Testing stability in a spatial unilateral autoregressive model

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Abstract

Least squares estimator of the stability parameter $\varrho := |\alpha| + |\beta|$ for a spatial unilateral autoregressive process $X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} + \varepsilon_{k,\ell}$ is investigated and asymptotic normality with a scaling factor $n^{5/4}$ is shown in the unstable case $\varrho = 1$. The result is in contrast to the unit root case of the AR(p) model $X_k = \alpha_1 X_{k-1} + \cdots + \alpha_p X_{k-p} + \varepsilon_k$, where the limiting distribution of the least squares estimator of the unit root parameter $\varrho := \alpha_1 + \cdots + \alpha_p$ is not normal.

Keywords Spatial unilateral autoregressive process; unstable model; unit root test.

Mathematics Subject Classification Primary 62M10; Secondary 62F12.

1 Introduction

Consider a spatial unilateral autoregressive process $\{X_{k,\ell} : k, \ell \in \mathbb{Z},\ k + \ell \geq 0\}$ defined by

$$X_{k,\ell} = \begin{cases} \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} + \varepsilon_{k,\ell}, & \text{for} \quad k + \ell \geq 1, \\ 0, & \text{for} \quad k + \ell = 0, \end{cases} \tag{1.1}$$

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where $\{\varepsilon_{k,\ell} : k, \ell \in \mathbb{Z},\, k+\ell \geq 1\}$ are independent random variables with $\mathsf{E}(\varepsilon_{k,\ell}) = 0$ and $\operatorname{Var}(\varepsilon_{k,\ell}) = 1$. This model is stable (asymptotically stationary) in case of $|\alpha| + |\beta| < 1$, unstable if $|\alpha| + |\beta| = 1$, and explosive if $|\alpha| + |\beta| > 1$ [see Whittle, 1954, Besag, 1972, Basu and Reinsel, 1993], hence $\varrho := |\alpha| + |\beta|$ can be considered as a *stability parameter*. The above classification is based on the rate of growth of $\operatorname{Var}(X_{k,\ell})$ as $k+\ell \to \infty$ with $k/\ell \to \text{const} > 0$, namely

$$\operatorname{Var}(X_{k,\ell}) \simeq \begin{cases} 1 & \text{if } |\alpha| + |\beta| < 1; \\ \sqrt{k+\ell} & \text{if } |\alpha| + |\beta| = 1,\ |\alpha| \notin \{0,1\}; \\ k+\ell & \text{if } |\alpha| + |\beta| = 1,\ |\alpha| \in \{0,1\}; \\ (|\alpha| + |\beta|)^{2(k+\ell)}/\sqrt{k+\ell} & \text{if } |\alpha| + |\beta| > 1, \end{cases}$$

where $a_n \simeq b_n$ means $\liminf_{n\to\infty} a_n b_n^{-1} > 0$ and $\limsup_{n\to\infty} a_n b_n^{-1} < \infty$. We remark that the last statement can be derived e.g. using normal approximation given in Baran et al. [2007, Theorem 2.4]. A detailed investigation of the exact asymptotic behaviour of the variances has been given by Paulauskas [2007].

For a finite set $H \subset \{(k,\ell) \in \mathbb{Z}^2 : k+\ell \geq 1\}$, the least squares estimator (LSE) $(\widehat{\alpha}^*_H, \widehat{\beta}^*_H)$ of the coefficients (α, β) based on the observations $\{X_{k,\ell} : (k,\ell) \in H\}$ can be obtained by minimizing the sum of squares

$$\sum_{(k,\ell)\in H} \big(X_{k,\ell} - \alpha X_{k-1,\ell} - \beta X_{k,\ell-1}\big)^2$$

with respect to α and β, and it has the form

$$\begin{bmatrix} \widehat{\alpha}^*_H \\ \widehat{\beta}^*_H \end{bmatrix} = (A^*_H)^{-1} b^*_H, \quad \text{where} \quad A^*_H := \sum_{(k,\ell)\in H} \begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \end{bmatrix} \begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \end{bmatrix}^\top, \quad b^*_H := \sum_{(k,\ell)\in H} X_{k,\ell} \begin{bmatrix} X_{k-1,\ell} \\ X_{k,\ell-1} \end{bmatrix}.$$

Model (1.1) has been studied in details by several authors. Baran et al. [2004] investigated the special case $\alpha = \beta$ and verified the asymptotic normality of the LSE of the unknown parameter in cases $|\alpha| < 1/2$ and $|\alpha| = 1/2$, corresponding to stable and unstable models, respectively. Later Paulauskas [2007] determined the exact asymptotic behaviour of the variances of the process (1.1), while in Baran et al. [2007] the asymptotic normality of the LSE of the coefficients (α, β) is proved both in stable and unstable cases. Finally, in Baran and Pap [2012] the more general Pickard model is investigated [Tory and Pickard, 1992], where a third parameter appears controlling the dependence of $X_{k,\ell}$ on $X_{k-1,\ell-1}$. Pickard processes, including also model (1.1), have important applications e.g. in agricultural trials [Martin, 1990, Cullis and Gleeson, 1991, Basu and Reinsel, 1994] or in image processing [Bustos et al., 2009, Vallejos and Ojeda, 2012].

The limiting behavior of the LSE of the stability parameter ϱ has not been treated yet, this gives the novelty of the current work. This parameter has a great importance since the asymptotic behaviour of the LSE of this quantity can be applied for testing stability of model (1.1).

We remark that a similar unit root parameter is well investigated in case of unstable AR(p) processes [see e.g. Hamilton, 1994, Section 17]. For example, in case of an AR(p) process $Y_k = \alpha_1 Y_{k-1} + \cdots + \alpha_p Y_{k-p} + \zeta_k$, $k \in \mathbb{N}$, with $Y_0 := 0$ and an i.i.d. sequence $\{\zeta_k : k \in \mathbb{N}\}$ having zero mean and positive variance, the LSE of the parameter $\varrho := \alpha_1 + \cdots + \alpha_p$ based on a sample $\{Y_1, \dots, Y_n\}$ takes the form $\widehat{\varrho}_n = \widehat{\alpha}_{1,n} + \cdots + \widehat{\alpha}_{p,n}$, where

$$\begin{bmatrix} \widehat{\alpha}_{1,n} \\ \vdots \\ \widehat{\alpha}_{p,n} \end{bmatrix} = \left(\sum_{k=1}^{n} \begin{bmatrix} Y_{k-1} \\ \vdots \\ Y_{k-p} \end{bmatrix} \begin{bmatrix} Y_{k-1} \\ \vdots \\ Y_{k-p} \end{bmatrix}^\top \right)^{-1} \sum_{k=1}^{n} Y_k \begin{bmatrix} Y_{k-1} \\ \vdots \\ Y_{k-p} \end{bmatrix}, \qquad n \in \mathbb{N},$$

and in the unit root case $\varrho = 1$ we have

$$n(\widehat{\varrho}_n - 1) \stackrel{\mathcal{D}}{\longrightarrow} \sum_{j=1}^{p} j\alpha_j \frac{\int_0^1 \mathcal{W}_t \mathrm{d}\mathcal{W}_t}{\int_0^1 \mathcal{W}_t^2 \mathrm{d}t} \qquad \text{as} \ \ n \to \infty,$$

where $(\mathcal{W}_t)_{t \geq 0}$ is a standard Wiener process [it can be proved in the same way as (17.7.25) in Hamilton, 1994].

