

Neumark operators and sharp reconstructions: The finite dimensional case

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A commutative positive operator valued (POV) measure F with real spectrum is characterized by the existence of a projection valued measure E (the sharp reconstruction of F) with real spectrum such that F can be interpreted as a randomization of E . This paper focuses on the relationships between this characterization of commutative POV measures and Neumark's extension theorem. In particular, we show that in the finite dimensional case there exists a relation between the Neumark operator corresponding to the extension of F and the sharp reconstruction of F . The relevance of this result to the theory of nonideal quantum measurement and to the definition of unsharpness is analyzed. © 2007 American Institute of Physics. [DOI: [10.1063/1.2437653](https://doi.org/10.1063/1.2437653)]

I. INTRODUCTION

A positive operator valued measure (POV measure) is a map F from the Borel σ -algebra of the reals $\mathcal{B}(\mathbb{R})$ into the set of bounded, positive, self-adjoint operators $\mathcal{F}(\mathcal{H})$ in a Hilbert space \mathcal{H} . POV measures were introduced in quantum mechanics by Davis,¹ Ludwig,² and Holevo³⁻⁵ in order to generalize the concept of observable of a physical system. Before the introduction of POV measures, the observables were described by projection valued measures (PV measures). In the new terminology, we distinguish between sharp observables, which are described by PV measures, and unsharp observables, which are described by POV measures.

Although the introduction of POV measures comes from the foundational analysis of quantum mechanics, POV measures find several applications, for example, in quantum stochastic processes, quantum optics,^{6,7} and relativistic quantum mechanics.⁸ A way to justify their use is to consider the process of repeated measurements of a quantum observable⁹ or to derive them as a consequence of the probabilistic structure of quantum mechanics.^{3-5,10}

The state of a physical system can be represented by a density operator ρ acting in \mathcal{H} . Holevo³⁻⁵ has shown that there exists a one-to-one correspondence between POV measures and affine maps from the set of states of a physical system into the set of probability measures on $\mathcal{B}(\mathbb{R})$. The affine map $\rho \mapsto \mu_{(\cdot)}^F(\rho)$ corresponding to the POV measure F is determined by the relation

$$\mu_{(\Delta)}^F(\rho) = \text{Tr}[\rho F(\Delta)] \quad \text{for all } \Delta \in \mathcal{B}(\mathbb{R}). \quad (1)$$

This allows one to interpret the real number $\mu_{(\Delta)}^F(\rho) = \text{Tr}[\rho F(\Delta)]$ as the probability that the outcomes of the measurement of an unsharp observable^{1-3,9,10} F is in Δ when the physical system under consideration is in the state ρ . Equation (1) generalizes the analogous equation

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$$\mu_{(\Delta)}^E(\rho) = \text{Tr}[\rho E(\Delta)],$$

which has the same meaning for a sharp observable^{1-3,9,10} $E: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$, from $\mathcal{B}(\mathbb{R})$ to the set of projection operators $\mathcal{E}(\mathcal{H})$. Therefore, the unsharp observables, represented by POV measures, generalize the sharp ones, represented by PV measures.

Several characterizations of POV measures¹¹⁻¹⁷ can be found in the literature.

This paper focuses on the problem of finding a relation between two of them. The first characterization is due to Neumark¹¹ and establishes that every POV measure $F: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\mathcal{H})$ can be extended to a PV measure $E^+: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\mathcal{H}^+)$ in an extended Hilbert space \mathcal{H}^+ in such a way that F is the projection of E^+ on \mathcal{H} (see Theorem 3). Neumark's theorem brings to a physical interpretation^{3,4} the measurement of nonorthogonal POV measures (see Corollary 1).

The second characterization can be found in Refs. 15-18 and concerns commutative POV measures. In Ref. 17 (see Theorem 2), it is shown that for each commutative POV measure F there exists a self-adjoint operator A with spectrum $\sigma(A)$, named the sharp reconstruction of F , such that for every $\lambda \in \sigma(A)$, there is a probability measure $\mu_{(\Delta)}^A(\lambda): \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ such that

$$F(\Delta) = \int \mu_{(\Delta)}^A(\lambda) dE_{\lambda}^A = \mu_{\Delta}^A(A), \quad \Delta \in \mathcal{B}(\mathbb{R}), \quad (2)$$

where E^A is the PV measure corresponding to A and the rules of the functional calculus have been used (see Sec. II and Ref. 19). The sharp reconstruction A is unique up to bijections. This characterization allows^{16,17} one to interpret a commutative POV measure F as a randomization of its sharp reconstruction A (see Theorem 2 and comments to the theorem).

In the present paper, we restrict ourselves, without loss of generality, to the case of POV measures with spectrum in $[0, 1]$ (see Sec. III and Appendix A). Let F be a commutative POV measure, $A = \int \lambda dE_{\lambda}^A$ its sharp reconstruction, E^+ a Neumark extension of F , and $A^+ = \int \lambda dE_{\lambda}^{E^+}$ the corresponding operator. We show (Theorem 4) that for any bounded, measurable function $f: [0, 1] \rightarrow \mathbb{R}$, there exists a bounded, measurable function $G_f: [0, 1] \rightarrow [0, 1]$ such that

$$P^+ f(A^+) |_{\mathcal{H}} = G_f(A),$$

where P^+ is the operator of projection onto \mathcal{H} , $G_f(A) = \int G_f(\lambda) dE_{\lambda}^A$, $f(A^+) = \int f(\lambda) dE_{\lambda}^{E^+}$, and $f(A^+) |_{\mathcal{H}}$ is the restriction of $f(A^+)$ to \mathcal{H} .

Moreover, we prove (Theorem 7) that, in the finite dimensional case, there exists a bounded, one-to-one function f such that G_f is one to one (this is shown to be true for POV measures both with finite and countably infinite outcome sets). This gives a notion of equivalence between sharp reconstructions and projections of Neumark operators which generalizes the one proposed in Ref. 18. We denote this equivalence by $A \leftrightarrow \text{Pr } A^+$ (see Definition 9). This result suggests that it is reasonable to look for an extension of Theorem 7 to the infinite dimensional case. Furthermore, it bears interesting implications to the theory of nonideal quantum measurement^{20,21} (see Corollary 3).

Finally, the properties of the sharp reconstruction proved in Theorems 6 and 7 allow us to comment on the differences between the definition of unsharpness proposed in Refs. 20 and 21 (see Definition 11) and that given in Refs. 22-25 (see Definition 12). In particular, we show that the two definitions do not coincide even in the case of commutative POV measures. Moreover, denoting by \mathcal{A}_1 and \mathcal{A}_2 the sets of sharp observables of which F is an unsharp version according to Definitions 11 and 12, respectively, it is possible to see that $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$. In fact, the sharp reconstruction belongs to $\mathcal{A}_1 \cap \mathcal{A}_2$. Furthermore, we show that Definition 12 can be modified in order to enlarge \mathcal{A}_2 to a set which contains $\mathcal{A}_1 \cup \mathcal{A}_2$ (see Definition 13 and Theorem 9).

The paper is organized as follows. In Sec. II we introduce some basic definitions, state the classical theorem of Neumark, and summarize the main results of Refs. 16 and 17. In Sec. III A we state Theorem 4 and present some examples of POV measures such that $A \leftrightarrow \text{Pr } A^+$. In Sec. III B we prove the main result of the paper, Theorem 7, and give some examples. Then, in the last section, we prove Corollary 3 and make some observations on the definition of an unsharp

observable. In Appendix A we show that we can restrict ourselves, without loss of generality, to POV measures with a bounded spectrum. In Appendix B we prove Theorem 4. In Appendix C we prove Lemma 1.

II. PRELIMINARIES

In this section we fix the basic notation and terminology, state the classical theorem of Neumark, and give the characterization of commutative POV measures obtained in Refs. 16 and 17.

We denote by $\mathcal{B}(\mathbb{R})$ the Borel σ -algebra of \mathbb{R} , by $\mathbf{0}$ and $\mathbf{1}$ the null and the identity operators, respectively, by $\mathcal{L}_s(\mathcal{H})$ the space of all bounded self-adjoint linear operators acting on a Hilbert space \mathcal{H} with scalar product $\langle \cdot, \cdot \rangle$, by $\mathcal{F}(\mathcal{H}) \subset \mathcal{L}_s(\mathcal{H})$ the subspace of all positive, bounded self-adjoint operators on \mathcal{H} , and by $\mathcal{E}(\mathcal{H}) \subset \mathcal{F}(\mathcal{H})$ the subspace of all projection operators on \mathcal{H} .

