have:

Corollary 3.12. *The leaves of the nullity foliations $\mathcal{N}_{\mathfrak{R}}$ and $\mathcal{N}_{R^*}$ are totally geodesic submanifolds with respect to the Chern connection.*

Theorem 3.13. *If $\mathfrak{R} = 0$, then the two distributions $\mathcal{N}_{R^*}$ and Ker_{R^*} coincide.*

Proof. By Proposition 3.3 (3), we always have $\mathcal{N}_{R^*} \subset \mathrm{Ker}_{R^*}$. Let $X \in \Gamma(\mathrm{Ker}_{R^*})$ and let Y, Z, W be vector fields on TM, then by (2.15), we have

$$\begin{aligned}\overset{*}{R}(Y,Z)X = 0 &\Longrightarrow g(\overset{*}{R}(Y,Z)X, JW) = 0\\ &\Longrightarrow \overset{*}{R}(Y,Z,X,W) = 0\\ &\Longrightarrow \overset{*}{R}(X,W,Y,Z) = 0\\ &\Longrightarrow g(\overset{*}{R}(X,W)Y, JZ) = 0\\ &\Longrightarrow \overset{*}{R}(X,W)Y = 0\\ &\Longrightarrow X \in \Gamma(\mathcal{N}_{R^*}),\end{aligned}$$

thus $\mathrm{Ker}_{R^*} \subset \mathcal{N}_{R^*}$. □

Theorem 3.14. *Let (M, E) be a complete Finsler manifold and U the open subset of M on which μ_{R^*} takes its minimum. If $\mathfrak{R}$ vanishes, then every integral manifold of the nullity foliation $\mathcal{N}_{R^*}$ in U is complete.*

Proof. The proof is inspired by [2], taking into account the fact that the two spaces $\mathcal{N}_{R^*}(z)$ and $\mathcal{N}_{R^*}(x)$, $x = \pi(z)$, are isomorphic. Let N be an integral manifold of the nullity foliation $\mathcal{N}_{R^*}$ in U. To prove that N is complete, it suffices to show that every geodesic $\gamma : [0, c) \to N$ on N can be extended to a geodesic $\widetilde{\gamma} : [0, \infty) \to N$ on N. Suppose that such a geodesic extension $\widetilde{\gamma}$ does not exist. As N is totally geodesic, by Corollary 3.12, γ is a geodesic on M and thus has a geodesic extension $\widetilde{\gamma} : [0, \infty) \to M$ such that $\gamma = \widetilde{\gamma} \cap N$. It follows that $p := \widetilde{\gamma}(c) \notin U$. Let $p_0 := \gamma(0) = \widetilde{\gamma}(0)$ and set $r_0 := \mu_{R^*}(p_0)$, the dimension of the nullity space $\mathcal{N}_{R^*}(p_0)$. Since μ_{R^*} is positive and minimal on U, then

$\mu_{R^*}(p) > r_0 > 0$. Now, consider a basis $B = \{e_1, ..., e_{r_0}, e_{r_0+1}, ..., e_n\}$ for $T_{p_0}M$ such that $\{e_1, ..., e_{r_0}\}$ is a basis for $\mathcal{N}_{R^*}(p_0)$ and e_1 is tangent to γ at $p_0 = \gamma(0)$. Using the system of differential equations

$$\frac{\overset{*}{D}F_i}{dt} = 0, \quad F_i(0) = e_i, \quad i = 1, 2, ..., n,$$

the basis B can be translated into a parallel frame $(F_1, ..., F_{r_0}, F_{r_0+1}, ..., F_n)$ along $\widetilde{\gamma}$. Then $(F_1, ..., F_{r_0})$ is a basis for the nullity space at every point $\widetilde{\gamma}(t)$ in $U \cap V$ for some neighborhood V of $\widetilde{\gamma}(t)$ on M. Since $\mu_{R^*}(p) > r_0$, there is a vector field F_a along $\widetilde{\gamma}$, for a fixed integer a in the range $r_0 + 1, ..., n$, such that for every $t \in [0, c)$, we have

$$F_a(\gamma(t)) \notin \mathcal{N}_{R^*}(\gamma(t)), \quad F_a(p) \in \mathcal{N}_{R^*}(p). \tag{3.5}$$

