TRANSFORMATIONS ON POSITIVE DEFINITE MATRICES PRESERVING GENERALIZED DISTANCE MEASURES

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ABSTRACT. We substantially extend and unify former results on the structure of surjective isometries of spaces of positive definite matrices obtained in the paper [14]. The isometries there correspond to certain geodesic distances in Finsler-type structures and to a recently defined interesting metric which also follows a non-Euclidean geometry. The novelty in our present paper is that here we consider not only true metrics but so-called generalized distance measures which are parameterized by unitarily invariant norms and continuous real functions satisfying certain conditions. Among the many possible applications, we shall see that using our new result it is easy to describe the surjective maps of the set of positive definite matrices that preserve the Stein's loss or several other types of divergences. We also present results concerning similar preserver transformations defined on the subset of all complex positive definite matrices with unit determinant.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

We begin with a short history of the problem we are considering in this paper. First of all we mention that in [14], the first author has described the structure of all surjective isometries of the space $\mathbb{P}_n$ of all $n \times n$ complex positive definite matrices with respect to any element of a large family of metrics. Those distances can be regarded as generalizations of the geodesic distance in the natural Riemannian structure on $\mathbb{P}_n$. To explain this, a few details follow. The set $\mathbb{P}_n$ is

an open subset of the normed linear space $\mathbb{H}_n$ of all $n \times n$ Hermitian matrices, hence it is a differentiable manifold which can naturally be equipped with a Riemannian structure as follows. For any $A \in \mathbb{P}_n$, the tangent space at A is identified with $\mathbb{H}_n$ on which we define an inner product by

$$\langle X, Y \rangle_A = \mathrm{Tr}(A^{-1/2} X A^{-1} Y A^{-1/2}), \qquad X, Y \in \mathbb{H}_n.$$

Clearly, the corresponding norm is

$$\|X\|_A = \|A^{-1/2} X A^{-1/2}\|_{HS}, \qquad X \in \mathbb{H}_n,$$

where $\|.\|_{HS}$ stands for the Hilbert-Schmidt norm (Frobenius norm) defined by $\|T\|_{HS}^2 = \mathrm{Tr}(T^* T)$, $T \in \mathbb{M}_n$. Here $\mathbb{M}_n$ denotes the linear space of all $n \times n$ complex matrices. In that way we obtain a Riemannian space whose geometry has been investigated deeply in the literature for many reasons. It is well known that in this space the geodesic distance $\delta_R(A, B)$ between $A, B \in \mathbb{P}_n$ is

(1) $$\delta_R(A, B) = \|\log A^{-1/2} B A^{-1/2}\|_{HS}.$$

That sort of distance measure appears in a more general setting, too. In fact, in a series of papers from the 1990's Corach and his collaborators studied the cone of invertible positive elements in general C^*-algebras equipped with a Finsler-type structure, see, e.g., [5], [6], [7]. They explored interesting and important connections among geodesics, operator means and operator inequalities. In the particular case of matrices (i.e., when the underlying C^*-algebra is just $\mathbb{M}_n$) the structure they studied is the following. At any point $A \in \mathbb{P}_n$, on the tangent space $\mathbb{H}_n$ a Finsler-type norm is given by

$$\|X\|_A = \|A^{-1/2} X A^{-1/2}\|, \qquad X \in \mathbb{H}_n,$$

where $\|.\|$ stands for the usual operator norm (spectral norm). The corresponding shortest path distance on $\mathbb{P}_n$ can be computed in a way similar to (1) but the Hilbert-Schmidt norm is replaced by the operator norm.

Proceeding further, we mention that in the paper [9], Fujii presented a common extension of the above two approaches in the setting of finite dimensional C^*-algebras. For the algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices this means the following. Let N be a unitarily invariant norm on $\mathbb{M}_n$. For each point $A \in \mathbb{P}_n$ and every vector $X \in \mathbb{H}_n$ define

$$N(X)_A = N(A^{-1/2} X A^{-1/2})$$

which gives a Finsler-type metric on the tangent space at A. Theorem 5 in [9] states that in the corresponding structure on $\mathbb{P}_n$ the shortest

path distance $d_N(A, B)$ between any pair $A, B \in \mathbb{P}_n$ of points is

$$(2) \qquad d_N(A, B) = N(\log A^{-1/2} B A^{-1/2}).$$

In [14] the first author has described the structure of all surjective isometries of $\mathbb{P}_n$ with respect to any such metric d_N. In the same paper another structural result has also been presented concerning the isometries of $\mathbb{P}_n$ with respect to a recently defined interesting metric originating from the so-called symmetric Stein divergence. The details in short are the following. For any pair $A, B \in \mathbb{P}_n$ of positive definite matrices the Stein's loss $l(A, B)$ is defined by

$$l(A, B) = \operatorname{Tr} AB^{-1} - \log \det AB^{-1} - n.$$

The Jensen-Shannon symmetrization of $l(A, B)$ is the quantity

$$S_{JS}(A, B) = \frac{1}{2}\left(l\left(A, \frac{A+B}{2}\right) + l\left(B, \frac{A+B}{2}\right)\right)$$

which is called symmetric Stein divergence. It is easy to see that we have

$$S_{JS}(A, B) = \log \det\left(\frac{A+B}{2}\right) - \frac{1}{2}\log \det AB, \qquad A, B \in \mathbb{P}_n.$$

In [15] Sra has proven that the square root of S_{JS}, i.e.,

$$\delta_S(A, B) = \sqrt{S_{JS}(A, B)}, \qquad A, B \in \mathbb{P}_n,$$

gives a true metric on $\mathbb{P}_n$. (As a matter of curiosity we mention that in [3] it was conjectured that δ_S not a metric, shortly after that in [2] the opposite was claimed, and finally, Sra has shown that δ_S is indeed a true metric on $\mathbb{P}_n$.) In [15] he has pointed out the importance of this new distance function. Among others, he has emphasized that δ_S is a useful substitute of the widely applied geodesic distance δ_R, it respects a non-Euclidean geometry of a rather similar kind, but, compared to the case of δ_R, the calculation of δ_S is easier, it is much less time and capacity demanding which is a really considerable advantage from the computational points view. In [14] the structure of all surjective isometries of the metric space $(\mathbb{P}_n, \delta_S)$ has also been determined.