Definition 1: A positive operator valued measure (for short, POV measure) is a map $F: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\mathcal{H})$ such that if $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(\mathbb{R})$, then

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n), \quad (3)$$

where the series converges in the weak operator topology.

Definition 2: A POV measure is said to be

1. normalized if

$$F(\mathbb{R}) = \mathbf{1}; \quad (4)$$

2. commutative if

$$[F(\Delta_1), F(\Delta_2)] = \mathbf{0}, \quad \forall \Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R}); \quad (5)$$

3. orthogonal if

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \quad \text{if } \Delta_1 \cap \Delta_2 = \emptyset. \quad (6)$$

In what follows we shall always refer to normalized POV measures defined on $\mathcal{B}(\mathbb{R})$.

Definition 3: A projection valued measure (for short, PV measure) is an orthogonal, normalized POV measure.

It is simple to see that for a PV measure E , we have $E(\Delta) = E(\Delta)^2$ for any $\Delta \in \mathcal{B}(\mathbb{R})$. Then, $E(\Delta)$ is a projection operator for every $\Delta \in \mathcal{B}(\mathbb{R})$, and the PV measure is a map $E: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$.

In quantum mechanics, nonorthogonal normalized POV measures are also called generalized or unsharp observables and PV measures standard or sharp observables.

We shall use the term “measurable” for the Borel measurable functions. For any vector $x \in \mathcal{H}$, the map

$$\langle F(\cdot)x, x \rangle: \mathcal{B}(\mathbb{R}) \rightarrow \mathbb{R}, \quad \Delta \mapsto \langle F(\Delta)x, x \rangle,$$

is a Lebesgue-Stieltjes measure. There exists a one-to-one correspondence²⁶ between POV measures F and POV functions $F_\lambda := F((-\infty, \lambda])$. In the following, we will use the symbol $d\langle F_\lambda x, x \rangle$ to mean integration with respect to the measure $\langle F(\cdot)x, x \rangle$.

We shall say that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is bounded with respect to a POV measure F if it is equal to a bounded function g almost everywhere (a.e.) with respect to F ; that is, if $f=g$ a.e. with respect to the measure $\langle F(\cdot)x, x \rangle$, $\forall x \in \mathcal{H}$. For any real, bounded, and measurable function f and for any $F \in \mathcal{F}(\mathcal{H})$, there is a unique²⁷ bounded self-adjoint operator $B \in \mathcal{L}_s(\mathcal{H})$ such that

$$\langle Bx, x \rangle = \int f(\lambda) d\langle F_\lambda x, x \rangle \quad \text{for each } x \in \mathcal{H}. \quad (7)$$

If Eq. (7) is satisfied, we write $B = \int f(\lambda) dF_\lambda$.

Definition 4: The spectrum $\sigma(F)$ of a POV measure F is the closed set

$$\{\lambda \in \mathbb{R} : F((\lambda - \delta, \lambda + \delta)) \neq 0, \forall \delta > 0\}.$$

By the spectral theorem,^{19,28} PV measures are in a one-to-one correspondence with self-adjoint operators. In fact, we recall the following theorem of functional analysis.

Theorem 1 (see Ref. 19): There is a one-to-one correspondence between self-adjoint operators B on a Hilbert space \mathcal{H} and PV measures E^B on \mathcal{H} , the correspondence being given by

$$B = \int \lambda dE_\lambda^B.$$

In the following, we do not distinguish between PV measures and the corresponding self-adjoint operators.

If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a measurable real-valued function, then we can define the self-adjoint operator¹⁹

$$f(B) = \int f(\lambda) dE_\lambda^B.$$

If f is bounded, then $f(B)$ is bounded.¹⁹

Definition 5: Two bounded self-adjoint operators A and B are said to be equivalent if there exists a bounded, one-to-one, measurable function f such that $A = f(B)$. In this case we write $A \leftrightarrow B$.

Definition 6: We say that the triplet $(F, B, \mu_{(\cdot)}^B(\lambda))$ satisfies the thesis of von Neumann's theorem²⁹⁻³¹ if $\mu_{(\Delta)}(B) = F(\Delta)$, for every $\Delta \in \mathcal{B}(\mathbb{R})$.

Summing up the results obtained in Refs. 15–17, we can state the following theorem.

Theorem 2 (see Refs. 16 and 17): A POV measure $F: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\mathcal{H})$ is commutative if and only if there exist a self-adjoint operator B and, for every $\lambda \in \sigma(B)$, a probability measure^{32,33} $\mu_{(\cdot)}^B(\lambda): \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ such that the triplet $(F, B, \mu_{(\cdot)}^B(\lambda))$ satisfies the thesis of von Neumann's theorem.

Moreover, there exists a couple $(A, \mu_{(\cdot)}^A(\lambda))$ such that (i) the triplet $(F, A, \mu_{(\cdot)}^A(\lambda))$ satisfies the thesis of von Neumann's theorem, and that (ii) for every triplet $(F, B, \mu_{(\cdot)}^B(\lambda))$ satisfying the thesis of von Neumann's theorem, there exists a real function g such that $A = g(B)$. The operator A is unique up to bijections.

Definition 7: The operator A defined by Theorem 2 (or, equivalently, the corresponding PV measure E^A) is called the sharp reconstruction of F .

Theorem 2 suggests interpreting^{16,17} the outcomes of the measurement of F as deriving from a randomization of the outcomes of the measurement of its sharp reconstruction E^A . Indeed, for every $\Delta \in \mathcal{B}(\mathbb{R})$ and $\lambda \in \sigma(A)$, $\mu_{(\Delta)}^A(\lambda)$ can be interpreted as the probability that the outcome of a measurement of F is in Δ when the outcome of the measurement of E^A is λ .

Theorem 3 (Neumark^{29,31}): Let F be a POV measure of the Hilbert space \mathcal{H} . Then there exist a Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ and a PV measure E^+ of the space \mathcal{H}^+ such that

$$F(\Delta) = P^+ E^+(\Delta)|_{\mathcal{H}},$$

where P^+ is the operator of projection onto \mathcal{H} .

Definition 8: Each operator $\int f(\lambda) dE^+$, where f is a one-to-one, measurable, real-valued function, is said to be a Naimark operator corresponding to F . The Neumark operator $\int \lambda dE^+$ is denoted by A^+ .

The following corollary yields a physical interpretation of the measurement of a nonorthogonal POV measure.

Corollary 1 (see Refs. 3 and 4): For any POV measure $F: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\mathcal{H})$, there exists a

Hilbert space \mathcal{H}_0 , a pure state S_0 in \mathcal{H}_0 , and a PV measure $E^+ : \mathcal{B}(R) \rightarrow \mathcal{E}(\mathcal{H}^+)$ in $\mathcal{H}^+ = \mathcal{H} \otimes \mathcal{H}_0$ such that

$$\mu_{(\Delta)}^{E^+}(S \otimes S_0) = \mu_{(\Delta)}^F(S), \quad \Delta \in \mathcal{B}(R), \quad (8)$$

for each state S in \mathcal{H} . The converse is also true; that is, for every triplet $(\mathcal{H}_0, S_0, E^+)$, where E^+ is a PV measure in the Hilbert space $\mathcal{H} \otimes \mathcal{H}_0$ and S_0 is a pure state in \mathcal{H}_0 , there exists a unique POV measure F satisfying Eq. (8).

Equation (8) establishes the existence of a pure state S_0 such that the measurements of the observables F and E^+ are statistically equivalent. Therefore the measurement of an unsharp observable in the Hilbert space \mathcal{H} is equivalent to the measurement of a sharp one in the Hilbert space $\mathcal{H} \otimes \mathcal{H}_0$, which represents the composition between the system and additional independent degrees of freedom described by \mathcal{H}_0 .