Now, let $\widehat{\widetilde{\gamma}}$ be the natural lift of $\widetilde{\gamma}$ to $\mathcal{T}M$ and $\{\widehat{F}_1, ..., \widehat{F}_{r_0}, \widehat{F}_{r_0+1}, ..., \widehat{F}_n\}$ the basis of $H_{\widehat{\widetilde{\gamma}}(t)}TM$ such that $\pi_*(\widehat{F}_i) = F_i$. Let ϕ^h_{ijk} be the functions defined by

$$\overset{*}{R}(\widehat{F}_i, \widehat{F}_j)\widehat{F}_k = \phi^h_{ijk} \frac{\partial}{\partial y^h}. \tag{3.6}$$

By (2.13), taking into account that $\mathfrak{R} = 0$, we have

$$(\overset{*}{D}_{hX}\overset{*}{R})(Y, Z) + (\overset{*}{D}_{hY}\overset{*}{R})(Z, X) + (\overset{*}{D}_{hZ}\overset{*}{R})(X, Y) = 0.$$

Plugging $\widehat{F}_1$, $\widehat{F}_i$ and $\widehat{F}_j$ instead of X, Y and Z, where $i, j = r_0 + 1, ..., n$, we get

$$(\overset{*}{D}_{\widehat{F}_1}\overset{*}{R})(\widehat{F}_i, \widehat{F}_j) + (\overset{*}{D}_{\widehat{F}_i}\overset{*}{R})(\widehat{F}_j, \widehat{F}_1) + (\overset{*}{D}_{\widehat{F}_j}\overset{*}{R})(\widehat{F}_1, \widehat{F}_i) = 0.$$

Since $\widehat{F}_1 \in \mathcal{N}_{R^*}$ and $\overset{*}{T}(hX, hY) = \mathfrak{R}(X, Y) = 0$, the last equality takes the form

$$\overset{*}{D}_{\widehat{F}_1}\overset{*}{R}(\widehat{F}_i, \widehat{F}_j) + \overset{*}{R}(\widehat{F}_j, [\widehat{F}_1, \widehat{F}_i]) + \overset{*}{R}(\widehat{F}_i, [\widehat{F}_j, \widehat{F}_1]) = 0.$$

Applying the above equation on $\widehat{F}_a$, we get

$$\overset{*}{D}_{\widehat{F}_1}\overset{*}{R}(\widehat{F}_i, \widehat{F}_j)\widehat{F}_a + \overset{*}{R}(\widehat{F}_j, [\widehat{F}_1, \widehat{F}_i])\widehat{F}_a + \overset{*}{R}(\widehat{F}_i, [\widehat{F}_j, \widehat{F}_1])\widehat{F}_a = 0. \tag{3.7}$$

Since, $[\widehat{F}_1, \widehat{F}_i]$ is horizontal, it can be written in the form $[\widehat{F}_1, \widehat{F}_i] = \xi^k_{1i}\widehat{F}_k + \xi^\mu_{1i}\widehat{F}_\mu$, where $k = r_0 + 1, ..., n$ and $\mu = 1, ..., r_0$. Consequently, by (3.6) and (3.7), noting that $\widehat{F}_\mu$ are null vector fields, we get

$$(\phi^h_{ija})' + \xi^k_{1i}\,\phi^h_{jka} - \xi^k_{1j}\,\phi^h_{ika} = 0 \tag{3.8}$$

Since F_a is a nullity vector field at p, then for the fixed index a, $\phi^h_{lma}(p) = 0$, where $l, m = r_0 + 1, ..., n$. Hence, the differential equations (3.8) with the initial condition $\phi^h_{lma}(p) = 0$ imply that the functions ϕ^h_{lma} vanish identically. As $\mathfrak{R} = 0$, Theorem 3.13 and (3.6) give rise to

$$F_a(\gamma(t)) \in \mathcal{N}_{R^*}(\gamma(t)), \text{ for all } t \in [0, c] \tag{3.9}$$

Now (3.5) and (3.9) lead to a contradiction. Consequently, γ can be extended to a geodesic $\widetilde{\gamma} : [0, \infty) \longrightarrow N$. $\square$

4. Nullity distribution of the Chern hv-curvature

In this section we investigate the nullity distribution of the hv-curvature $\overset{*}{P}$ of the Chern connection. We show, by a counterexample, that the nullity distribution $\mathcal{N}_{P^*}$ is not completely integrable. We find a sufficient condition for $\mathcal{N}_{P^*}$ to be completely integrable.