This was the short history of the former results in [14]. Now, a few sentences about the new results we are going to exhibit. First of all, our main aim here is to give a far reaching and common generalization of the above mentioned results in [14]. Our idea comes from the following observation. The metrics d_N, δ_S can be regarded as particular distance measures of the form

$$(3) \qquad d_{N,f}(A, B) = N(f(A^{-1/2} B A^{-1/2})), \qquad A, B \in \mathbb{P}_n,$$

where N is a unitarily invariant norm on $\mathbb{M}_n$ and $f \colon]0, \infty[\to \mathbb{R}$ is an appropriate real function. We emphasize that $d_{N,f}$ is not a true metric in general only a so-called generalized distance measure. By this concept in this paper we mean a function $d : X \times X \to [0, \infty[$ (X is any set) which has the definiteness property (for arbitrary $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$), but neither the symmetry of d nor the triangle inequality for d is assumed.

In Theorem 1 below we determine the structure of all surjective maps on $\mathbb{P}_n$ that leave $d_{N,f}(.,.)$ invariant. To demonstrate that our new result really extends the ones we have obtained in [14], observe that the metric d_N considered in [14] (and also defined in (2)) coincides with $d_{N,\log}$ defined in (3). As for δ_S, for any $A, B \in \mathbb{P}_n$ we have

$$\delta_S(A, B)^2 = S_{JS}(A, B) = \operatorname{Tr} \log \frac{Y + I}{2\sqrt{Y}} = \left\| \log \frac{Y + I}{2\sqrt{Y}} \right\|_1$$

with $Y = A^{-1/2} B A^{-1/2}$, where $\| \cdot \|_1$ denotes the trace-norm on $\mathbb{M}_n$. (Here we use the notation $(Y + I)/(2\sqrt{Y})$ for the matrix $((Y + I)/2)(\sqrt{Y})^{-1}$; it should not cause any confusion since the terms $(Y + I)/2$ and $\sqrt{Y}$ commute.) Indeed, on the one hand, observe that $\log((Y + I)/(2\sqrt{Y}))$ is a positive semidefinite matrix for every positive definite Y and hence its trace equals its trace-norm. On the other hand, one can compute

$$\operatorname{Tr} \log \frac{Y + I}{2\sqrt{Y}} = \operatorname{Tr} \left(\log \frac{Y + I}{2} - \frac{1}{2} \log Y \right)$$

$$= \log \det \frac{A^{-1/2}(B + A)A^{-1/2}}{2} - \frac{1}{2} \log \det B + \frac{1}{2} \log \det A$$

$$= \log \det \frac{A + B}{2} - \log \det A - \frac{1}{2} \log \det B + \frac{1}{2} \log \det A = S_{JS}(A, B).$$

We now present our main result which is a far reaching generalization of the mentioned structural theorems obtained in [14].

Theorem 1. *Let N be a unitarily invariant norm on $\mathbb{M}_n$. Assume $f \colon]0, \infty[\to \mathbb{R}$ is a continuous function such that*

(a1) *$f(y) = 0$ holds if and only if $y = 1$;*
(a2) *there exists a number $K > 1$ such that*

$$|f(y^2)| \geq K|f(y)|, \qquad y \in]0, \infty[.$$

Define, as above, $d_{N,f} \colon \mathbb{P}_n \times \mathbb{P}_n \to [0, \infty[$ by

$$d_{N,f}(A, B) = N(f(A^{-1/2} B A^{-1/2})), \qquad A, B \in \mathbb{P}_n.$$

Assume that $n \geq 3$. If $\phi \colon \mathbb{P}_n \to \mathbb{P}_n$ is a surjective map which leaves $d_{N,f}(.,.)$ invariant, i.e., which satisfies

(4) $\qquad d_{N,f}(\phi(A), \phi(B)) = d_{N,f}(A, B), \qquad A, B \in \mathbb{P}_n,$

then there exist an invertible matrix $T \in \mathbb{M}_n$ and a real number c such that ϕ is of one of the following forms

- (f1) $\phi(A) = (\det A)^c TAT^*, \quad A \in \mathbb{P}_n$;
- (f2) $\phi(A) = (\det A)^c TA^{-1}T^*, \quad A \in \mathbb{P}_n$;
- (f3) $\phi(A) = (\det A)^c TA^{\mathrm{tr}}T^*, \quad A \in \mathbb{P}_n$;
- (f4) $\phi(A) = (\det A)^c T(A^{\mathrm{tr}})^{-1}T^*, \quad A \in \mathbb{P}_n$.

Here and in the sequel ^{tr} stands for the transpose of matrices.

Apparently, the function $d_{N,f}(.,.)$ appearing in the theorem is a generalized distance measure in the sense we introduced above.

As we have already observed, the function f in (4) which corresponds to the metric d_N in (2) is the logarithmic function while the function f corresponding to S_{JS} is the one defined by $f(y) = \log((y+1)/(2\sqrt{y}))$, $y > 0$. It is easy to check that both functions have the properties (a1), (a2) listed in the theorem (the constant K being 2 in both cases).

In what follows we point out that Theorem 1 can be applied to many other generalized distance measures. First of all, we mention the Stein's loss. One can easily see that for any $A, B \in \mathbb{P}_n$ we have

$$l(A, B) = \mathrm{Tr}(Y^{-1} - \log Y^{-1} - 1) = \|Y^{-1} - \log Y^{-1} - 1\|_1,$$

where $Y = A^{-1/2}BA^{-1/2}$. The latter equality follows from the fact that the matrix $Y^{-1} - \log Y^{-1} - 1$ is positive semidefinite for every positive definite Y which is the consequence of the inequality $y^{-1} - \log y^{-1} - 1 \geq 0$, $y > 0$. Therefore, we can write $l(A, B) = d_{N,f}(A, B)$, where N is the trace-norm and $f(y) = y^{-1} - \log y^{-1} - 1$, $y > 0$. One can check that this function satisfies the conditions (a1), (a2) (with constant $K = 2$) in Theorem 1.