For unstable spatial unilateral autoregressive processes the situation is different, since the LSE of the stability parameter turns out to be asymptotically normal, see Corollary 1.3. However, we should remark that this result cannot be derived from the asymptotic behaviour of the estimators of the original parameters α and β given in Baran et al. [2007, Theorem 1.1], since the corresponding limiting distribution is a degenerated two-dimensional normal which is concentrated on a one-dimensional hyperplane. A similar problem arises when one tries to apply the more general result of Baran and Pap [2012, Theorem 1.1].

With the help of the stability parameter ϱ model (1.1) can also be written in the form

$$X_{k,\ell} = \begin{cases} \alpha\big(X_{k-1,\ell} - \operatorname{sign}(\alpha\beta)X_{k,\ell-1}\big) + \varrho\operatorname{sign}(\beta)X_{k,\ell-1} + \varepsilon_{k,\ell}, & \text{for} \quad k+\ell \geq 1, \\ 0, & \text{for} \quad k+\ell = 0. \end{cases} \tag{1.2}$$

This reparametrization can be called the canonical form of Sims et al. [1990] [see also Hamilton, 1994, 17.7.6]. Observe that (1.2) gives four different models according to the signs of α and β. Hence, in order to derive estimators of the parameters (α, ϱ) one should have information about these signs. Moreover, case $\alpha\beta = 0$, when the limiting behaviour of the LSE of (α, β) is different, should be excluded from investigations.

For a set $H \subset \{(k,\ell) \in \mathbb{Z}^2 : k+\ell \geq 1\}$, the least squares estimator $(\widehat{\alpha}_H, \widehat{\varrho}_H)$ of (α, ϱ) based on the observations $\{X_{k,\ell} : (k,\ell) \in H\}$ can be obtained by minimizing the sum of squares

$$\sum_{(k,\ell)\in H} \Big[X_{k,\ell} - \alpha\big(X_{k-1,\ell} - \operatorname{sign}(\alpha\beta)X_{k,\ell-1}\big) - \varrho\operatorname{sign}(\beta)X_{k,\ell-1}\Big]^2$$

with respect to α and ϱ, and it has the form

$$\begin{bmatrix} \widehat{\alpha}_H \\ \widehat{\varrho}_H \end{bmatrix} = A_H^{-1} b_H,$$

where

$$A_H := \sum_{(k,\ell)\in H} \begin{bmatrix} X_{k-1,\ell} - \operatorname{sign}(\alpha\beta)X_{k,\ell-1} \\ \operatorname{sign}(\beta)X_{k,\ell-1} \end{bmatrix} \begin{bmatrix} X_{k-1,\ell} - \operatorname{sign}(\alpha\beta)X_{k,\ell-1} \\ \operatorname{sign}(\beta)X_{k,\ell-1} \end{bmatrix}^\top = BA_H^* B^\top,$$

$$b_H := \sum_{(k,\ell)\in H} X_{k,\ell} \begin{bmatrix} X_{k-1,\ell} - \operatorname{sign}(\alpha\beta)X_{k,\ell-1} \\ \operatorname{sign}(\beta)X_{k,\ell-1} \end{bmatrix} = Bb_H^*, \qquad \text{with} \qquad B := \begin{bmatrix} 1 & -\operatorname{sign}(\alpha\beta) \\ 0 & \operatorname{sign}(\beta) \end{bmatrix}.$$

Obviously, this estimator is well defined if $\alpha\beta \neq 0$ and then we have

$$\begin{bmatrix} \widehat{\alpha}_H \\ \widehat{\varrho}_H \end{bmatrix} = \big(B^\top\big)^{-1} \begin{bmatrix} \widehat{\alpha}_H^* \\ \widehat{\beta}_H^* \end{bmatrix}.$$

Now, let us define an estimator of β by $\widehat{\beta}_H := \big(\widehat{\varrho}_H - \operatorname{sign}(\alpha)\widehat{\alpha}_H\big)\operatorname{sign}(\beta)$. Short calculation shows that $\widehat{\alpha}_H = \widehat{\alpha}_H^*$ and $\widehat{\beta}_H = \widehat{\beta}_H^*$.

For $k, \ell \in \mathbb{Z}$ with $k + \ell \geq 1$, consider the triangle

$$T_{k,\ell} := \{(i,j) \in \mathbb{Z}^2 : i + j \geq 1, \ i \leq k \ \text{ and } \ j \leq \ell\}.$$

For simplicity, we shall write $T_n := T_{n,n}$ for $n \in \mathbb{N}$. Concerning the asymptotic behaviour of the LSE of parameters of model (1.2) in the unstable case one can formulate the following theorem.

Theorem 1.1 *Let* $\{\varepsilon_{k,\ell} : k, \ell \in \mathbb{Z}, \, k + \ell \geq 1\}$ *be independent random variables with* $\mathsf{E}(\varepsilon_{k,\ell}) = 0$, $\operatorname{Var}(\varepsilon_{k,\ell}) = 1$ *and* $\sup\{\mathsf{E}(\varepsilon_{k,\ell}^4) : k, \ell \in \mathbb{Z}, \, k + \ell \geq 1\} < \infty$. *If* $|\alpha| + |\beta| = 1$ *and* $\alpha\beta \neq 0$ *then*

$$\begin{bmatrix} (nm)^{1/2}(\widehat{\alpha}_{T_{n,m}} - \alpha) \\ (nm)^{5/8}(\widehat{\varrho}_{T_{n,m}} - 1) \end{bmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}\big(0, \Sigma_\alpha\big) \qquad \text{as } \ n, m \to \infty \ \text{ with } \ m/n \to \text{const} > 0, \tag{1.3}$$

where

$$\Sigma_\alpha := \begin{bmatrix} \varphi_\alpha & 0 \\ 0 & \psi_\alpha \end{bmatrix} \qquad \text{with} \qquad \varphi_\alpha := \frac{|\alpha|(1-|\alpha|)}{2} \quad \text{and} \quad \psi_\alpha := \frac{15\sqrt{\pi|\alpha|(1-|\alpha|)}}{2^{9/2}}.$$