Definition 4.1. Let $\overset{*}{P}$ be the hv-curvature of the Chern connection. The nullity space of $\overset{*}{P}$ at a point $z \in TM$ is a subspace of $H_z(TM)$ defined by

$$\mathcal{N}_{P^*}(z) := \{v \in H_z(TM) | \overset{*}{P}_z(v, w) = 0, \text{ for all } w \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_{P^*}(z)$, denoted by $\mu_{P^*}(z)$, is the nullity index of $\overset{*}{P}$ at z.

Proposition 4.2. *The nullity distribution of* $\overset{*}{P}$ *satisfies:*

(1) $\mathcal{N}_{P^*} \neq \phi$.

(2) *If* $X \in \Gamma(\mathcal{N}_{P^*})$, *then* $[C, X] \in \Gamma(\mathcal{N}_{P^*})$.

(3) *If* $X \in \Gamma(\mathcal{N}_{P^*})$, *then* $\mathcal{C}'(X, Y) = 0$, *for all* $Y \in \mathfrak{X}^h(\mathcal{T}M)$.

(4) *If* $\mu_{P^*} = n$, *then* $\mathcal{N}_{R^*} = \mathcal{N}_{R^\circ}$,

where $\mathcal{N}_{R^\circ}$ *is the nullity distribution of the h-curvature of the Berwald connection [13].*

A Finsler manifold is said to be Landsbergian if the Landsberg tensor $\mathcal{C}'$ vanishes or, equivalently, $P = 0$ [11]. If the nullity index μ_{P^*} takes its maximum, then by Proposition 4.2 (3), $\mathcal{C}' = 0$. Consequently, a Finsler manifold (M, E) is Landsbergian if the nullity index μ_{P^*} achieves its maximum.

Theorem 4.3. *A Finsler manifold* (M, E) *is Landsbergian if and only if the canonical spray* S *is a nullity vector field for the the distribution* $\mathcal{N}_{P^*}$.

Proof. By (2.10), we have

$$\begin{aligned} (M, E) \text{ is Landsbergian} &\Longleftrightarrow \mathcal{C}' = 0 \\ &\Longleftrightarrow \overset{*}{P}(X, Y)S = 0 \text{ for all } X, Y \in \mathfrak{X}(TM) \\ &\Longleftrightarrow \overset{*}{P}(S, Y)X = 0 \text{ for all } X, Y \in \mathfrak{X}(TM) \\ &\Longleftrightarrow S \in \Gamma(\mathcal{N}_{P^*}), \end{aligned}$$

as was to be shown. □

Remark 4.4. The above theorem shows that the canonical spray S does not belong to the nullity distribution $\mathcal{N}_{P^*}$ except in the Landsbergian case. This is in contrast to the case of Cartan connection, where the canonical spray always belongs to the nullity distribution of the Cartan hv-curvature P.

The nullity distribution $\mathcal{N}_{P^*}$ is not completely integrable in general, as is illustrated by the following example.

Example 4.5. Let $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^1, y^2, y^3 \neq 0, y^3 \neq 4y^2\} \subset TM$, where $M := \mathbb{R}^3$. Define F on U by

$$F(x, y) := \sqrt[4]{e^{-x^1x^2}(y^1)^2(y^3)^2 e^{-\frac{y^3}{y^2}}}.$$

By Maple program and NF-package we can perform the following calculations.

```
> F0 := (exp(-x1x2)*y1^2*y3^2*exp(-y3/(y2)))^(1/2);
```

$$F0 := \sqrt{\mathrm{e}^{-x1x2}\, y1^2 y3^2 \mathrm{e}^{-\frac{y3}{y2}}}$$

Barthel connection

```
> show(N[i,-j]);
```

$N^{x1}_{x1} = -\frac{1}{2}x2y1 \qquad N^{x2}_{x2} = -\frac{4x1\,y2^3(3y2-y3)}{(-y3+4y2)^2 y3} \qquad N^{x2}_{x3} = \frac{2x1\,y2^4(2y2-y3)}{(-y3+4y2)^2 y3^2}$

$N^{x3}_{x2} = -\frac{x1\,y3(2y2-y3)y2}{(-y3+4y2)^2} \qquad N^{x3}_{x3} = -\frac{2x1\,y2^3}{(-y3+4y2)^2}$

