Asymptotic behavior of critical primitive multi-type branching processes with immigration

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Abstract

Under natural assumptions a Feller type diffusion approximation is derived for critical multi-type branching processes with immigration when the offspring mean matrix is primitive (in other words, positively regular). Namely, it is proved that a sequence of appropriately scaled random step functions formed from a sequence of critical primitive multi-type branching processes with immigration converges weakly towards a squared Bessel process supported by a ray determined by the Perron vector of the offspring mean matrix.

1 Introduction

Branching processes have a number of applications in biology, finance, economics, queueing theory etc., see e.g. Haccou, Jagers and Vatutin [1]. Many aspects of applications in epidemiology, genetics and cell kinetics were presented at the 2009 Badajoz Workshop on Branching Processes, see [2].

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In this paper, let $\mathbb{Z}_+$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the set of non-negative integers, positive integers, real numbers, non-negative real numbers and positive real numbers, respectively. Every random variable will be defined on a fixed probability space $(\Omega, \mathcal{A}, \mathrm{P})$.

Let $(X_k)_{k\in\mathbb{Z}_+}$ be a single-type Galton–Watson branching process with immigration and with initial value $X_0 = 0$. Suppose that it is critial, i.e., the offspring mean equals 1. Wei and Winnicki [3] proved a functional limit theorem $\mathcal{X}^{(n)} \xrightarrow{\mathcal{L}} \mathcal{X}$ as $n \to \infty$, where $\mathcal{X}_t^{(n)} := n^{-1} X_{\lfloor nt \rfloor}$ for $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$, and $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is a (nonnegative) diffusion process with initial value $\mathcal{X}_0 = 0$ and with generator

$$Lf(x) = m_\varepsilon f'(x) + \frac{1}{2} V_\xi x f''(x), \qquad f \in C_{\mathrm{c}}^\infty(\mathbb{R}_+), \tag{1.1}$$

where m_ε is the immigration mean, V_ξ is the offspring variance, and $C_{\mathrm{c}}^\infty(\mathbb{R}_+)$ denotes the space of infinitely differentiable functions on $\mathbb{R}_+$ with compact support. The process $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ can also be characterized as the unique strong solution of the stochastic differential equation (SDE)

$$\mathrm{d}\mathcal{X}_t = m_\varepsilon \,\mathrm{d}t + \sqrt{V_\xi \mathcal{X}_t^+}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\mathcal{X}_0 = 0$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process, and x^+ denotes the positive part of $x \in \mathbb{R}$. Note that this so-called square-root process is also known as Cox–Ingersoll–Ross model in financial mathematics (see Musiela and Rutkowski [4, p. 290]). In fact, $(4V_\xi^{-1}\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is the square of a $4V_\xi^{-1}m_\varepsilon$-dimensional Bessel process started at 0 (see Revuz and Yor [5, XI.1.1]).

Moreover, for critical Galton–Watson branching processes without immigration, Feller [6] proved the following diffusion approximation (see also Ethier and Kurtz [7, Theorem 9.1.3]). Consider a sequence of critical Galton–Watson branching processes

$\big(X_k^{(n)}\big)_{k\in\mathbb{Z}_+}$, $n \in \mathbb{N}$, without immigration, with the same offspring distribution, and with initial value $X_0^{(n)}$ independent of the offspring variables such that $n^{-1}X_0^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \mu$ as $n \to \infty$. Then $\mathcal{X}^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{X}$ as $n \to \infty$, where $\mathcal{X}_t^{(n)} := n^{-1}X_{\lfloor nt\rfloor}^{(n)}$ for $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, and $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is a (nonnegative) diffusion process with initial distribution μ and with generator given by (1.1) with $m_\varepsilon = 0$. Furthermore, independently of each other, Lebedev [8] and Sriram [9] generalized the result of Wei and Winnicki for a sequence of branching processes with immigration which is nearly critical in the sense that $m_\xi^{(n)} = 1 + \alpha n^{-1} + \mathrm{o}(n^{-1})$ as $n \to \infty$ with $\alpha \in \mathbb{R}$, where $m_\xi^{(n)}$ is the offspring mean of the process $\big(X_k^{(n)}\big)_{k\in\mathbb{Z}_+}$. They proved that, as $n \to \infty$, $\mathcal{X}^{(n)}$ converges towards a diffusion process with initial value $\mathcal{X}_0 = 0$ and with generator $L_\alpha f(x) = (\alpha x + m_\varepsilon)f'(x) + \frac{1}{2}V_\xi x f''(x)$, $f \in C_\mathrm{c}^\infty(\mathbb{R}_+)$.

A multi-type branching process $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ is referred to respectively as subcritical, critical or supercritical if $\varrho(\boldsymbol{m_\xi}) < 1$, $\varrho(\boldsymbol{m_\xi}) = 1$ or $\varrho(\boldsymbol{m_\xi}) > 1$, where $\varrho(\boldsymbol{m_\xi})$ denotes the spectral radius of the offspring mean matrix $\boldsymbol{m_\xi}$ (see, e.g., Athreya and Ney [10] or Quine [11]). Joffe and Métivier [12, Theorem 4.3.1] studied a sequence $(\boldsymbol{X}_k^{(n)})_{k\in\mathbb{Z}_+}$ of critical multi-type branching processes with the same offspring distributions but without immigration if the offspring mean matrix is primitive and $n^{-1}\boldsymbol{X}_0^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \boldsymbol{\mu}$ as $n \to \infty$. They determined the limiting behavior of the martingale part $(\boldsymbol{\mathcal{M}}^{(n)})_{n\in\mathbb{N}}$ given by $\boldsymbol{\mathcal{M}}_t^{(n)} := n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}\boldsymbol{M}_k^{(n)}$ with $\boldsymbol{M}_k^{(n)} := \boldsymbol{X}_k^{(n)} - \mathrm{E}(\boldsymbol{X}_k^{(n)} \mid \boldsymbol{X}_0^{(n)}, \dots, \boldsymbol{X}_{k-1}^{(n)})$ (see (3.4)). Joffe and Métivier [12, Theorem 4.2.2] also studied a sequence $(\boldsymbol{X}_k^{(n)})_{k\in\mathbb{Z}_+}$, $n \in \mathbb{N}$, of multi-type branching processes without immigration which is nearly critical of special type, namely, when the offspring mean matrices $\boldsymbol{m}_{\boldsymbol{\xi}}^{(n)}$, $n \in \mathbb{N}$, satisfy $\boldsymbol{m}_{\boldsymbol{\xi}}^{(n)} = \boldsymbol{I}_p + n^{-1}\boldsymbol{C} + \mathrm{o}(n^{-1})$ as $n \to \infty$, and they proved that the sequence $(n^{-1}\boldsymbol{X}_{\lfloor nt\rfloor}^{(n)})_{t\in\mathbb{R}_+}$ converges towards a diffusion process.

The aim of the present paper is to obtain a joint generalization of the above mentioned results for critical multi-type branching processes with immigration. We succeeded to determine the asymptotic behavior of a sequence of critical multi-type branching processes with immigration and with the same offspring and immigration distributions if the offspring mean matrix is primitive and $n^{-1}\boldsymbol{X}_0^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \boldsymbol{\mu}$ as $n \to \infty$, where $\boldsymbol{\mu}$ is concentrated on the ray $\mathbb{R}_+ \cdot \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$, where $\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$ is the Perron eigenvector of the offspring mean matrix $\boldsymbol{m}_{\boldsymbol{\xi}}$ (see Theorem 3.1). It turned out that the limiting diffusion process is always one-dimensional in the sense that for all $t \in \mathbb{R}_+$, the distribution of $\boldsymbol{\mathcal{X}}_t$ is also concentrated on the ray $\mathbb{R}_+ \cdot \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$. In fact, $\boldsymbol{\mathcal{X}}_t = \mathcal{X}_t \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$, $t \in \mathbb{R}_+$, where $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is again a squared Bessel process which is a continuous time and continuous state branching process with immigration. In the single-type case, Li [13] proved a result on the convergence of a sequence of discrete branching processes with immigration to a continuous branching process with immigration using appropriate time scaling which is different from our scaling. Later, Ma [14] extended Li's result for two-type branching processes. They proved the convergence of the sequence of infinitesimal generators of single(two)-type branching processes with immigration towards the generator of the limiting diffusion process which is a well-known technique in case of time-homogeneous Markov processes, see, e.g., Ethier and Kurtz [7]. Contrarily, our approach is based on the martingale method. It is interesting to note that Kesten and Stigum [15] considered a supercritical multi-type branching process without immigration, with a fixed initial distribution and with primitive offspring mean matrix, and they proved that $\varrho(\boldsymbol{m}_\xi)^{-n}\boldsymbol{X}_n \to \boldsymbol{W}$ almost surely as $n \to \infty$, where the random vector $\boldsymbol{W}$ is also concentrated on the ray $\mathbb{R}_+ \cdot \boldsymbol{u}_{\boldsymbol{m}_\xi}$ (see also Kurtz, Lyons, Pemantle and Peres [16]).

2 Multi-type branching processes with immigration

We will investigate a sequence $\big(\boldsymbol{X}_k^{(n)}\big)_{k\in\mathbb{Z}_+}$, $n \in \mathbb{N}$, of critical p-type branching processes with immigration sharing the same offspring and immigration distributions, but having possibly different initial distributions. For each $n \in \mathbb{N}$, $k \in \mathbb{Z}_+$, and $i \in \{1, \ldots, p\}$, the number of individuals of type i in the k^{th} generation of the n^{th} process is denoted by $X_{k,i}^{(n)}$. By $\xi_{k,j,i,\ell}^{(n)}$ we denote the number of type ℓ offspring produced by the j^{th} individual who is of type i belonging to the $(k-1)^{\text{th}}$ generation of the n^{th} process. The number of type i immigrants in the k^{th} generation of the n^{th} process will be denoted by $\varepsilon_{k,i}^{(n)}$. Consider the random vectors

$$\boldsymbol{X}_k^{(n)} := \begin{bmatrix} X_{k,1}^{(n)} \\ \vdots \\ X_{k,p}^{(n)} \end{bmatrix}, \qquad \boldsymbol{\xi}_{k,j,i}^{(n)} := \begin{bmatrix} \xi_{k,j,i,1}^{(n)} \\ \vdots \\ \xi_{k,j,i,p}^{(n)} \end{bmatrix}, \qquad \boldsymbol{\varepsilon}_k^{(n)} := \begin{bmatrix} \varepsilon_{k,1}^{(n)} \\ \vdots \\ \varepsilon_{k,p}^{(n)} \end{bmatrix}.$$

Then, for $n, k \in \mathbb{N}$, we have

$$\boldsymbol{X}_k^{(n)} = \sum_{i=1}^{p} \sum_{j=1}^{X_{k-1,i}^{(n)}} \boldsymbol{\xi}_{k,j,i}^{(n)} + \boldsymbol{\varepsilon}_k^{(n)}. \tag{2.1}$$

Here $\Big\{\boldsymbol{X}_0^{(n)}, \boldsymbol{\xi}_{k,j,i}^{(n)}, \boldsymbol{\varepsilon}_k^{(n)} : k, j \in \mathbb{N}, i \in \{1, \ldots, p\}\Big\}$ are supposed to be independent for all $n \in \mathbb{N}$. Moreover, $\big\{\boldsymbol{\xi}_{k,j,i}^{(n)} : k, j, n \in \mathbb{N}\big\}$ for each $i \in \{1, \ldots, p\}$, and $\{\boldsymbol{\varepsilon}_k^{(n)} : k, n \in \mathbb{N}\}$ are supposed to consist of identically distributed vectors.