Remark 1.2 Observe that (1.3) implies $(nm)^{1/2}(\widehat{\varrho}_{T_{n,m}} - 1) \stackrel{\mathsf{P}}{\longrightarrow} 0$, hence by

$$\begin{aligned}\begin{bmatrix} (nm)^{1/2}(\widehat{\alpha}_{T_{n,m}} - \alpha) \\ (nm)^{1/2}(\widehat{\beta}_{T_{n,m}} - \beta) \end{bmatrix} &= \begin{bmatrix} (nm)^{1/2}(\widehat{\alpha}_{T_{n,m}} - \alpha) \\ \operatorname{sign}(\beta)\big((nm)^{1/2}(\widehat{\varrho}_{T_{n,m}} - 1) - \operatorname{sign}(\alpha)(nm)^{1/2}(\widehat{\alpha}_{T_{n,m}} - \alpha)\big) \end{bmatrix} \\ &= (nm)^{1/2}(\widehat{\alpha}_{T_{n,m}} - \alpha) \begin{bmatrix} 1 \\ -\operatorname{sign}(\alpha\beta) \end{bmatrix} + (nm)^{1/2}(\widehat{\varrho}_{T_{n,m}} - 1) \begin{bmatrix} 0 \\ \operatorname{sign}(\beta) \end{bmatrix},\end{aligned}$$

as a byproduct, we obtain

$$\begin{bmatrix} (nm)^{1/2}(\widehat{\alpha}_{T_{n,m}} - \alpha) \\ (nm)^{1/2}(\widehat{\beta}_{T_{n,m}} - \beta) \end{bmatrix} \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \varphi_\alpha \begin{bmatrix} 1 & -\operatorname{sign}(\alpha\beta) \\ -\operatorname{sign}(\alpha\beta) & 1 \end{bmatrix}\right),$$

as $n, m \to \infty$ with $m/n \to \text{const} > 0$, which has already been proved in Baran et al. [2007], but with a far more complicated method than here.

Now, the limiting distribution of the stability parameter in the unstable case can directly be derived from Theorem 1.1.

Corollary 1.3 *Let* $\{\varepsilon_{k,\ell} : k,\ell \in \mathbb{Z},\, k+\ell \geq 1\}$ *be independent random variables with* $\mathsf{E}(\varepsilon_{k,\ell}) = 0$, $\operatorname{Var}(\varepsilon_{k,\ell}) = 1$ *and* $\sup\{\mathsf{E}(\varepsilon_{k,\ell}^4) : k,\ell \in \mathbb{Z},\, k+\ell \geq 1\} < \infty$. *If* $|\alpha| + |\beta| = 1$ *and* $\alpha\beta \neq 0$ *then*

$$(nm)^{5/8}(\widehat{\varrho}_{T_{n,m}} - 1) \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, 15\sqrt{\pi|\alpha|(1-|\alpha|)/2^9}\Big)$$

as $n,m \to \infty$ *with* $m/n \to \mathrm{const} > 0$.

Remark 1.4 In the stable case $|\alpha| + |\beta| < 1$ and $\alpha\beta \neq 0$ the asymptotic behaviour of $\widehat{\varrho}_{T_n}$ is a direct consequence of the first statement of Theorem 1.1 of Baran et al. [2007], namely we have

$$(nm)^{1/2}\big(\widehat{\varrho}_{T_{n,m}} - |\alpha| - |\beta|\big) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \big(1 + \kappa_{\alpha,\beta}\operatorname{sign}(\alpha\beta)\big)^{-1}\sigma_{\alpha,\beta}^{-2}\right),$$

as $n,m \to \infty$ with $m/n \to \mathrm{const} > 0$, where

$$\sigma_{\alpha,\beta}^2 := \big((1+\alpha+\beta)(1+\alpha-\beta)(1-\alpha+\beta)(1-\alpha-\beta)\big)^{-1/2},$$

$$\kappa_{\alpha,\beta} := \frac{(1-\alpha^2-\beta^2)\sigma_{\alpha,\beta}^2 - 1}{2\alpha\beta\sigma_{\alpha,\beta}^2}.$$

Remark 1.5 Instead of the triangular domain of observations $T_{n,m}$ one can also consider rectangles of form $\{(k,\ell) \in \mathbb{Z}^2 : 0 \leq k \leq n,\ 0 \leq \ell \leq m\}$. The corresponding results differ only in the constants of the limiting distributions, the rates of convergence remain the same. However, we prefer the use of triangular domains because it allows a direct application of the results of Baran et al. [2007]. It is still an open question, whether one can get rid of condition $m/n \to \mathrm{const} > 0$ and consider e.g. $\min\{m,n\} \to \infty$ instead [see e.g. Davydov and Paulauskas, 2008]. Unfortunately, the method presented here cannot be used under such condition, because the Martingale Central Limit Theorem applied in the proof of Proposition 1.7 does not allow this generalization.

For the sake of simplicity, we carry out the proof of Theorem 1.1 only for $m = n$. The general case can be handled with slight modifications. Let us write

$$\begin{bmatrix} \widehat{\alpha}_{T_n} - \alpha \\ \widehat{\varrho}_{T_n} - 1 \end{bmatrix} = A_{T_n}^{-1} d_{T_n}, \qquad \text{where} \qquad d_{T_n} := \sum_{(k,\ell)\in T_n} \varepsilon_{k,\ell} \begin{bmatrix} X_{k-1,\ell} - \operatorname{sign}(\alpha\beta) X_{k,\ell-1} \\ \operatorname{sign}(\beta) X_{k,\ell-1} \end{bmatrix},$$

and by the continuous mapping theorem (1.3) is a consequence of the convergence

$$\big(\widetilde{A}_{T_n}, \widetilde{d}_{T_n}\big) := \left(\begin{bmatrix} n^{-1} & 0 \\ 0 & n^{-5/4} \end{bmatrix} A_{T_n} \begin{bmatrix} n^{-1} & 0 \\ 0 & n^{-5/4} \end{bmatrix}, \begin{bmatrix} n^{-1} & 0 \\ 0 & n^{-5/4} \end{bmatrix} d_{T_n} \right) \stackrel{\mathcal{D}}{\longrightarrow} \big(\widetilde{A}, \widetilde{d}\big) \tag{1.4}$$

as $n \to \infty$, with

$$\widetilde{A} := \begin{bmatrix} 1/\varphi_\alpha & 0 \\ 0 & 1/\psi_\alpha \end{bmatrix} \qquad \text{and} \qquad \widetilde{d} \stackrel{\mathcal{D}}{=} \mathcal{N}(0, \widetilde{A}).$$

Obviously, (1.4) can be verified by proving the following two propositions.

Proposition 1.6 *Under the conditions of Theorem 1.1*

$$\widetilde{A}_{T_n} \stackrel{\mathsf{P}}{\longrightarrow} \widetilde{A} \qquad \text{as } n \to \infty. \tag{1.5}$$

Proposition 1.7 *Under the conditions of Theorem 1.1*

$$\widetilde{d}_{T_n} \stackrel{\mathcal{D}}{\longrightarrow} \mathcal{N}(0, \widetilde{A}) \qquad \text{as } n \to \infty.$$