Chern hv-curvature $\overset{*}{P}$:

```
> definetensor(Pchern[i,-h,-j,-k] = tddiff(Gammastar[i,-h,-j],
> Y[k]),symm[2,3]);
> show(Pchern[h, -i, -j, -k]);
```

$$Pchern^{x2}_{x2x2x2} = -\frac{12x1y2\left(-y3^3+8y3^2y2-24y2^2y3+24y2^3\right)}{y3(-y3+4y2)^4}$$

$$Pchern^{x2}_{x2x2x3} = \frac{12x1y2^2\left(-y3^3+8y3^2y2-24y2^2y3+24y2^3\right)}{y3^2(-y3+4y2)^4}$$

$$Pchern^{x3}_{x2x2x2} = \frac{6x1y3\left(y3^2-4y2y3+8y2^2\right)}{(-y3+4y2)^4}$$

$$Pchern^{x3}_{x2x2x3} = -\frac{6x1y2\left(y3^2-4y2y3+8y2^2\right)}{(-y3+4y2)^4}$$

$$Pchern^{x2}_{x2x3x2} = \frac{6x1y2^2\left(-28y2^2y3+32y2^3+8y3^2y2-y3^3\right)}{y3^2(-y3+4y2)^4}$$

$$Pchern^{x2}_{x2x3x3} = -\frac{6x1y2^3\left(-28y2^2y3+32y2^3+8y3^2y2-y3^3\right)}{y3^3(-y3+4y2)^4}$$

$Pchern^{x3}_{x2x3x2} = -\frac{12x1y2^2y3}{(-y3+4y2)^4} \qquad Pchern^{x3}_{x2x3x3} = \frac{12x1y2^3}{(-y3+4y2)^4}$

$Pchern^{x2}_{x3x3x2} = -\frac{48x1y2^5(2y2-y3)}{y3^3(-y3+4y2)^4} \qquad Pchern^{x2}_{x3x3x3} = \frac{48x1y2^6(2y2-y3)}{y3^4(-y3+4y2)^4}$

$Pchern^{x3}_{x3x3x2} = -\frac{6x1y2^2\left(-8y2y3+8y2^2+y3^2\right)}{y3(-y3+4y2)^4} \qquad Pchern^{x3}_{x3x3x3} = \frac{6x1y2^3\left(-8y2y3+8y2^2+y3^2\right)}{y3^2(-y3+4y2)^4}$

$\overset{*}{P}$-nullity vectors:

```
> definetensor(PchernW[h, -i, -k] = Pchern[h, -i, -j, -k]*w[j]);
> show(PchernW[h, -i, -k]);
```

$$PchernW^{x2}_{x2x2} = -\frac{12x1y2\left(8y3^2y2-y3^3-24y2^2y3+24y2^3\right)w^{x2}}{y3(-y3+4y2)^4} + \frac{6x1y2^2\left(32y2^3+8y3^2y2-28y2^2y3-y3^3\right)w^{x3}}{y3^2(-y3+4y2)^4}$$

$$PchernW^{x2}_{x2x3} = \frac{12x1y2^2\left(8y3^2y2-y3^3-24y2^2y3+24y2^3\right)w^{x2}}{y3^2(-y3+4y2)^4} - \frac{6x1y2^3\left(32y2^3-28y2^2y3+8y3^2y2-y3^3\right)w^{x3}}{y3^3(-y3+4y2)^4}$$

$$PchernW^{x2}_{x3x2} = \frac{6x1y2^2\left(-28y2^2y3+32y2^3+8y3^2y2-y3^3\right)w^{x2}}{y3^2(-y3+4y2)^4} - \frac{48x1y2^5(2y2-y3)w^{x3}}{y3^3(-y3+4y2)^4}$$

$$PchernW^{x2}_{x3x3} = -\frac{6y2^3\left(-28x1y2^2y3+32y2^3+8y3^2y2-y3^3\right)w^{x2}}{y3^3(-y3+4y2)^4} + \frac{48x1y2^6(2y2-y3)w^{x3}}{y3^4(-y3+4y2)^4}$$

$$PchernW^{x3}_{x2x2} = \frac{6x1y3\left(y3^2-4y2y3+8y2^2\right)w^{x2}}{(-y3+4y2)^4} - \frac{12x1y2^2y3w^{x3}}{(-y3+4y2)^4}$$