We suppose $\operatorname{E}\big(\|\boldsymbol{\xi}_{1,1,i}^{(1)}\|^2\big) < \infty$ for all $i \in \{1, \ldots, p\}$ and $\operatorname{E}\big(\|\boldsymbol{\varepsilon}_1^{(1)}\|^2\big) < \infty$.

Introduce the notations

$$\boldsymbol{m}_{\boldsymbol{\xi}_i} := \operatorname{E}\big(\boldsymbol{\xi}_{1,1,i}^{(1)}\big) \in \mathbb{R}_+^p, \qquad \boldsymbol{m}_{\boldsymbol{\xi}} := \begin{bmatrix} \boldsymbol{m}_{\boldsymbol{\xi}_1} & \cdots & \boldsymbol{m}_{\boldsymbol{\xi}_d} \end{bmatrix} \in \mathbb{R}_+^{p\times p}, \qquad \boldsymbol{m}_{\boldsymbol{\varepsilon}} := \operatorname{E}\big(\boldsymbol{\varepsilon}_1^{(1)}\big) \in \mathbb{R}_+^p,$$

$$\boldsymbol{V}_{\boldsymbol{\xi}_i} := \operatorname{Var}\big(\boldsymbol{\xi}_{1,1,i}^{(1)}\big) \in \mathbb{R}^{p\times p}, \qquad \boldsymbol{V}_{\boldsymbol{\varepsilon}} := \operatorname{Var}\big(\boldsymbol{\varepsilon}_1^{(1)}\big) \in \mathbb{R}^{p\times p}.$$

Note that many authors define the offspring mean matrix as $\boldsymbol{m}_{\boldsymbol{\xi}}^\top$. For $k \in \mathbb{Z}_+$, let $\mathcal{F}_k^{(n)} := \sigma\big(\boldsymbol{X}_0^{(n)}, \boldsymbol{X}_1^{(n)}, \ldots, \boldsymbol{X}_k^{(n)}\big)$. By (2.1),

$$\operatorname{E}\big(\boldsymbol{X}_k^{(n)} \,\big|\, \mathcal{F}_{k-1}^{(n)}\big) = \sum_{i=1}^p X_{k-1,i}^{(n)} \boldsymbol{m}_{\boldsymbol{\xi}_i} + \boldsymbol{m}_{\boldsymbol{\varepsilon}} = \boldsymbol{m}_{\boldsymbol{\xi}} \boldsymbol{X}_{k-1}^{(n)} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}. \tag{2.2}$$

Consequently,

$$\operatorname{E}\big(\boldsymbol{X}_k^{(n)}\big) = \boldsymbol{m}_{\boldsymbol{\xi}} \operatorname{E}\big(\boldsymbol{X}_{k-1}^{(n)}\big) + \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k, n \in \mathbb{N}, \tag{2.3}$$

which implies

$$\operatorname{E}\big(\boldsymbol{X}_k^{(n)}\big) = \boldsymbol{m}_{\boldsymbol{\xi}}^k \operatorname{E}\big(\boldsymbol{X}_0^{(n)}\big) + \sum_{j=0}^{k-1} \boldsymbol{m}_{\boldsymbol{\xi}}^j \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k, n \in \mathbb{N}. \tag{2.4}$$

Hence, the offspring mean matrix $\boldsymbol{m}_{\boldsymbol{\xi}}$ plays a crucial role in the asymptotic behavior of the sequence $\big(\boldsymbol{X}_k^{(n)}\big)_{k\in\mathbb{Z}_+}$.

In what follows we recall some known facts about primitive nonnegative matrices. A matrix $\boldsymbol{A} \in \mathbb{R}_+^{p\times p}$ is called primitive if there exists $m \in \mathbb{N}$ such that $\boldsymbol{A}^m \in \mathbb{R}_{++}^{p\times p}$. A matrix $\boldsymbol{A} \in \mathbb{R}_+^{p\times p}$ is primitive if and only if it is irreducible and has only one eigenvalue of maximum modulus; see, e.g., Horn and Johnson [17, Definition 8.5.0, Theorem 8.5.2]. If a matrix $\boldsymbol{A} \in \mathbb{R}_+^{p\times p}$ is primitive then, by the Frobenius–Perron theorem (see, e.g., Horn and Johnson [17, Theorems 8.2.11 and 8.5.1]), the following assertions hold:

- $\varrho(\boldsymbol{A}) \in \mathbb{R}_{++}$, $\varrho(\boldsymbol{A})$ is an eigenvalue of $\boldsymbol{A}$, the algebraic and geometric multiplicities of $\varrho(\boldsymbol{A})$ equal 1 and the absolute values of the other eigenvalues of $\boldsymbol{A}$ are less than $\varrho(\boldsymbol{A})$.

- Corresponding to the eigenvalue $\varrho(\boldsymbol{A})$ there exists a unique (right) eigenvector $\boldsymbol{u}_{\boldsymbol{A}} \in \mathbb{R}^p_{++}$, called Perron vector, such that the sum of the coordinates of $\boldsymbol{u}_{\boldsymbol{A}}$ is 1.

- Further,
$$\varrho(\boldsymbol{A})^{-n}\boldsymbol{A}^n \to \boldsymbol{\Pi}_{\boldsymbol{A}} := \boldsymbol{u}_{\boldsymbol{A}}\boldsymbol{v}_{\boldsymbol{A}}^\top \in \mathbb{R}^{p\times p}_{++} \qquad \text{as} \quad n \to \infty,$$
where $\boldsymbol{v}_{\boldsymbol{A}} \in \mathbb{R}^p_{++}$ is the unique left eigenvector corresponding to the eigenvalue $\varrho(\boldsymbol{A})$ with $\boldsymbol{u}_{\boldsymbol{A}}^\top \boldsymbol{v}_{\boldsymbol{A}} = 1$.

- Moreover, there exist $c_{\boldsymbol{A}}, r_{\boldsymbol{A}} \in \mathbb{R}_{++}$ with $r_{\boldsymbol{A}} < 1$ such that
$$\|\varrho(\boldsymbol{A})^{-n}\boldsymbol{A}^n - \boldsymbol{\Pi}_{\boldsymbol{A}}\| \leqslant c_{\boldsymbol{A}} r_{\boldsymbol{A}}^n \qquad \text{for all} \quad n \in \mathbb{N}, \tag{2.5}$$
where $\|\boldsymbol{B}\|$ denotes the operator norm of a matrix $\boldsymbol{B} \in \mathbb{R}^{p\times p}$ defined by $\|\boldsymbol{B}\| := \sup_{\|\boldsymbol{x}\|=1} \|\boldsymbol{B}\boldsymbol{x}\|$.

A multi-type branching process with immigration will be called primitive if its offspring mean matrix $\boldsymbol{m}_{\boldsymbol{\xi}}$ is primitive. Note that many authors call it positively regular.

3 Convergence results

A function $f : \mathbb{R}_+ \to \mathbb{R}^p$ is called *càdlàg* if it is right continuous with left limits. Let $\mathsf{D}(\mathbb{R}_+, \mathbb{R}^p)$ and $\mathsf{C}(\mathbb{R}_+, \mathbb{R}^p)$ denote the space of all $\mathbb{R}^p$-valued càdlàg and continuous functions on $\mathbb{R}_+$, respectively. Let $\mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p)$ denote the Borel σ-algebra in $\mathsf{D}(\mathbb{R}_+, \mathbb{R}^p)$ for the metric defined in Jacod and Shiryaev [18, Chapter VI, (1.26)] (with this metric $\mathsf{D}(\mathbb{R}_+, \mathbb{R}^p)$ is a complete and separable metric space). For $\mathbb{R}^p$-valued stochastic processes $(\boldsymbol{\mathcal{Y}}_t)_{t\in\mathbb{R}_+}$ and $(\boldsymbol{\mathcal{Y}}^{(n)}_t)_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, with càdlàg paths we write

$\boldsymbol{\mathcal{Y}}^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \boldsymbol{\mathcal{Y}}$ if the distribution of $\boldsymbol{\mathcal{Y}}^{(n)}$ on the space $(\mathsf{D}(\mathbb{R}_+, \mathbb{R}^p), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p))$ converges weakly to the distribution of $\boldsymbol{\mathcal{Y}}$ on the space $(\mathsf{D}(\mathbb{R}_+, \mathbb{R}^p), \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p))$ as $n \to \infty$.

For each $n \in \mathbb{N}$, consider the random step processes

$$\boldsymbol{\mathcal{X}}_t^{(n)} := n^{-1} \boldsymbol{X}_{\lfloor nt \rfloor}^{(n)}, \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

For a vector $\boldsymbol{\alpha} = (\alpha_i)_{i=1,\dots,p} \in \mathbb{R}_+^p$, we will use notation $\boldsymbol{\alpha} \odot \boldsymbol{V}_{\boldsymbol{\xi}} := \sum_{i=1}^p \alpha_i \boldsymbol{V}_{\boldsymbol{\xi}_i} \in \mathbb{R}^{p \times p}$, which is a positive semi-definite matrix, a mixture of the variance matrices $\boldsymbol{V}_{\xi_1}, \dots, \boldsymbol{V}_{\xi_p}$.