The aim of the following discussion is to show that it suffices to prove Propositions 1.6 and 1.7 for $\alpha > 0$ and $\beta > 0$ implying $\varrho = \alpha + \beta$. In this case we have

$$\widetilde{A}_{T_n} = \sum_{(k,\ell)\in T_n} \begin{bmatrix} n^{-2}\big(X_{k-1,\ell} - X_{k,\ell-1}\big)^2 & n^{-9/4}\big(X_{k-1,\ell} - X_{k,\ell-1}\big)X_{k,\ell-1} \\ n^{-9/4}\big(X_{k-1,\ell} - X_{k,\ell-1}\big)X_{k,\ell-1} & n^{-5/2}X_{k,\ell-1}^2 \end{bmatrix}, \tag{1.6}$$

$$\widetilde{d}_{T_n} = \sum_{(k,\ell)\in T_n} \begin{bmatrix} n^{-1}\big(X_{k-1,\ell} - X_{k,\ell-1}\big)\varepsilon_{k,\ell} \\ n^{-5/4}X_{k,\ell-1}\varepsilon_{k,\ell} \end{bmatrix}. \tag{1.7}$$

Model equation (1.1) implies that random variable $X_{k,\ell}$ can be expressed as a linear combination of the variables $\{\varepsilon_{i,j} : (i,j) \in T_{k,\ell}\}$, namely,

$$X_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \binom{k+\ell-i-j}{k-i} \alpha^{k-i}\beta^{\ell-j}\varepsilon_{i,j} \tag{1.8}$$

for $k, \ell \in \mathbb{Z}$ with $k + \ell \geq 1$. If $\alpha + \beta = 1$ we can also write

$$X_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \mathsf{P}\big(S^{(\alpha)}_{k+\ell-i-j} = k - i\big)\varepsilon_{i,j}, \tag{1.9}$$

where $S^{(\alpha)}_n$ is a binomial random variable with parameters (n, α).

Let $\alpha < 0$, $\beta < 0$ implying $\varrho = -\alpha - \beta$ and put $\varepsilon^*_{k,\ell} := (-1)^{k+\ell}\varepsilon_{k,\ell}$ for $k, \ell \in \mathbb{Z}$ with $k + \ell \geq 1$. Then $\{\varepsilon^*_{k,\ell} : k, \ell \in \mathbb{Z},\, k + \ell \geq 1\}$ are independent random variables with $\mathsf{E}\,(\varepsilon^*_{k,\ell}) = 0$, and $\operatorname{Var}(\varepsilon^*_{k,\ell}) = 1$. Consider the zero start triangular spatial AR process $\{X^*_{k,\ell} : k, \ell \in \mathbb{Z},\, k + \ell \geq 0\}$ defined by

$$X^*_{k,\ell} = \begin{cases} -\alpha X^*_{k-1,\ell} - \beta X^*_{k,\ell-1} + \varepsilon^*_{k,\ell}, & \text{for } k + \ell \geq 1, \\ 0, & \text{for } k + \ell = 0. \end{cases}$$

In this case (1.2) takes the form

$$X^*_{k,\ell} = \begin{cases} -\alpha\big(X^*_{k-1,\ell} - X^*_{k,\ell-1}\big) + \varrho X^*_{k,\ell-1} + \varepsilon^*_{k,\ell}, & \text{for } k + \ell \geq 1, \\ 0, & \text{for } k + \ell = 0. \end{cases}$$

Then, by representation (1.8),

$$X^*_{k,\ell} = \sum_{(i,j)\in T_{k,\ell}} \binom{k+\ell-i-j}{k-i} (-\alpha)^{k-i}(-\beta)^{\ell-j}\varepsilon^*_{i,j} = (-1)^{k+\ell}X_{k,\ell}$$

for $k, \ell \in \mathbb{Z}$ with $k + \ell \geq 0$. Hence,

$$\begin{aligned} A^*_{T_n} &= \sum_{(k,\ell)\in T_n} \begin{bmatrix} n^{-2}\big(X^*_{k-1,\ell} - X^*_{k,\ell-1}\big)^2 & -n^{-9/4}\big(X^*_{k-1,\ell} - X^*_{k,\ell-1}\big)X^*_{k,\ell-1} \\ -n^{-9/4}\big(X^*_{k-1,\ell} - X^*_{k,\ell-1}\big)X^*_{k,\ell-1} & n^{-5/2}\big(X^*_{k,\ell-1}\big)^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \widetilde{A}_{T_n} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \\ d^*_{T_n} &= \sum_{(k,\ell)\in T_n} \begin{bmatrix} -n^{-1}\big(X^*_{k-1,\ell} - X^*_{k,\ell-1}\big)\varepsilon^*_{k,\ell} \\ n^{-5/4}X^*_{k,\ell-1}\varepsilon^*_{k,\ell} \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \widetilde{d}_{T_n}, \end{aligned}$$

where $\widetilde{A}_{T_n}$ and $\widetilde{d}_{T_n}$ have forms (1.6) and (1.7), respectively. Consequently, in order to prove Propositions 1.6 and 1.7 for $\alpha < 0$ and $\beta < 0$ it suffices to prove them for $\alpha > 0$ and $\beta > 0$.

Cases $\alpha < 0$, $\beta > 0$ and $\alpha > 0$, $\beta < 0$ implying $\varrho = -\alpha + \beta$ and $\varrho = \alpha - \beta$, respectively, can be handled in a similar way.

2 Results on the covariance structure

In order to prove Propositions 1.6 and 1.7 one has to know the asymptotic behaviour of the covariances of the process $X_{k,\ell}$. By representation (1.8) we obtain that for all $k_1, \ell_1, k_2, \ell_2 \in \mathbb{Z}$ with $k_1 + \ell_1 \geq 0$ and $k_2 + \ell_2 \geq 0$, and for all $\alpha, \beta \in \mathbb{R}$,

$$\operatorname{Cov}\big(X_{k_1,\ell_1}, X_{k_2,\ell_2}\big) = \sum_{(i,j)\in T_{k_1\wedge k_2,\ell_1\wedge \ell_2}} \binom{k_1+\ell_1-i-j}{k_1-i}\binom{k_2+\ell_2-i-j}{k_2-i}\alpha^{k_1+k_2-2i}\beta^{\ell_1+\ell_2-2j}, \quad (2.1)$$

where $k \wedge \ell := \min\{k, \ell\}$ and an empty sum is defined to be equal to 0. Observe, if $0 < \alpha < 1$ and $\beta = 1 - \alpha$ then by representation (1.9) covariance (2.1) can be expressed in the form

$$\operatorname{Cov}\big(X_{k_1,\ell_1}, X_{k_2,\ell_2}\big) = \sum_{m=1}^{k_1\wedge k_2+\ell_1\wedge \ell_2} \mathsf{P}\big(S^{(\alpha,1-\alpha)}_{k_1+\ell_1-m,k_2+\ell_2-m} = k_1 + \ell_2 - m\big),$$

where for $\nu, \mu \in (0, 1)$ real numbers $S^{(\mu,\nu)}_{k,\ell} := \xi^{(\mu)}_k + \eta^{(\nu)}_\ell$, and $\xi^{(\mu)}_k$ and $\eta^{(\nu)}_\ell$ are independent binomial random variables with parameters (k, μ) and (ℓ, ν), respectively. Now, Lemmas 2.4 and 2.6 of Baran and Pap [2012] directly imply that there exists a constant $D_{\mu,\nu} > 0$ such that for all $k, \ell \geq 0$, $k + \ell \geq 1$, $0 \leq i \leq k + \ell$ and $0 \leq j \leq k + \ell - 1$ we have

$$\mathsf{P}\big(S^{(\mu,\nu)}_{k,\ell} = i\big) \leq \frac{D_{\mu,\nu}}{\sqrt{k+\ell}} \quad \text{and} \quad \Big|\mathsf{P}\big(S^{(\mu,\nu)}_{k,\ell} = j+1\big) - \mathsf{P}\big(S^{(\mu,\nu)}_{k,\ell} = j\big)\Big| \leq \frac{D_{\mu,\nu}}{k+\ell}. \quad (2.2)$$

Hence, one can determine the magnitudes of the covariances and prove the following lemma.

