

New Finsler Package

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Abstract

The book “Handbook of Finsler geometry” has been included with a CD containing an elegant Maple package, FINSLER, for calculations in Finsler geometry. Using this package, an example concerning a Finsler generalization of Einstein’s vacuum field equations was treated. In this example, the calculation of the components of the hv-curvature of Cartan connection leads to wrong expressions. On the other hand, the FINSLER package works only in dimension four. We introduce a new Finsler package in which we fix the two problems and solve them. Moreover, we extend this package to compute not only the geometric objects associated with Cartan connection but also those associated with Berwald, Chern and Hashiguchi connections in any dimension. These improvements have been illustrated by a concrete example. Furthermore, the problem of simplifying tensor expressions is treated. This paper is intended to make calculations in Finsler geometry more easier and simpler.

Keywords: Maple program, FINSLER package, computer algebra, Finsler space, fundamental Finsler connections, tensor simplification.

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1. Introduction

Antonelli et. al. have had a good contribution in Finsler geometry computations using MAPLE (cf. [3], [4],[9]). Rutz and Portugal [11] have introduced the remarkable FINSLER package [10] (it is also included in a CD with the “Handbook of Finsler geometry” [2]). They illustrated how to use this package by an example related to general relativity.

During the preparation of our paper [13], searching for some Finsler counterexamples, we have encountered some problems concerning FINSLER package. In fact, we studied an example in which the coefficients of Berwald connection are functions of positional argument x^i only. Hence, the space under consideration is Berwaldian and is thus Landesbergian. It is well known that for a Landesberg space the hv-curvature P_{ijk}^h of Cartan connection vanishes. But according to the package, the program calculated non-vanishing components of P_{ijk}^h . After a deep reading of the source code (Finsler.mpl), we discovered an error in the definition of P_{ijk}^h (similar error is found in “Handbook of Finsler geometry, II”, page 1154). Another problem with this package is that of dimension. If one considers a Finsler space of dimension three, the package can not compute the components of the h-curvature R_{ijk}^h and hv-curvature P_{ijk}^h of Cartan connection.

In our modified package we solve the above two mentioned problems. We illustrate our modification and extension of the FINSLER package by treating a concrete example of a three dimensional Finsler space. We calculate the curvature tensors of the four fundamental connections of Finsler geometry, namely, Cartan, Berwald, Chern and Hashiguchi connections. The geometric objects, not defined in the FINSLER package, can be added in a similar manner. We also propose a technique for simplifying tensor expressions.

2. Notations and preliminaries

In this section, we give a brief introduction to Finsler connections. For more details, we refer, for example, to [2], [5], [6] and [12].

Let (M, F) be a Finsler manifold. Let (x^i) be the coordinates of any point of M and (y^i) a supporting element at this point. Partial differentiation with respect to x^i (resp. y^i) will be denoted by ∂_i (resp. $\dot{\partial}_i$). We use the following notations:

- $l_i := \dot{\partial}_i F = g_{ij} l^j = g_{ij} \frac{y^j}{F}$: the normalized supporting element; $l^i := \frac{y^i}{F}$,
- $l_{ij} := \dot{\partial}_i l_j$,
- $h_{ij} := F l_{ij} = g_{ij} - l_i l_j$: the angular metric tensor,
- $C_{ijk} := \frac{1}{2} \dot{\partial}_k g_{ij} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2$: the Cartan tensor,
- $C_{jk}^i := g^{ri} C_{rjk}$: the (h)hv-torsion tensor,
- $\gamma_{jk}^i(x, y) := \frac{1}{2} g^{ir} (\partial_j g_{kr} + \partial_k g_{jr} - \partial_r g_{jk})$: the Christoffel symbols with respect to ∂_i ,
- $G^i(x, y) := \frac{1}{2} \gamma_{jk}^i y^j y^k$: the components of the canonical spray associated with (M, F) ,
- $N_j^i := \dot{\partial}_j G^i$: the Barthel (or Cartan nonlinear) connection associated with (M, F) ,
- $G_{jh}^i := \dot{\partial}_h N_j^i = \dot{\partial}_h \dot{\partial}_j G^i$: the coefficients of Berwald connection,
- $\delta_i := \partial_i - N_i^r \dot{\partial}_r$: the basis vector fields of the horizontal bundle,

$\Gamma_{jk}^i(x, y) := \frac{1}{2}g^{ir}(\delta_j g_{kr} + \delta_k g_{jr} - \delta_r g_{jk})$: the Christoffel symbols with respect to δ_i .

A **Finsler connection** [1] on M is a triple $F\Gamma = (\mathbf{F}_{jk}^i(x, y), \mathbf{N}_j^i(x, y), \mathbf{C}_{jk}^i(x, y))$ such that, under a change of coordinates $(x^i) \rightarrow (\tilde{x}^k)$, the geometric objects $\mathbf{F}_{jk}^i(x, y)$, $\mathbf{N}_j^i(x, y)$ and \mathbf{C}_{jk}^i transform respectively as follows:

$$\begin{aligned}\tilde{\mathbf{F}}_{ij}^k &= \frac{\partial \tilde{x}^k}{\partial x^l} \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} \mathbf{F}_{pq}^l + \frac{\partial^2 x^p}{\partial \tilde{x}^i \partial \tilde{x}^j} \frac{\partial \tilde{x}^k}{\partial x^p}, \\ \tilde{\mathbf{N}}_j^i &= \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \tilde{x}^j} \mathbf{N}_q^p + \frac{\partial x^p}{\partial \tilde{x}^j} \frac{\partial^2 \tilde{x}^i}{\partial x^p \partial x^q} y^q, \quad \tilde{\mathbf{C}}_{ij}^k = \frac{\partial \tilde{x}^k}{\partial x^l} \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} \mathbf{C}_{pq}^l.\end{aligned}$$

Moreover, $F\Gamma$ defines two types of covariant derivatives:

$$\begin{aligned}X_{j|k}^i &:= \delta_k X_j^i + X_j^r \mathbf{F}_{rk}^i - X_r^i \mathbf{F}_{jk}^r, \\ X_j^i|_k &:= \dot{\partial}_k X_j^i + X_j^r \mathbf{C}_{rk}^i - X_r^i \mathbf{C}_{jk}^r.\end{aligned}$$