Theorem 3.1 *Let $\big(\boldsymbol{X}_k^{(n)}\big)_{k \in \mathbb{Z}_+}$, $n \in \mathbb{N}$, be a sequence of critical primitive p-type branching processes with immigration sharing the same offspring and immigration distributions, but having possibly different initial distributions, such that $n^{-1} \boldsymbol{X}_0^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{X}_0 \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$, where $\mathcal{X}_0$ is a nonnegative random variable with distribution μ. Suppose $\mathrm{E}\big(\|\boldsymbol{X}_0^{(n)}\|^2\big) = \mathrm{O}(n^2)$, $\mathrm{E}\big(\|\boldsymbol{\xi}_{1,1,i}^{(1)}\|^4\big) < \infty$ for all $i \in \{1, \dots, p\}$ and $\mathrm{E}\big(\|\boldsymbol{\varepsilon}_1^{(1)}\|^4\big) < \infty$. Then*

$$\boldsymbol{\mathcal{X}}^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \mathcal{X} \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \qquad \text{as } n \to \infty, \tag{3.1}$$

where $(\mathcal{X}_t)_{t \in \mathbb{R}_+}$ is the unique weak solution (in the sense of probability law) of the SDE

$$\mathrm{d}\mathcal{X}_t = \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{m}_{\boldsymbol{\varepsilon}} \,\mathrm{d}t + \sqrt{\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top (\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \odot \boldsymbol{V}_{\boldsymbol{\xi}}) \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \mathcal{X}_t^+} \,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+, \tag{3.2}$$

with initial distribution μ, where $(\mathcal{W}_t)_{t \in \mathbb{R}_+}$ is a standard Wiener process.

Remark 1 Theorem 3.1 will remain true under the weaker assumptions $\mathrm{E}\big(\|\boldsymbol{\xi}_{1,1,i}^{(1)}\|^2\big) < \infty$ for all $i \in \{1, \dots, p\}$ and $\mathrm{E}\big(\|\boldsymbol{\varepsilon}_1^{(1)}\|^2\big) < \infty$. In fact, the higher moment assumptions in the theorem are needed only for facilitating of checking the conditional Lindeberg condition, namely, condition (ii) of Theorem A.3 for proving convergence (3.4) of the martingale part. One can check the conditional Lindeberg condition under the weaker moment assumptions of Theorem 3.1 by the method of Ispány and Pap [19], see also this method in Barczy et al. [20]. If $d \geqslant 2$ then it is not clear if one might get rid of the assumption $\operatorname{E}(\|\boldsymbol{X}_0^{(n)}\|^2) = \operatorname{O}(n^2)$ in Theorem 3.1.

Remark 2 Under the assumptions of Theorem 3.1, by the same method, one can also prove $\widetilde{\boldsymbol{\mathcal{X}}}^{(n)} \xrightarrow{\mathcal{L}} \widetilde{\mathcal{X}} \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$ as $n \to \infty$, where $\widetilde{\boldsymbol{\mathcal{X}}}_t^{(n)} := n^{-1}\big(\boldsymbol{X}_{\lfloor nt \rfloor}^{(n)} - \boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt \rfloor} \boldsymbol{X}_0^{(n)}\big)$, $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, and $(\widetilde{\mathcal{X}}_t)_{t \in \mathbb{R}_+}$ is the unique strong solution of the SDE (3.2) with initial value $\widetilde{\mathcal{X}}_0 = 0$.

Remark 3 The SDE (3.2) has a unique strong solution $(\mathcal{X}_t^{(x_0)})_{t \in \mathbb{R}_+}$ for all initial values $\mathcal{X}_0^{(x_0)} = x_0 \in \mathbb{R}$. Indeed, since $|\sqrt{x} - \sqrt{y}| \leqslant \sqrt{|x-y|}$ for $x, y \geqslant 0$, the coefficient functions $\mathbb{R} \ni x \mapsto \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{m}_{\boldsymbol{\varepsilon}} \in \mathbb{R}_+$ and $\mathbb{R} \ni x \mapsto \sqrt{\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top (\boldsymbol{u}_{\boldsymbol{\xi}} \odot \boldsymbol{V}_{\boldsymbol{\xi}}) \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}} x^+}$ satisfy conditions of part (ii) of Theorem 3.5 in Chapter IX in Revuz and Yor [5] or the conditions of Proposition 5.2.13 in Karatzas and Shreve [21]. Further, by the comparison theorem (see, e.g., Revuz and Yor [5, Theorem 3.7, Chapter IX]), if the initial value $\mathcal{X}_0^{(x_0)} = x_0$ is nonnegative, then $\mathcal{X}_t^{(x)}$ is nonnegative for all $t \in \mathbb{R}_+$ with probability one. Hence $\mathcal{X}_t^+$ may be replaced by $\mathcal{X}_t$ under the square root in (3.2).

Proof of Theorem 3.1. In order to prove (3.1), for each $n \in \mathbb{N}$, introduce the sequence

$$\boldsymbol{M}_k^{(n)} := \boldsymbol{X}_k^{(n)} - \operatorname{E}\big(\boldsymbol{X}_k^{(n)} \,\big|\, \mathcal{F}_{k-1}^{(n)}\big) = \boldsymbol{X}_k^{(n)} - \boldsymbol{m}_{\boldsymbol{\xi}} \boldsymbol{X}_{k-1}^{(n)} - \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k \in \mathbb{N}, \tag{3.3}$$

which is a sequence of martingale differences with respect to the filtration $\big(\mathcal{F}_k^{(n)}\big)_{k \in \mathbb{Z}_+}$. Consider the random step processes

$$\boldsymbol{\mathcal{M}}_t^{(n)} := n^{-1}\Bigg(\boldsymbol{X}_0^{(n)} + \sum_{k=1}^{\lfloor nt \rfloor} \boldsymbol{M}_k^{(n)}\Bigg), \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

First we will verify convergence

$$\boldsymbol{\mathcal{M}}^{(n)} \xrightarrow{\mathcal{L}} \boldsymbol{\mathcal{M}} \qquad \text{as} \quad n \to \infty, \tag{3.4}$$

where $(\boldsymbol{\mathcal{M}}_t)_{t\in\mathbb{R}_+}$ is the unique weak solution of the SDE

$$\mathrm{d}\boldsymbol{\mathcal{M}}_t = \sqrt{(\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}(\boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}}))^+ \odot \boldsymbol{V}_{\boldsymbol{\xi}}}\,\mathrm{d}\boldsymbol{\mathcal{W}}_t, \qquad t \in \mathbb{R}_+, \tag{3.5}$$

with initial distribution $\boldsymbol{\mu} :\stackrel{\mathcal{L}}{=} \mathcal{X}_0 \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$, where $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ is a standard p-dimensional Wiener process, $\boldsymbol{x}^+$ denotes the positive part of $\boldsymbol{x} \in \mathbb{R}^p$, and for a positive semi-definite matrix $\boldsymbol{A} \in \mathbb{R}^{p\times p}$, $\sqrt{\boldsymbol{A}}$ denotes its unique symmetric positive semi-definite square root.

From (3.3) we obtain the recursion

$$\boldsymbol{X}_k^{(n)} = \boldsymbol{m}_{\boldsymbol{\xi}} \boldsymbol{X}_{k-1}^{(n)} + \boldsymbol{M}_k^{(n)} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \qquad k \in \mathbb{N}, \tag{3.6}$$

implying

$$\boldsymbol{X}_k^{(n)} = \boldsymbol{m}_{\boldsymbol{\xi}}^k \boldsymbol{X}_0^{(n)} + \sum_{j=1}^{k} \boldsymbol{m}_{\boldsymbol{\xi}}^{k-j} (\boldsymbol{M}_j^{(n)} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}), \qquad k \in \mathbb{N}. \tag{3.7}$$

Applying a version of the continuous mapping theorem (see Appendix) together with (3.4) and (3.7), in Section 4 we show that

$$\boldsymbol{\mathcal{X}}^{(n)} \xrightarrow{\mathcal{L}} \boldsymbol{\mathcal{X}} \qquad \text{as} \quad n \to \infty, \tag{3.8}$$

where $\boldsymbol{\mathcal{X}}_t := \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}(\boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}})$, $t \in \mathbb{R}_+$. Using $\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}} = \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top$ and $\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} = 1$ we get that the process $\mathcal{Y}_t := \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{\mathcal{X}}_t$, $t \in \mathbb{R}_+$, satisfies $\mathcal{Y}_t = \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}(\boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}}) = \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top(\boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}})$, $t \in \mathbb{R}_+$, hence $\boldsymbol{\mathcal{X}}_t = \mathcal{Y}_t \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$. By Itô's formula we obtain that $(\mathcal{Y}_t)_{t\in\mathbb{R}_+}$ satisfies the SDE (3.2) (see the analysis of the process $(\mathcal{P}_t^{(\boldsymbol{y}_0)})_{t\in\mathbb{R}_+}$ in the first equation of (4.1) and in equation (4.2)) such that $\mathcal{Y}_0 = \boldsymbol{v}_{\boldsymbol{m}_\xi}^\top \boldsymbol{\mathcal{X}}_0 = \boldsymbol{v}_{\boldsymbol{m}_\xi}^\top (\mathcal{X}_0 \boldsymbol{u}_{\boldsymbol{m}_\xi}) = \mathcal{X}_0$, thus we conclude the statement of Theorem 3.1. $\Box$

Remark 4 By Itô's formula, the limit process $(\boldsymbol{\mathcal{X}}_t)_{t\in\mathbb{R}_+}$ in (3.1) can also be characterized as a weak solution of the SDE

$$\mathrm{d}\boldsymbol{\mathcal{X}}_t = \boldsymbol{\Pi}_{\boldsymbol{m}_\xi} \boldsymbol{m}_{\boldsymbol{\varepsilon}} \, \mathrm{d}t + \boldsymbol{\Pi}_{\boldsymbol{m}_\xi} \sqrt{\boldsymbol{\mathcal{X}}_t^+ \odot \boldsymbol{V}_{\boldsymbol{\xi}}} \, \mathrm{d}\boldsymbol{\mathcal{W}}_t, \qquad t \in \mathbb{R}_+, \tag{3.9}$$

with initial distribution $\boldsymbol{\Pi}_{\boldsymbol{m}_\xi} \boldsymbol{\mathcal{M}}_0 = \boldsymbol{\Pi}_{\boldsymbol{m}_\xi} (\mathcal{X}_0 \boldsymbol{u}_{\boldsymbol{m}_\xi}) = \mathcal{X}_0 \boldsymbol{u}_{\boldsymbol{m}_\xi}$, since $\boldsymbol{\Pi}_{\boldsymbol{m}_\xi} \boldsymbol{u}_{\boldsymbol{m}_\xi} = \boldsymbol{u}_{\boldsymbol{m}_\xi} \boldsymbol{v}_{\boldsymbol{m}_\xi}^\top \boldsymbol{u}_{\boldsymbol{m}_\xi} = \boldsymbol{u}_{\boldsymbol{m}_\xi}$.