Lemma 2.1 *[Baran et al., 2007, Lemma 2.1] If* $|\alpha| + |\beta| = 1$ *and* $0 < |\alpha| < 1$ *then*

$$\big|\operatorname{Cov}\big(X_{k_1,\ell_1}, X_{k_2,\ell_2}\big)\big| \leq C_\alpha \sqrt{k_1 + \ell_1 + k_2 + \ell_2}$$

with some constant $C_\alpha > 0$.

Now, for $n \in \mathbb{N}$, $s, t \in \mathbb{R}$, $s + t \geq 0$, let us introduce piecewise constant random fields

$$Z_{1,0}^{(n)}(s,t) := n^{-1/4} X_{[ns]+1,[nt]} \quad \text{and} \quad Z_{0,1}^{(n)}(s,t) := n^{-1/4} X_{[ns],[nt]+1}.$$

Concerning the asymptotic behaviour of their covariances one can verify the following result.

Proposition 2.2 *[Baran et al., 2007, Proposition 2.2] Let $s_1, t_1, s_2, t_2 \in \mathbb{R}$ with $s_1+t_1 > 0$, $s_2 + t_2 > 0$. If $0 < \alpha < 1$ and $\beta = 1 - \alpha$ then*

$$\begin{bmatrix} \operatorname{Cov}(Z_{1,0}^{(n)}(s_1,t_1), Z_{1,0}^{(n)}(s_2,t_2)) & \operatorname{Cov}(Z_{1,0}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)) \\ \operatorname{Cov}(Z_{1,0}^{(n)}(s_2,t_2), Z_{0,1}^{(n)}(s_1,t_1)) & \operatorname{Cov}(Z_{0,1}^{(n)}(s_1,t_1), Z_{0,1}^{(n)}(s_2,t_2)) \end{bmatrix} \to z_\alpha(s_1,t_1,s_2,t_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as $n \to \infty$, where

$$z_\alpha(s_1,t_1,s_2,t_2) = \begin{cases} \frac{\sqrt{s_1+s_2+t_1+t_2} - \sqrt{|s_1-s_2|+|t_1-t_2|}}{\sqrt{2\pi\alpha(1-\alpha)}} & \text{if } (1-\alpha)(s_1-s_2) = \alpha(t_1-t_2), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, if $(1-\alpha)(s_1 - s_2) \neq \alpha(t_1 - t_2)$ then the convergence to 0 has an exponential rate.

Further, one can also estimate the difference of two neighbouring covariances.

Proposition 2.3 *[Baran et al., 2007, Proposition 2.5] If $0 < \alpha < 1$ and $\beta = 1 - \alpha$ then there exists a constant $K_\alpha > 0$ such that*

$$\left| \operatorname{Cov}(Z_{i,j}^{(n)}(s_1,t_1), Z_{j,i}^{(n)}(s_2,t_2)) - \operatorname{Cov}(Z_{i,j}^{(n)}(s_1,t_1), Z_{i,j}^{(n)}(s_2,t_2)) \right| \leq K_\alpha n^{-1/2}$$

for all $n \in \mathbb{N}$, $s_1, t_1, s_2, t_2 \in \mathbb{R}$, with $s_1 + t_1 > 0$, $s_2 + t_2 > 0$ and $(i,j) \in \{(0,1),(1,0)\}$.

Finally, in order to estimate covariances we make use of he following lemma which is a generalization of Baran et al. [2004, Lemma 11].

Lemma 2.4 *[Baran et al., 2007, Lemma 2.8] Let $\xi_1, \ldots, \xi_N$ be independent random variables with $\mathsf{E}(\xi_i) = 0$, $\mathsf{E}(\xi_i^2) = 1$ for all $i = 1, \ldots, N$, and $M_4 := \max_{1 \leq i \leq N} \mathsf{E}(\xi_i^4) < \infty$. Let $a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2}$, $c_1, \ldots, c_{n_3}, d_1, \ldots, d_{n_4} \in \mathbb{R}$, $n_1, n_2, n_3, n_4 \leq N$ and*

$$X := \sum_{i=1}^{n_1} a_i \xi_i, \quad Y := \sum_{j=1}^{n_2} b_j \xi_j, \quad Z := \sum_{i=1}^{n_3} c_i \xi_i, \quad W := \sum_{j=1}^{n_4} d_j \xi_j.$$

Then

$$\operatorname{Cov}(XY,ZW)=\sum_{i=1}^{n_1\wedge n_2\wedge n_3\wedge n_4}\big(\mathsf{E}(\xi_i^4)-3\big)\,a_ib_ic_id_i+\operatorname{Cov}(X,Z)\operatorname{Cov}(Y,W)+\operatorname{Cov}(X,W)\operatorname{Cov}(Y,Z).$$

Moreover, if $a_i, b_i, c_i, d_i \geq 0$ *then*

$$0 \leq \operatorname{Cov}(XY,ZW) \leq M_4\operatorname{Cov}(X,Z)\operatorname{Cov}(Y,W) + M_4\operatorname{Cov}(X,W)\operatorname{Cov}(Y,Z),$$

and

$$0 \leq \mathsf{E}(XYZW) \leq M_4\big(\mathsf{E}(XZ)\,\mathsf{E}(YW) + \mathsf{E}(XW)\,\mathsf{E}(YZ) + \mathsf{E}(XY)\,\mathsf{E}(ZW)\big).$$

3 Proof of Proposition 1.6

Let $\alpha, \beta \in (0,1)$ with $\alpha+\beta=1$ and

$$S_{n,1} := \sum_{(k,\ell)\in T_n}\big(X_{k-1,\ell}-X_{k,\ell-1}\big)^2, \quad S_{n,2} := \sum_{(k,\ell)\in T_n}\big(X_{k-1,\ell}-X_{k,\ell-1}\big)X_{k,\ell-1}, \quad S_{n,3} := \sum_{(k,\ell)\in T_n} X_{k,\ell-1}^2.$$

Thus,

$$\widetilde{A}_{T_n} = \begin{bmatrix} n^{-2}S_{n,1} & n^{-9/4}S_{n,2} \\ n^{-9/4}S_{n,2} & n^{-5/2}S_{n,3} \end{bmatrix}$$

and (1.5) follows from

$$n^{-2}S_{n,1} \xrightarrow{\mathsf{L}_2} \frac{1}{\varphi_\alpha} = \frac{2}{\alpha(1-\alpha)}, \quad n^{-9/4}S_{n,2} \xrightarrow{\mathsf{L}_2} 0, \quad n^{-5/2}S_{n,3} \xrightarrow{\mathsf{L}_2} \frac{1}{\psi_\alpha} = \frac{2^{9/2}}{15\sqrt{\pi\alpha(1-\alpha)}}. \tag{3.1}$$