Beside the Jensen-Shannon symmetrization S_{JS} of the Stein's loss l appearing above, in the literature they have investigated in details the so-called Jeffrey's Kullback-Leibler divergence defined by

$$S_{JKL}(A, B) = \frac{l(A, B) + l(B, A)}{2}, \qquad A, B \in \mathbb{P}_n$$

which represents the most natural symmetrization of the function l. The advantages offered by this generalized distance measure (which is not a true metric) are similar to those by S_{JS} (more precisely, by δ_S): it has many of the properties of the geodesic distance δ_R but its calculation does not require matrix eigenvalue computations, or

logarithms, see [4]. It can be easily seen that for any $A, B \in \mathbb{P}_n$ we have

$$S_{JKL}(A, B) = \mathrm{Tr}\left(\frac{Y + Y^{-1} - 2I}{2}\right) = \left\|\frac{Y + Y^{-1} - 2I}{2}\right\|_1,$$

where $Y = A^{-1/2}BA^{-1/2}$. Again, to see the last equality we note that the matrix $(Y + Y^{-1} - 2I)/2$ is positive semidefinite for every positive definite Y. Therefore, we can write $S_{JKL}(A, B) = d_{N,f}(A, B)$, where N is the trace-norm and $f(y) = (y + y^{-1} - 2)/2$, $y > 0$. Easy computations show that f satisfies the conditions (a1), (a2) (with constant $K = 2$) in Theorem 1.

To present further examples, we recall that in the paper [2] Chebbi and Moakher introduced and studied a one-parameter family of divergences which is related to the Stein's loss. For any parameter $-1 < \alpha < 1$ they defined the so-called log-determinant α-divergence D_{LD}^α by

$$D_{LD}^\alpha(A, B) = \frac{4}{1 - \alpha^2} \log \frac{\det\left(\frac{1-\alpha}{2}A + \frac{1+\alpha}{2}B\right)}{(\det A)^{(1-\alpha)/2}(\det B)^{(1+\alpha)/2}}, \qquad A, B \in \mathbb{P}_n.$$

For $\alpha = \pm 1$ they defined

$$D_{LD}^{-1}(A, B) = \mathrm{Tr}(A^{-1}B - I) - \log\det(A^{-1}B), \qquad A, B \in \mathbb{P}_n;$$
$$D_{LD}^{1}(A, B) = \mathrm{Tr}(B^{-1}A - I) - \log\det(B^{-1}A), \qquad A, B \in \mathbb{P}_n.$$

We clearly have

$$D_{LD}^{-1}(A, B) = l(B, A) = \|Y - \log Y - 1\|_1$$

and

$$D_{LD}^{1}(A, B) = l(A, B) = \|Y^{-1} - \log Y^{-1} - 1\|_1,$$

where $Y = A^{-1/2}BA^{-1/2}$. Furthermore, for $-1 < \alpha < 1$, one can easily check that

$$D_{LD}^\alpha(A, B) = \frac{4}{1 - \alpha^2} \mathrm{Tr} \log \left(\frac{(1-\alpha)I + (1+\alpha)Y}{2Y^{(1+\alpha)/2}}\right)$$

holds with $Y = A^{-1/2}BA^{-1/2}$. It can be shown by elementary calculus that

$$(5) \qquad \log\left(\frac{(1-\alpha) + (1+\alpha)y}{2y^{(1+\alpha)/2}}\right) \geq 0$$

for all $y > 0$. Therefore, the matrix

$$\log\left(\frac{(1-\alpha)I + (1+\alpha)Y}{2Y^{(1+\alpha)/2}}\right)$$

is positive semidefinite for any positive definite Y and we obtain that D_{LD}^α can be written as $D_{LD}^\alpha = d_{N,f}$, where N is the trace-norm and f is the function of the real variable y that appears in (5). It is not difficult to check that this f also satisfies the conditions (a1), (a2) (again, with constant $K = 2$). To sum up, above we have shown that the field of possible applications of Theorem 1 is really large, a number of generalized distance measures fulfill its assumptions.

Also relating to the applications of our main theorem, we must point out that in the particular choices of the unitarily invariant norm N and real function f, after the use of Theorem 1 one may need to make further steps in order to determine the precise structure of particular distance measure preservers. In accordance with this we present the complete structural result for the measures we have discussed above.

Theorem 2. *Let $div(.,.)$ denote any of the functions $l(.,.)$, $D_{LD}^\alpha(.,.)$, $-1 \le \alpha \le 1$. A surjective map $\phi \colon \mathbb{P}_n \to \mathbb{P}_n$ preserves $div(.,.)$, i.e., satisfies*

$$div(\phi(A), \phi(B)) = div(A, B), \qquad A, B \in \mathbb{P}_n,$$

if and only if there exists an invertible matrix $T \in \mathbb{M}_n$ such that ϕ is of one of the forms

$$\phi(A) = TAT^*, \quad A \in \mathbb{P}_n;$$
$$\phi(A) = TA^{\mathrm{tr}}T^*, \quad A \in \mathbb{P}_n.$$

A surjective map $\phi \colon \mathbb{P}_n \to \mathbb{P}_n$ preserves $S_{JKL}(.,.)$, if and only if there exists an invertible matrix $T \in \mathbb{M}_n$ such that ϕ is of one of the forms

$$\phi(A) = TAT^*, \quad A \in \mathbb{P}_n;$$
$$\phi(A) = TA^{-1}T^*, \quad A \in \mathbb{P}_n;$$
$$\phi(A) = TA^{\mathrm{tr}}T^*, \quad A \in \mathbb{P}_n;$$
$$\phi(A) = T(A^{\mathrm{tr}})^{-1}T^*, \quad A \in \mathbb{P}_n.$$

The proof follows from Theorem 1 and from rather simple calculations, hence we shall not present it.

In connection with the problem of defining the geometric mean of a finite collection of positive definite matrices, in [11] Moakher studied the submanifold $\mathbb{P}_n^1$ of $\mathbb{P}_n$ which consists of all $n \times n$ positive definite matrices with determinant 1. Moreover, in the paper [8] the authors examined the same structure for its interesting connections to the space of so-called diffusion tensors. In fact, they also studied the set $\mathbb{P}_n^c$ of all positive definite matrices with constant determinant c which, for any positive c, is a so-called totally geodesic submanifold of $\mathbb{P}_n$. These facts motivate us to complete our main result by describing the corresponding generalized distance measure preservers also on $\mathbb{P}_n^c$. In fact, following the approach given in [14] we first determine the structure

of all continuous Jordan triple endomorphisms of $\mathbb{P}_n^1$ (i.e., continuous maps respecting the Jordan triple product ABA). Finally, in our last result we shall describe the structure of all surjective transformations on $\mathbb{P}_n^1$ which leave a given generalized distance measure $d_{N,f}$ invariant.