Remark 5 The generator of $(\boldsymbol{\mathcal{M}}_t)_{t\in\mathbb{R}_+}$ is given by

$$\begin{aligned} L_t f(\boldsymbol{x}) &= \frac{1}{2} \big\langle [(\boldsymbol{\Pi}_{\boldsymbol{m}_\xi} (\boldsymbol{x} + t\boldsymbol{m}_{\boldsymbol{\varepsilon}})) \odot \boldsymbol{V}_{\boldsymbol{\xi}}] \nabla, \nabla \big\rangle f(\boldsymbol{x}) \\ &= \frac{1}{2} (\boldsymbol{x} + t\boldsymbol{m}_{\boldsymbol{\varepsilon}})^\top \boldsymbol{\Pi}_{\boldsymbol{m}_\xi}^\top \sum_{i=1}^p \sum_{j=1}^p \boldsymbol{V}_{\xi,i,j} \, \partial_i \partial_j f(\boldsymbol{x}), \qquad t \in \mathbb{R}_+, \quad f \in C_{\mathrm{c}}^\infty(\mathbb{R}^p), \end{aligned}$$

where $\boldsymbol{V}_{\xi,i,j} := (\mathrm{Cov}(\xi_{1,1,\ell,i}, \xi_{1,1,\ell,j}))_{\ell=1,\dots,d} \in \mathbb{R}_+^p$. (Joffe and Métivier [12, Theorem 4.3.1] also obtained this generator with $\boldsymbol{m}_{\boldsymbol{\varepsilon}} = \boldsymbol{0}$ deriving (3.4) for processes without immigration.)

4 Proof of $\boldsymbol{\mathcal{M}}^{(n)} \xrightarrow{\mathcal{L}} \boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{X}}^{(n)} \xrightarrow{\mathcal{L}} \boldsymbol{\mathcal{X}}$

First we prove $\boldsymbol{\mathcal{M}}^{(n)} \xrightarrow{\mathcal{L}} \boldsymbol{\mathcal{M}}$ applying Theorem A.3 for $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{M}}$, $\boldsymbol{U}_0^{(n)} = n^{-1} \boldsymbol{X}_0^{(n)}$ and $\boldsymbol{U}_k^{(n)} = n^{-1} \boldsymbol{M}_k^{(n)}$ for $n, k \in \mathbb{N}$, and with coefficient function $\gamma : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^{p\times p}$ of the SDE (3.5) given by $\gamma(t, \boldsymbol{x}) = \sqrt{(\boldsymbol{\Pi}_{\boldsymbol{m}_\xi} (\boldsymbol{x} + t\boldsymbol{m}_{\boldsymbol{\varepsilon}}))^+ \odot \boldsymbol{V}_{\boldsymbol{\xi}}}$. The aim of the following discussion is to show that the SDE (3.5) has a unique strong solution $\big(\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)}\big)_{t\in\mathbb{R}_+}$ with initial value $\boldsymbol{\mathcal{M}}_0^{(\boldsymbol{y}_0)} = \boldsymbol{y}_0$ for all $\boldsymbol{y}_0 \in \mathbb{R}^p$. First suppose that the

SDE (3.5), which can also be written in the form

$$\mathrm{d}\boldsymbol{\mathcal{M}}_t = \sqrt{(\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top(\boldsymbol{\mathcal{M}}_t + t\boldsymbol{m}_{\boldsymbol{\varepsilon}}))^+(\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \odot \boldsymbol{V}_{\boldsymbol{\xi}})}\,\mathrm{d}\boldsymbol{\mathcal{W}}_t,$$

has a strong solution $\big(\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)}\big)_{t\in\mathbb{R}_+}$ with $\boldsymbol{\mathcal{M}}_0^{(\boldsymbol{y}_0)} = \boldsymbol{y}_0$. Then, by Itô's formula, the process $\big(\mathcal{P}_t^{(\boldsymbol{y}_0)}, \boldsymbol{\mathcal{Q}}_t^{(\boldsymbol{y}_0)}\big)_{t\in\mathbb{R}_+}$, defined by

$$\mathcal{P}_t^{(\boldsymbol{y}_0)} := \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top(\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)} + t\boldsymbol{m}_{\boldsymbol{\varepsilon}}), \qquad \boldsymbol{\mathcal{Q}}_t^{(\boldsymbol{y}_0)} := \boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)} - \mathcal{P}_t^{(\boldsymbol{y}_0)}\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$$

is a strong solution of the SDE

$$\text{(4.1)} \qquad \begin{cases} \mathrm{d}\mathcal{P}_t = \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{m}_{\boldsymbol{\varepsilon}}\,\mathrm{d}t + \sqrt{\mathcal{P}_t^+}\,\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \sqrt{\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \odot \boldsymbol{V}_{\boldsymbol{\xi}}}\,\mathrm{d}\boldsymbol{\mathcal{W}}_t, \\[2ex] \mathrm{d}\boldsymbol{\mathcal{Q}}_t = -\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \boldsymbol{m}_{\boldsymbol{\varepsilon}}\,\mathrm{d}t + \sqrt{\mathcal{P}_t^+}\,\big(\boldsymbol{I}_p - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\big)\sqrt{\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \odot \boldsymbol{V}_{\boldsymbol{\xi}}}\,\mathrm{d}\boldsymbol{\mathcal{W}}_t \end{cases}$$

with initial value $\big(\mathcal{P}_0^{(\boldsymbol{y}_0)}, \boldsymbol{\mathcal{Q}}_0^{(\boldsymbol{y}_0)}\big) = \big(\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{y}_0, (\boldsymbol{I}_d - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}})\boldsymbol{y}_0\big)$, where $\boldsymbol{I}_p$ denotes the p-dimensional unit matrix. The SDE (4.1) has a unique strong solution $\big(\mathcal{P}_t^{(p_0)}, \boldsymbol{\mathcal{Q}}_t^{(\boldsymbol{q}_0)}\big)_{t\in\mathbb{R}_+}$, with an arbitrary initial value $\big(\mathcal{P}_0^{(p_0)}, \boldsymbol{\mathcal{Q}}_0^{(\boldsymbol{q}_0)}\big) = (p_0, \boldsymbol{q}_0) \in \mathbb{R}_+ \times \mathbb{R}^p$, since the first equation of (4.1) can be written in the form

$$\text{(4.2)} \qquad \mathrm{d}\mathcal{P}_t = b\,\mathrm{d}t + \sqrt{\mathcal{P}_t^+}\,\mathrm{d}\widetilde{\mathcal{W}}_t$$

with $b := \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{m}_{\boldsymbol{\varepsilon}} \in \mathbb{R}_+$ and

$$\widetilde{\mathcal{W}}_t := \boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \sqrt{\boldsymbol{u}_{m_\xi} \odot \boldsymbol{V}_{\boldsymbol{\xi}}}\,\boldsymbol{\mathcal{W}}_t = \sqrt{\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top(\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} \odot \boldsymbol{V}_{\boldsymbol{\xi}})\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}}\,\mathcal{W}_t,$$

where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard one-dimensional Wiener process. (Equation (4.2) can be discussed as equation (3.2) in Remark 3.) If $(\mathcal{P}_t^{(\boldsymbol{y}_0)}, \boldsymbol{\mathcal{Q}}_t^{(\boldsymbol{y}_0)})_{t\in\mathbb{R}_+}$ is the unique strong solution of the SDE (4.1) with the initial value $\big(\mathcal{P}_0^{(\boldsymbol{y}_0)}, \boldsymbol{\mathcal{Q}}_0^{(\boldsymbol{y}_0)}\big) = \big(\boldsymbol{v}_{\boldsymbol{m}_{\boldsymbol{\xi}}}^\top \boldsymbol{y}_0, (\boldsymbol{I}_p -$

$\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}})\boldsymbol{y}_0\big)$, then, again by Itô's formula,

$$\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)} := \mathcal{P}_t^{(\boldsymbol{y}_0)} \boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}} + \boldsymbol{\mathcal{Q}}_t^{(\boldsymbol{y}_0)}, \qquad t \in \mathbb{R}_+,$$

is a strong solution of (3.5) with $\boldsymbol{\mathcal{M}}_0^{(\boldsymbol{y}_0)} = \boldsymbol{y}_0$. Consequently, (3.5) admits a unique strong solution $\big(\boldsymbol{\mathcal{M}}_t^{(\boldsymbol{y}_0)}\big)_{t\in\mathbb{R}_+}$ with $\boldsymbol{\mathcal{M}}_0^{(\boldsymbol{y}_0)} = \boldsymbol{y}_0$ for all $\boldsymbol{y}_0 \in \mathbb{R}^p$.