The last two statements of (3.1) have already been proved, see Baran et al. [2007, Proposition 1.2 and Section 6, pp. 40-41]. In order to verify the remaining statement one has to show

$$\lim_{n\to\infty} n^{-2}\mathsf{E}\big(S_{n,1}\big) = \frac{1}{\varphi_\alpha} \qquad \text{and} \qquad \lim_{n\to\infty} n^{-4}\operatorname{Var}\big(S_{n,1}\big) = 0. \tag{3.2}$$

It is easy to see that

$$n^{-2}\mathsf{E}\big(S_{n,1}\big) = \iint_T \sqrt{n}\Big(\operatorname{Var}\big(Z_{0,1}^{(n)}(s,t)\big) + \operatorname{Var}\big(Z_{1,0}^{(n)}(s,t)\big) - 2\operatorname{Cov}\big(Z_{0,1}^{(n)}(s,t), Z_{1,0}^{(n)}(s,t)\big)\Big)\,\mathrm{d}s\,\mathrm{d}t,$$

where $T := \{(s,t) \in \mathbb{R}^2 : s+t \geq 0,\ s \leq 1,\ t \leq 1\}$, and using (2.1) one can prove

$$\lim_{n\to\infty} \sqrt{n}\Big(\mathrm{Var}\big(Z^{(n)}_{i,j}(s,t)\big) - \mathrm{Cov}\big(Z^{(n)}_{i,j}(s,t), Z^{(n)}_{j,i}(s,t)\big)\Big) = \frac{1}{2\alpha(1-\alpha)},$$

where $(i,j) \in \big\{(0,1),(1,0)\big\}$. The details can be found in Baran et al. [2007, Section 5, pp. 36-37]. Hence, Proposition 2.3 and the dominated convergence theorem (DCT) imply the first statement of (3.2).

Now, by Lemma 2.4

$$\begin{aligned}\mathrm{Var}\big(S_{n,1}\big) &= \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} \mathrm{Cov}\big((X_{k_1-1,\ell_1} - X_{k_1,\ell_1-1})^2, (X_{k_2-1,\ell_2} - X_{k_2,\ell_2-1})^2\big) \\ &\leq \sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} \Big(2M_4 L^{(1)}_{k_1,\ell_1,k_2,\ell_2} + (M_4-3)^+ L^{(2)}_{k_1,\ell_1,k_2,\ell_2}\Big) + \mathcal{O}(n^3),\end{aligned}$$

where

$$\begin{aligned} L^{(1)}_{k_1,\ell_1,k_2,\ell_2} &:= \mathrm{Cov}\big(X_{k_1-1,\ell_1} - X_{k_1,\ell_1-1}, X_{k_2-1,\ell_2} - X_{k_2,\ell_2-1}\big)^2, \\ L^{(2)}_{k_1,\ell_1,k_2,\ell_2} &:= \sum_{(i,j)\in T_{k_1\wedge k_2-1,\ell_1\wedge\ell_2-1}} \Big(\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1-i\big) - \mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1-1-i\big)\Big)^2 \\ &\qquad\qquad \times \Big(\mathsf{P}\big(S^{(\alpha)}_{k_2+\ell_2-1-i-j} = k_2-i\big) - \mathsf{P}\big(S^{(\alpha)}_{k_2+\ell_2-1-i-j} = k_2-1-i\big)\Big)^2 \\ &\leq \sum_{(i,j)\in T_{k_1\wedge k_2-1,\ell_1\wedge\ell_2-1}} \Big(\mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1-i\big)^2 + \mathsf{P}\big(S^{(\alpha)}_{k_1+\ell_1-1-i-j} = k_1-1-i\big)^2\Big) \\ &\qquad\qquad \times \Big(\mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j} = \ell_2-1-j\big)^2 + \mathsf{P}\big(S^{(1-\alpha)}_{k_2+\ell_2-1-i-j} = \ell_2-j\big)^2\Big). \end{aligned} \tag{3.3}$$

Obviously,

$$\begin{aligned} n^{-4} &\sum_{(k_1,\ell_1)\in T_n} \sum_{(k_2,\ell_2)\in T_n} L^{(1)}_{k_1,\ell_1,k_2,\ell_2} \\ &= \iint\limits_T \iint\limits_T \Big(\sqrt{n}\mathrm{Cov}\big(Z^{(n)}_{0,1}(s_1,t_1) - Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{0,1}(s_2,t_2) - Z^{(n)}_{1,0}(s_2,t_2)\big)\Big)^2 \mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2, \end{aligned}$$

where due to Propositions 2.2, 2.3 and DCT the right hand side converges to 0 as $n \to \infty$.

Further, the second inequality of (2.2) implies

$$L^{(2)}_{k_1,\ell_1,k_2,\ell_2} \leq \sum_{(i,j)\in T_{k_1\wedge k_2-1,\ell_1\wedge\ell_2-1}} \frac{D^4_{\alpha,\alpha}}{(k_1+\ell_1-1-i-j)^2(k_2+\ell_2-1-i-j)^2}$$
$$\leq \sum_{m=1}^{k_1\wedge k_2+\ell_1\wedge\ell_2-2} \frac{D^4_{\alpha,\alpha}}{(k_1\wedge k_2+\ell_1\wedge\ell_2-1-m)^3} < \frac{\pi^2 D^4_{\alpha,\alpha}}{6} < \infty,$$

so

$$n^{-4}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} L^{(2)}_{k_1,\ell_1,k_2,\ell_2} = \iint_T \iint_T L^{(2)}_{[ns_1],[nt_1],[ns_2],[nt_2]} \mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2 \leq \frac{2\pi^2 D^4_{\alpha,\alpha}}{3}.$$

Finally, e.g.

$$\sum_{(i,j)\in T_{[ns_1]\wedge[ns_2]-1,[nt_1]\wedge[nt_2]-1}} \mathsf{P}\big(S^{(\alpha)}_{[ns_1]+[nt_1]-1-i-j} = [ns_1]-i\big)^2 \mathsf{P}\big(S^{(1-\alpha)}_{[ns_2]+[nt_2]-1-i-j} = [nt_2]-1-j\big)^2$$
$$\leq \sqrt{n}\operatorname{Cov}\big(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2)\big),$$

which by Proposition 2.2 converges to 0 as $n\to\infty$ if $(1-\alpha)(s_1-s_2)\neq\alpha(t_1-t_2)$. Similar results can be derived for the remaining three terms of the right hand side of (3.3), so by the DCT

$$\lim_{n\to\infty} n^{-4}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} L^{(2)}_{k_1,\ell_1,k_2,\ell_2} = 0, \tag{3.4}$$

which completes the proof. □

4 Proof of Proposition 1.7

Again, let $\alpha,\beta\in(0,1)$ with $\alpha+\beta=1$ and denote by $d^{(i)}_n$, $i=1,2$, the components of d_{T_n}. First we show that $(d_{T_n})_{n\geq1}$ is a square integrable two dimensional martingale with respect to filtration $(\mathcal{F}_n)_{n\geq1}$, where $\mathcal{F}_n$ denotes the σ–algebra generated by random variables $\{\varepsilon_{k,\ell} : (k,\ell)\in T_n\}$.