Theorem 3. *Assume $n \geq 3$. Let $\phi \colon \mathbb{P}_n^1 \to \mathbb{P}_n^1$ be a continuous map which is a Jordan triple endomorphism, i.e., ϕ is a continuous map which satisfies*

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \qquad A, B \in \mathbb{P}_n^1.$$

Then there is a unitary matrix $U \in \mathbb{M}_n$ such that ϕ is of one of the following forms

   (g1) $\phi(A) = UAU^*, \quad A \in \mathbb{P}_n^1;$
   (g2) $\phi(A) = UA^{-1}U^*, \quad A \in \mathbb{P}_n^1;$
   (g3) $\phi(A) = UA^{\mathrm{tr}}U^*, \quad A \in \mathbb{P}_n^1;$
   (g4) $\phi(A) = U(A^{\mathrm{tr}})^{-1}U^*, \quad A \in \mathbb{P}_n^1;$
   (g5) $\phi(A) = I, \quad A \in \mathbb{P}_n^1.$

The theorem immediately gives us the following structural result on the continuous Jordan triple automorphisms of $\mathbb{P}_n^1$.

Corollary 4. *Assume $n \geq 3$. Let $\phi \colon \mathbb{P}_n^1 \to \mathbb{P}_n^1$ be a continuous Jordan triple automorphism, i.e., a continuous bijective map which satisfies*

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \qquad A, B \in \mathbb{P}_n^1.$$

Then ϕ is of one of the forms (g1)-(g4).

Our result on the form of surjective transformations of $\mathbb{P}_n^1$ leaving a generalized distance measure $d_{N,f}$ invariant reads as follows.

Theorem 5. *Let N be a unitarily invariant norm on $\mathbb{M}_n$ and $f \colon]0, \infty[\to \mathbb{R}$ be a continuous function which satisfies the conditions (a1), (a2). Assume that $n \geq 3$. Let $\phi \colon \mathbb{P}_n^1 \to \mathbb{P}_n^1$ be a surjective map which preserves $d_{N,f}(.,.)$, i.e., which satisfies*

$$d_{N,f}(\phi(A), \phi(B)) = d_{N,f}(A, B), \qquad A, B \in \mathbb{P}_n^1.$$

Then there exists an invertible matrix $T \in \mathbb{M}_n$ with $|\det T| = 1$ such that ϕ is of one of the following forms

   (h1) $\phi(A) = TAT^*, \quad A \in \mathbb{P}_n^1;$
   (h2) $\phi(A) = TA^{-1}T^*, \quad A \in \mathbb{P}_n^1;$
   (h3) $\phi(A) = TA^{\mathrm{tr}}T^*, \quad A \in \mathbb{P}_n^1;$
   (h4) $\phi(A) = T(A^{\mathrm{tr}})^{-1}T^*, \quad A \in \mathbb{P}_n^1.$

Using this theorem one can easily obtain the structure of $d_{N,f}$-preserving surjective maps of the spaces $\mathbb{P}_n^c$ as follows. Observe that for any $d_{N,f}$-preserving surjective map ϕ of $\mathbb{P}_n^c$ and for the number $\lambda = \sqrt[n]{c}$, the transformation ψ defined by $\psi(A) = (1/\lambda)\phi(\lambda A)$, $A \in \mathbb{P}_n^1$ is a $d_{N,f}$-preserving surjective map of $\mathbb{P}_n^1$. Hence, Theorem 5 is applied and we have the following corollary.

Corollary 6. *Let N, f be as in the previous theorem and assume $n \geq 3$ and c is a positive real number. If $\phi \colon \mathbb{P}_n^c \to \mathbb{P}_n^c$ is a surjective map which preserves $d_{N,f}(.,.)$, i.e., which satisfies*

$$d_{N,f}(\phi(A), \phi(B)) = d_{N,f}(A, B), \qquad A, B \in \mathbb{P}_n^c,$$

then there exists an invertible matrix $T \in \mathbb{M}_n$ with $|\det T| = 1$ such that ϕ is of one of the following forms

$$\phi(A) = TAT^*, \quad A \in \mathbb{P}_n^c;$$
$$\phi(A) = \lambda^2 TA^{-1}T^*, \quad A \in \mathbb{P}_n^c;$$
$$\phi(A) = TA^{\mathrm{tr}}T^*, \quad A \in \mathbb{P}_n^c;$$
$$\phi(A) = \lambda^2 T(A^{\mathrm{tr}})^{-1}T^*, \quad A \in \mathbb{P}_n^c,$$

where $\lambda = \sqrt[n]{c}$.

2. Proofs

In this section we present the proofs of our results. We begin with some auxiliary statements. The most important one among them, Proposition 11, shows that on certain substructures of groups surjective transformations that preserve a given generalized distance measure d which is compatible with the group operation, necessarily preserve locally the so-called inverted Jordan triple product (i.e., they respect the operation $xy^{-1}x$). We point out that results of this kind (which can be considered as noncommutative versions of the famous Mazur-Ulam theorem) are first appeared in the paper [12]. In fact, below we closely follow the approach presented in Sections 2 and 3 of that paper but here we have to make several small modifications according to our present need.

In what follows, after a simple definition we shall exhibit statements that are similar to Lemma 2.3 and Theorem 2.4 in [12] and then we shall introduce conditions similar to the ones $B(.,.)$ and $C(.,.)$ in Definitions 3.2 and 3.4 in that paper. Finally, we shall obtain Proposition 11, a statement similar to Corollary 3.10 in [12] which is the basic tool in the proof of our main result.

Definition 7. *Let X be a set and $d : X \times X \to [0, \infty[$ be any function. We say that a map $\varphi \colon X \to X$ is d-preserving if*

$$d(\varphi(x), \varphi(y)) = d(x, y)$$

holds for any $x, y \in X$. We say that φ is d-reversing if

$$d(\varphi(x), \varphi(y)) = d(y, x)$$

holds for any $x, y \in X$.