Now we show that conditions (i) and (ii) of Theorem A.3 hold. We have to check that, for each $T > 0$,

$$\sup_{t\in[0,T]} \left\| \frac{1}{n^2} \sum_{k=1}^{\lfloor nt\rfloor} \mathrm{E}\big[\boldsymbol{M}_k^{(n)}(\boldsymbol{M}_k^{(n)})^\top \,\big|\, \mathcal{F}_{k-1}^{(n)}\big] - \int_0^t (\boldsymbol{\mathcal{R}}_s^{(n)})_+ \,\mathrm{d}s \odot \boldsymbol{V}_{\boldsymbol{\xi}} \right\| \xrightarrow{\mathrm{P}} 0, \tag{4.3}$$

$$\frac{1}{n^2} \sum_{k=1}^{\lfloor nT\rfloor} \mathrm{E}\big(\|\boldsymbol{M}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{M}_k^{(n)}\|>n\theta\}} \,\big|\, \mathcal{F}_{k-1}^{(n)}\big) \xrightarrow{\mathrm{P}} 0 \qquad \text{for all } \theta > 0 \tag{4.4}$$

as $n \to \infty$, where the process $(\boldsymbol{\mathcal{R}}_t^{(n)})_{t\in\mathbb{R}_+}$ is defined by

$$\boldsymbol{\mathcal{R}}_t^{(n)} := \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\big(\boldsymbol{\mathcal{M}}_t^{(n)} + t\boldsymbol{m}_{\boldsymbol{\varepsilon}}\big), \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}. \tag{4.5}$$

By (3.3),

$$\begin{aligned}
\boldsymbol{\mathcal{R}}_t^{(n)} &= \boldsymbol{\Pi}_{m_\xi}\left(n^{-1}\left(\boldsymbol{X}_0^{(n)} + \sum_{k=1}^{\lfloor nt\rfloor}(\boldsymbol{X}_k^{(n)} - \boldsymbol{m}_{\boldsymbol{\xi}}\boldsymbol{X}_{k-1}^{(n)} - \boldsymbol{m}_{\boldsymbol{\varepsilon}})\right) + t\boldsymbol{m}_{\boldsymbol{\varepsilon}}\right)\\
&= n^{-1}\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\boldsymbol{X}_{\lfloor nt\rfloor}^{(n)} + n^{-1}(nt - \lfloor nt\rfloor)\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\boldsymbol{m}_{\boldsymbol{\varepsilon}},
\end{aligned}$$

where we used that $\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\boldsymbol{m}_{\boldsymbol{\xi}} = (\lim_{n\to\infty} \boldsymbol{m}_{\boldsymbol{\xi}}^n)\boldsymbol{m}_{\boldsymbol{\xi}} = \lim_{n\to\infty} \boldsymbol{m}_{\boldsymbol{\xi}}^{n+1} = \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$ implies $\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}(\boldsymbol{I}_p - \boldsymbol{m}_{\boldsymbol{\xi}}) = \boldsymbol{0}$. Thus $(\boldsymbol{\mathcal{R}}_t^{(n)})_+ = \boldsymbol{\mathcal{R}}_t^{(n)}$, and

$$\int_0^t (\boldsymbol{\mathcal{R}}_s^{(n)})_+ \,\mathrm{d}s = \frac{1}{n^2} \sum_{\ell=0}^{\lfloor nt\rfloor-1} \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\boldsymbol{X}_\ell^{(n)} + \frac{nt - \lfloor nt\rfloor}{n^2}\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\boldsymbol{X}_{\lfloor nt\rfloor}^{(n)} + \frac{\lfloor nt\rfloor + (nt - \lfloor nt\rfloor)^2}{2n^2}\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\boldsymbol{m}_{\boldsymbol{\varepsilon}}.$$

Using (A.4), we obtain

$$\frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor} \mathrm{E}\big[\boldsymbol{M}_k^{(n)}(\boldsymbol{M}_k^{(n)})^\top \,\big|\, \mathcal{F}_{k-1}^{(n)}\big] = \frac{\lfloor nt\rfloor}{n^2}\boldsymbol{V}_{\boldsymbol{\varepsilon}} + \frac{1}{n^2}\sum_{k=1}^{\lfloor nt\rfloor} \boldsymbol{X}_{k-1}^{(n)} \odot \boldsymbol{V}_{\boldsymbol{\xi}}.$$

Hence, in order to show (4.3), it suffices to prove

$$\sup_{t\in[0,T]} \frac{1}{n^2}\sum_{k=0}^{\lfloor nt\rfloor-1} \|(\boldsymbol{I}_p - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}})\boldsymbol{X}_k^{(n)}\| \stackrel{\mathrm{P}}{\longrightarrow} 0, \qquad \sup_{t\in[0,T]} \frac{1}{n^2}\|\boldsymbol{X}_{\lfloor nt\rfloor}^{(n)}\| \stackrel{\mathrm{P}}{\longrightarrow} 0 \tag{4.6}$$

as $n\to\infty$. Using (3.7) and $\boldsymbol{\Pi}_{\boldsymbol{m}_\xi}\boldsymbol{m}_{\boldsymbol{\xi}} = \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$, we obtain

$$(\boldsymbol{I}_d - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}})\boldsymbol{X}_k^{(n)} = \big(\boldsymbol{m}_{\boldsymbol{\xi}}^k - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\big)\,\boldsymbol{X}_0^{(n)} + \sum_{j=1}^{k}\Big(\boldsymbol{m}_{\boldsymbol{\xi}}^{k-j} - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\Big)(\boldsymbol{M}_j^{n)} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}).$$

Hence by (2.5),

$$\begin{aligned}\sum_{k=0}^{\lfloor nt\rfloor-1} \|(\boldsymbol{I}_d - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}})\boldsymbol{X}_k^{(n)}\| &\leqslant c_{\boldsymbol{m}_{\boldsymbol{\xi}}} \sum_{k=0}^{\lfloor nt\rfloor-1} r_{\boldsymbol{m}_{\boldsymbol{\xi}}}^k \|\boldsymbol{X}_0^{(n)}\| + c_{\boldsymbol{m}_{\boldsymbol{\xi}}} \sum_{k=1}^{\lfloor nt\rfloor-1}\sum_{j=1}^{k} r_{\boldsymbol{m}_{\boldsymbol{\xi}}}^{k-j}\|\boldsymbol{M}_j^{(n)} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}\| \\ &\leqslant \frac{c_{\boldsymbol{m}_{\boldsymbol{\xi}}}}{1-r_{\boldsymbol{m}_{\boldsymbol{\xi}}}}\Bigg(\|\boldsymbol{X}_0^{(n)}\| + \lfloor nt\rfloor\cdot\|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\| + \sum_{j=1}^{\lfloor nt\rfloor-1}\|\boldsymbol{M}_j^{(n)}\|\Bigg).\end{aligned}$$

Moreover, by (3.7) and (A.8),

$$\begin{aligned}\|\boldsymbol{X}_{\lfloor nt\rfloor}\| &\leqslant \|\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}\|\cdot\|\boldsymbol{X}_0^{(n)}\| + \sum_{j=1}^{\lfloor nt\rfloor}\|\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-j}\|\cdot\|\boldsymbol{M}_j^{(n)} + \boldsymbol{m}_{\boldsymbol{\varepsilon}}\| \\ &\leqslant C_{\boldsymbol{m}_{\boldsymbol{\xi}}}\Bigg(\|\boldsymbol{X}_0^{(n)}\| + \lfloor nt\rfloor\cdot\|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\| + \sum_{j=1}^{\lfloor nt\rfloor}\|\boldsymbol{M}_j^{(n)}\|\Bigg),\end{aligned}$$

where $C_{\boldsymbol{m}_{\boldsymbol{\xi}}}$ is defined by (A.8). Consequently, in order to prove (4.6), it suffices to show

$$\frac{1}{n^2}\sum_{j=1}^{\lfloor nT\rfloor}\|\boldsymbol{M}_j^{(n)}\| \stackrel{\mathrm{P}}{\longrightarrow} 0, \qquad \frac{1}{n^2}\|\boldsymbol{X}_0^{(n)}\| \stackrel{\mathrm{P}}{\longrightarrow} 0 \qquad \text{as } n\to\infty.$$

In fact, assumption $n^{-1}\boldsymbol{X}_0^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \boldsymbol{\mu}$ implies the second convergence, while Lemma

A.2 yields $n^{-2}\sum_{j=1}^{\lfloor nT\rfloor}\operatorname{E}(\|\boldsymbol{M}_j^{(n)}\|)\to 0$, thus we obtain (4.3).

Next we check condition (4.4). We have

$$\operatorname{E}\big(\|\boldsymbol{M}_k^{(n)}\|^2\mathbb{1}_{\{\|\boldsymbol{M}_k^{(n)}\|>n\theta\}}\,\big|\,\mathcal{F}_{k-1}^{(n)}\big)\leqslant n^{-2}\theta^{-2}\operatorname{E}\big(\|\boldsymbol{M}_k^{(n)}\|^4\,\big|\,\mathcal{F}_{k-1}^{(n)}\big).$$

Moreover, $n^{-4}\sum_{k=1}^{\lfloor nT\rfloor}\operatorname{E}\big(\|\boldsymbol{M}_k^{(n)}\|^4\big)\to 0$ as $n\to\infty$, since $\operatorname{E}\big(\|\boldsymbol{M}_k^{(n)}\|^4\big)=\operatorname{O}\big((k+n)^2\big)$ by Lemma A.2. Hence we obtain (4.4).

Now we turn to prove (3.8) applying Lemma A.4. By (3.7), $\boldsymbol{\mathcal{X}}^{(n)}=\Psi_n(\boldsymbol{\mathcal{M}}^{(n)})$, where the mapping $\Psi_n:\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p)\to\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p)$ is given by

$$\Psi_n(f)(t):=\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}f(0)+\sum_{j=1}^{\lfloor nt\rfloor}\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-j}\left(f\left(\frac{j}{n}\right)-f\left(\frac{j-1}{n}\right)+n^{-1}\boldsymbol{m}_{\boldsymbol{\varepsilon}}\right)$$

for $f\in\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p)$, $t\in\mathbb{R}_+$, $n\in\mathbb{N}$. Further, $\boldsymbol{\mathcal{X}}=\Psi(\boldsymbol{\mathcal{M}})$, where the mapping $\Psi:\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p)\to\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p)$ is given by

$$\Psi(f)(t):=\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}(f(t)+t\boldsymbol{m}_{\boldsymbol{\varepsilon}}),\qquad f\in\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p),\qquad t\in\mathbb{R}_+.$$

Measurability of the mappings Ψ_n, $n\in\mathbb{N}$, and Ψ can be checked as in Barczy et al. [20].

The aim of the following discussion is to show that the set $C:=\{f\in\mathsf{C}(\mathbb{R}_+,\mathbb{R}^p):\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}f(0)=f(0)\}$ satisfies $C\in\mathcal{D}_\infty(\mathbb{R}_+,\mathbb{R}^p)$, $C\subset C_{\Psi,(\Psi_n)_{n\in\mathbb{N}}}$ and $\operatorname{P}(\boldsymbol{\mathcal{M}}\in C)=1$. Note that $f\in C$ implies $f(0)\in\mathbb{R}\cdot\boldsymbol{u}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$.