In order to do this we give a useful decomposition of $d_{T_n}-d_{T_{n-1}}$, where $d_{T_0} := (0,0)^\top$.

By representation (1.8),

$$d_n^{(1)} - d_{n-1}^{(1)} = \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell}\Bigg(\sum_{(i,j)\in T_{k-1,\ell}} \mathsf{P}\big(S_{k+\ell-1-i-j}^{(\alpha)} = k-1-i\big)\varepsilon_{i,j} - \sum_{(i,j)\in T_{k,\ell-1}} \mathsf{P}\big(S_{k+\ell-1-i-j}^{(\alpha)} = k-i\big)\varepsilon_{i,j}\Bigg),$$

$$d_n^{(2)} - d_{n-1}^{(2)} = \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in T_{k,\ell-1}} \mathsf{P}\big(S_{k+\ell-1-i-j}^{(\alpha)} = k-i\big)\varepsilon_{i,j}.$$

Collecting first the terms containing only $\varepsilon_{i,j}$ with $(i,j)\in T_n\setminus T_{n-1}$, and then the rest, we obtain decomposition

$$d_{T_n} - d_{T_{n-1}} = d_{n,1} + \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell}\, d_{n,2,k,\ell}, \tag{4.1}$$

where $d_{n,1} = \big(\delta_{n,1}^{(1)} - \delta_{n,1}^{(2)}, \delta_{n,1}^{(2)}\big)^\top$ and $d_{n,2,k,\ell} = \big(\delta_{n,2,k-1,\ell} - \delta_{n,2,k,\ell-1}, \delta_{n,2,k,\ell-1}\big)^\top$ with

$$\delta_{n,1}^{(1)} := \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in T_{k-1,\ell}\setminus T_{n-1}} \mathsf{P}\big(S_{k+\ell-1-i-j}^{(\alpha)} = k-1-i\big)\varepsilon_{i,j} = \sum_{k=-n+2}^{m} \sum_{i=-n+1}^{k-1} \alpha^{k-1-i}\varepsilon_{k,n}\varepsilon_{i,n},$$

$$\delta_{n,1}^{(2)} := \sum_{(k,\ell)\in T_n\setminus T_{n-1}} \varepsilon_{k,\ell} \sum_{(i,j)\in T_{k,\ell-1}\setminus T_{n-1}} \mathsf{P}\big(S_{k+\ell-1-i-j}^{(\alpha)} = k-i\big)\varepsilon_{i,j} = \sum_{\ell=-n+2}^{n} \sum_{j=-n+1}^{\ell-1} \beta^{\ell-1-j}\varepsilon_{n,\ell}\varepsilon_{n,j},$$

$$\delta_{n,2,k,\ell} := \sum_{(i,j)\in T_{k,\ell}\cap T_{n-1}} \mathsf{P}\big(S_{k+\ell-i-j}^{(\alpha)} = k-i\big)\varepsilon_{i,j}.$$

The components of $d_{n,1}$ are quadratic forms of the variables $\{\varepsilon_{i,j} : (i,j)\in T_n\setminus T_{n-1}\}$, hence $d_{n,1}$ is independent of $\mathcal{F}_{n-1}$. Besides this the terms $\delta_{n,2,k,\ell}$ are linear combinations of the variables $\{\varepsilon_{i,j} : (i,j)\in T_{n-1}\}$, thus vectors $d_{n,2,k,\ell}$ are measurable with respect to $\mathcal{F}_{n-1}$. Consequently,

$$\mathsf{E}(d_{T_n} - d_{T_{n-1}} \mid \mathcal{F}_{n-1}) = \mathsf{E}(d_{n,1}) + \sum_{(k,\ell)\in T_n\setminus T_{n-1}} d_{n,2,k,\ell}\, \mathsf{E}(\varepsilon_{k,\ell} \mid \mathcal{F}_{n-1}) = 0.$$

Hence $(d_{T_n})_{n\geq 1}$ is a square integrable martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$ and obviously the same is valid for $(\widetilde{d}_{T_n})_{n\geq 1}$.

By the Martingale Central Limit Theorem [Jacod and Shiryaev, 1987], in order to prove the statement of Proposition 1.7, it suffices to show that the conditional variances of the martingale differences converge in probability and to verify the conditional Lindeberg condition. To be precise, the statement is a consequence of the following two propositions, where $\mathbb{1}_H$ denotes the indicator function of a set H.

Proposition 4.1

$$\sum_{m=1}^{n} \mathsf{E}\Big(\big(\widetilde{d}_{T_m} - \widetilde{d}_{T_{m-1}}\big)\big(\widetilde{d}_{T_m} - \widetilde{d}_{T_{m-1}}\big)^\top \,\Big|\, \mathcal{F}_{m-1}\Big) \stackrel{\mathsf{P}}{\longrightarrow} \widetilde{A} \qquad \text{as } n \to \infty.$$

Proposition 4.2 *For all* $\delta > 0$,

$$\sum_{m=1}^{n} \mathsf{E}\Big(\big\|\widetilde{d}_{T_m} - \widetilde{d}_{T_{m-1}}\big\|^2 \mathbb{1}_{\{\|\widetilde{d}_{T_m} - \widetilde{d}_{T_{m-1}}\| \geq \delta\}} \,\Big|\, \mathcal{F}_{m-1}\Big) \stackrel{\mathsf{P}}{\longrightarrow} 0 \qquad \text{as } n \to \infty.$$

Proof of Proposition 4.1. Considering separately the entries of $\big(\widetilde{d}_{T_m} - \widetilde{d}_{T_{m-1}}\big)\big(\widetilde{d}_{T_m} - \widetilde{d}_{T_{m-1}}\big)^\top$ one can see that the statement of the proposition is a consequence of

$$n^{-2} \sum_{m=1}^{n} \mathsf{E}\Big(\big(d_m^{(1)} - d_{m-1}^{(1)}\big)^2 \,\Big|\, \mathcal{F}_{m-1}\Big) \stackrel{\mathsf{L}_2}{\longrightarrow} \frac{1}{\varphi_\alpha}, \tag{4.2}$$

$$n^{-5/2} \sum_{m=1}^{n} \mathsf{E}\Big(\big(d_m^{(2)} - d_{m-1}^{(2)}\big)^2 \,\Big|\, \mathcal{F}_{m-1}\Big) \stackrel{\mathsf{L}_2}{\longrightarrow} \frac{1}{\psi_\alpha}, \tag{4.3}$$

$$n^{-9/4} \sum_{m=1}^{n} \mathsf{E}\Big(\big(d_m^{(1)} - d_{m-1}^{(1)}\big)\big(d_m^{(2)} - d_{m-1}^{(2)}\big) \,\Big|\, \mathcal{F}_{m-1}\Big) \stackrel{\mathsf{L}_2}{\longrightarrow} 0 \tag{4.4}$$

as $n \to \infty$. Limits (4.2) and (4.3) have already been proved, see Baran et al. [2007, Section 6, pp. 40-41 and Proposition 4.1]. A more detailed proof can be found in Baran et al. [2005, Propositions 6.1 and 4.1].