Proposition 1: Let us consider the extension E^+ of a POV measure F and the Neumark operator $A^+ = \int \lambda dE_\lambda^+$ corresponding to E^+ . Let f be a measurable function which is bounded with respect to E^+ . Then

$$P^+f(A^+)_{|\mathcal{H}} = \int f(\lambda) dF_\lambda,$$

and $P^+f(A^+)_{|\mathcal{H}}$ is a bounded self-adjoint operator. □

Proof:

$$\langle P^+f(A^+)x, y \rangle = \int_{-\infty}^{\infty} f(\lambda) d\langle E_\lambda^+x, P^+y \rangle = \int_{-\infty}^{\infty} f(\lambda) d\langle F_\lambda x, y \rangle$$

for every $x, y \in \mathcal{H}$.

The boundedness and the self-adjointness of $P^+f(A^+)_{|\mathcal{H}}$ come, respectively, from the boundedness and the real valuedness of f with respect to E^+ (see Theorem 10 in Ref. 27).

Definition 9: Whenever there exists a one-to-one, bounded, measurable function $f: \sigma(A^+) \rightarrow \mathbb{R}$ such that the sharp reconstruction A of a commutative POV measure F is equivalent to $P^+f(A^+)_{|\mathcal{H}}$, we write $A \leftrightarrow \text{Pr } A^+$ and say that the sharp reconstruction A is equivalent to the projection of a Neumark operator corresponding to F .

III. NAIMARK OPERATORS AND SHARP RECONSTRUCTIONS

In the present section we analyze the relationships between Neumark's theorem and Theorem 2. In Sec. III A, generalizing a result obtained in Ref. 18, we show that for every commutative POV measure F and for every bounded and measurable function f , there exists a function G_f such that $G_f(A) = P^+f(A^+)_{|\mathcal{H}}$, where A and A^+ are respectively the sharp reconstruction of F and the Neumark operator $\int \lambda dE_\lambda^+$ corresponding to the extension E^+ of F . We also give some examples where $A \leftrightarrow \text{Pr } A^+$. In Sec. III B we prove that, in the finite dimensional case, a positive answer can be given to the problem of the equivalence between the sharp reconstruction A and the projection of the Neumark operator, i.e., there exists always a one-to-one, bounded, and measurable function f such that $A \leftrightarrow P^+f(A^+)_{|\mathcal{H}}$.

We recall that, as shown in Ref. 15, the sharp reconstruction A of a given POV measure F coincides with the sharp reconstruction of the POV measure \bar{F} defined by $\bar{F}(\Delta) = F(f(\Delta \cap (0, 1)))$, where f is a one-to-one and measurable function from $(0, 1)$ to \mathbb{R} and $\sigma(\bar{F}) \subset [0, 1]$. In Appendix A we prove that $A \leftrightarrow \text{Pr } A^+$ if and only if $A \leftrightarrow \text{Pr } \bar{A}^+$, having denoted by A^+ and \bar{A}^+ the Neumark operators $\int \lambda dE_\lambda^+$ and $\int \lambda d\bar{E}_\lambda^+$ associated with F and \bar{F} , respectively. Therefore, in what follows, we restrict ourselves, without loss of generality, to POV measures with spectrum in $[0, 1]$.

A. The general case

In the present section we generalize Theorem 5 of Ref. 18. In particular, in Appendix B we prove the following theorem.

Theorem 4: *Let F be a commutative POV measure with spectrum in $[0, 1]$ and A the sharp reconstruction of F . Let E^+ be the extension of F whose existence is asserted by Neumark's theorem, and A^+ the Neumark operator $\int \lambda dE_\lambda^+$. Then, to each bounded and measurable function $f: [0, 1] \rightarrow \mathbb{R}$, there corresponds a bounded function $G_f: [0, 1] \rightarrow \mathbb{R}$ such that*

$$B := P^+ f(A^+) |_{\mathcal{H}} = G_f(A).$$

In Ref. 18, the case $f=\lambda$ was analyzed and it was given an example (see Example 1 below) where the operators A and B coincide up to a bijection, thanks to the injectivity of G_f .

Example 1: *Let us consider the POV measure*

$$F(\Delta) = \begin{cases} \Phi & \text{if } 1 \in \Delta \text{ and } 0 \notin \Delta \\ C = I - \Phi & \text{if } 1 \notin \Delta \text{ and } 0 \in \Delta \\ I & \text{if } 1, 0 \in \Delta \\ 0 & \text{if } 1 \notin \Delta \text{ and } 0 \notin \Delta, \end{cases} \quad (9)$$

where Φ is a bounded self-adjoint operator such that $0 \leq \Phi \leq I$. We can easily find a family of probability measures $\omega_{(\cdot)}(\lambda): \mathcal{B}([0, 1]) \rightarrow [0, 1]$, $\lambda \in [0, 1]$, such that $\omega_\Delta(\Phi) = F(\Delta)$. It is sufficient to choose

$$\omega_\Delta(\lambda) = \begin{cases} \lambda & \text{if } 1 \in \Delta \text{ and } 0 \notin \Delta \\ 1 - \lambda & \text{if } 1 \notin \Delta \text{ and } 0 \in \Delta \\ 1 & \text{if } 1, 0 \in \Delta \\ 0 & \text{if } 1 \notin \Delta \text{ and } 0 \notin \Delta. \end{cases} \quad (10)$$

Therefore the triplet $(F, \Phi, \omega_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. Now we show that Φ coincides with the projection $B := P^+ A^+ |_{\mathcal{H}} = \int \text{id} F_t$ of the Neumark operator A^+ corresponding to the extension E^+ of F . Indeed, we can follow the proof of Theorem 4, with the operator A replaced by the operator Φ , and $f=\lambda$. Then, we get $G_f^{(\Phi)}(\Phi) = B$, where $G_f^{(\Phi)}(\lambda) = \int_0^{1+} \text{id}_t[\omega_t(\lambda)]$. Moreover,

$$G_f^{(\Phi)}(\lambda) = \int_0^{1+} \text{id}_t[\omega_t(\lambda)] = \lambda,$$

hence,

$$B = \Phi.$$

Now, let us consider the sharp reconstruction A corresponding to F . By applying Theorem 4 with $f=\lambda$, we get

$$G_f(A) = B = \Phi.$$

Since the triplet $(F, \Phi, \omega_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem, it follows (by Theorem 2) that there exists a function $g: \sigma(\Phi) \rightarrow [0, 1]$ such that $g(\Phi) = A$. In Ref. 16 it was shown that g is injective. Therefore, $g(\Phi) = g[G_f(A)] = A$ and $g[G_f(\lambda)] = \lambda$ a.e. with respect to the PV measure E^A corresponding to A . This means that $G_f(\lambda)$ is injective in $\sigma(A)$ so that the sharp reconstruction A of F and the projection Φ of the Neumark operator A^+ are equivalent, hence $A \leftrightarrow \text{Pr } A^+$.

Next we recall an example used by Grabowski²³ to analyze the concept of unsharp observable, which we use as an example of a POV measure whose sharp reconstruction coincides (up to a bijection) with the projection of the Neumark operator A^+ (hence $A \leftrightarrow \text{Pr } A^+$).

Example 2: Let $\mathcal{H}=\mathbb{C}^3$ be the Hilbert space for a system with spin $J=1$, and E_{-1}, E_0, E_1 the projections corresponding to the eigenvectors of the spin observable $J_3=\sum_{m=-j}^j mE_m$. Let us consider the position operator $Q:L^2(\mathbb{R})\rightarrow L^2(\mathbb{R})$, of a particle in \mathbb{R} , the corresponding PV measure $E(\cdot):\mathcal{B}(\mathbb{R})\rightarrow\mathcal{E}(L^2(\mathbb{R}))$, and a vector $\psi\in L^2(\mathbb{R})$ such that $\langle\psi,Q\psi\rangle=0$. Let us define the commutative POV measure

$$F(\Delta)=\sum_{m=-1}^1\langle\psi,E(\Delta+m)\psi\rangle E_m$$

in \mathbb{C}^3 . The projection of the Neumark operator coincides with J_3 . Indeed,

$$\int_{\mathbb{R}}\lambda dF_{\lambda}=\sum_m E_m\int_{\mathbb{R}}\lambda|\psi(\lambda-m)|^2 d\lambda=\sum_m mE_m=J_3.$$

Because of the maximality of J_3 (that is, if $J_3=g(A)$, then g is one-to-one) and of Theorem 4, the sharp reconstruction A must be equivalent to J_3 , $A\leftrightarrow J_3$.