First note that $C=\mathsf{C}(\mathbb{R}_+,\mathbb{R}^p)\cap\pi_0^{-1}\big((\boldsymbol{I}_p-\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}})^{-1}(\{\boldsymbol{0}\})\big)$, where $\pi_0:\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p)\to\mathbb{R}^p$ denotes the projection defined by $\pi_0(f):=f(0)$ for $f\in\mathsf{D}(\mathbb{R}_+,\mathbb{R}^p)$. Using that $\mathsf{C}(\mathbb{R}_+,\mathbb{R}^p)\in\mathcal{D}_\infty$ (see, e.g., Ethier and Kurtz [7, Problem 3.11.25]), the mapping $\mathbb{R}^p\ni\boldsymbol{x}\mapsto(\boldsymbol{I}_p-\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}})\boldsymbol{x}\in\mathbb{R}^p$ is measurable and that π_0 is measurable (see, e.g., Ethier and Kurtz [7, Proposition 3.7.1]), we obtain $C\in\mathcal{D}_\infty(\mathbb{R}_+,\mathbb{R}^p)$.

Fix a function $f \in C$ and a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathsf{D}(\mathbb{R}^p)$ with $f_n \xrightarrow{\text{lu}} f$. By the definition of Ψ, we have $\Psi(f) \in \mathsf{C}(\mathbb{R}^p)$. Further, we can write

$$\begin{aligned}\Psi_n(f_n)(t) &= \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\left(f_n\left(\frac{\lfloor nt\rfloor}{n}\right) + \frac{\lfloor nt\rfloor}{n}\boldsymbol{m}_{\boldsymbol{\varepsilon}}\right) + \big(\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor} - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\big) f(0) \\ &\quad + \sum_{j=1}^{\lfloor nt\rfloor}\big(\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-j} - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\big)\left(f_n\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) + \frac{1}{n}\boldsymbol{m}_{\boldsymbol{\varepsilon}}\right),\end{aligned}$$

hence we have

$$\begin{aligned}\|\Psi_n(f_n)(t) - \Psi(f)(t)\| &\leqslant \|\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\|\left(\left\|f_n\left(\frac{\lfloor nt\rfloor}{n}\right) - f(t)\right\| + \frac{1}{n}\|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\|\right) \\ &\quad + \left\|\left(\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor} - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\right) f_n(0)\right\| + \sum_{j=1}^{\lfloor nt\rfloor}\|\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-j} - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\|\left(\left\|f_n\left(\frac{j}{n}\right) - f_n\left(\frac{j-1}{n}\right)\right\| + \frac{1}{n}\|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\|\right).\end{aligned}$$

For all $T > 0$ and $t \in [0, T]$,

$$\begin{aligned}\left\|f_n\left(\frac{\lfloor nt\rfloor}{n}\right) - f(t)\right\| &\leqslant \left\|f_n\left(\frac{\lfloor nt\rfloor}{n}\right) - f\left(\frac{\lfloor nt\rfloor}{n}\right)\right\| + \left\|f\left(\frac{\lfloor nt\rfloor}{n}\right) - f(t)\right\| \\ &\leqslant \omega_T(f, n^{-1}) + \sup_{t\in[0,T]}\|f_n(t) - f(t)\|,\end{aligned}$$

where $\omega_T(f,\cdot)$ is the modulus of continuity of f on $[0,T]$, and we have $\omega_T(f, n^{-1}) \to 0$ since f is continuous (see, e.g., Jacod and Shiryaev [18, VI.1.6]). In a similar way,

$$\left\|f_n\left(\frac{j}{n}\right) - f_n\left(\frac{j-1}{n}\right)\right\| \leqslant \omega_T(f, n^{-1}) + 2\sup_{t\in[0,T]}\|f_n(t) - f(t)\|.$$

By (2.5),

$$\sum_{j=1}^{\lfloor nt\rfloor}\|\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor-j} - \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\| \leqslant \sum_{j=1}^{\lfloor nT\rfloor} c_{\boldsymbol{m}_{\boldsymbol{\xi}}} r_{\boldsymbol{m}_{\boldsymbol{\xi}}}^{\lfloor nt\rfloor-j} \leqslant \frac{c_{\boldsymbol{m}_{\boldsymbol{\xi}}}}{1 - r_{\boldsymbol{m}_{\boldsymbol{\xi}}}}.$$

Further,

$$\left\|\left(\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}-\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\right)f_n(0)\right\| \leqslant \left\|\left(\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}-\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\right)\left(f_n(0)-f(0)\right)\right\| + \left\|\left(\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}-\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\right)f(0)\right\|$$
$$\leqslant c_{\boldsymbol{m}_{\boldsymbol{\xi}}} \sup_{t\in[0,T]} \|f_n(t)-f(t)\|,$$

since $\left(\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}-\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\right)f(0)=\boldsymbol{0}$ for all $t\in\mathbb{R}_+$. Indeed, $\boldsymbol{m}_{\boldsymbol{\xi}}\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}} = \boldsymbol{m}_{\boldsymbol{\xi}} \lim_{n\to\infty}\boldsymbol{m}_{\boldsymbol{\xi}}^n = \lim_{n\to\infty}\boldsymbol{m}_{\boldsymbol{\xi}}^{n+1} = \boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}$ and $f(0)=\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}f(0)$ imply $\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}f(0)=\boldsymbol{m}_{\boldsymbol{\xi}}^{\lfloor nt\rfloor}\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}f(0)=\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}f(0)$. Thus we conclude $C\subset C_{\Psi,\,(\Psi_n)_{n\in\mathbb{N}}}$.

By the definition of a weak solution (see, e.g., Jacod and Shiryaev [18, Definition 2.24, Chapter III]), $\boldsymbol{\mathcal{M}}$ has almost sure continuous sample paths, so we have $\mathrm{P}(\boldsymbol{\mathcal{M}}\in C)=1$. Consequently, by Lemma A.4, we obtain $\boldsymbol{\mathcal{X}}^{(n)}=\Psi_n(\boldsymbol{\mathcal{M}}^{(n)})\stackrel{\mathcal{L}}{\longrightarrow}\Psi(\boldsymbol{\mathcal{M}})\stackrel{\mathcal{L}}{=}\boldsymbol{\mathcal{X}}$ as $n\to\infty$. □

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Appendix

In the proof of Theorem 3.1 we will use some facts about the first and second order moments of the sequences $(\boldsymbol{X}_k^{(n)})_{k\in\mathbb{Z}_+}$ and $(\boldsymbol{M}_k^{(n)})_{k\in\mathbb{N}}$.

Lemma A.1 *Under the assumptions of Theorem 3.1 we have for all* $k, n \in \mathbb{N}$

$$\operatorname{E}(\boldsymbol{X}_k^{(n)}) = \boldsymbol{m}_{\boldsymbol{\xi}}^k \operatorname{E}(\boldsymbol{X}_0^{(n)}) + \sum_{j=0}^{k-1} \boldsymbol{m}_{\boldsymbol{\xi}}^j \boldsymbol{m}_{\boldsymbol{\varepsilon}}, \tag{A.1}$$

$$\begin{aligned}\operatorname{Var}(\boldsymbol{X}_k^{(n)}) &= \sum_{j=0}^{k-1} \boldsymbol{m}_{\boldsymbol{\xi}}^j \left[\boldsymbol{V}_{\boldsymbol{\varepsilon}} + (\boldsymbol{m}_{\boldsymbol{\xi}}^{k-j-1} \operatorname{E}(\boldsymbol{X}_0^{(n)})) \odot \boldsymbol{V}_{\xi}\right] (\boldsymbol{m}_{\boldsymbol{\xi}}^\top)^j \\ &\quad + \boldsymbol{m}_{\boldsymbol{\xi}}^k (\operatorname{Var}(\boldsymbol{X}_0^{(n)}))(\boldsymbol{m}_{\boldsymbol{\xi}}^\top)^k + \sum_{j=0}^{k-2} \boldsymbol{m}_{\boldsymbol{\xi}}^j \sum_{\ell=0}^{k-j-2} \left[(\boldsymbol{m}_{\boldsymbol{\xi}}^\ell \boldsymbol{m}_{\boldsymbol{\varepsilon}}) \odot \boldsymbol{V}_{\xi}\right] (\boldsymbol{m}_{\boldsymbol{\xi}}^\top)^j.\end{aligned} \tag{A.2}$$

Moreover,

$$\operatorname{E}\big(\boldsymbol{M}_k^{(n)} \,\big|\, \mathcal{F}_{k-1}^{(n)}\big) = \boldsymbol{0} \qquad \text{for } k, n \in \mathbb{N}, \tag{A.3}$$

$$\operatorname{E}\big[\boldsymbol{M}_k^{(n)} (\boldsymbol{M}_\ell^{(n)})^\top \,\big|\, \mathcal{F}_{\max\{k,\ell\}-1}^{(n)}\big] = \begin{cases} \boldsymbol{V}_{\boldsymbol{\varepsilon}} + \boldsymbol{X}_{k-1}^{(n)} \odot \boldsymbol{V}_{\boldsymbol{\xi}} & \text{if } k = \ell, \\ \boldsymbol{0} & \text{if } k \neq \ell. \end{cases} \tag{A.4}$$

Further,

$$\operatorname{E}(\boldsymbol{M}_k^{(n)}) = \boldsymbol{0} \qquad \text{for } k \in \mathbb{N}, \tag{A.5}$$

$$\operatorname{E}\big[\boldsymbol{M}_k^{(n)} (\boldsymbol{M}_\ell^{(n)})^\top\big] = \begin{cases} \boldsymbol{V}_{\boldsymbol{\varepsilon}} + \operatorname{E}(\boldsymbol{X}_{k-1}^{(n)}) \odot \boldsymbol{V}_{\boldsymbol{\xi}} & \text{if } k = \ell, \\ \boldsymbol{0} & \text{if } k \neq \ell. \end{cases} \tag{A.6}$$

Proof. We have already proved (A.1), see (2.4). The equality $\boldsymbol{M}_k^{(n)} = \boldsymbol{X}_k^{(n)} - \operatorname{E}\big(\boldsymbol{X}_k^{(n)} \,\big|\, \mathcal{F}_{k-1}^{(n)}\big)$ clearly implies (A.3) and (A.5). By (2.1) and (3.3),

(A.7)

$$\boldsymbol{M}_k^{(n)} = \boldsymbol{X}_k^{(n)} - \sum_{i=1}^p X_{k-1,i}^{(n)} \operatorname{E}(\boldsymbol{\xi}_{1,1,i}^{(1)}) - \boldsymbol{m}_{\boldsymbol{\varepsilon}} = (\boldsymbol{\varepsilon}_k - \operatorname{E}(\boldsymbol{\varepsilon}_k)) + \sum_{i=1}^p \sum_{j=1}^{X_{k-1,i}^{(n)}} (\boldsymbol{\xi}_{k,j,i}^{(n)} - \operatorname{E}(\boldsymbol{\xi}_{k,j,i}^{(n)})).$$

For each $k, n \in \mathbb{N}$, the random vectors $\big\{\boldsymbol{\xi}_{k,j,i}^{(n)} - \operatorname{E}(\boldsymbol{\xi}_{k,j,i}^{(n)}),\, \boldsymbol{\varepsilon}_k^{(n)} - \operatorname{E}(\boldsymbol{\varepsilon}_k^{(n)}) : j \in \mathbb{N},\, i \in \{1, \ldots, p\}\big\}$ are independent of each others, independent of $\mathcal{F}_{k-1}^{(n)}$, and have zero mean, thus in case $k = \ell$ we conclude (A.4) and hence (A.6). If $k < \ell$ then $\mathrm{E}\big[\boldsymbol{M}_k^{(n)}(\boldsymbol{M}_\ell^{(n)})^\top \,\big|\, \mathcal{F}_{\ell-1}^{(n)}\big] = \boldsymbol{M}_k^{(n)} \mathrm{E}\big[(\boldsymbol{M}_\ell^{(n)})^\top \,\big|\, \mathcal{F}_{\ell-1}^{(n)}\big] = \boldsymbol{0}$ by (A.3), thus we obtain (A.4) and (A.6) in case $k \neq \ell$.