Let $F\Gamma = (\mathbf{F}_{jk}^i, \mathbf{N}_j^i, \mathbf{C}_{jk}^i)$ be an arbitrary Finsler connection. The (h)h-, (h)hv-, (v)h-, (v)hv- and (v)v-torsion tensors of $F\Gamma$ are given respectively by [7]:

$$\begin{aligned}\mathbf{T}_{jk}^i &= \mathbf{F}_{jk}^i - \mathbf{F}_{kj}^i, & \mathbf{C}_{jk}^i &= \text{the connection parameters } \mathbf{C}_{jk}^i, \\ \mathbf{R}_{jk}^i &= \delta_k \mathbf{N}_j^i - \delta_j \mathbf{N}_k^i, & \mathbf{P}_{jk}^i &= \dot{\partial}_k \mathbf{N}_j^i - \mathbf{F}_{jk}^i, & \mathbf{S}_{jk}^i &= \mathbf{C}_{jk}^i - \mathbf{C}_{kj}^i.\end{aligned}$$

and the h-, hv- and v-curvature tensors of $F\Gamma$ are given respectively by [7]:

$$\begin{aligned}\mathbf{R}_{hjk}^i &= \mathfrak{A}_{(j,k)}\{\delta_k \mathbf{F}_{hj}^i + \mathbf{F}_{hj}^m \mathbf{F}_{mk}^i\} + \mathbf{C}_{hm}^i \mathbf{R}_{jk}^m, \\ \mathbf{P}_{hjk}^i &= \dot{\partial}_k \mathbf{F}_{hj}^i - \mathbf{C}_{hk|j}^i + \mathbf{C}_{hm}^i \mathbf{P}_{jk}^m, & \mathbf{S}_{hjk}^i &= \mathfrak{A}_{(j,k)}\{\dot{\partial}_k \mathbf{C}_{hj}^i + \mathbf{C}_{hk}^m \mathbf{C}_{mj}^i\},\end{aligned}$$

where $\mathfrak{A}_{(j,k)}\{A_{jk}\} := A_{jk} - A_{kj}$.

The **Cartan connection** is given by $C\Gamma = (\Gamma_{jk}^i, N_j^i, C_{jk}^i)$, where Γ_{jk}^i , N_j^i and C_{jk}^i are as defined above. The (h)hv-, (v)h- and (v)hv-torsion tensors of $C\Gamma$ are:

$$C_{jk}^i = \frac{1}{2}g^{ir} \dot{\partial}_k g_{rj}, \quad R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i, \quad P_{jk}^i = \dot{\partial}_k N_j^i - \Gamma_{jk}^i.$$

The h-, hv- and v-curvature tensors of $C\Gamma$ are:

$$\begin{aligned}R_{hjk}^i &= \mathfrak{A}_{(j,k)}\{\delta_k \Gamma_{hj}^i + \Gamma_{hj}^m \Gamma_{mk}^i\} + C_{hm}^i R_{jk}^m, \\ P_{hjk}^i &= \dot{\partial}_k \Gamma_{hj}^i - C_{hk|j}^i + C_{hm}^i P_{jk}^m, & S_{hjk}^i &= \mathfrak{A}_{(j,k)}\{C_{hk}^m C_{mj}^i\}.\end{aligned}$$

The **Berwald connection** is given by $B\Gamma = (G_{jk}^i, N_j^i, 0)$. The associated geometric objects will be marked by a circle. The (v)h-torsion tensor of $B\Gamma$ is given by:

$$\mathring{R}_{jk}^i = R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i.$$

The h-, and hv-curvature tensors of $B\Gamma$ are:

$$\mathring{P}_{hjk}^i = \mathfrak{A}_{(j,k)}\{\delta_k G_{hj}^i + G_{hj}^m G_{mk}^i\}, \quad \mathring{P}_{hjk}^i = \dot{\partial}_k G_{hj}^i.$$

The **Chern (Rund) connection** is given by $R\Gamma = (\Gamma_{jk}^i, N_j^i, 0)$. The associated geometric objects will be marked by a star. The (v)h- and (v)hv-torsion tensors of $R\Gamma$ are:

$$\overset{\star}{R}_{jk}^i = R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i, \quad \overset{\star}{P}_{jk}^i = P_{jk}^i = \dot{\partial}_k N_j^i - \Gamma_{jk}^i.$$

The h- and hv-curvature tensors of $R\Gamma$ are:

$$\overset{\star}{R}_{hjk}^i = \mathfrak{A}_{(j,k)} \{ \delta_k \Gamma_{hj}^i + \Gamma_{hj}^m \Gamma_{mk}^i \}, \quad \overset{\star}{P}_{hjk}^i = \dot{\partial}_k \Gamma_{hj}^i.$$

The **Hashiguchi connection** is given by $H\Gamma = (G_{jk}^i, N_j^i, C_{jk}^i)$. The associated geometric objects will be marked by an asterisk. The (h)hv- and (v)h-torsion tensors of $H\Gamma$ are:

$$\overset{\star}{C}_{jk}^i = C_{jk}^i, \quad \overset{\star}{R}_{jk}^i = R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i.$$

The h-, hv- and v-curvature tensors of $H\Gamma$ are:

$$\overset{\star}{R}_{hjk}^i = \mathfrak{A}_{(j,k)} \{ \delta_k G_{hj}^i + G_{hj}^m G_{mk}^i \} + C_{hm}^i R_{jk}^m,$$

$$\overset{\star}{P}_{hjk}^i = \dot{\partial}_k G_{hj}^i - C_{hk|j}^i, \quad \overset{\star}{S}_{hjk}^i = \mathfrak{A}_{(j,k)} \{ C_{hk}^m C_{mj}^i \}.$$

Table 1: Fundamental linear connections [12]

Connection		Cartan	Berwald	Chern (Rund)	Hashiguchi
	$(\mathbf{F}_{ij}^h, \mathbf{N}_i^h, \mathbf{C}_{ij}^h)$	$(\Gamma_{ij}^h, N_i^h, C_{ij}^h)$	$(G_{ij}^h, N_i^h, 0)$	$(\Gamma_{ij}^h, N_i^h, 0)$	$(G_{ij}^h, N_i^h, C_{ij}^h)$
Torsions	(h)h-torsion \mathbf{T}_{jk}^i	0	0	0	0
	(h)hv-torsion \mathbf{C}_{jk}^i	C_{jk}^i	0	0	C_{jk}^i
	(v)h-torsion \mathbf{R}_{jk}^i	R_{jk}^i	$\overset{\circ}{R}_{jk}^i = R_{jk}^i$	$\overset{\star}{R}_{jk}^i = R_{jk}^i$	$\overset{\star}{R}_{jk}^i = R_{jk}^i$
	(v)hv-torsion \mathbf{P}_{jk}^i	$P_{jk}^i = C_{jk h}^i y^h$	0	$\overset{\star}{P}_{jk}^i = P_{jk}^i$	0
	(v)v-torsion \mathbf{S}_{jk}^i	0	0	0	0
Curvatures	h-curvature \mathbf{R}_{ijk}^h	R_{ijk}^h	$\overset{\circ}{R}_{ijk}^h$	$\overset{\star}{R}_{ijk}^h$	$\overset{\star}{R}_{ijk}^h$
	hv-curvature \mathbf{P}_{ijk}^h	P_{ijk}^h	$\overset{\circ}{P}_{ijk}^h$	$\overset{\star}{P}_{ijk}^h$	$\overset{\star}{P}_{ijk}^h$
	v-curvature \mathbf{S}_{ijk}^h	S_{ijk}^h	0	0	$\overset{\star}{S}_{ijk}^h = S_{ijk}^h$
Covariant derivatives	h-cov. der.	$K_{j k}^i$	$K_{j k}^i$	$K_{j k}^i = K_{j k}^i$	$K_{j k}^i = K_{j k}^i$
	v-cov. der.	$K_{j k}^i$	$K_{j k}^i = \dot{\partial}_k K_j^i$	$K_{j k}^i = K_{j k}^i$	$K_{j k}^i = K_{j k}^i$