B. The finite dimensional case

In this section we show that, in the finite dimensional case, the sharp reconstruction of a given commutative POV measure is equivalent to the projection of the Neumark operator in the sense specified by Definition 9.

Therefore, in what follows, we restrict ourselves to the finite dimensional case and consider an n -dimensional Hilbert space \mathcal{H} . Definition 1 becomes

Definition 10: For a finite or countable outcome set $K\subset\mathbb{R}$, a POV measure F is an application $F:K\rightarrow\{F_k\}_{k\in K}$, also denoted by $\{F_k\}_{k\in K}$, where $\{F_k\}_{k\in K}$ is a set of positive self-adjoint operators acting on a finite dimensional Hilbert space \mathcal{H} such that

$$\sum_{k\in K} F_k = \mathbf{1}.$$

In Ref. 34, a procedure is given for obtaining the sharp reconstruction A corresponding to a commutative POV measure F with a finite outcome set. Being interested in the spectral measure corresponding to A , we outline the procedure for its construction, which we generalize to the case of an infinite but countable outcome set.

Let us consider a commutative POV measure $F:\{k_1,\dots,k_m,\dots\}\rightarrow\{F_{k_1},\dots,F_{k_m},\dots\}$ in an n -dimensional Hilbert space \mathcal{H} . Let $E_j^{(i)}$, $j=1,2,\dots,n$, be a set of n one-dimensional projections corresponding to a base for \mathcal{H} which diagonalizes the operator F_{k_i} , and let $\lambda_j^{(i)}$ be the corresponding eigenvalues. Notice that the eigenvalues are not necessarily distinct. Moreover, because of the commutativity relations

$$[F_{k_i},F_{k_j}]=\mathbf{0}, \quad i,j=1,\dots,m,\dots,$$

we can assume

$$E_j := E_j^{(i)} = E_j^{(l)}, \quad i,l=1,\dots,m,\dots, \quad j=1,\dots,n,$$

so that

$$F_{k_i} = \sum_{j=1}^n \lambda_j^{(i)} E_j. \tag{11}$$

Here, the real number $\lambda_j^{(i)}$ is the eigenvalue of F_{k_i} which corresponds to the projection E_j .

Next, for any $j\in\{1,\dots,n\}$, let us consider the sequence $\{\lambda_j^{(1)},\dots,\lambda_j^{(m)},\dots\}$. There exists a set of projection operators E_1^A,E_2^A,\dots,E_N^A , where $N\leq n$, such that the sequences $\{\lambda_j^{(i)}\}_{i=1,\dots,m,\dots}, j$

$=\{1, \dots, N\}$, corresponding to the projections $E_j^A, j=1, \dots, N$, are distinct, i.e., for every couple of indices (j, l) there exists an index i such that $\lambda_j^{(i)} \neq \lambda_l^{(i)}$. Indeed, if the sequences $\{\lambda_j^{(i)}\}_{i=1, \dots, m, \dots}$ and $\{\lambda_l^{(i)}\}_{i=1, \dots, m, \dots}$, corresponding to the projections E_j and E_l , are equal, we can replace in Eq. (11) E_j with the projection $E_j + E_l$ and skip the l th term. Iterating this procedure, after relabeling the indices, we get $F_{k_i} = \sum_{j=1}^N \lambda_j^{(i)} E_j^A$ for every $i \in \mathbb{N}$, for an integer $N \leq n$. The resulting sequences $\{\lambda_j^{(i)}\}_{i=1, \dots, m, \dots}, j=\{1, \dots, N\}$, corresponding to the projections $E_j^A, j=1, \dots, N$, are distinct. Moreover, if $n(i)$ is the number of distinct eigenvalues of F_{k_i} , we have $n(i) \leq N$ for every $i \in \mathbb{N}$.

The sharp reconstruction A of F is defined³⁴ (up to bijections) as follows:

$$A = \lambda_1 E_1^A + \lambda_2 E_2^A + \dots + \lambda_N E_N^A,$$

where $\{\lambda_i\}_{i=1, \dots, N} = \sigma(A)$. The POV measure F can be interpreted as a randomization of A . Indeed, the functions

$$f_{k_i}^A: \{\lambda_1, \lambda_2, \dots, \lambda_N\} \rightarrow \{\lambda_1^{(i)}, \lambda_2^{(i)}, \dots, \lambda_N^{(i)}\} \in \mathbb{R},$$

$$\lambda_j \mapsto \lambda_j^{(i)}$$

are such that

- (i) $f_{k_i}^A(A) = F_{k_i}$
- (ii) $\sum_{i=1}^{\infty} f_{k_i}^A(\lambda) = 1$ for every $\lambda \in \sigma(A)$.

Item (i) is quite obvious while item (ii) comes from Definition 10. Notice that the functions $f_{k_i}^A$ are not generally one-to-one.

The following theorem, which corresponds to the first part of Theorem 2, summarizes what was said above.

Theorem 5: *A POV measure $F: \{k_1, \dots, k_m, \dots\} \rightarrow \{F_{k_1}, \dots, F_{k_m}, \dots\}$ is commutative if and only if there exist a PV measure $E^A: \{\lambda_1, \dots, \lambda_N\} \rightarrow \{E_1^A, \dots, E_N^A\}$ and a set of functions $f_{k_i}^A: \{\lambda_1, \dots, \lambda_N\} \rightarrow \{\lambda_1^{(i)}, \dots, \lambda_N^{(i)}\}, i=1, \dots, m, \dots$, such that*

- (i) $f_{k_i}^A(A) = F_{k_i}$;
- (ii) $\sum_{i=1}^{\infty} f_{k_i}^A(\lambda) = 1$ for every $\lambda \in \{\lambda_1, \dots, \lambda_N\}$;
- (iii) for every couple (λ_i, λ_j) there exists an index $l \in \mathbb{N}$ such that $f_{k_l}^A(\lambda_i) \neq f_{k_l}^A(\lambda_j)$, where $A = \sum_{j=1}^N \lambda_j E_j^A$ is the self-adjoint operator corresponding to E^A .

Following Martens and de Muynck^{20,21} (see also Definition 11), we summarize the relationship between POV measures and sharp reconstruction, expressed by items (i) and (ii) in Theorem 5, by writing

$$E^A \rightarrow F.$$

In the finite dimensional case, the second part of Theorem 2 becomes

Theorem 6: *If B and $\{f_{k_i}^B\}_{i=1, \dots, m, \dots}$ are respectively a self-adjoint operator and a family of functions $f_{k_i}^B: \sigma(B) \rightarrow \{\lambda_1^{(i)}, \dots, \lambda_N^{(i)}\}$ such that*

- (i) $f_{k_i}^B(B) = F_{k_i}$;
- (ii) $\sum_{i=1}^{\infty} f_{k_i}^B(\lambda) = 1$ for every $\lambda \in \sigma(B)$;

then there exists a function g such that

$$g(B) = A,$$

where A is the sharp reconstruction of F .

Now we can proceed to prove the main result of the paper (Theorem 7). By Neumark's theorem, the POV measure $\{F_{k_j}\}_{j=1, \dots, m, \dots}$ can be extended to a PV measure $\{E_j^+\}_{j=1, \dots, m, \dots}$ in a

Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ such that $P^+E_j^+|_{\mathcal{H}} = F_{k_j}$. We show that there exists a one-to-one function f such that the projection $B = P^+B^+|_{\mathcal{H}} = \sum_j f(k_j)F_{k_j}$ of the Neumark operator $B^+ = f(A^+) = \sum_j f(k_j)E_j^+$ is equivalent to A , i.e., there exists a one-to-one function G_f such that $B = G_f(A)$.

The following lemma, which we prove in Appendix C, is the key to the proof of Theorem 7.