By (3.7) and (A.1), we conclude

$$\boldsymbol{X}_k^{(n)} - \mathrm{E}(\boldsymbol{X}_k^{(n)}) = \boldsymbol{m}_{\boldsymbol{\xi}}^k(\boldsymbol{X}_0^{(n)} - \mathrm{E}(\boldsymbol{X}_0^{(n)})) + \sum_{j=1}^{k} \boldsymbol{m}_{\boldsymbol{\xi}}^{k-j} \boldsymbol{M}_j^{(n)}.$$

Now by (A.6),

$$\begin{aligned}
\mathrm{Var}(\boldsymbol{X}_k^{(n)}) &= \boldsymbol{m}_{\boldsymbol{\xi}}^k \,\mathrm{E}\big[(\boldsymbol{X}_0^{(n)} - \mathrm{E}(\boldsymbol{X}_0^{(n)}))(\boldsymbol{X}_0^{(n)} - \mathrm{E}(\boldsymbol{X}_0^{(n)}))^\top\big](\boldsymbol{m}_{\boldsymbol{\xi}}^\top)^k \\
&\quad + \sum_{j=1}^{k}\sum_{\ell=1}^{k} \big(\boldsymbol{m}_{\boldsymbol{\xi}}^\top\big)^{k-j} \,\mathrm{E}\big[\boldsymbol{M}_j^n(\boldsymbol{M}_\ell^n)^\top\big]\, (\boldsymbol{m}_{\boldsymbol{\xi}})^{k-\ell} \\
&= \boldsymbol{m}_{\boldsymbol{\xi}}^k \,\mathrm{Var}(\boldsymbol{X}_0^{(n)})(\boldsymbol{m}_{\boldsymbol{\xi}}^\top)^k + \sum_{j=1}^{k} \boldsymbol{m}_{\boldsymbol{\xi}}^{k-j} \,\mathrm{E}\big[\boldsymbol{M}_j^{(n)}(\boldsymbol{M}_j^{(n)})^\top\big]\, (\boldsymbol{m}_{\boldsymbol{\xi}}^\top)^{k-j}.
\end{aligned}$$

Finally, using the expression in (A.6) for $\mathrm{E}\big[\boldsymbol{M}_j^{(n)}(\boldsymbol{M}_j^{(n)})^\top\big]$ we obtain (A.2). □

Lemma A.2 *Under the assumptions of Theorem 3.1 we have*

$$\mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|) = \mathrm{O}(k+n), \qquad \mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|^2) = \mathrm{O}((k+n)^2),$$
$$\mathrm{E}(\|\boldsymbol{M}_k^{(n)}\|) = \mathrm{O}((k+n)^{1/2}), \qquad \mathrm{E}(\|\boldsymbol{M}_k^{(n)}\|^4) = \mathrm{O}((k+n)^2).$$

Proof. By (A.1),

$$\|\,\mathrm{E}(\boldsymbol{X}_k^{(n)})\| \leqslant \|\boldsymbol{m}_{\boldsymbol{\xi}}^k\| \cdot \mathrm{E}(\|\boldsymbol{X}_0^{(n)}\|) + \sum_{j=0}^{k-1} \|\boldsymbol{m}_{\boldsymbol{\xi}}^j\| \cdot \|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\| \leqslant C_{\boldsymbol{m}_{\boldsymbol{\xi}}}(\sqrt{C}n + \|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\|k),$$

where

$$\text{(A.8)} \qquad C_{\boldsymbol{m}_{\boldsymbol{\xi}}} := \sup_{j\in\mathbb{Z}_+} \|\boldsymbol{m}_{\boldsymbol{\xi}}^j\| < \infty, \qquad C := \sup_{n\in\mathbb{N}} n^{-2}\, \mathrm{E}(\|\boldsymbol{X}_0^{(n)}\|^2) < \infty,$$

since (2.5) implies $C_{\boldsymbol{m}_{\boldsymbol{\xi}}} \leqslant c_{\boldsymbol{m}_{\boldsymbol{\xi}}} + \|\boldsymbol{\Pi}_{\boldsymbol{m}_{\boldsymbol{\xi}}}\|$. Hence, we obtain $\mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|) \leqslant p\|\mathrm{E}(\boldsymbol{X}_k^{(n)})\| = \mathrm{O}(k+n)$.

We have

$$\mathrm{E}(\|\boldsymbol{M}_k^{(n)}\|) \leqslant \sqrt{\mathrm{E}(\|\boldsymbol{M}_k^{(n)}\|^2)} = \sqrt{\mathrm{E}\big[\mathrm{tr}(\boldsymbol{M}_k^{(n)}(\boldsymbol{M}_k^{(n)})^\top)\big]} = \sqrt{\mathrm{tr}\big[\boldsymbol{V}_{\boldsymbol{\varepsilon}} + \mathrm{E}(\boldsymbol{X}_{k-1}^{(n)}) \odot \boldsymbol{V}_{\boldsymbol{\xi}}\big]}$$
$$\leqslant \sqrt{\mathrm{tr}(\boldsymbol{V}_{\boldsymbol{\varepsilon}})} + \sqrt{\mathrm{tr}\big[\mathrm{E}(\boldsymbol{X}_{k-1}^{(n)}) \odot \boldsymbol{V}_{\boldsymbol{\xi}}\big]},$$

hence we obtain $\mathrm{E}(\|\boldsymbol{M}_k^{(n)}\|) = \mathrm{O}((k+n)^{1/2})$ from $\mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|) = \mathrm{O}(k+n)$.

We have

$$\mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|^2) = \mathrm{E}\big[\mathrm{tr}(\boldsymbol{X}_k^{(n)}(\boldsymbol{X}_k^{(n)})^\top)\big] = \mathrm{tr}(\mathrm{Var}(\boldsymbol{X}_k^{(n)})) + \mathrm{tr}\big[\mathrm{E}(\boldsymbol{X}_k^{(n)})\,\mathrm{E}(\boldsymbol{X}_k^{(n)})^\top\big],$$

where $\mathrm{tr}\big[\mathrm{E}(\boldsymbol{X}_k^{(n)})\,\mathrm{E}(\boldsymbol{X}_k^{(n)})^\top\big] = \|\mathrm{E}(\boldsymbol{X}_k^{(n)})\|^2 \leqslant \big[\mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|)\big]^2 = \mathrm{O}((k+n)^2)$. Moreover, $\mathrm{tr}(\mathrm{Var}(\boldsymbol{X}_k^{(n)})) = \mathrm{O}((k+n)^2)$. Indeed, by (A.2) and (A.8),

$$\begin{aligned}\|\mathrm{Var}(\boldsymbol{X}_k^{(n)})\| &\leqslant \sum_{j=0}^{k-1}\Big(\|\boldsymbol{V}_{\boldsymbol{\varepsilon}}\| + \|\boldsymbol{V}_{\boldsymbol{\xi}}\| \cdot \|\boldsymbol{m}_{\boldsymbol{\xi}}^{k-j-1}\| \cdot \mathrm{E}(\|\boldsymbol{X}_0^{(n)}\|)\Big)\|\boldsymbol{m}_{\boldsymbol{\xi}}^j\|^2 \\ &\quad + \|\mathrm{Var}(X_0^{(n)})\| \cdot \|\boldsymbol{m}_{\boldsymbol{\xi}}^k\|^2 + \|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\| \cdot \|\boldsymbol{V}_{\boldsymbol{\xi}}\| \sum_{j=0}^{k-2}\|\boldsymbol{m}_{\boldsymbol{\xi}}^j\|^2 \sum_{\ell=0}^{k-j-2}\|\boldsymbol{m}_{\boldsymbol{\xi}}^\ell\| \\ &\leqslant \Big(\|\boldsymbol{V}_{\boldsymbol{\varepsilon}}\| + C_{\boldsymbol{m}_{\boldsymbol{\xi}}}\|\boldsymbol{V}_{\boldsymbol{\xi}}\| \cdot \mathrm{E}(\|\boldsymbol{X}_0^{(n)}\|)\Big) C_{\boldsymbol{m}_{\boldsymbol{\xi}}}^2 k \\ &\quad + \big(\mathrm{E}(\|\boldsymbol{X}_0^{(n)}\|^2) + \big[\mathrm{E}(\|X_0^n\|)\big]^2\big)C_{\boldsymbol{m}_{\boldsymbol{\xi}}}^2 + C_{\boldsymbol{m}_{\boldsymbol{\xi}}}^3\|\boldsymbol{m}_{\boldsymbol{\varepsilon}}\| \cdot \|\boldsymbol{V}_{\boldsymbol{\xi}}\| k^2,\end{aligned}$$

where $\|\boldsymbol{V}_{\boldsymbol{\xi}}\| := \sum_{i=1}^p \|\boldsymbol{V}_{\boldsymbol{\xi}_i}\|$, hence we obtain $\mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|^2) = O((k+n)^2)$.