3. Notes on the FINSLER package

In [11], Rutz and Portugal discussed and applied the FINSLER package they introduced in [10]. This package is an extension of the RIEMANN package [9]. The FINSLER package is included in a CD with the book [2], where an interesting example related to general relativity, namely, a family of metrics known as the Schwarzschild solution to Einstein's field equations, has been treated. Important geometric objects and, in particular, the three curvature tensors of Cartan connection have been computed.

When performing some applications using the FINSLER package, we have encountered some problems. To show one of these problems, let us consider the following example. Let $M = \mathbb{R}^4$, $U = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : x_1 \neq 0; y_4 \neq 0, y_1^2 + y_2^2 + y_3^2 \neq 0\}$. Let F be the Finsler structure defined on the open subset U of TM by:

$$F = \sqrt{x_1 y_4 \sqrt{y_1^2 + y_2^2 + y_3^2}}.$$

Based on this package, the non-vanishing coefficients of Berwald connection are as follows:

$$G_{11}^1 = G_{12}^2 = G_{13}^3 = \frac{1}{x_1}, \quad G_{22}^1 = G_{33}^1 = -\frac{1}{x_1}.$$

This shows that the coefficients of Berwald connection are functions of the positional argument x^i only. Hence, the space under consideration is Berwaldian and is thus Landesbergian. Consequently, the hv-curvature P_{ijk}^h of Cartan connection should vanish identically. However, the FINSLER package calculated non-vanishing components of P_{ijk}^h .

After a deep study of the source code (Finsler.mpl), we have discovered some wrong indices in the definition of P_{ijk}^h . (Similar error is found in [2], page 1154). Another problem with this package is the problem of dimension. If one considers a three dimensional Finsler space, the package can not compute the components of the hh-curvature R_{ijk}^h and hv-curvature P_{ijk}^h of Cartan connection. The package response is that these objects are outside dimension.

Summing up, we have two problems with the Rutz and Portugal's package. The first is the wrong calculations of the curvature P_{ijk}^h . The second is the disability of computing R_{ijk}^h and P_{ijk}^h in dimensions different from 4.

4. Improvement of the package

In this section, we solve the two above mentioned problems. Moreover, we extend the package in order to compute various geometric objects associated not only with Cartan connection but also with the other fundamental connections in Finsler geometry. And this is for any dimension. Other geometric objects can be similarly added to the package. We illustrate these tasks using a concrete example.

Rutz and Portugal have illustrated how to use the package [11]. However, let us recall some instructions to make the use of this package easier. When we write, for example, $N[i,-j]$ we mean N_j^i , i.e., a positive (resp. negative) index means that it is a contravariant (resp. covariant) index. If one wants to lower or raise an index by the metric or the inverse

metric, he just changes its sign from positive to negative or vice versa. The command $tdiff(N[i,-j], X[k])$ means $\partial_k N_j^i$, the command $tddiff(N[i,-j], Y[k])$ means $\dot{\partial}_k N_j^i$ and the command $Hdiff(N[i,-j], X[k])$ means $\delta_k N_j^i$.

In addition to the definitions of geometric objects existing already in the FINSLER package, we add other definitions by using the command *definetensor*. We rewrite the correct expression of P_{ijk}^h and tackle the issue of dimension.

Now, let us illustrate what have been said before using a concrete example.

Let $M = \mathbb{R}^3$, $U = \{(x1, x2, x3; y1, y2, y3) \in \mathbb{R}^3 \times \mathbb{R}^3 : x3 \neq 0; y2 \neq 0, y1^2 + y3^2 \neq 0\} \subset TM$. Let F be the Finsler structure defined on U by

$$F = \sqrt{\frac{x3y1^3}{y2} + y3^2}.$$

It should first be noted that, according to Table 1, we have only three independent torsions, namely, C_{ij}^h , R_{ij}^h and P_{ij}^h . So, we will compute these torsions for Cartan connection and we will not repeat their calculation for the other connections.

Following the instructions of the FINSLER package, the following calculations can be performed.

```
> restart;
> libname := libname, 'c:/Finsler';
> with(Finsler);
      [Dcoordinates, Hdiff, K, connection, init, metricfunction, tddiff]
> Dimension := 3:
> coordinates(x1,x2,x3):
> Dcoordinates(y1,y2,y3):
      'The coordinates are:'
          X1 = x1
          X2 = x2
          X3 = x3
      'The d-coordinates are:'
          Y1 = y1
          Y2 = y2
          Y3 = y3
```

Finsler structure F:

```
> F := sqrt(x3*y1^3/y2+y3^2);
```

$$F := \sqrt{\frac{x3y1^3}{y2} + y3^2}$$

Plotting the Finsler structure in a special domain:

```
> plot3d(subs(x1=5,y3=5,F), y1 = -2..2, y2 = -2..2,
> axes=BOXED,style=patch);
```

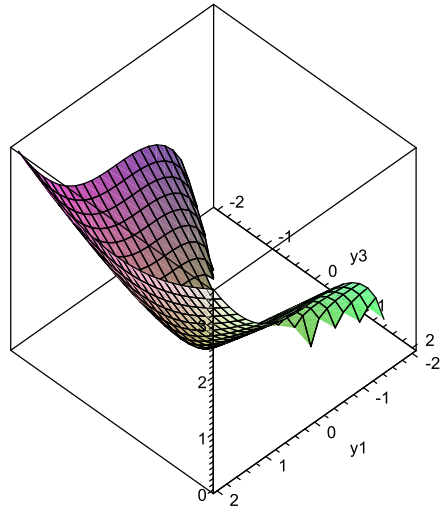


Figure 1

Metric tensor g_{ij} :

> F0 := y1^3*x3/y2+y3^2;

$$F0 := \frac{x^3 y^1^3}{y^2} + y^3$$

> metricfunction(F0):

‘The components of the metric are:’

$$\begin{aligned} g_{x1x1} &= \frac{3 x^3 y^1}{y^2} & g_{x1x2} &= -\frac{3}{2} \frac{x^3 y^1^2}{y^2} \\ g_{x2x2} &= \frac{x^3 y^1^3}{y^2^3} & g_{x3x3} &= 1 \end{aligned}$$

Inverse metric tensor g^{ij} :