$$PchernW^{x3}_{x2x3} = -\frac{6x1y2\left(y3^2-4y2y3+8y2^2\right)w^{x2}}{(-y3+4y2)^4} + \frac{12x1y2^3w^{x3}}{(-y3+4y2)^4}$$

$$PchernW^{x3}_{x3x2} = -\frac{12x1y2^2y3w^{x2}}{(-y3+4y2)^4} - \frac{6x1y2^2\left(-8y2y3+8y2^2+y3^2\right)w^{x3}}{y3(-y3+4y2)^4}$$

$$PchernW^{x3}_{x3x3} = \frac{12x1y2^3w^{x2}}{(-y3+4y2)^4} + \frac{6x1y2^3\left(-8y2y3+8y2^2+y3^2\right)w^{x3}}{y3^2(-y3+4y2)^4}$$

Putting $PchernW^h_{ij} = 0$, we get a system of algebraic equations. This system has a solution if $y_3 = 2y_2$ and $x^1 > 0$: $W^1 = s$, $W^2 = t$, $W^3 = 2t$, $s, t \in \mathbb{R}$. Hence, a $\overset{*}{P}$-nullity vector must have the form $W = sh_1 + t(h_2 + 2h_3)$, where the horizontal basis vector fields h_1, h_2, h_3 are given by $h_1 = \frac{\partial}{\partial x_1} + \frac{x_2 y_1}{2}\frac{\partial}{\partial y_1}$, $h_2 = \frac{\partial}{\partial x_2} + \frac{x_1 y_2}{2}\frac{\partial}{\partial y_2}$, $h_3 = \frac{\partial}{\partial x_3} + \frac{x_1 y_2}{2}\frac{\partial}{\partial y_3}$. Now, take $X, Y \in \mathcal{N}_{P^*}$ such that $X = h_1$, $Y = h_2 + 2h_3$. Hence, the bracket $[X, Y] = [h_1, h_2 + 2h_3] = -\frac{y_1}{2}\frac{\partial}{\partial y_1} + \frac{y_2}{2}\frac{\partial}{\partial y_2} + \frac{y_2}{2}\frac{\partial}{\partial y_3}$ is vertical and, consequently, $\mathcal{N}_{P^*}$ is not completely integrable.

Theorem 4.6. *Let μ_{P^*} be constant on an open subset U of TM. The nullity distribution $\mathcal{N}_{P^*}$ is completely integrable on U if and only if $\mathfrak{R}(X, Y) = 0$ and $(\overset{*}{D}_{JZ}\overset{*}{R})(X, Y) = 0$, for all $X, Y \in \Gamma(\mathcal{N}_{P^*})$.*

Proof. Necessity. Let $\mathcal{N}_{P^*}$ be completely integrable. Then, if $X, Y \in \Gamma(\mathcal{N}_{P^*})$, the bracket $[hX, hY]$ is horizontal, thus, $\mathfrak{R}(X, Y) = 0$. Also, by (2.14) and the fact that $\overset{*}{P}([hX, hY], Z) = (\overset{*}{D}_{hX}\overset{*}{P})(Y, Z) - (\overset{*}{D}_{hY}\overset{*}{P})(X, Z) = 0$ (by (2.6)), we have
$(\overset{*}{D}_{JZ}\overset{*}{R})(X, Y) = 0$, for all $X, Y \in \Gamma(\mathcal{N}_{P^*})$, for all $Z \in \mathfrak{X}(TM)$.

Sufficiency. Let $\mathfrak{R}(X, Y) = 0$ and $(\overset{*}{D}_{JZ}\overset{*}{R})(X, Y) = 0$ for all $X, Y \in \Gamma(\mathcal{N}_{P^*})$. As $0 = \mathfrak{R}(X, Y) = -v[hX, hY] = -v[X, Y]$, the bracket $[X, Y]$ is horizontal. Making use of (2.6) and (2.14), we get

$$\begin{aligned}(\overset{*}{D}_{hX}\overset{*}{P})(Y, Z) - (\overset{*}{D}_{hY}\overset{*}{P})(X, Z) = 0 &\Longrightarrow \overset{*}{P}(\overset{*}{D}_X Y - \overset{*}{D}_Y X, Z) = 0\\ &\Longrightarrow \overset{*}{P}([X, Y] + \mathfrak{R}(X, Y), Z) = 0\\ &\Longrightarrow \overset{*}{P}([X, Y], Z) = 0\\ &\Longrightarrow [X, Y] \in \Gamma(\mathcal{N}_{P^*}).\end{aligned}$$

Hence $\mathcal{N}_{P^*}$ is completely integrable. □

By the property $\overset{*}{P}(X, Y)Z = \overset{*}{P}(Z, Y)X$ we have the following result.