Lemma 8. *Let X be a set and $d : X \times X \to [0, \infty[$ be an arbitrary function. Assume $\varphi : X \to X$ is a bijective d-reversing map, $b \in X$, and $K > 1$ is a constant such that*

$$d(x, \varphi(x)) \geq K d(x, b), \qquad x \in X.$$

If $\sup\{d(x, b) | x \in X\} < \infty$, then for every bijective d-reversing map $f \colon X \to X$ we have $d(f(b), b) = 0$.

Proof. Let

$$\lambda = \sup\{d(f(b), b) | f \colon X \to X \text{ is a bijective } d\text{-reversing map}\}.$$

Then $0 \leq \lambda < \infty$. For an arbitrary bijective d-reversing map $f \colon X \to X$, consider $\tilde{f} = f^{-1} \circ \varphi \circ f$. Then $\tilde{f}$ is also a bijective d-reversing transformation and

$$\lambda \geq d(\tilde{f}(b), b) = d(f(b), \varphi(f(b)) \geq K d(f(b), b).$$

By the definition of λ we get $\lambda \geq K\lambda$ which implies that $\lambda = 0$ and this completes the proof. $\qquad \square$

Proposition 9. *Let X be a set, $d : X \times X \to [0, \infty[$ be any function. Let $a, b \in X$ and assume that $\varphi \colon X \to X$ is a bijective d-reversing map such that $\varphi(b) = b$ and $\varphi \circ \varphi$ is the identity on X. We set*

$$L = \{x \in X | d(a, x) = d(x, \varphi(a)) = d(a, b)\}.$$

Suppose that $\sup\{d(x, b) | x \in L\} < \infty$ and there exists a constant $K > 1$ such that

$$d(x, \varphi(x)) \geq K d(x, b), \qquad x \in L.$$

If $T \colon X \to X$ is a bijective d-preserving map, $\psi \colon X \to X$ is a bijective d-reversing map, moreover $\psi(T(a)) = T(\varphi(a))$ and $\psi(T(\varphi(a))) = T(a)$ hold, then we have

$$d(\psi(T(b)), T(b)) = 0.$$

Proof. Since $\varphi(b) = b$ and φ is a d-reversing map, we have

$$d(a, b) = d(\varphi(b), \varphi(a)) = d(b, \varphi(a)),$$

which implies that $b \in L$. Let

$$L' = \{y \in X | d(T(a), y) = d(y, T(\varphi(a))) = d(a, b)\}.$$

Using the bijectivity and the d-preserving property of T one can easily check that $T(L) = L'$. Furthermore, in a similar way, by the bijectivity and the d-reversing property of the maps φ, ψ we obtain that $\varphi(L) = L$ and $\psi(L) = L$. Consider now the transformation $\tilde{T} = T^{-1} \circ \psi \circ T$. Plainly, the restrictions of the maps $\tilde{T}$ and φ to L are bijective d-reversing maps of L. Since $\sup\{d(x, b) | x \in L\} < \infty$, applying the previous lemma we deduce that

$$0 = d(\tilde{T}(b), b) = d(\psi(T(b)), T(b)).$$

$\square$

In the following we need some notions. Let G be a group. The operation $(x, y) \mapsto xy^{-1}x$ is called inverted Jordan triple product. A non-empty subset X of G is called a twisted subgroup if it is closed under that operation, i.e. $xy^{-1}x \in X$ holds for every pair $x, y \in X$. We say that X is 2-divisible if for each $a \in X$ the equation $x^2 = a$ has a solution $x \in X$. We say that X is 2-torsion free if the unit element e of G belongs to X and the equality $x^2 = e$ implies $x = e$.

We shall need the following technical lemma. We remark that its proof has appeared as a part of the proof of Corollary 3.10 in [12].

Lemma 10. *Let X be a twisted subgroup of a group which is 2-divisible and 2-torsion free and let $c \in X$. The only solution $x \in X$ of the equation $cx^{-1}c = x$ is $x = c$.*

Proof. Since X is 2-divisible there exists an element $g \in X$ such that $g^2 = c$. From $g^2 x^{-1} g^2 = x$ it follows that $g^2 x^{-1} g^2 x^{-1} = e$ and then multiplying by g^{-1} from the left and by g from the right, we have

$$e = gx^{-1}g^2x^{-1}g = (gx^{-1}g)^2.$$

By the 2-torsion free property of X we deduce that $gx^{-1}g = e$. This implies $x^{-1} = g^{-2}$ and hence $x = g^2 = c$. $\square$

We next introduce some conditions for a pair a, b of elements that belong to a twisted subgroup of a group. We shall use them in the next proposition.

Let X be a twisted subgroup of a group G, let $d : X \times X \to [0, \infty[$ be any function and pick $a, b \in X$. We say that the pair a, b satisfies the condition

(b1) if the equality

$$d(bx^{-1}b, by^{-1}b) = d(y, x)$$

holds for any $x, y \in X$;

(b2) if $\sup\{d(x, b) | x \in L_{a,b}\} < \infty$, where

$$L_{a,b} = \{x \in X | d(a, x) = d(x, ba^{-1}b) = d(a, b)\};$$

(b3) if there exists a constant $K > 1$ such that

$$d(x, bx^{-1}b) \geq Kd(x, b), \qquad x \in L_{a,b};$$

(b4) if there exists an element $c \in X$ with $ca^{-1}c = b$ such that

$$d(cx^{-1}c, cy^{-1}c) = d(y, x)$$

holds for any $x, y \in X$.

Now we present our basic tool in the proof of Theorem 1.