> show(g[i,j]);

$$\begin{aligned} g^{x1x1} &= \frac{4y^2}{3x^3y^1} & g^{x1x2} &= \frac{2y^2^2}{x^3y^1^2} \\ g^{x2x2} &= \frac{4y^2^3}{x^3y^1^3} & g^{x3x3} &= 1 \end{aligned}$$

Supporting element l_i :

> show(l[-i]);

$$\begin{aligned} l_{x1} &= \frac{3}{2} \frac{x^3 y^1^2}{y^2 \sqrt{\frac{x^3 y^1^3 + y^2 y^3^2}{y^2}}} & l_{x2} &= -\frac{1}{2} \frac{x^3 y^1^3}{y^2 \sqrt{\frac{x^3 y^1^3 + y^2 y^3^2}{y^2}}} \\ l_{x3} &= \frac{y^3}{\sqrt{\frac{x^3 y^1^3 + y^2 y^3^2}{y^2}}} \end{aligned}$$

Angular metric tensor h_{ij} :

> definetensor(h[-i,-j] = g[-i,-j]-1[-i]*1[-j],symm);
 > show(h[-i, -j]);

$$\begin{aligned} h_{x_1x_1} &= \frac{3x^3y^1(x^3y^1^3 + 4y^2y^3^2)}{4y^2(x^3y^1^3 + y^2y^3^2)} & h_{x_1x_2} &= -\frac{3x^3y^1^2(x^3y^1^3 + 2y^2y^3^2)}{4y^2(x^3y^1^3 + y^2y^3^2)} \\ h_{x_1x_3} &= -\frac{3x^3y^1^2y^3}{2x^3y^1^3 + y^2y^3^2} & h_{x_2x_2} &= \frac{1x^3y^1^3(3x^3y^1^3 + 4y^2y^3^2)}{4y^2^3(x^3y^1^3 + y^2y^3^2)} \\ h_{x_2x_3} &= \frac{1x^3y^1^3y^3}{2y^2(x^3y^1^3 + y^2y^3^2)} & h_{x_3x_3} &= \frac{x^3y^1^3}{x^3y^1^3 + y^2y^3^2} \end{aligned}$$

Cartan tensor C_{ijk} :

> show(C[-i,-j,-k]);

$$\begin{aligned} C_{x_1x_1x_1} &= \frac{3x^3}{2y^2} & C_{x_1x_1x_2} &= -\frac{3x^3y^1}{2y^2^2} \\ C_{x_1x_2x_2} &= \frac{3x^3y^1^2}{2y^2^3} & C_{x_2x_2x_2} &= -\frac{3x^3y^1^3}{2y^2^4} \end{aligned}$$

Spray coefficients G^i :

> show(G[i]);

$$\begin{aligned} G^{x^1} &= \frac{1y^1y^3}{2x^3} & G^{x^2} &= \frac{1y^2y^3}{2x^3} \\ G^{x^3} &= -\frac{1y^1^3}{4y^2} \end{aligned}$$

Nonlinear connection (Barthel connection) N_j^i :

> show(N[i,-j]);

$$\begin{aligned} N_{x_1}^{x^1} &= \frac{1y^3}{2x^3} & N_{x_3}^{x^1} &= \frac{1y^1}{2x^3} \\ N_{x_2}^{x^2} &= \frac{1y^3}{2x^3} & N_{x_3}^{x^2} &= \frac{1y^2}{2x^3} \\ N_{x_1}^{x^3} &= -\frac{3y^1^2}{4y^2} & N_{x_2}^{x^3} &= \frac{1y^1^3}{4y^2} \end{aligned}$$

Coefficients of Berwald connection G_{jk}^i :

> show(G[i,-j,-k]);

$$\begin{aligned} G_{x_1x_1}^{x^3} &= -\frac{3y^1}{2y^2} & G_{x_1x_2}^{x^3} &= \frac{3y^1^2}{4y^2^2} \\ G_{x_1x_3}^{x^1} &= \frac{1}{2x^3} & G_{x_2x_2}^{x^3} &= -\frac{1y^1^3}{2y^2^3} \\ G_{x_2x_3}^{x^2} &= \frac{1}{2x^3} \end{aligned}$$

Coefficients of Cartan connection Γ_{jk}^i :

> show(Gammastar[i, -j, -k]);

$$Gammastar_{x_1x_1}^{x_1} = \frac{1}{2} \frac{y^3}{x^3 y_1}$$

$$Gammastar_{x_1x_1}^{x_2} = \frac{3}{2} \frac{y_2 y^3}{x^3 y_1^2}$$

$$Gammastar_{x_1x_1}^{x_3} = -\frac{3}{2} \frac{y_1}{y_2}$$

$$Gammastar_{x_1x_2}^{x_1} = -\frac{1}{2} \frac{y^3}{x^3 y_2}$$

$$Gammastar_{x_1x_2}^{x_2} = -\frac{3}{2} \frac{y^3}{x^3 y_1}$$

$$Gammastar_{x_1x_2}^{x_3} = \frac{3}{4} \frac{y_1^2}{y_2^2}$$

$$Gammastar_{x_1x_3}^{x_1} = \frac{1}{2x^3}$$

$$Gammastar_{x_2x_2}^{x_1} = \frac{1}{2} \frac{y_1 y^3}{x^3 y_2^2}$$

$$Gammastar_{x_2x_2}^{x_2} = \frac{3}{2} \frac{y^3}{x^3 y_2}$$

$$Gammastar_{x_2x_2}^{x_3} = -\frac{1}{2} \frac{y_1^3}{y_2^3}$$

$$Gammastar_{x_2x_3}^{x_2} = \frac{1}{2x^3}$$

Torsion tensors of Cartan connection

• (h)hv-torsion C_{ij}^h :

> show(C[i, -j, -k]);

$$C_{x_1x_1}^{x_1} = -\frac{1}{y_1}$$

$$C_{x_1x_2}^{x_1} = \frac{1}{y_2}$$

$$C_{x_2x_2}^{x_1} = -\frac{y_1}{y_2^2}$$

$$C_{x_2x_1x_1}^{x_2} = -\frac{3y_2}{y_1^2}$$

$$C_{x_1x_2}^{x_2} = \frac{3}{y_1}$$

$$C_{x_2x_2}^{x_2} = -\frac{3}{y_2}$$

• (v)h-torsion R_{ij}^h :

> definetensor(RN[i, -j, -k]=Hdiff(N[i, -j], X[k]) - Hdiff(N[i, -k], X[j]));