Lemma 1 (see Appendix C): *A matrix of real numbers*

$$\begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_N^{(1)} & \lambda_N^{(2)} & \dots & \lambda_N^{(m)} & \dots \end{pmatrix} \tag{12}$$

such that

- (i) for every couple of indices (i, j) there exists an index $l \in \mathbb{N}$ such that $\lambda_i^{(l)} \neq \lambda_j^{(l)}$;
- (ii) $\sum_{i=1}^{\infty} \lambda_j^{(i)} = 1$;

defines a compact operator $T: l_{\infty} \rightarrow \mathbb{C}^N$ with the property that there exists a real vector $\{k_1, k_2, \dots, k_m, \dots; k_i \neq k_j, i \neq j\} \in l_{\infty}$ such that the elements of the image vector

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} := \begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_N^{(1)} & \lambda_N^{(2)} & \dots & \lambda_N^{(m)} & \dots \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \\ \vdots \end{pmatrix} \tag{13}$$

are distinct real numbers, i.e., $a_i \neq a_j$ if $i \neq j$.

Theorem 7: Let $F: \{k_1, \dots, k_m, \dots\} \rightarrow \{F_{k_1}, \dots, F_{k_m}, \dots\}$ be a commutative POV measure with spectrum in $[0, 1]$, $A = \lambda_1 E_1^A + \dots + \lambda_N E_N^A$ its sharp reconstruction, $E^+: \{k_1, \dots, k_m, \dots\} \rightarrow \{E_{k_1}^+, \dots, E_{k_m}^+, \dots\}$ an extension of F whose existence is asserted by Neumark's theorem, and A^+ the Neumark operator $\sum_{j=1}^{\infty} k_j E_j^+$. Then $A \leftrightarrow \text{Pr } A^+$.

Proof: Let us consider a bounded function $f: \{k_1, \dots, k_m, \dots\} \rightarrow \mathbb{R}$ and the bounded operator $B^+ = f(A^+) = \sum_j f(k_j)E_j^+$. By Lemma 1, we get

$$\begin{aligned} P^+B^+|_{\mathcal{H}} &= P^+ \sum_j f(k_j)E_j^+|_{\mathcal{H}} = \sum_j f(k_j)F_{k_j} = \sum_{j=1}^{\infty} f(k_j) \sum_{i=1}^N \lambda_i^{(j)} E_i^A \\ &= \sum_{i=1}^N \left(\sum_{j=1}^{\infty} \lambda_i^{(j)} f(k_j) \right) E_i^A = \sum_{i=1}^N G_f(\lambda_i) E_i^A = G_f(A), \end{aligned} \tag{14}$$

where A is the sharp reconstruction of F and

$$G_f(\lambda_i) := \sum_{j=1}^{\infty} \lambda_i^{(j)} f(k_j) \leq \sup_j |f(k_j)|, \quad \lambda_i \in \sigma(A), i = 1, \dots, N.$$

In matrix form, we can write

$$\begin{pmatrix} G_f(\lambda_1) \\ G_f(\lambda_2) \\ \vdots \\ \vdots \\ G_f(\lambda_N) \end{pmatrix} = \begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_N^{(1)} & \lambda_N^{(2)} & \dots & \lambda_N^{(m)} & \dots \end{pmatrix} \begin{pmatrix} f(k_1) \\ f(k_2) \\ \vdots \\ \vdots \\ f(k_m) \\ \vdots \end{pmatrix}. \tag{15}$$

Moreover, by Theorem 5 [items (ii) and (iii)], we have the following:

- (i) for every couple of indices (i, j) there exists an index $l \in \mathbb{N}$ such that $\lambda_i^{(l)} \neq \lambda_j^{(l)}$;
- (ii) $\sum_{j=1}^{\infty} \lambda_i^{(j)} = 1$ for every $i \in \{1, 2, \dots, m\}$.

By Lemma 1, there exists a vector $\{\lambda_1^+, \dots, \lambda_m^+, \dots; \lambda_i^+ \neq \lambda_j^+, i \neq j\} \in l_\infty$ such that the function f defined by $f(k_1) = \lambda_1^+, \dots, f(k_m) = \lambda_m^+, \dots$, and the function G_f are one-to-one, i.e., $f(k_i) \neq f(k_j)$ and $G_f(\lambda_i) \neq G_f(\lambda_j)$ if $i \neq j$. By Eq. (14), $P^+f(A^+) = P^+B_{|\mathcal{H}}^+ = G_f(A)$ and then $A \leftrightarrow \text{Pr } A^+$. \square

Example 3: Let $\mathcal{H} = \mathbb{C}^2$ be the Hilbert space for a system with spin $J = 1/2$. Let P_+, P_- be the projections corresponding to the eigenvectors of the spin observable $J_z = 1/2P_+ - 1/2P_-$. Let us consider the commutative POV measure $F: \{1/2, -1/2\} \rightarrow \{F_1 = (1 - \epsilon)P_+ + \delta P_-, F_2 = \epsilon P_+ + (1 - \delta)P_-\}$, $\epsilon + \delta \neq 1$, which can be interpreted^{25,35} as the representation of the measurement of the spin in the z direction where a “spin up” is registered as “spin down” with probability ϵ and a spin down is registered as spin up with probability δ . The sharp reconstruction of F is

$$A = 1P_+ + 2P_-.$$

The functions f_1 and f_2 connecting F and A are defined as follows:

$$f_1(1) = 1 - \epsilon, \quad f_1(2) = \delta,$$

$$f_2(1) = \epsilon, \quad f_2(2) = 1 - \delta.$$

By Neumark’s theorem there exist an extended Hilbert space \mathcal{H}^+ and an orthogonal resolution of the identity $\{E_1^+, E_2^+\}$ in \mathcal{H}^+ such that $P^+E_i^+|_{\mathcal{H}} = F_i$. It is easy to see that the Neumark operator $A^+ = 1/2E_1^+ - 1/2E_2^+$, corresponding to the POV measure F , is such that its projection $P^+A^+|_{\mathcal{H}}$ coincides, up to a bijection, with the sharp reconstruction A .

Indeed,

$$\begin{aligned} P^+A^+|_{\mathcal{H}} &= 1/2P^+E_1^+ - 1/2P^+E_2^+ = 1/2F_1 - 1/2F_2 = 1/2[(1 - \epsilon)P_+ + \delta P_-] - 1/2[\epsilon P_+ + (1 - \delta)P_-] \\ &= (1/2 - \epsilon)P_+ + (-1/2 + \delta)P_- = f(A), \end{aligned}$$

where

$$f(1) = 1/2 - \epsilon, \quad f(2) = -1/2 + \delta.$$

Example 4: Let $\mathcal{H} = \mathbb{C}^3$ be the Hilbert space for a system with spin $J = 1$. Let E_{-1}, E_0, E_1 be the projections corresponding to the eigenvectors of the spin observable $J_3 = \sum_{m=-1}^1 mE_m$. Let us consider the POV measure $\{1, 2, 3\} \rightarrow \{F_1 = 1/2E_{-1} + 1/2E_0 + 1/4E_1, F_2 = 1/5E_{-1} + 1/5E_0 + 1/4E_1, F_3 = 3/10E_{-1} + 3/10E_0 + 1/2E_1\}$. The corresponding sharp reconstruction is $A = 1(E_{-1} + E_0) + 2E_1$. The projection of the Neumark operator A^+ corresponding to F is $P^+A^+|_{\mathcal{H}} = 1F_1 + 2F_2 + 3F_3 = 9/5(E_{-1} + E_0) + 9/4E_1 = f(A)$, where f is the one-to-one function such that $f(1) = 9/5$ and $f(2) = 9/4$. Notice that A is a function of J_3 [$A = g(J_3)$, where $g(-1) = 1, g(0) = 1$, and $g(1) = 2$].

IV. NEUMARK’S THEOREM AND NONIDEAL QUANTUM MEASUREMENT

In this section we briefly recall the definition of nonideal quantum measurement^{20,21} and its connection with Corollary 1, and analyze some implications of Theorems 6 and 7 to this connection (Corollary 3). Moreover, we comment (Definition 13 and Theorem 9) on the relationships between different Definitions of unsharpness.