Proposition 11. *Let G be a group and $X \subset G$ a twisted subgroup which is 2-divisible and 2-torsion free. Assume that the function $d:$ $X \times X \to [0, \infty[$ is a generalized distance measure, i.e., it has the property that for any $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$. Let $T: X \to X$ be a surjective d-preserving map. Pick $a, b \in X$ such that the pair a, b satisfies the conditions (b1)-(b3) and the pair $T(a), T(ba^{-1}b)$ satisfies the condition (b4). Then we have*

$$T(ba^{-1}b) = T(b)T(a)^{-1}T(b).$$

Proof. First observe that any d-preserving function is automatically injective. Let $\varphi(x) = bx^{-1}b$ for every $x \in X$. Then φ is a bijective d-reversing map on X and it satisfies the conditions appearing in Proposition 9, i.e., it fixes b and $\varphi \circ \varphi$ is the identity. Since (b4) holds for the pair $T(a), T(ba^{-1}b)$, there exists an element $c \in X$ such that

$$(6) \qquad\qquad cT(a)^{-1}c = T(ba^{-1}b)$$

and $d(cx^{-1}c, cy^{-1}c) = d(y, x)$ holds for all $x, y \in X$. Let the map $\psi: X \to X$ be defined by $\psi(x) = cx^{-1}c$ for every $x \in X$. Clearly, ψ is a bijective d-reversing map on X and by (6) we have that $\psi(T(a)) = T(\varphi(a))$ and also that $\psi(T(\varphi(a))) = T(a)$ holds. Now we are in a position to apply Proposition 9 and we get that $d(\psi(T(b)), T(b)) = 0$ which implies $T(b) = cT(b)^{-1}c$. Using Lemma 10 we infer that $c = T(b)$. Finally, by (6) we obtain

$$T(ba^{-1}b) = T(b)T(a)^{-1}T(b).$$

$\square$

After these preliminaries we can present the proof of Theorem 1.

Proof of Theorem 1. Let N, f be as in the formulation of the theorem and let $\phi \colon \mathbb{P}_n \to \mathbb{P}_n$ be a surjective map which preserves the generalized distance measure $d_{N,f}(.,.)$, i.e., assume

$$d_{N,f}(\phi(A), \phi(B)) = d_{N,f}(A, B), \qquad A, B \in \mathbb{P}_n.$$

We are going to apply Proposition 11. To do this, we show that all conditions appearing there are satisfied for $\mathbb{P}_n$ and for any pair A, B of its elements.

First, $X = \mathbb{P}_n$ is a twisted subgroup of the group of all invertible matrices which is clearly 2-divisible and 2-torsion free. Next, we assert that the equalities

$$(7) \quad d_{N,f}(A^{-1}, B^{-1}) = d_{N,f}(B, A), \quad d_{N,f}(TAT^*, TBT^*) = d_{N,f}(A, B)$$

hold for all $A, B \in \mathbb{P}_n$ and invertible matrix $T \in \mathbb{M}_n$. Indeed, let $A, B \in \mathbb{P}_n$ and consider the polar decomposition $B^{-1/2}A^{1/2} = U|B^{-1/2}A^{1/2}|$. We see that $|A^{1/2}B^{-1/2}|^2 = U|B^{-1/2}A^{1/2}|^2U^*$ and then compute

$$d_{N,f}(A^{-1}, B^{-1}) = N(f(A^{1/2}B^{-1}A^{1/2})) = N(f(|B^{-1/2}A^{1/2}|^2))$$
$$= N(f(U^*|A^{1/2}B^{-1/2}|^2U)) = N(U^*f(|A^{1/2}B^{-1/2}|^2)U)$$
$$= N(f(B^{-1/2}AB^{-1/2})) = d_{N,f}(B, A).$$

Now, for an arbitrary invertible matrix $T \in \mathbb{M}_n$ we deduce

$$((TAT^*)^{-1/2}TBT^*(TAT^*)^{-1/2})^2$$
$$= (TAT^*)^{-1/2}TBT^*(TAT^*)^{-1}TBT^*(TAT^*)^{-1/2}.$$

For $X = A^{-1/2}BT^*(TAT^*)^{-1/2}$ we have

$$XX^* = A^{-1/2}BA^{-1}BA^{-1/2} = (A^{-1/2}BA^{-1/2})^2.$$

Hence, using the polar decomposition $X = V|X|$, we compute

$$(TAT^*)^{-1/2}TBT^*(TAT^*)^{-1/2}$$
$$= ((TAT^*)^{-1/2}TBT^*(TAT^*)^{-1}TBT^*(TAT^*)^{-1/2})^{1/2}$$
$$= ((TAT^*)^{-1/2}TBA^{-1}BT^*(TAT^*)^{-1/2})^{1/2}$$
$$= (X^*X)^{1/2} = |X| = V^*|X^*|V = V^*(A^{-1/2}BA^{-1/2})V.$$

It readily follows that $d_{N,f}(TAT^*, TBT^*) = d_{N,f}(A, B)$ holds for any $A, B \in \mathbb{P}_n$ completing the proof of (7).

Let us now select two arbitrary elements A, B of $\mathbb{P}_n$. By (7), the condition (b1) is satisfied for the pair A, B. As for condition (b2), let

us consider the set $\mathcal{H}$ of those elements $X \in \mathbb{P}_n$ for which we have

$$d_{N,f}(A, X) = N(f(A^{-1/2}XA^{-1/2}))$$
$$= N(f(A^{-1/2}BA^{-1/2})) = d_{N,f}(A, B).$$

(Clearly, $L_{A,B} \subset \mathcal{H}$.) We show that the corresponding set of numbers

$$N(f(X^{-1/2}BX^{-1/2})) = d_{N,f}(X, B)$$
$$= d_{N,f}(B^{-1}, X^{-1}) = N(f(B^{1/2}X^{-1}B^{1/2}))$$

is bounded. Indeed, since $N(f(A^{-1/2}XA^{-1/2}))$ is constant on $\mathcal{H}$ and N is equivalent to the operator norm $||\cdot||$, the set

$$\{\|f(A^{-1/2}XA^{-1/2})\| : X \in \mathcal{H}\}$$

is bounded. It is easy to see that (a1), (a2) imply

$$\lim_{y \to 0} f(y), \lim_{y \to \infty} f(y) \in \{-\infty, \infty\}.$$

Then it follows easily that there are positive numbers m, M such that $mI \leq A^{-1/2}XA^{-1/2} \leq MI$ holds for all $X \in \mathcal{H}$. Clearly, we then have another pair m', M' of positive numbers such that $m'I \leq X \leq M'I$ and finally another one m'', M'' such that $m''I \leq B^{1/2}X^{-1}B^{1/2} \leq M''I$ holds for all $X \in \mathcal{H}$. By continuity, f is bounded on the interval $[m'', M'']$ and this implies that the set

$$\{N(f(B^{1/2}X^{-1}B^{1/2})) : X \in \mathcal{H}\}$$

is bounded. We conclude that the condition (b2) is fulfilled.