> show(RN[i, -j, -k]);

$$RN_{x_1x_2}^{x_1} = -\frac{1}{8} \frac{y_1^3}{x^3 y_2^2}$$

$$RN_{x_1x_2}^{x_2} = -\frac{3}{8} \frac{y_1^2}{x^3 y_2}$$

$$RN_{x_1x_3}^{x_1} = -\frac{1}{4} \frac{y^3}{x^3}$$

$$RN_{x_1x_3}^{x_3} = \frac{3}{8} \frac{y_1^2}{x^3 y_2}$$

$$RN_{x_2x_3}^{x_2} = -\frac{1}{4} \frac{y^3}{x^3}$$

$$RN_{x_2x_3}^{x_3} = -\frac{1}{8} \frac{y_1^3}{x^3 y_2^2}$$

• (v)hv-torsion P_{ij}^h :

> definetensor(PT[i, -j, -k] = G[i, -j, -k] - Gammastar[i, -j, -k]);

> show(PT[i, -j, -k]);

$$PT_{x_1x_1}^{x_1} = -\frac{1}{2} \frac{y^3}{x^3 y_1}$$

$$PT_{x_1x_1}^{x_2} = -\frac{3}{2} \frac{y_2 y^3}{x^3 y_1^2}$$

$$PT_{x_1x_2}^{x_1} = \frac{1}{2} \frac{y^3}{x^3 y_2}$$

$$PT_{x_1x_2}^{x_2} = \frac{3}{2} \frac{y^3}{x^3 y_1}$$

$$PT_{x_2x_2}^{x_1} = -\frac{1}{2} \frac{y_1 y^3}{x^3 y_2^2}$$

$$PT_{x_2x_2}^{x_2} = -\frac{3}{2} \frac{y^3}{x^3 y_2}$$

Curvature tensors of Cartan connection

- **h-curvature tensor** R_{ijk}^h :

```
> definetensor(RC[i,-h,-j,-k] = Hdiff(Gammastar[i,-h,-j],X[k])
> -Hdiff(Gammastar[i,-h,-k], X[j])+Gammastar[m,-h,-j]
> *Gammastar[i,-m,-k]-Gammastar[m,-h,-k]*Gammastar[i,-m,-j]
> +C[i,-h,-m]*RG[m,-j,-k], antisymm[3,4]):
> show(RC[i,-h,-j,-k]);
```

$$\begin{aligned}
 RC_{x_1x_1x_2}^{x_1} &= -\frac{3}{8} \frac{y_1^2}{x_3 y_2^2} & RC_{x_2x_1x_2}^{x_1} &= \frac{1}{4} \frac{y_1^3}{x_3 y_2^3} \\
 RC_{x_1x_1x_2}^{x_2} &= -\frac{3}{4} \frac{y_1}{x_3 y_2} & RC_{x_2x_1x_2}^{x_2} &= \frac{3}{8} \frac{y_1^2}{x_3 y_2^2} \\
 RC_{x_3x_1x_3}^{x_2} &= -\frac{1}{4x_3^2} & RC_{x_1x_1x_3}^{x_3} &= \frac{3}{4} \frac{y_1}{x_3 y_2} \\
 RC_{x_2x_1x_3}^{x_3} &= -\frac{3}{8} \frac{y_1^2}{x_3 y_2^2} & RC_{x_3x_2x_3}^{x_2} &= -\frac{1}{4x_3^2} \\
 RC_{x_1x_2x_3}^{x_3} &= -\frac{3}{8} \frac{y_1^2}{x_3 y_2^2} & RC_{x_2x_2x_3}^{x_3} &= \frac{1}{4} \frac{y_1^3}{x_3 y_2^3}
 \end{aligned}$$

- **hv-curvature tensor** P_{ijk}^h :

```
> definetensor(FT[i,-j,-k,-h] = Hdiff(C[i,-j,-k], X[h])
> +Gammastar[i,-h,-u]*C[u,-k,-j]-Gammastar[u,-k,-h]*C[i,-u,-j]
> -Gammastar[u,-h,-j]*C[i,-u,-k]):
> definetensor(PC[i,-h,-j,-k] = tddiff(Gammastar[i,-h,-j],Y[k])
> -FT[i,-h,-k,-j]+C[i,-h,-m]*PT[m,-j,-k]);
```

$$\begin{aligned}
 PC_{x_3x_1x_1}^{x_1} &= -\frac{1}{2x_3 y_1} & PC_{x_3x_1x_2}^{x_1} &= \frac{1}{2x_3 y_2} \\
 PC_{x_3x_2x_1}^{x_1} &= \frac{1}{2x_3 y_2} & PC_{x_3x_2x_2}^{x_1} &= -\frac{1}{2} \frac{y_1}{x_3 y_2^2} \\
 PC_{x_3x_1x_1}^{x_2} &= -\frac{3}{2} \frac{y_2}{x_3 y_1^2} & PC_{x_3x_1x_2}^{x_2} &= \frac{3}{2x_3 y_1} \\
 PC_{x_3x_2x_1}^{x_2} &= \frac{3}{2x_3 y_1} & PC_{x_3x_2x_2}^{x_2} &= -\frac{3}{2x_3 y_2} \\
 PC_{x_1x_1x_1}^{x_3} &= -\frac{3}{4y_2} & PC_{x_1x_1x_2}^{x_3} &= \frac{3}{4} \frac{y_1}{y_2^2} \\
 PC_{x_1x_2x_1}^{x_3} &= \frac{3}{4} \frac{y_1}{y_2^2} & PC_{x_1x_2x_2}^{x_3} &= -\frac{3}{4} \frac{y_1^2}{y_2^3} \\
 PC_{x_2x_1x_1}^{x_3} &= \frac{3}{4} \frac{y_1}{y_2^2} & PC_{x_2x_1x_2}^{x_3} &= -\frac{3}{4} \frac{y_1^2}{y_2^3} \\
 PC_{x_2x_2x_1}^{x_3} &= -\frac{3}{4} \frac{y_1^2}{y_2^3} & PC_{x_2x_2x_2}^{x_3} &= \frac{3}{4} \frac{y_1^3}{y_2^4}
 \end{aligned}$$

- **v-curvature tensor** S_{ijk}^h :

```
> definetensor(S[i,-h,-j,-k] = C[m,-h,-k]*C[i,-m,-j]
> -C[m,-h,-j]*C[i,-m,-k]):
```

> show(S[i,-h,-j,-k]);

$$S_{ijk}^h = 0$$

Curvature tensors of Berwald connection

- h-curvature tensor $\overset{\circ}{R}_{ijk}^h$:

> definetensor(RB[i,-h,-j,-k]= Hdiff(G[i,-h,-j], X[k])
 > -Hdiff(G[i,-h,-k], X[j])+G[m,-h,-j]*G[i,-m,-k]
 > -G[m,-h,-k]*G[i,-m,-j], antisymm[3, 4]):
 > show(RB[-i,h,-j,-k]);

$$\begin{aligned} RB_{x_1x_1x_2}^{x_1} &= -\frac{3}{8} \frac{y1^2}{x^3 y^2} & RB_{x_2x_1x_2}^{x_1} &= \frac{1}{4} \frac{y1^3}{x^3 y^2^3} \\ RB_{x_1x_1x_2}^{x_2} &= -\frac{3}{4} \frac{y1}{x^3 y^2} & RB_{x_2x_1x_2}^{x_2} &= \frac{3}{8} \frac{y1^2}{x^3 y^2^2} \\ RB_{x_3x_1x_3}^{x_2} &= -\frac{1}{4x^3} & RB_{x_1x_1x_3}^{x_3} &= \frac{3}{4} \frac{y1}{x^3 y^2} \\ RB_{x_2x_1x_3}^{x_3} &= -\frac{3}{8} \frac{y1^2}{x^3 y^2^2} & RB_{x_3x_2x_3}^{x_2} &= -\frac{1}{4x^3} \\ RB_{x_1x_2x_3}^{x_3} &= -\frac{3}{8} \frac{y1^2}{x^3 y^2} & RB_{x_2x_2x_3}^{x_3} &= \frac{1}{4} \frac{y1^3}{x^3 y^2^3} \end{aligned}$$

- hv-curvature tensor $\overset{\circ}{P}_{ijk}^h$:

> definetensor(PB[i,-h,-j,-k]= tddiff(G[i,-h,-j], Y[k])):
 > show(PB[h,-i,-j,-k]);

$$\begin{aligned} PB_{x_1x_1x_1}^{x_3} &= -\frac{3}{2y^2} & PB_{x_1x_1x_2}^{x_3} &= \frac{3}{2} \frac{y1}{y^2^2} \\ PB_{x_1x_2x_2}^{x_3} &= -\frac{3}{2} \frac{y1^2}{y^2^3} & PB_{x_2x_2x_2}^{x_3} &= \frac{3}{2} \frac{y1^3}{y^2^4} \end{aligned}$$

Curvature tensors of Chern connection

- h-curvature tensor $\overset{\star}{R}_{ijk}^h$:

> definetensor(Rchern[i,-h,-j,-k] = Hdiff(Gammastar[i,-h,-j], X[k])
 > -Hdiff(Gammastar[i,-h,-k], X[j])+Gammastar[m,-h,-j]*Gammastar[i,-m,-k]
 > -Gammastar[m,-h,-k]*Gammastar[i,-m,-j], antisymm[3,4]):
 > show(Rchern[i,-h,-j,-k]);

$$\begin{aligned}
Rchern_{x1x1x2}^{x1} &= -\frac{1}{8} \frac{y1^2}{x3 y2^2} & Rchern_{x2x1x2}^{x2} &= -\frac{3}{8} \frac{y1^2}{x3 y2^2} \\
Rchern_{x1x1x3}^{x1} &= -\frac{1}{4} \frac{y3}{x3^2 y1} & Rchern_{x2x1x3}^{x1} &= \frac{1}{4} \frac{y3}{x3^2 y2} \\
Rchern_{x3x1x3}^{x2} &= -\frac{1}{4x3^2} & Rchern_{x1x1x3}^{x2} &= -\frac{3}{4} \frac{y2 y3}{x3^2 y1^2} \\
Rchern_{x2x1x3}^{x2} &= \frac{3}{4} \frac{y3}{x3^2 y1} & Rchern_{x1x1x3}^{x3} &= \frac{3}{4} \frac{y1}{x3 y2} \\
Rchern_{x2x1x3}^{x3} &= -\frac{3}{8} \frac{y1^2}{x3 y2^2} & Rchern_{x1x2x3}^{x1} &= \frac{1}{4} \frac{y3}{x3^2 y2} \\
Rchern_{x2x2x3}^{x1} &= -\frac{1}{4} \frac{y1 y3}{x3^2 y2^2} & Rchern_{x1x2x3}^{x2} &= \frac{3}{4} \frac{y3}{x3^2 y1} \\
Rchern_{x2x2x3}^{x2} &= -\frac{3}{4} \frac{y3}{x3^2 y2} & Rchern_{x3x2x3}^{x2} &= -\frac{1}{4x3^2} \\
Rchern_{x1x2x3}^{x3} &= -\frac{3}{8} \frac{y1^2}{x3 y2^2} & Rchern_{x2x2x3}^{x3} &= \frac{1}{4} \frac{y1^3}{x3 y2^3}
\end{aligned}$$

• hv-curvature tensor $\overset{\star}{P}_{ijk}^h$:

- > definetensor(Pchern[i,-h,-j,-k]=tddiff(Gammastar[i,-h,-j],Y[k])):
- > show(Pchern[h,-i,-j,-k]);

$$\begin{aligned}
Pchern_{x1x1x1}^{x1} &= -\frac{1}{2} \frac{y3}{x3 y1^2} & Pchern_{x1x1x3}^{x1} &= \frac{1}{2x3 y1} \\
Pchern_{x1x1x1}^{x2} &= -\frac{3y2 y3}{x3 y1^3} & Pchern_{x1x1x2}^{x2} &= \frac{3}{2} \frac{y3}{x3 y1^2} \\
Pchern_{x1x1x3}^{x2} &= \frac{3}{2} \frac{y2}{x3 y1^2} & Pchern_{x1x1x1}^{x3} &= -\frac{3}{2y2} \\
Pchern_{x1x1x2}^{x3} &= \frac{3}{2} \frac{y1}{y2^2} & Pchern_{x1x2x2}^{x1} &= \frac{1}{2} \frac{y3}{x3 y2^2} \\
Pchern_{x1x2x3}^{x1} &= -\frac{1}{2x3 y2} & Pchern_{x1x2x1}^{x2} &= \frac{3}{2} \frac{y3}{x3 y1^2} \\
Pchern_{x1x2x3}^{x2} &= -\frac{3}{2x3 y1} & Pchern_{x1x2x1}^{x3} &= \frac{3}{2} \frac{y1}{y2^2} \\
Pchern_{x1x2x2}^{x3} &= -\frac{3}{2} \frac{y1^2}{y2^3} & Pchern_{x2x2x1}^{x1} &= \frac{1}{2} \frac{y3}{x3 y2^2} \\
Pchern_{x2x2x2}^{x1} &= -\frac{y1 y3}{x3 y2^3} & Pchern_{x2x2x3}^{x1} &= \frac{1}{2} \frac{y1}{x3 y2^2} \\
Pchern_{x2x2x2}^{x2} &= -\frac{3}{2} \frac{y3}{x3 y2^2} & Pchern_{x2x2x3}^{x2} &= \frac{3}{2x3 y2} \\
Pchern_{x2x2x1}^{x3} &= -\frac{3}{2} \frac{y1^2}{y2^3} & Pchern_{x2x2x2}^{x3} &= \frac{3}{2} \frac{y1^3}{y2^4}
\end{aligned}$$