The concept of nonideal quantum measurement of a PV measure E by means of a POV measure F is defined as follows:

Definition 11 (see Refs. 20 and 21): *The POV measure $F:K \rightarrow \{F_k\}_{k \in K}$ with a finite or countably infinite outcome set K is said to be an unsharp version of the PV measure $E:L \rightarrow \{E_l\}_{l \in L}$ if there exists a set of non-negative real numbers $\{\lambda_l^{(k)}\}_{k \in K, l \in L}$ such that*

- (i) $\sum_{k \in K} \lambda_l^{(k)} = 1.$
- (ii) $F_k = \sum_{l \in L} \lambda_l^{(k)} E_l.$

Remark 1: *Notice that Definition 11 is equivalent to requiring that for each operator $B = \sum_{l \in L} \gamma_l E_l$, with $\{\gamma_l\}_{l \in L}$ set of distinct real numbers, there exists a set of functions $\{f_k^B\}_{k \in K}$ such that $f_k^B(B) = F_k$ [e.g., $f_k^B(\gamma_l) = \lambda_l^{(k)}$].*

In this paper we do not distinguish between a self-adjoint operator B and the corresponding PV measure E^B . Therefore, we say that an unsharp observable, represented by a POV measure F , is the unsharp version of a sharp observable, represented by a self-adjoint operator B , if F is the unsharp version of the PV measure E^B corresponding to B . In particular, a commutative POV measure F is the unsharp version of its sharp reconstruction A .

Example 5: *The POV measure F in Example 4 is an unsharp version of the spin observable $J_3 = \sum_{m=-1}^1 m E_m$. Indeed, by setting*

$$\begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \lambda_1^{(3)} \\ \lambda_0^{(1)} & \lambda_0^{(2)} & \lambda_0^{(3)} \\ \lambda_{-1}^{(1)} & \lambda_{-1}^{(2)} & \lambda_{-1}^{(3)} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/5 & 3/10 \\ 1/2 & 1/5 & 3/10 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

we get $E \rightarrow F$, where $E = \{E_{-1}, E_0, E_1\}$. Moreover, by Remark 1, there exists a family of functions f_i such that $f_i(J_3) = F_i$, then by Theorem 6, there is a function g such that $g(J_3) = A$, where $A = E_{-1} + E_0 + 2E_1$ is the sharp reconstruction of F .

Before giving the connection between Definition 11 and Corollary 1, we state the latter in the finite dimensional case.

Corollary 2 (see Refs. 3, 9, and 21): *For every POV measure $\{F_k\}_{k \in K}$ on \mathcal{H} , with a finite or countable outcome set K , there is a Hilbert space \mathcal{H}' , a density operator ρ' on \mathcal{H}' , and a PV measure $\{E_k^+\}_{k \in K}$ on $\mathcal{H} \otimes \mathcal{H}'$ such that*

$$F_k = \text{Tr}_{\mathcal{H}'}(\rho' E_k^+).$$

The connection mentioned above is summarized by the following theorem.

Theorem 8 (see Refs. 20 and 21): *Let $\bar{A} = \sum_a a \bar{E}_a$ be a self-adjoint operator on \mathcal{H} and $A^+ = \sum_k k E_k^+$ a self-adjoint operator on $\mathcal{H} \otimes \mathcal{H}'$. If there exist a self-adjoint operator $T = \sum_l l E_l'$ on \mathcal{H}' and a function $k(a, l)$ such that $A^+ = k(A, T)$, then for every density operator ρ' on \mathcal{H}' ,*

$$\bar{E} \rightarrow F,$$

where \bar{E} and F are respectively the PV measure corresponding to \bar{A} and the POV measure defined by $F_k = \text{Tr}_{\mathcal{H}'}(\rho' E_k^+)$.

The following corollary is a consequence of Theorems 6 and 7.

Corollary 3: *Under the hypothesis of Theorem 8, let us consider the density operator ρ' , the corresponding POV measure $F_k = \text{Tr}_{\mathcal{H}'}(\rho' E_k^+)$, and its sharp reconstruction A . If the function $f(a) = \sum_k k(a, l) \text{Tr}_{\mathcal{H}'}(\rho' E_l')$ is one-to-one, then $A \leftrightarrow \bar{A}$. Conversely, if $A \leftrightarrow \bar{A}$ then there exists a one-to-one function h such that the function $r(a) = \sum_l h(k(a, l)) \text{Tr}_{\mathcal{H}'}(\rho' E_l')$ is one-to-one.*

Proof: Assume f is one-to-one. Since $\bar{E} \rightarrow F$, there exists (by Theorem 6) a function g such that

$$g(\bar{A}) = A,$$

where $A = \sum \lambda_n E_n$ is the sharp reconstruction of F . Moreover, proceeding as in Eq. (14) in theorem 7, we get a function G such that

$$G(A) = \sum_k k F_k. \quad (16)$$

Hence,

$$G[g(\bar{A})] = \sum_k k F_k = \text{Tr}_{\mathcal{H}'}[\rho' A^+] = \sum_a \left(\sum_l k(a, l) \text{Tr}_{\mathcal{H}'}(\rho' E_l') \right) \bar{E}_a = f(\bar{A}),$$

which shows that f is one-to-one if and only if both g and G are one-to-one. The thesis comes from the fact that f is one-to-one. Conversely, if $A \leftrightarrow \bar{A}$, then there is a one-to-one function g such that $A = g(\bar{A})$. Furthermore, by Theorem 7, there exist two one-to-one functions h and G_h such that

$$G_h(A) = \sum_k h(k) F_k = \text{Tr}_{\mathcal{H}'}[\rho' h(A^+)] = r(\bar{A}),$$

where

$$r(a) = \sum_l h(k(a, l)) \text{Tr}_{\mathcal{H}'}(\rho' E_l'),$$

which means that $r = g^{-1} \circ G_h$ is one-to-one (the symbol \circ denotes the operation of composition between functions). \square

We recall that another definition of “unsharpness” is the following:²²⁻²⁵

Definition 12: *The observable represented by the POV measure $F: K \rightarrow \{F_k\}_{k \in K}$ is an unsharp version of the observable represented by a self-adjoint operator B , if there exists a sequence of real numbers $\{\gamma_k\}_{k \in K}$ such that*

$$B = \sum_k \gamma_k F_k.$$

Example 6: *From Example 3 we have that the observables P_+ , P_- , and $f(A)$ can be written as $P_+ = (1 - \delta)/(1 - \epsilon - \delta)F_1 + [-\delta/(1 - \epsilon - \delta)]F_2$, $P_- = [-\epsilon/(1 - \epsilon - \delta)]F_1 + (1 - \epsilon)/(1 - \epsilon - \delta)F_2$, and $f(A) = 1/2F_1 - 1/2F_2$. Therefore, F is an unsharp version of P_+ , P_- and $f(A)$. Moreover, this shows that all the observables which are function of the sharp reconstruction A can be represented as a sum of the kind $\sum \gamma_i F_i$. Notice that F is not an unsharp version of P_+ and P_- in the sense of Definition 11.*

Grabowski²³ conjectured that Definition 12 is equivalent to Definition 11, but Uffink²⁵ observed that this is false since, according to Definition 11, any unsharp version of a sharp observable B must be commutative and this is not true for Definition 12. Now we show that also in the case where we restrict ourselves to commutative POV measures, Grabowski’s conjecture is false. Indeed, let us consider the POV measure F in Example 4. From Example 5 it follows that $A = g(J_3)$, where A is the sharp reconstruction of F and g is not one-to-one since $g(-1) = g(0) = 1$. Assume F to be the unsharp version of J_3 in the sense of Definition 12, i.e.,

$$J_3 = \sum_{i=1}^3 \gamma_i F_i.$$

By proceeding as in Eq. (14), we get a function $G(j) = \sum_i \lambda_j^{(i)} \gamma_i$ such that

$$G(A) = \sum_{i=1}^3 \gamma_i F_i.$$

Then

$$G(A) = J_3 \text{ and } g(J_3) = A,$$

hence

$$G(g(J_3)) = J_3,$$

which means that g is one-to-one and contradicts the hypothesis. Therefore, F is an unsharp version of J_3 according to Definition 11 but not according to Definition 12.

The following definition is a generalization of both Definitions 11 and 12.