Theorem 4.7. *The nullity distribution $\mathcal{N}_{P^*}$ and the kernel distribution Ker_{P^*} coincide.*

A Finsler manifold in which the Chern hv-curvature tensor $\overset{*}{P}$ vanishes is called a Berwald space [11]. It is well known that every Berwald space is a Landsberg space, but it is not known whether the converse is true. In [10], Shen introduced a class of non-regular Finsler metrics which is Landsbergian and not Berwaldian. The calculations are not easy, especially, if one wants to study some concrete examples. Here, by using Maple program together with the results of [10] and [15], we give a simple class of proper non-regular non Berwaldian Landsbergian spaces.

Example 4.8. Let $M = \mathbb{R}^3$, $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^2 > 0, y^3 > 0\} \subset TM$. Define F on U by

$$F(x,y) := f(x^1)\sqrt{(y^1)^2 + y^2y^3 + y^1\sqrt{y^2y^3}}\, e^{\frac{1}{\sqrt{3}}\arctan\left(\frac{2y^1}{\sqrt{3y^2y^3}}+\frac{1}{\sqrt{3}}\right)}.$$

The idea is to compute the Landsberg tensor L_{ijk} and the Berwald tensor G^h_{ijk} which are locally given by

$$L_{ijk} := \frac{F}{2}\frac{\partial F}{\partial y^h}G^h_{ijk}, \quad G^h_{ijk} := \frac{\partial^3 G^h}{\partial y^i \partial y^j \partial y^k}.$$

Then, we show that the Landsberg tensor vanishes identically while there are some non vanishing components of the Berwald tensor (for simplicity we consider only one nonzero component and check it at a point) .

```
> restart
```

$$F := f(x1)\sqrt{y1^2 + y2\, y3 + y1\sqrt{y2\, y3}}\; e^{\frac{\sqrt{3}}{3}\arctan\left(\frac{2}{3}\frac{y1\sqrt{3}}{\sqrt{y2\, y3}}+\frac{\sqrt{3}}{3}\right)}$$

```
> simplify(G1)
```

$$G1 := \frac{1}{2}\frac{\left(y1^2 - y2\, y3\right)\frac{d}{dx1}f(x1)}{f(x1)}$$

```
> simplify(G2)
```

$$G2 := \frac{1}{2}\left(\frac{d}{dx1}f(x1)\right) y2^2 y3\, (92\, y2^5 y3^5 y1^3 + 408\, y2^3 y3^3 y1^7 + 230\, y2^2 y3^2 y1^9$$
$$+48\, y2\, y3\, y1^{11} + 8\, y2^6 y3^6 y1 + 306\, y2^4 y3^4 y1^5 + 2\, y1^{13} + (y2^6 y3^6 + 33\, y2^5 y3^5 y1^2$$
$$+190\, y2^4 y3^4 y1^4 + 121\, y2\, y3\, y1^{10} + 342\, y2^2 y3^2 y1^8 + 393\, y2^3 y3^3 y1^6$$
$$+13\, y1^{12})\sqrt{y2\, y3})/(f(x1)\,(50\, y2^5 y3^5 y1^3 + 126\, y2^3 y3^3 y1^7 + 50\, y2^2 y3^2 y1^9$$
$$+6\, y2\, y3\, y1^{11} + 6\, y2^6 y3^6 y1 + 126\, y2^4 y3^4 y1^5 + (y2^6 y3^6 + 21\, y2^5 y3^5 y1^2 + y1^{12}$$
$$+90\, y2^4 y3^4 y1^4 + 21\, y2\, y3\, y1^{10} + 90\, y2^2 y3^2 y1^8 + 141\, y2^3 y3^3 y1^6)\sqrt{y2\, y3})\sqrt{y2\, y3})$$

```
> simplify(G3)
```

$$G3 := \frac{1}{2}\left(\frac{d}{dx1}f(x1)\right) y3^2 y2\, (408\, y2^3 y3^3 y1^7 + 230\, y2^2 y3^2 y1^9 + 8\, y2^6 y3^6 y1$$
$$+2\, y1^{13} + (33\, y1^2 y2^5 y3^5 + 393\, y1^6 y2^3 y3^3 + 342\, y1^8 y3^2 y2^2 + 121\, y1^{10} y3\, y2$$