Curvature tensors of Hashiguchi connection

- h-curvature tensor $\overset{*}{R}_{ijk}^h$:

```
> definetensor(RH[i,-h,-j,-k] = Hdiff(G[i,-h,-j], X[k])
> -Hdiff(G[i,-h,-k], X[j])+G[m,-h,-j]*G[i,-m,-k]-G[m,-h,-k]*G[i,-m,-j]
> +C[i,-h,-m]*RG[m,-j,-k], antisymm[3, 4]):
> show(RH[i,-h,-j,-k]);
```

$$\begin{array}{ll}
 RH_{x_1x_1x_2}^{x_1} = -\frac{5}{8} \frac{y_1^2}{x_3 y_2^2} & RH_{x_2x_1x_2}^{x_1} = \frac{1}{2} \frac{y_1^3}{x_3^2 y_2^3} \\
 RH_{x_1x_1x_2}^{x_2} = -\frac{3}{2} \frac{y_1}{x_3 y_2} & RH_{x_2x_1x_2}^{x_2} = \frac{9}{8} \frac{y_1^2}{x_3^2 y_2^2} \\
 RH_{x_1x_1x_3}^{x_1} = \frac{1}{4} \frac{y_3}{x_3 y_1^2} & RH_{x_2x_1x_3}^{x_1} = -\frac{1}{4} \frac{y_3}{x_3^2 y_2} \\
 RH_{x_3x_1x_3}^{x_2} = -\frac{1}{4x_3^2} & RH_{x_1x_1x_3}^{x_2} = \frac{3}{4} \frac{y_2 y_3}{x_3^2 y_1^2} \\
 RH_{x_2x_1x_3}^{x_2} = -\frac{3}{4} \frac{y_3}{x_3 y_1^2} & RH_{x_1x_1x_3}^{x_3} = \frac{3}{4} \frac{y_1}{x_3 y_2} \\
 RH_{x_2x_1x_3}^{x_3} = -\frac{3}{8} \frac{y_1^2}{x_3 y_2^2} & RH_{x_1x_2x_3}^{x_1} = -\frac{1}{4} \frac{y_3}{x_3^2 y_2} \\
 RH_{x_2x_2x_3}^{x_1} = \frac{1}{4} \frac{y_1 y_3}{x_3^2 y_2^2} & RH_{x_1x_2x_3}^{x_2} = -\frac{3}{4} \frac{y_3}{x_3 y_1^2} \\
 RH_{x_2x_2x_3}^{x_2} = \frac{3}{4} \frac{y_3}{x_3^2 y_2} & RH_{x_3x_2x_3}^{x_2} = -\frac{1}{4x_3^2} \\
 RH_{x_1x_2x_3}^{x_3} = -\frac{3}{8} \frac{y_1^2}{x_3 y_2^2} & RH_{x_2x_2x_3}^{x_3} = \frac{1}{4} \frac{y_1^3}{x_3^2 y_2^3}
 \end{array}$$

- hv-curvature tensor $\overset{*}{P}_{ijk}^h$:

```
> definetensor(PH1[i,-h,-k,-j]=Hdiff(C[i,-h,-k], X[j])+G[i,-m,-j]
> *C[m,-h,-k]-G[m,-h,-j]*C[i,-m,-k]-G[m,-k,-j]*C[i,-h,-m]):
> definetensor(PH[i,-h,-j,-k] =
> tddiff(G[i,-h,-j], Y[k])-PH1[i,-h,-k,-j], symm[2,4]);
> show(PH[h,-i,-j,-k]);
```

$$\begin{aligned}
PH_{x_1x_1x_1}^{x_1} &= \frac{1}{2} \frac{y^3}{x^3 y_1^2} & PH_{x_1x_1x_1}^{x_2} &= \frac{3y_2 y_3}{x^3 y_1^3} \\
PH_{x_1x_2x_1}^{x_2} &= -\frac{3}{2} \frac{y^3}{x^3 y_1^2} & PH_{x_1x_1x_1}^{x_3} &= -\frac{3}{4y_2} \\
PH_{x_1x_2x_1}^{x_3} &= \frac{3}{4} \frac{y_1}{y_2^2} & PH_{x_1x_2x_2}^{x_1} &= -\frac{1}{2} \frac{y^3}{x^3 y_2^2} \\
PH_{x_1x_1x_2}^{x_2} &= -\frac{3}{2} \frac{y^3}{x^3 y_1^2} & PH_{x_1x_1x_2}^{x_3} &= \frac{3}{4} \frac{y_1}{y_2^2} \\
PH_{x_1x_2x_2}^{x_3} &= -\frac{3}{4} \frac{y_1^2}{y_2^3} & PH_{x_1x_1x_3}^{x_1} &= -\frac{1}{2x^3 y_1} \\
PH_{x_1x_2x_3}^{x_1} &= \frac{1}{2x^3 y_2} & PH_{x_1x_1x_3}^{x_2} &= -\frac{3}{2} \frac{y_2}{x^3 y_1^2} \\
PH_{x_1x_2x_3}^{x_2} &= \frac{3}{2x^3 y_1} & PH_{x_2x_1x_2}^{x_1} &= -\frac{1}{2} \frac{y^3}{x^3 y_2^2} \\
PH_{x_2x_2x_2}^{x_1} &= \frac{y_1 y^3}{x^3 y_2^3} & PH_{x_2x_2x_2}^{x_2} &= \frac{3}{2} \frac{y^3}{x^3 y_2^2} \\
PH_{x_2x_1x_2}^{x_3} &= -\frac{3}{4} \frac{y_1^2}{y_2^3} & PH_{x_2x_2x_2}^{x_3} &= \frac{3}{4} \frac{y_1^3}{y_2^4} \\
PH_{x_2x_1x_3}^{x_1} &= \frac{1}{2x^3 y_2} & PH_{x_2x_2x_3}^{x_1} &= -\frac{1}{2} \frac{y_1}{x^3 y_2^2} \\
PH_{x_2x_1x_3}^{x_2} &= \frac{3}{2x^3 y_1} & PH_{x_2x_2x_3}^{x_2} &= -\frac{3}{2x^3 y_2}
\end{aligned}$$

The v-curvature of Hashiguchi connection is the same as the v-curvature of Cartan connection.

Remark 4.1. According to the above consideration, if we calculate the hv-curvature P of Cartan connection, in the example mentioned in Section 3, we find that the components P_{ijk}^h vanish identically as expected.

5. Tensor simplification

It is well known that the simplification of tensor expressions is not an easy task [8]. However, we have noted that if we have a complicated formula of a geometric object, such as P_{ijk}^h , we can significantly simplify its expression as follows. We let the package compute the tensor $P_{hijk} := g_{rh} P_{ijk}^r$ (instead of P_{ijk}^h) and ask it to show the tensor P_{ijk}^h .

To illustrate this technique let us consider the following example.