Relating to condition (b3) we assert that $N(f(C^2)) \geq KN(f(C))$ holds for every $C \in \mathbb{P}_n$. To see this, we recall the famous fact that any unitarily invariant norm on $\mathbb{M}_n$ is induced by some symmetric gauge function on $\mathbb{R}^n$. By a well-known result of Ky Fan [10], for given finite sequences $0 \leq a_n \leq \ldots \leq a_1$ and $0 \leq b_n \leq \ldots \leq b_1$ of numbers we have $\Phi(a_1, \ldots, a_n) \leq \Phi(b_1, \ldots, b_n)$ for all symmetric gauge functions Φ on $\mathbb{R}^n$ if and only if the inequality

$$\sum_{i=1}^{k} a_k \leq \sum_{i=1}^{k} b_k$$

holds for every $1 \leq k \leq n$. By (a2) it then follows that

(8) $\Psi\left(|f(\lambda_1)|^2, \ldots, |f(\lambda_n)|^2\right) \geq K\Psi\left(|f(\lambda_1)|, \ldots, |f(\lambda_n)|\right),$

where Ψ is the symmetric gauge function corresponding to N and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of an arbitrary positive definite matrix $C \in \mathbb{P}_n$. Consequently, we obtain the desired inequality $N(f(C^2)) \geq KN(f(C))$.

Next, selecting any $X \in \mathbb{P}_n$ and setting $Y = X^{-1/2}BX^{-1/2}$, we easily deduce that

$$d_{N,f}(X, BX^{-1}B) = N(f(X^{-1/2}BX^{-1}BX^{-1/2}))$$

$$= N(f(Y^2)) \geq KN(f(Y)) = KN(f(X^{-1/2}BX^{-1/2})) = Kd_{N,f}(X, B).$$

Therefore, the condition (b3) is also satisfied. Consequently, all assumptions (b1)-(b3) are fulfilled for any pair $A, B \in \mathbb{P}_n$.

We assert that the same holds in relation with condition (b4), too. To see this, observe that for any pair $A, B \in \mathbb{P}_n$ we can find $C \in \mathbb{P}_n$ such that $CA^{-1}C = B$. Indeed, the geometric mean of A and B, that is, the positive definite matrix $C = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ is a solution of that equation. The remaining invariance property of $d_{N,f}$ in (b4) has already been verified in (7).

Taking all the information what we have into account, we can now apply Proposition 11 and obtain that $\phi \colon \mathbb{P}_n \to \mathbb{P}_n$ is a bijective map which satisfies

$$\phi(BA^{-1}B) = \phi(B)\phi(A)^{-1}\phi(B)$$

for all $A, B \in \mathbb{P}_n$. We prefer to write

$$\phi(AB^{-1}A) = \phi(A)\phi(B)^{-1}\phi(A), \qquad A, B \in \mathbb{P}_n.$$

Consider the transformation $\psi \colon \mathbb{P}_n \to \mathbb{P}_n$ defined by

$$\psi(A) = \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}, \qquad A \in \mathbb{P}_n.$$

It is easy to see that ψ is a bijective map on $\mathbb{P}_n$ which satisfies

$$\psi(AB^{-1}A) = \psi(A)\psi(B)^{-1}\psi(A), \qquad A, B \in \mathbb{P}_n$$

and has the additional property that $\psi(I) = I$. Substituting $A = I$ in the above displayed equation we obtain that $\psi(B^{-1}) = \psi(B)^{-1}$ which implies that ψ is a Jordan triple automorphism of $\mathbb{P}_n$, i.e., a bijective map satisfying

$$\psi(ABA) = \psi(A)\psi(B)\psi(A), \qquad A, B \in \mathbb{P}_n.$$

We next prove that ψ is continuous in the operator norm. Clearly, ψ preserves $d_{N,f}(.,.)$ which is a consequence of the second invariance property in (7). Let (X_n) be a sequence in $\mathbb{P}_n$ which tends to $X \in \mathbb{P}_n$ with respect to the operator norm topology. Then $X^{-1/2}X_nX^{-1/2} \to I$, and hence

$$d_{N,f}(X, X_n) = N(f(X^{-1/2}X_nX^{-1/2})) \to N(f(I)) = 0.$$

Since ψ preserves the generalized distance measure $d_{N,f}(.,.)$, we infer that

$$N(f(\psi(X)^{-1/2}\psi(X_n)\psi(X)^{-1/2})) = d_{N,f}(\psi(X), \psi(X_n)) \to 0.$$

It follows that $f(\psi(X)^{-1/2}\psi(X_n)\psi(X)^{-1/2}) \to 0$ in the operator norm. By the continuity of f and the property (a1), it is easy to verify that we necessarily have

$$\psi(X)^{-1/2}\psi(X_n)\psi(X)^{-1/2} \to I,$$

i.e., $\psi(X_n) \to \psi(X)$ in the operator norm and we obtain the continuity of ψ.

The structure of continuous Jordan triple automorphisms of $\mathbb{P}_n$ has been determined in [14]. Applying Corollary 2 in that paper we have a unitary matrix U and a scalar $c \neq -1/n$ such that ψ is of one of the forms

(i) $\psi(A) = (\det A)^c U A U^*$, $A \in \mathbb{P}_n$;
(ii) $\psi(A) = (\det A)^c U A^{-1} U^*$, $A \in \mathbb{P}_n$;
(iii) $\psi(A) = (\det A)^c U A^{\mathrm{tr}} U^*$, $A \in \mathbb{P}_n$;
(iv) $\psi(A) = (\det A)^c U (A^{\mathrm{tr}})^{-1} U^*$, $A \in \mathbb{P}_n$.