By (A.7),

$$\|\boldsymbol{M}_k^{(n)}\| \leqslant \|\boldsymbol{\varepsilon}_k^{(n)} - \mathrm{E}(\boldsymbol{\varepsilon}_k^{(n)})\| + \sum_{i=1}^p \Bigg\|\sum_{j=1}^{X_{k-1,i}^{(n)}} (\boldsymbol{\xi}_{k,j,i}^{(n)} - \mathrm{E}(\boldsymbol{\xi}_{k,j,i}^{(n)}))\Bigg\|,$$

hence

$$\mathrm{E}(\|\boldsymbol{M}_k^{(n)}\|^4) \leqslant (p+1)^3\,\mathrm{E}(\|\boldsymbol{\varepsilon}_1^{(1)} - \mathrm{E}(\boldsymbol{\varepsilon}_1^{(1)})\|^4) + (p+1)^3 \sum_{i=1}^{p} \mathrm{E}\left(\left\| \sum_{j=1}^{X_{k-1,i}^{(n)}} (\boldsymbol{\xi}_{k,j,i}^{(n)} - \mathrm{E}(\boldsymbol{\xi}_{k,j,i}^{(n)})) \right\|^4\right).$$

Here

$$\begin{aligned}
\mathrm{E}\left(\left\| \sum_{j=1}^{X_{k-1,i}^{(n)}} (\boldsymbol{\xi}_{k,j,i}^{(n)} - \mathrm{E}(\boldsymbol{\xi}_{k,j,i}^{(n)})) \right\|^4\right) &= \mathrm{E}\left[\left(\sum_{\ell=1}^{p}\left(\sum_{j=1}^{X_{k-1,i}^{(n)}} (\xi_{k,j,i,\ell}^{(n)} - \mathrm{E}(\xi_{k,j,i,\ell}^{(n)}))\right)^2\right)^2\right] \\
&\leqslant p \sum_{\ell=1}^{p} \mathrm{E}\left[\left(\sum_{j=1}^{X_{k-1,i}^{(n)}} (\xi_{k,j,i,\ell}^{(n)} - \mathrm{E}(\xi_{k,j,i,\ell}^{(n)}))\right)^4\right],
\end{aligned}$$

where

$$\begin{aligned}
&\mathrm{E}\left[\left(\sum_{j=1}^{X_{k-1,i}^{(n)}} (\xi_{k,j,i,\ell} - \mathrm{E}(\xi_{k,j,i,\ell}))\right)^4 \,\middle|\, \mathcal{F}_{k-1}^{(n)}\right] \\
&\qquad = X_{k-1,i}^{(n)}\,\mathrm{E}[(\xi_{1,1,i,\ell}^{(1)} - \mathrm{E}(\xi_{1,1,i,\ell}^{(1)}))^4] + X_{k-1,i}^{(n)}(X_{k-1,i}^{(n)} - 1)\big(\mathrm{E}[(\xi_{1,1,i,\ell}^{(1)} - \mathrm{E}(\xi_{1,1,i,\ell}^{(1)}))^2]\big)^2
\end{aligned}$$

with $\big(\mathrm{E}[(\xi_{1,1,i,\ell}^{(1)} - \mathrm{E}(\xi_{1,1,i,\ell}^{(1)}))^2]\big)^2 \leqslant \mathrm{E}[(\xi_{1,1,i,\ell}^{(1)} - \mathrm{E}(\xi_{1,1,i,\ell}^{(1)}))^4]$, hence

$$\mathrm{E}\left[\left(\sum_{j=1}^{X_{k-1,i}^{(n)}} (\xi_{k,j,i,\ell}^{(n)} - \mathrm{E}(\xi_{k,j,i,\ell}^{(n)}))\right)^4\right] \leqslant \mathrm{E}[(\xi_{1,1,i,\ell}^{(1)} - \mathrm{E}(\xi_{1,1,i,\ell}^{(1)}))^4]\,\mathrm{E}[(X_{k-1,i}^{(n)})^2]$$

Consequently, $\mathrm{E}(\|\boldsymbol{X}_k^{(n)}\|^2) = \mathrm{O}((k+n)^2)$ implies $\mathrm{E}(\|\boldsymbol{M}_k^{(n)}\|^4) = \mathrm{O}((k+n)^2)$. □

Next we recall a result about convergence of random step processes towards a diffusion process, see Ispány and Pap [22, Corollary 2.2].

Theorem A.3 *Let* $\boldsymbol{\gamma} : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^{p\times r}$ *be a continuous function. Assume that uniqueness in the sense of probability law holds for the SDE*

(A.9) $$\mathrm{d}\boldsymbol{\mathcal{U}}_t = \gamma(t, \boldsymbol{\mathcal{U}}_t)\,\mathrm{d}\boldsymbol{\mathcal{W}}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\boldsymbol{\mathcal{U}}_0 = \boldsymbol{u}_0$ *for all* $\boldsymbol{u}_0 \in \mathbb{R}^p$, *where* $(\boldsymbol{\mathcal{W}}_t)_{t\in\mathbb{R}_+}$ *is an* r*-dimensional standard Wiener process. Let* $\boldsymbol{\mu}$ *be a probability measure on* $(\mathbb{R}^p, \mathcal{B}((\mathbb{R}^p))$, *and let* $(\boldsymbol{\mathcal{U}}_t)_{t\in\mathbb{R}_+}$ *be a solution of* (A.9) *with initial distribution* $\boldsymbol{\mu}$.

For each $n \in \mathbb{N}$, *let* $(\boldsymbol{U}_k^{(n)})_{k\in\mathbb{Z}_+}$ *be a sequence of* p*-dimensional martingale differences with respect to a filtration* $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$. *Let*

$$\boldsymbol{\mathcal{U}}_t^{(n)} := \sum_{k=0}^{\lfloor nt\rfloor} \boldsymbol{U}_k^{(n)}, \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Suppose $\operatorname{E}\big(\|\boldsymbol{U}_k^{(n)}\|^2\big) < \infty$ *for all* $n, k \in \mathbb{N}$, *and* $\boldsymbol{U}_0^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \boldsymbol{\mu}$. *Suppose that, for each* $T > 0$,

(i) $\sup\limits_{t\in[0,T]} \left\| \sum\limits_{k=1}^{\lfloor nt\rfloor} \operatorname{E}\Big[\boldsymbol{U}_k^{(n)}(\boldsymbol{U}_k^{(n)})^\top \mid \mathcal{F}_{k-1}^{(n)}\Big] - \int_0^t \boldsymbol{\gamma}(s, \boldsymbol{\mathcal{U}}_s^{(n)})\boldsymbol{\gamma}(s, \boldsymbol{\mathcal{U}}_s^{(n)})^\top \mathrm{d}s \right\| \stackrel{\mathrm{P}}{\longrightarrow} 0,$

(ii) $\sum\limits_{k=1}^{\lfloor nT\rfloor} \operatorname{E}\big(\|\boldsymbol{U}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{U}_k^{(n)}\|>\theta\}} \,\big|\, \mathcal{F}_{k-1}^{(n)}\big) \stackrel{\mathrm{P}}{\longrightarrow} 0$ *for all* $\theta > 0$,

where $\stackrel{\mathrm{P}}{\longrightarrow}$ *denotes convergence in probability. Then* $\boldsymbol{\mathcal{U}}^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \boldsymbol{\mathcal{U}}$ *as* $n \to \infty$.

Now we recall a version of the continuous mapping theorem.

For functions f and f_n, $n \in \mathbb{N}$, in $\mathsf{D}(\mathbb{R}_+, \mathbb{R}^p)$, we write $f_n \stackrel{\mathrm{lu}}{\longrightarrow} f$ if $(f_n)_{n\in\mathbb{N}}$ converges to f locally uniformly, i.e., if $\sup_{t\in[0,T]} \|f_n(t) - f(t)\| \to 0$ as $n \to \infty$ for all $T > 0$. For measurable mappings $\Phi : \mathsf{D}(\mathbb{R}_+, \mathbb{R}^p) \to \mathsf{D}(\mathbb{R}_+, \mathbb{R}^q)$ and $\Phi_n : \mathsf{D}(\mathbb{R}_+, \mathbb{R}^p) \to \mathsf{D}(\mathbb{R}_+, \mathbb{R}^q)$, $n \in \mathbb{N}$, we will denote by $C_{\Phi,(\Phi_n)_{n\in\mathbb{N}}}$ the set of all functions $f \in \mathsf{C}(\mathbb{R}_+, \mathbb{R}^p)$ for which $\Phi_n(f_n) \to \Phi(f)$ whenever $f_n \stackrel{\mathrm{lu}}{\longrightarrow} f$ with $f_n \in \mathsf{D}(\mathbb{R}_+, \mathbb{R}^p)$, $n \in \mathbb{N}$.

Lemma A.4 *Let* $(\boldsymbol{\mathcal{U}}_t)_{t\in\mathbb{R}_+}$ *and* $(\boldsymbol{\mathcal{U}}_t^{(n)})_{t\in\mathbb{R}_+}$, $n \in \mathbb{N}$, *be* $\mathbb{R}^p$*-valued stochastic processes with càdlàg paths such that* $\boldsymbol{\mathcal{U}}^{(n)} \stackrel{\mathcal{L}}{\longrightarrow} \boldsymbol{\mathcal{U}}$. *Let* $\Phi : \mathsf{D}(\mathbb{R}_+, \mathbb{R}^p) \to \mathsf{D}(\mathbb{R}_+, \mathbb{R}^q)$ *and* $\Phi_n : \mathsf{D}(\mathbb{R}_+, \mathbb{R}^p) \to \mathsf{D}(\mathbb{R}_+, \mathbb{R}^q)$, $n \in \mathbb{N}$, *be measurable mappings such that there exists*

$C \subset C_{\Phi,(\Phi_n)_{n\in\mathbb{N}}}$ *with* $C \in \mathcal{D}_\infty(\mathbb{R}_+, \mathbb{R}^p)$ *and* $\mathrm{P}(\boldsymbol{\mathcal{U}} \in C) = 1$. *Then* $\Phi_n(\boldsymbol{\mathcal{U}}^{(n)}) \stackrel{\mathcal{L}}{\longrightarrow} \Phi(\boldsymbol{\mathcal{U}})$.

Lemma A.4 can be considered as a consequence of Theorem 3.27 in Kallenberg [23], and we note that a proof of this lemma can also be found in Ispány and Pap [22, Lemma 3.1].

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