Definition 13: *The observable represented by the POV measure $F: K \rightarrow \{F_k\}_{k \in K}$ is an unsharp version of the observable represented by the operator B if there exist a function h and a sequence of real numbers $\{\gamma_k\}_{k \in K}$ such that*

$$h(B) = \sum_k \gamma_k F_k.$$

Theorem 9: *If F is an unsharp version of B according to Definitions 11 or 12, then it is an unsharp version of B according to Definition 13.*

Proof: Let E^B be the PV measure corresponding to the self-adjoint operator B . If $E^B \rightarrow F$ then there exists a function g such that $g(B) = A$, where A is the sharp reconstruction of F . Moreover, proceeding as in Eq. (14), we get, for any bounded function $f: K \rightarrow \mathbb{R}$, a function G_f such that $G_f(A) = \sum_k f(k) F_k$, hence

$$h(B) := G_f(g(B)) = \sum_k f(k) F_k = \sum_k \gamma_k F_k,$$

where we have set $\gamma_k := f(k)$. Clearly, if F is an unsharp version of B according to Definition 12, then it is an unsharp version of B according to Definition 13 [it is sufficient to choose $h(\lambda) = \lambda$]. \square

In order to outline the relationships between Definitions 11 and 12, we introduce the sets \mathcal{A}_1 , \mathcal{A}_2 , and \mathcal{A}'_2 as follows:

- (i) by Definition 11 (see Theorem 6), the set of sharp observables of which F is an unsharp version is the set

$$\mathcal{A}_1 = \{B \in \mathcal{L}_s(H) \mid \text{there exists a function } h \text{ such that } h(B) = A\};$$

- (ii) by Definition 12 (see Theorem 7), the set \mathcal{A}_2 of sharp observables of which F is an unsharp version is a subset of

$$\mathcal{A}'_2 = \{B \in \mathcal{L}_s(H) \mid \text{there exists a function } g \text{ such that } B = g(A)\},$$

where A is the sharp reconstruction of F .

The two sets \mathcal{A}_1 and \mathcal{A}'_2 have a nonempty intersection; e.g., each self-adjoint operator B such that $A = g(B)$, with g one-to-one, belongs to $\mathcal{A}_1 \cap \mathcal{A}'_2$. Moreover, we have proved that $\mathcal{A}_1 \cap \mathcal{A}_2 \neq \emptyset$ and, in particular, that the sharp reconstruction of F is contained in $\mathcal{A}_1 \cap \mathcal{A}_2$. Now it is clear why Definition 13 is a generalization of both Definitions 11 and 12; it enlarges \mathcal{A}_2 to a set \mathcal{A} such that $\mathcal{A}_1 \cup \mathcal{A}_2 \subseteq \mathcal{A} \subseteq \mathcal{A}_1 \cup \mathcal{A}'_2$ (notice that for the POV measure in Example 3, $\mathcal{A}_2 = \mathcal{A}'_2$ and then

$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2'$). A problem to be faced in future investigations is to search for a common meaning of the concepts of unsharpness given by Definitions 11 and 12. However, it is worth noticing that the observables in \mathcal{A}_2 can be recovered from F (by appropriately choosing the coefficients in the sum $\sum \gamma_k F_k$), while for the observables in \mathcal{A}_1 , this is true only for those observables which are equivalent to the sharp reconstruction A of F . Conversely, F can be recovered by each observable $B \in \mathcal{A}_1$ since to each $B \in \mathcal{A}_1$ there corresponds a set of functions f_k^B such that $f_k^B(B) = F_k$. Then, we can say that in \mathcal{A}_2 there are observables which contain less information than F and observables which contain the same information as F , while in \mathcal{A}_1 there are observables which contain more information than F and observables which contain the same information as F .

APPENDIX A: POV MEASURES WITH SPECTRUM IN [0, 1]

Let $\bar{F}: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\mathcal{H})$, $\bar{F}(\Delta) = F[f(\Delta \cap (0, 1))]$, be the POV measure with spectrum in $[0, 1]$ corresponding to the POV measure $F: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F}(\mathcal{H})$ as stated in Sec. III. If the PV measure $E^+: \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$ is the extension of F , whose existence is asserted by Neumark's theorem, then $\bar{E}^+(\Delta) = E^+[f(\Delta \cap (0, 1))]$ is the extension of \bar{F} . Indeed, $P^+\bar{E}^+(\Delta)|_{\mathcal{H}} = P^+E^+[f(\Delta \cap (0, 1))]|_{\mathcal{H}} = F[f(\Delta \cap (0, 1))] = \bar{F}(\Delta)$.

Theorem 10: *Let F and \bar{F} be two commutative POV measures such that $\bar{F}(\Delta) = F[f(\Delta \cap (0, 1))]$; let E^+ and \bar{E}^+ be the corresponding extensions as stated above. Then $A \leftrightarrow \text{Pr } A^+$ holds if and only if $A \leftrightarrow \text{Pr } \bar{A}^+$ holds.*

Proof: By the change of measure principle,²⁸ we get

$$\begin{aligned} \bar{A}^+ &= \int_{-\infty}^{\infty} \lambda d\bar{E}_\lambda^+ = \int_{-\infty}^{\infty} \lambda \bar{E}^+([\lambda - d\lambda, \lambda]) = \int_{-\infty}^{\infty} \lambda E^+[f([\lambda - d\lambda, \lambda]) \cap (0, 1)] \\ &= \int_{(0,1)} \lambda E^+[f([\lambda - d\lambda, \lambda])] = \int_{-\infty}^{\infty} f^{-1}(\lambda) dE_\lambda^+ = f^{-1}(A^+). \end{aligned} \tag{A1}$$

If $A \leftrightarrow \text{Pr } A^+$, then there exist two one-to-one, bounded measurable functions $g(\lambda)$ and $h(\lambda)$ such that $g(A) = P^+h(A^+) = P^+h(f(\bar{A}^+))$. Therefore, there exists a one-to-one, bounded, measurable function $H(\lambda) = h(f(\lambda))$ such that $g(A) = P^+H(\bar{A}^+)$, which proves that $A \leftrightarrow \text{Pr } \bar{A}^+$. Conversely, if $A \leftrightarrow \text{Pr } \bar{A}^+$, then there exist two one-to-one, bounded, measurable functions $g(\lambda)$ and $H(\lambda)$ such that $g(A) = P^+H(\bar{A}^+) = P^+H(f^{-1}(A^+))$. Therefore, there exist two one-to-one, bounded, measurable functions $g(\lambda)$ and $h(\lambda) = H(f^{-1}(\lambda))$ such that $g(A) = P^+h(A^+)$, which proves that $A \leftrightarrow \text{Pr } A^+$. ■

APPENDIX B: PROOF OF THEOREM 4

Proof: The function $f: [0, 1] \rightarrow \mathbb{R}$ (we denote by m and M , respectively, the infimum and the supremum of f in $[0, 1]$) and the POV measure F uniquely define²⁷ a self-adjoint bounded operator by means of the relation

$$\langle Bx, x \rangle = \int_{[0,1]} f(t) d_t \langle F_t x, x \rangle. \tag{B1}$$

One has

$$\begin{aligned} \langle Bx, x \rangle &= \int_{[0,1]} f(t) d_t \langle F_t x, x \rangle = \int_{[0,1]} f(t) d_t \left[\int_{[0,1]} \mu_t^A(\lambda) d_\lambda \langle E_\lambda^A x, x \rangle \right] \\ &= \int_{[0,1]} \left[\int_{[0,1]} f(t) d_t [\mu_t^A(\lambda)] \right] d_\lambda \langle E_\lambda^A x, x \rangle = \langle G_f(A)x, x \rangle \quad \text{for every } x \in \mathcal{H}, \end{aligned} \tag{B2}$$

where

$$G_f(\lambda) = \int_{[0,1]} f(t) d_t [\mu_t^A(\lambda)], \tag{B3}$$

E^A and $\mu_{(\cdot)}^A(\lambda)$ are respectively the PV measure corresponding to the sharp reconstruction A and the probability measure whose existence is asserted by Theorem 2, and we have denoted by $d_t \int \mu_t^A(\lambda) d_\lambda \langle E_\lambda^A x, x \rangle$ the integration with respect to the measure $\omega(\cdot) = \int \mu_{(\cdot)}^A(\lambda) d_\lambda \langle E_\lambda^A x, x \rangle$.