By the definition of the transformation ψ we get that ϕ is necessarily of one of the forms (f1)-(f4) and the proof of the theorem is complete. $\square$

As already mentioned in the first section of the paper, when trying to determine the precise structure of bijective maps of $\mathbb{P}_n$ preserving a generalized distance measure with particular N and f, one should not stop at applying Theorem 1 but proceed further and check which ones of the possibilities (f1)-(f4) and for which parameters c and T give transformations that really have the desired preserver property (4). In fact, as for T, we can tell that for any invertible matrix $T \in \mathbb{M}_n$ the map $A \mapsto TAT^*$ satisfies (4). This follows from the second equality in (7). Concerning the inverse operation $A \mapsto A^{-1}$, there are cases where it does not show up. In fact, by the first equality in (7) that map is $d_{N,f}$-reversing, hence when $d_{N,f}$ is not symmetric, the inverse is surely not $d_{N,f}$-preserving. For example, this is the case with the Stein's loss $l(.,.)$. However, the transpose is always $d_{N,f}$-preserving. Indeed, it follows from the facts that the transpose operation commutes with the inverse operation, with the square root, with the map $A \mapsto f(A)$, and furthermore $N(C^{\mathrm{tr}}) = N(C)$ holds for every self-adjoint matrix C. For the above reasons, the map $A \mapsto (A^{\mathrm{tr}})^{-1}$ sometimes shows up, sometimes does not. This is the case with the determinant function too as can be seen, for example, in Theorem 3 in [14].

We now turn to the proof of Theorem 3. The structure of continuous Jordan triple endomorphisms of $\mathbb{P}_n$ has been described in Theorem 1 in [14]. We are going to apply that result in the proof below.

Proof of the Theorem 3. Let $\phi\colon \mathbb{P}_n^1 \to \mathbb{P}_n^1$ be a continuous Jordan triple endomorphism of $\mathbb{P}_n^1$, i.e., a continuous map which satisfies

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \qquad A, B \in \mathbb{P}_n^1.$$

Consider the transformation $\psi\colon \mathbb{P}_n \to \mathbb{P}_n$ defined by

$$\psi(A) = \sqrt[n]{\det(A)}\,\phi\left(\frac{A}{\sqrt[n]{\det(A)}}\right), \qquad A \in \mathbb{P}_n.$$

One can check trivially that ψ is a Jordan triple endomorphism of $\mathbb{P}_n$ which extends ϕ. Applying Theorem 1 in [14], it follows that there exist a unitary matrix $U \in \mathbb{M}_n$ and a real number c such that ψ is of one of the forms (i)-(iv) appearing at the end of the previous proof, or there exist a set $\{P_1, \ldots, P_n\}$ of mutually orthogonal rank-one projections in $\mathbb{M}_n$ and a set $\{c_1, \ldots, c_n\}$ of real numbers such that ψ is of the form

(v) $\psi(A) = \sum_{i=1}^n (\det A)^{c_i} P_i, \qquad A \in \mathbb{P}_n.$

Since $\{P_1, \ldots, P_n\}$ is a set of n mutually orthogonal rank-one projections, thus their sum equals the identity. Consequently, in this latter case ψ sends matrices with unit determinant to the identity. This implies that ϕ is really of one of the forms (g1)-(g5). The proof of the theorem is complete. $\qquad\square$

In what remains we present the key steps of the proof of Theorem 5. In fact, we use an approach very similar to the one we followed in the proof of Theorem 1 above, hence the details are omitted.

Sketch of the proof of Theorem 5. Let $\phi\colon \mathbb{P}_n^1 \to \mathbb{P}_n^1$ be a surjective map which preserves the generalized distance measure $d_{N,f}(.,.)$, i.e., which satisfies

$$d_{N,f}(\phi(A), \phi(B)) = d_{N,f}(A, B), \qquad A, B \in \mathbb{P}_n^1.$$

We claim that all conditions appearing in Proposition 11 are satisfied.

Clearly, the set $\mathbb{P}_n^1$ is a 2-divisible and 2-torsion free twisted subgroup of the group of all invertible matrices. Referring back to the proof of Theorem 1, the invariance properties (7) of $d_{N,f}$ hold true on the set $\mathbb{P}_n^1$, too. Similarly, the conditions (b1)-(b4) are satisfied for every pair A, B of elements of the subset $\mathbb{P}_n^1$ of $\mathbb{P}_n$. This means that we can apply Proposition 11 and we then obtain that ϕ is an inverted Jordan triple automorphism of $\mathbb{P}_n^1$, i.e.,

$$\phi(AB^{-1}A) = \phi(A)\phi(B)^{-1}\phi(A), \qquad A, B \in \mathbb{P}_n^1.$$

Next, we consider the transformation $\psi\colon \mathbb{P}_n^1 \to \mathbb{P}_n^1$ defined by

$$\psi(A) = \phi(I)^{-1/2}\phi(A)\phi(I)^{-1/2}, \qquad A \in \mathbb{P}_n^1.$$

It turns to be a Jordan triple automorphism of $\mathbb{P}_n^1$ which also preserves the generalized distance measure $d_{N,f}$. Following the argument presented in the proof of Theorem 1 gives that ψ is continuous. Therefore, by Corollary 4, we get that ψ is of one of the forms (g1)-(g4). Finally we conclude that ϕ is of one of the forms (h1)-(h4) and this completes the proof. □

Remark 12. In several applications the set of symmetric positive definite real matrices plays more important role than that of the positive definite complex matrices. See, e.g., [11] and [8]. In accordance with this, we remark that the main results of this paper, Theorems 1 and 5, along with Theorem 2 and Corollaries 4, 6, remain valid also in the real case. Indeed, a careful examination of our arguments above shows that all steps in the proofs can be unaltered, the only thing we really need to deal with is the structure of all continuous Jordan triple automorphisms of the set of all $n \times n$ symmetric positive definite real matrices ($n \geq 3$). In the complex case, those transformations have been described by the first author in Corollary 2 in [14]. In the real case, we can follow steps similar to the ones given in the proofs of Lemmas 5-7 and Theorem 1 in that paper. In fact, the mentioned lemmas can be shown in the same way as in [14] (the proof of Lemma 7 is given in [13]), but as for Theorem 1 and its Corollary 2 we need to use the result of Chan and Lim which describes the structure of all bijective commutativity preserving linear maps on the space of $n \times n$ symmetric real matrices [1]. Apparently, this means that in the real case we have a structural result only for Jordan triple automorphisms and not for all continuous Jordan triple endomorphisms.

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