Let $M = \mathbb{R}^3$, $U = \{(x_1, x_2, x_3; y_1, y_2, y_3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y_1 \neq 0, y_2 \neq 0, y_3 \neq 0\}$. Let F be the Finsler structure defined on U by

$$F = (x_1 y_2^3 + y_1^2 y_3)^{1/3}.$$

For example, let us compute the component S_{112}^1 of the v-curvature tensor S_{ijk}^h of Cartan connection.

- > `definetensor(SC[i,-h,-j,-k] = C[m,-h,-k]*C[i,-m,-j]`
- > `-C[m,-h,-j]*C[i,-m,-k]) :`

$$SC_{hjk}^i = C_{hk}^m C_{mj}^i - C_{hj}^m C_{mk}^i$$

- > `show(SC[i,-h,-j,-k]) ;`

$$\begin{aligned} SC_{x1x1x2}^{x1} = & -\frac{1}{18} \frac{y^3 y1 (-x1 y2^3 + y3 y1^2) x1 y2^2 (y3 y1^2 - 3x1 y2^3)}{(x1 y2^3 + y3 y1^2)^4} \\ & -\frac{2}{27} \frac{y^3 y1^3 x1 y2^2 (y3 y1^2 + 3 x1 y2^3)}{(x1 y2^3 + y3 y1^2)^4} \\ & +\frac{1}{36} \frac{y1 x1 y2^2 (-x1 y2^3 + y3 y1^2) y3 (-3 x1 y2^3 + 5y3 y1^2)}{(x1 y2^3 + y3 y1^2)^4} \\ & +\frac{1}{54} \frac{y1 x1 y2^2 (4y3^2 y1^4 + 21y3 y1^2 x1 y2^3 + 9x1^2 y2^6) y3}{(x1 y2^3 + y3 y1^2)^4} \end{aligned}$$

The above expression is complicated. But, in fact, if we lower the index i in the above definition and use the command `show(SC[i,-h,-j,-k])`, then we have the following simplification.

- > `definetensor(SC[-i,-h,-j,-k] = C[m,-h,-k]*C[-i,-m,-j]`
- > `-C[m,-h,-j]*C[-i,-m,-k]) :`

$$SC_{ihjk} = C_{hk}^m C_{imj} - C_{hj}^m C_{imk}$$

- > `show(SC[i,-h,-j,-k]) ;`

$$SC_{x1x1x2}^{x1} = \frac{1}{12} \frac{y^3 y1 x1 y2^2}{(x1 y2^3 + y3 y1^2)^2},$$

which is very simple compared with its expression before simplification.

Remark 5.1. Be careful when you lower or raise an index, this index should be lowerable or raisable. For example, in the definition of P_{ijk}^h we encounter the term $\dot{\partial}_k \Gamma_{hj}^i$ (cf. §1). The index i in this term can not be lowered since $g_{im}(\dot{\partial}_k \Gamma_{hj}^m) \neq \dot{\partial}_k(g_{im} \Gamma_{hj}^m)$. So we can not use the command `tddiff(Gammastar[i,-h,-j], Y[k])`. Such a problem can be treated as illustrated below:

- > `definetensor(FT[i,-j,-k,-h]=Hdiff(C[i,-j,-k], X[h])`
- > `+Gammastar[i,-h,-u]*C[u,-k,-j]-Gammastar[u,-k,-h]*C[i,-u,-j]`
- > `-Gammastar[u,-h,-j]*C[i,-u,-k]) ;`
- > `definetensor(PC[i,-h,-j,-k] = tddiff(Gammastar[i,-h,-j], Y[k])`
- > `-FT[i,-h,-k,-j]+C[i,-h,-m]*PT[m,-j,-k]) ;`

$$PC_{hjk}^i = tddiff_k (Gammastar_{hj}^i) - FT_{hjk}^i + C_{hm}^i PT_{jk}^m$$

- > `show(PC[i,-h,-j,-k]) ;`

$$\begin{aligned} PC_{x1x1x1}^{x1} = & \frac{1}{72} \frac{1}{(x1 y2^3 + y1^2 y3)^2 y1} \left(\frac{3x1 y2^6 (-y1^2 y3 + 3x1 y2^3) y1^2 y3}{(x1 y2^3 + y1^2 y3)^2} \right. \\ & + \frac{10y1^2 (y1^2 y3 + 3x1 y2^3) y2^6 y3 x1}{(x1 y2^3 + y1^2 y3)^2} + \frac{3 x1^2 y2^9 (-y1^2 y3 + 3x1 y2^3)}{(x1 y2^3 + y1^2 y3)^2} \\ & + \frac{5(y1^2 y3 + 3x1 y2^3) y2^9 x1^2}{(x1 y2^3 + y1^2 y3)^2} + \frac{3 y2^3 y1^4 (-y1^2 y3 + 3x1 y2^3) y3^2}{(x1 y2^3 + y1^2 y3)^2} \\ & \left. + \frac{5y2^3 y1^4 (y1^2 y3 + 3x1 y2^3) y3^2}{(x1 y2^3 + y1^2 y3)^2} - 15x1 y2^6 + y1^2 y3 y2^3 \right) \end{aligned}$$

This component can be simplified using the above mentioned technique.

```

> definetensor(FT[i,-j,-k,-h] = Hdiff(C[i,-j,-k], X[h])
> +Gammastar[i,-h, u]*C[u,-k,-j]-Gammastar[u,-k,-h]*C[i,-u,-j]
> -Gammastar[u,-h,-j]*C[i,-u,-k]);
> definetensor(ST[i,-h,-j,-k] = tddiff(Gammastar[i,-h,-j], Y[k]));
> definetensor(PC[-i,-h,-j,-k] = g[-m,-i]*ST[m,-h,-j,-k]
> -FT[-i,-h,-k,-j]+C[-i,-h,-m]*PT[m,-j,-k]);

```

$$PC_{ihjk} = g_{mi} ST_{hjk}^m - FT_{ihkj} + C_{ihm} PT_{jk}^m$$

```

> show(PC[i,-h,-j,-k]);

```

$$PC_{x1x1x1}^{x1} = \frac{1}{16} \frac{y2^3}{y1(x1y2^3+y3y1^2)},$$

which is simpler compared with its expression before simplification.

6. Conclusion

In this paper, we have achieved four objectives concerning the FINSLER package [10], [11]:

- The wrong calculation of the components of the hv-curvature tensor P_{ijk}^h of Cartan connection has been corrected
- Modifications have been made so that the h- and hv-curvatures of Cartan connection (and other geometric objects) could be computed in all dimensions (not only dimension 4).
- The package has been extended to compute not only the geometric objects associated with Cartan connection but also those associated with other fundamental connections of Finsler geometry. Other definitions can be added similarly to the package.
- A technique for simplifying tensor expressions has been introduced.

Thanks to the FINSLER package, one is able to study various examples and counterexamples in Finsler and Riemannian geometries. For example, in [13] and [14], we have studied interesting counterexamples in Finsler geometry.

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