In order to justify the change in the order of integration in Eq. (B2), we proceed as follows. First, we notice that

$$\omega(\cdot) = \int_{[0,1]} \mu_{(\cdot)}^A(\lambda) d_\lambda \langle E_\lambda^A x, x \rangle = \langle F(\cdot)x, x \rangle$$

is, for every $x \in \mathcal{H}$, a Lebesgue-Stieltjes measure. Therefore, by the definition of Lebesgue-Stieltjes integral,³⁶

$$\begin{aligned} \int_{[0,1]} f(t) d_t \omega(t) &= \lim_{\substack{n \rightarrow \infty \\ |\delta_n| \rightarrow 0}} \sum_{k=1}^n f_{k-1}^{(n)} \omega\{t \in [0,1] : f(t) \in (f_{k-1}^{(n)}, f_k^{(n)}]\} \\ &= \lim_{\substack{n \rightarrow \infty \\ |\delta_n| \rightarrow 0}} \sum_{k=1}^n f_{k-1}^{(n)} \int_{[0,1]} \mu_{(E_{k-1}^{(n)})}^A(\lambda) \langle E_\lambda x, x \rangle = \lim_{\substack{n \rightarrow \infty \\ |\delta_n| \rightarrow 0}} \int_{[0,1]} \sum_{k=1}^n f_{k-1}^{(n)} \mu_{(E_{k-1}^{(n)})}^A(\lambda) \langle E_\lambda x, x \rangle, \end{aligned} \tag{B4}$$

where a sequence of subdivisions was introduced, $\delta_n = \{[f_0, f_1^{(n)}], (f_1^{(n)}, f_2^{(n)}], \dots, (f_{n-1}^{(n)}, f_n]\}$, $m = f_0 < f_1 < \dots < f_n = M$, of the interval $[m, M]$, such that $|\delta_n| = \max_{1 \leq k \leq n} \{f_k^{(n)} - f_{k-1}^{(n)}\} \rightarrow 0$ when $n \rightarrow \infty$ and it was set $E_{k-1}^{(n)} = \{t \in [0, 1] : f(t) \in (f_{k-1}^{(n)}, f_k^{(n)})\}$.

Now let us consider the sequence of functions

$$H_n(\lambda) = \sum_{k=1}^n f_{k-1}^{(n)} \mu_{(E_{k-1}^{(n)})}^A(\lambda).$$

One has

$$H_n(\lambda) \leq \sup\{|f|\} \mu_{([0,1])}(\lambda) = M < \infty$$

for each $\lambda \in [0, 1]$ and $n \in \mathbb{N}$.

Moreover, by the integrability of f with respect to $\mu_{(\cdot)}^A(\lambda)$,

$$\lim_{n \rightarrow \infty} H_n(\lambda) = \int_{[0,1]} f(t) d_t \mu_t^A(\lambda) = G_f(\lambda).$$

By Theorem 11 in Ref. 27,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{[0,1]} \sum_{k=1}^n f_{k-1}^{(n)} \mu_{(E_{k-1}^{(n)})}(\lambda) \langle E_\lambda x, x \rangle &= \lim_{n \rightarrow \infty} \int_{[0,1]} H_n(\lambda) \langle E_\lambda x, x \rangle = \int_{[0,1]} \lim_{n \rightarrow \infty} H_n(\lambda) \langle E_\lambda x, x \rangle \\ &= \int_{[0,1]} \left[\int_{[0,1]} f(t) d_t \mu_t^A(\lambda) \right] \langle E_\lambda^A x, x \rangle \\ &= \int_{[0,1]} G_f(\lambda) \langle E_\lambda^A x, x \rangle = \langle G_f(A)x, x \rangle. \end{aligned}$$

The polarization identity completes the proof. ■

APPENDIX C: PROOF OF LEMMA 1

Proof: By item (ii),

$$a_i = \sum_{j=1}^{\infty} \lambda_i^{(j)} k_j \leq \sup_j |k_j| \sum_j \lambda_i^{(j)} = \sup_j |k_j| < \infty,$$

which means that T is defined everywhere on l_∞ and bounded. The compactness of T derives from (see Ref. 29, p. 58)

$$\sum_{i,j} |\lambda_i^{(j)}|^2 = N < \infty.$$

Now, we proceed by induction on N .

Step 1: The thesis is true for $N=2$. If $N=2$, Eq. (13) becomes

$$\begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \\ \vdots \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{C1}$$

We start from a real vector $\{k_1, k_2, \dots, k_m, \dots; k_i \neq k_j, i \neq j\} \in l_\infty$. Suppose $a_1 = a_2$. By item (ii), we can assume, without loss of generality, $\lambda_1^{(1)} \neq \lambda_2^{(1)}$ so that by replacing k_1 with $k'_1 \neq k_i, i \in \mathbb{N}$, we get

$$\begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \end{pmatrix} \begin{pmatrix} k'_1 \\ k_2 \\ \vdots \\ k_m \\ \vdots \end{pmatrix} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \end{pmatrix}, \tag{C2}$$

where $\tilde{a}_1 \neq \tilde{a}_2$. Indeed,

$$\tilde{a}_i - a_i = (k_1 - k'_1) \lambda_i^{(1)}, \quad i = 1, 2,$$

then

$$\tilde{a}_1 - \tilde{a}_2 = (a_1 - a_2) + (k_1 - k'_1)(\lambda_1^{(1)} - \lambda_2^{(1)}) = (k_1 - k'_1)(\lambda_1^{(1)} - \lambda_2^{(1)}) \neq 0.$$

Step 2: Induction on N . Suppose that the thesis is true for $N=n$. The case $N=n+1$ reads

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n+1} \end{pmatrix} := \begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{n+1}^{(1)} & \lambda_{n+1}^{(2)} & \dots & \lambda_{n+1}^{(m)} & \dots \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \\ \vdots \end{pmatrix}. \tag{C3}$$

Consider the subsystem

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_n^{(1)} & \lambda_n^{(2)} & \dots & \lambda_n^{(m)} & \dots \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ \vdots \\ k_m \end{pmatrix}. \quad (\text{C4})$$

By the induction hypothesis, there exists a real vector $\{k_1, \dots, k_m, \dots | k_i \neq k_j, i \neq j\} \in l_\infty$ such that the image vector satisfies the thesis of the lemma, i.e., $a_i \neq a_j$ if $i \neq j, i, j = 1, \dots, n$. Let us return to consider system (C3) and suppose, without loss of generality, $a_{n+1} = a_1$. By item (i), we can assume, without loss of generality, $\lambda_1^{(1)} \neq \lambda_{n+1}^{(1)}$. By replacing k_1 with a number k'_1 such that

$$k'_1 \neq \begin{cases} k_j, & j \in \mathbb{N} \\ k_1 - \frac{(a_j - a_i)}{(\lambda_j^{(1)} - \lambda_i^{(1)})} & \text{if } \lambda_j^{(1)} \neq \lambda_i^{(1)}, i, j = 1, 2, \dots, n+1, \end{cases} \quad (\text{C5})$$

we get

$$\begin{pmatrix} \lambda_1^{(1)} & \lambda_1^{(2)} & \dots & \lambda_1^{(m)} & \dots \\ \lambda_2^{(1)} & \lambda_2^{(2)} & \dots & \lambda_2^{(m)} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_{n+1}^{(1)} & \lambda_{n+1}^{(2)} & \dots & \lambda_{n+1}^{(m)} & \dots \end{pmatrix} \begin{pmatrix} k'_1 \\ k_2 \\ \vdots \\ \vdots \\ k_m \end{pmatrix} = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \vdots \\ \tilde{a}_{n+1} \end{pmatrix}, \quad (\text{C6})$$

where $\tilde{a}_i \neq \tilde{a}_j, i \neq j$. Indeed,

$$\tilde{a}_i - a_i = (k'_1 - k_1)\lambda_i^{(1)} \quad (\text{C7})$$

and, by subtracting Eq. (C7) from

$$\tilde{a}_j - a_j = (k'_1 - k_1)\lambda_j^{(1)},$$

we get

$$\tilde{a}_j - \tilde{a}_i = (k'_1 - k_1)(\lambda_j^{(1)} - \lambda_i^{(1)}) + (a_j - a_i). \quad (\text{C8})$$

By imposing $\tilde{a}_j - \tilde{a}_i \neq 0$ whenever $\lambda_j^{(1)} \neq \lambda_i^{(1)}$, we get the second solution of Eq. (C5). Moreover, if $\lambda_j^{(1)} = \lambda_i^{(1)}$ (which is false if $i=1$ and $j=n+1$), then $\tilde{a}_j - \tilde{a}_i = a_j - a_i \neq 0$ for each choice of k'_1 . ■

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