$+190\, y1^4 y3^4 y2^4 + 13\, y1^{12} + y2^6 y3^6)\sqrt{y2\, y3} + 92\, y2^5 y3^5 y1^3$

$+306\, y2^4 y3^4 y1^5 + 48\, y2\, y3\, y1^{11})/(f\,(x1)\,(50\, y2^2 y3^2 y1^9 + 6\, y2\, y3\, y1^{11}$

$+126\, y2^3 y3^3 y1^7 + 6\, y2^6 y3^6 y1 + (90\, y1^4 y3^4 y2^4 + 141\, y1^6 y2^3 y3^3 \sqrt{y2\, y3}$

$+21\, y1^{10} y3\, y2 + 90\, y1^8 y3^2 y2^2 + 21\, y1^2 y2^5 y3^5 + y1^{12} + y2^6 y3^6)\sqrt{y2\, y3}$

$+126\, y2^4 y3^4 y1^5 + 50\, y2^5 y3^5 y1^3)\sqrt{y2\, y3})$

```
> y1 := y[1]; 1; y2 := y[2]; 1; y3 := y[3]
```

$$y1 := y_1$$
$$y2 := y_2$$
$$y3 := y_3$$

```
> printlevel := 3;
> for i to 3 do
> for j to i do
> for k to j do
> Landsberg[i,j,k] := simplify((diff(F,y1))*(diff(G1,y[i],y[j],y[k]))
> +(diff(F,y2))*(diff(G2,y[i],y[j],y[k]))
> +(diff(F,y3))*(diff(G3,y[i],y[j],y[k])));
> end do;
> end do;
> end do;
```

$$Landsberg_{1,1,1} := 0$$
$$Landsberg_{2,1,1} := 0$$
$$Landsberg_{2,2,1} := 0$$
$$Landsberg_{2,2,2} := 0$$
$$Landsberg_{3,1,1} := 0$$
$$Landsberg_{3,2,1} := 0$$
$$Landsberg_{3,2,2} := 0$$
$$Landsberg_{3,3,1} := 0$$
$$Landsberg_{3,3,2} := 0$$
$$Landsberg_{3,3,3} := 0$$

```
> Berwald[2, 2, 2] := simplify(diff(G2, y[2], y[2], y[2]))
```

$Berwald_{2,2,2} := \frac{-3}{16}\frac{d\,f(x1)}{dx1} y_2 y_3{}^3((123286440\, y_2{}^5 y_3{}^5 y_1{}^{37} + 6190070040\, y_2{}^8 y_3{}^8 y_1{}^{31}$

$+13029127584\, y_2{}^9 y_3{}^9 y_1{}^{29} + 21263575256\, y_2{}^{13} y_3{}^{13} y_1{}^{21} + 13029127584\, y_2{}^{14} y_3{}^{14}$

$y_1{}^{19} + 6190070040\, y_2{}^{15} y_3{}^{15} y_1{}^{17} + 2252056776\, y_2{}^7 y_3{}^7 y_1{}^{33} + 2576\, y_2{}^{22} y_3{}^{22} y_1{}^3$

$+1621224\, y_2{}^{20} y_3{}^{20} y_1{}^7 + 21263575256\, y_2{}^{10} y_3{}^{10} y_1{}^{27} + 27114249960\, y_2{}^{12} y_3{}^{12} y_1{}^{23}$

$+27114249960\, y_2{}^{11} y_3{}^{11} y_1{}^{25} + 2576\, y_2 y_3 y_1{}^{45} + y_2{}^{24} y_3{}^{24} + 17363896\, y_2{}^4 y_3{}^4 y_1{}^{39}$

$+24\, y_1{}^{47} + 91080\, y_2{}^{21} y_3{}^{21} y_1{}^5 + 17363896\, y_2{}^{19} y_3{}^{19} y_1{}^9 + 91080\, y_2{}^2 y_3{}^2 y_1{}^{43})\sqrt{y_2 y_3}$

$+412896\, y_2{}^{21} y_3{}^{21} y_1{}^6 + y_1{}^{48} + 412896\, y_2{}^3 y_3{}^3 y_1{}^{42} + 300\, y_2{}^{23} y_3{}^{23} y_1{}^2 + 16974\, y_2{}^{22} y_3{}^{22} y_1{}^4$

$+5612805\, y_2{}^{20} y_3{}^{20} y_1{}^8 + 48497064\, y_2{}^{19} y_3{}^{19} y_1{}^{10} + 287134346\, y_2{}^{18} y_3{}^{18} y_1{}^{12}$

$+1222297740\, y_2{}^{17} y_3{}^{17} y_1{}^{14} + 3864164634\, y_2{}^{16} y_3{}^{16} y_1{}^{16} + 9276875476\, y_2{}^{15} y_3{}^{15} y_1{}^{18}$

$+17172595110\, y_2{}^{14} y_3{}^{14} y_1{}^{20} + 24755608584\, y_2{}^{13} y_3{}^{13} y_1{}^{22} + 27948336381\, y_2{}^{12} y_3{}^{12} y_1{}^{24}$

$+24755608584\, y_2{}^{11} y_3{}^{11} y_1{}^{26} + 17172595110\, y_2{}^{10} y_3{}^{10} y_1{}^{28} + 9276875476\, y_2{}^9 y_3{}^9 y_1{}^{30}$

$+3864164634\, y_2{}^8 y_3{}^8 y_1{}^{32} + 1222297740\, y_2{}^7 y_3{}^7 y_1{}^{34} + 287134346\, y_2{}^6 y_3{}^6 y_1{}^{36}$

$+300\, y_3 y_2 y_1{}^{46} + 48497064\, y_2{}^5 y_3{}^5 y_1{}^{38} + 5612805\, y_2{}^4 y_3{}^4 y_1{}^{40} + 16974\, y_3{}^2 y_2{}^2 y_1{}^{44}$

$+(615939264\, y_2{}^6 y_3{}^6 y_1{}^{35} + 123286440\, y_2{}^{18} y_3{}^{18} y_1{}^{11} + 615939264\, y_2{}^{17} y_3{}^{17} y_1{}^{13}$

$+1621224\, y_2{}^3 y_3{}^3 y_1{}^{41} + 2252056776\, y_2{}^{16} y_3{}^{16} y_1{}^{15} + 24\, y_2{}^{23} y_3{}^{23} y_1)\sqrt{y_2 y_3})$

$/(\sqrt{y_2 y_3} f(x1)\,(50\, y_2{}^5 y_3{}^5 y_1{}^3 + 126\, y_2{}^3 y_3{}^3 y_1{}^7 + 50\, y_2{}^2 y_3{}^2 y_1{}^9$

$+6\, y_2 y_3 y_1{}^{11} + 6\, y_2{}^6 y_3{}^6 y_1 + 126\, y_2{}^4 y_3{}^4 y_1{}^5 + (y_2{}^6 y_3{}^6 + 21\, y_2{}^5 y_3{}^5 y_1{}^2$

$+90\, y_2{}^4 y_3{}^4 y_1{}^4 + 21\, y_2 y_3 y_1{}^{10} + 90\, y_2{}^2 y_3{}^2 y_1{}^8 + 141\, y_2{}^3 y_3{}^3 y_1{}^6 + y_1{}^{12})\sqrt{y_2 y_3})^4$

```
> y[1] := 0; 1; y[2] := 1; 1; y[3] := 1;
```

$$y_1 := 0$$
$$y_2 := 1$$
$$y_3 := 1$$

```
> simplify(Berwald[2, 2, 2])
```

$$\frac{-3}{16}\frac{\frac{d}{dx1} f(x1)}{f(x1)}$$

By Example 4.8, for any non constant positive smooth function f on $\mathbb{R}$, the Landsberg tensor of (M, F) vanishes (or equivalently, the hv-curvature P of the Cartan connection vanishes) and hence the class is Landsbergian. On the other hand, the hv-curvature $\overset{*}{P}$ of the Chern connection does not vanish and hence the class is not Berwaldian. So we can confirm:

Theorem 4.9. *There are non-regular Landsberg spaces which are not Berwaldian.*

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