Now, let $U_m := \mathsf{E}\big((d_m^{(1)} - d_{m-1}^{(1)})(d_m^{(2)} - d_{m-1}^{(2)}) \,\big|\, \mathcal{F}_{m-1}\big)$ and we have

$$d_m^{(1)} - d_{m-1}^{(1)} = \sum_{(k,\ell)\in T_m \setminus T_{m-1}} (X_{k-1,\ell} - X_{k,\ell-1})\varepsilon_{k,\ell}, \qquad d_m^{(2)} - d_{m-1}^{(2)} = \sum_{(k,\ell)\in T_m \setminus T_{m-1}} X_{k,\ell-1}\varepsilon_{k,\ell}.$$

Representation (1.8) and independence of the error terms $\varepsilon_{i,j}$ imply

$$\begin{aligned}
\mathsf{E}\Big(\big(d_m^{(1)} - d_{m-1}^{(1)}\big)\big(d_m^{(2)} - d_{m-1}^{(2)}\big)\Big) &= \sum_{(k,\ell)\in T_m \setminus T_{m-1}} \mathsf{E}\big((X_{k-1,\ell} - X_{k,\ell-1})X_{k,\ell-1}\big)\,\mathsf{E}\big(\varepsilon_{k,\ell}^2\big) \\
&= \sum_{(k,\ell)\in T_m \setminus T_{m-1}} \mathsf{E}\big((X_{k-1,\ell} - X_{k,\ell-1})X_{k,\ell-1}\big) = \mathsf{E}(S_{m,2}) - \mathsf{E}(S_{m-1,2}),
\end{aligned}$$

so using the second statement of (3.1) we obtain

$$n^{-9/4}\sum_{m=1}^{n}\mathsf{E}(U_m)=n^{-9/4}\mathsf{E}(S_{n,2})\to 0 \qquad\text{as}\ \ n\to\infty.$$

Further, decomposition (4.1), independence of $\delta_{m,1}^{(1)},\ \delta_{m,1}^{(2)}$ and $\big\{\varepsilon_{k,\ell},\ (k,\ell)\in T_m\setminus T_{m-1}\big\}$ from $\mathcal{F}_{m-1}$, and measurability of $\delta_{m,2,k,\ell}$ with respect to $\mathcal{F}_{m-1}$ imply

$$U_m=\mathsf{E}\Big(\big(\delta_{m,1}^{(1)}-\delta_{m,1}^{(2)}\big)\delta_{m,1}^{(2)}\Big)+\sum_{(k,\ell)\in T_m\setminus T_{m-1}}\big(\delta_{m,2,k-1,\ell}-\delta_{m,2,k,\ell-1}\big)\delta_{m,2,k,\ell-1}.$$

In this way, to complete the proof of (4.4) one has to show

$$n^{-9/2}\mathrm{Var}\bigg(\sum_{m=1}^{n}U_m\bigg)=n^{-9/2}\mathrm{Var}\bigg(\sum_{m=1}^{n}\sum_{(k,\ell)\in T_m\setminus T_{m-1}}(\delta_{m,2,k-1,\ell}-\delta_{m,2,k,\ell-1})\delta_{m,2,k,\ell-1}\bigg)\to 0 \tag{4.5}$$

as $n\to\infty$.

Now, consider

$$\begin{aligned}
&\mathrm{Var}\bigg(\sum_{m=1}^{n}\sum_{(k,\ell)\in T_m\setminus T_{m-1}}(\delta_{m,2,k-1,\ell}-\delta_{m,2,k,\ell-1})\delta_{m,2,k,\ell-1}\bigg)\\
&\quad=\sum_{m_1=1}^{n}\sum_{(k_1,\ell_1)\in T_{m_1}\setminus T_{m_1-1}}\sum_{m_2=1}^{n}\sum_{(k_2,\ell_2)\in T_{m_2}\setminus T_{m_2-1}}G_{m_1,m_2,k_1,\ell_1,k_2,\ell_2} &(4.6)\\
&\quad=\sum_{m_1=1}^{n}\sum_{m_2=1}^{n}\Bigg(\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2}G_{m_1,m_2,k_1,m_1,k_2,m_2}+\sum_{k_1=-m_1+1}^{m_1}\sum_{\ell_2=-m_2+1}^{m_2-1}G_{m_1,m_2,k_1,m_1,m_2,\ell_2}\\
&\qquad+\sum_{\ell_1=-m_1+1}^{m_1-1}\sum_{k_2=-m_2+1}^{m_2}G_{m_1,m_2,m_1,\ell_1,k_2,m_2}+\sum_{\ell_1=-m_1+1}^{m_1-1}\sum_{\ell_2=-m_2+1}^{m_2-1}G_{m_1,m_2,m_1,\ell_1,m_2,\ell_2}\Bigg),
\end{aligned}$$

where

$$\begin{aligned}
&G_{m_1,m_2,k_1,\ell_1,k_2,\ell_2}\\
&\quad:=\mathrm{Cov}\big((\delta_{m_1,2,k_1-1,\ell_1}-\delta_{m_1,2,k_1,\ell_1-1})\delta_{m_1,2,k_1,\ell_1-1},(\delta_{m_2,2,k_2-1,\ell_2}-\delta_{m_2,2,k_2,\ell_2-1})\delta_{m_2,2,k_2,\ell_2-1}\big).
\end{aligned}$$

By representation (1.8) of $X_{k,\ell}$ and definition of $\delta_{m,2,k,\ell}$ we have

$$\begin{aligned}
\delta_{m,2,k-1,m}&=X_{k-1,m}-\sum_{i=-m+2}^{k-1}\alpha^{k-1-i}\varepsilon_{i,m}, &&-m+2\le k\le m,\\
\delta_{m,2,k,m-1}&=X_{k,m-1}, &&-m+1\le k\le m,
\end{aligned}$$

$$\delta_{m,2,m,\ell-1} = X_{m,\ell-1} - \sum_{j=-m+2}^{\ell-1} (1-\alpha)^{\ell-1-j} \varepsilon_{m,j}, \qquad -m+2 \le \ell \le m-1,$$

$$\delta_{m,2,m-1,\ell} = X_{m-1,\ell}, \qquad -m+1 \le \ell \le m-1.$$

Hence, e.g.

$$\begin{aligned}
\sum_{k_1=-m_1+1}^{m_1} \sum_{k_2=-m_2+1}^{m_2} & G_{m_1,m_2,k_1,m_1,k_2,m_2} \\
&= \sum_{k_1=-m_1+1}^{m_1} \sum_{k_2=-m_2+1}^{m_2} \operatorname{Cov}\Bigg(\Bigg(X_{k_1-1,m_1} - X_{k_1,m_1-1} - \sum_{i_1=-m_1+2}^{k_1-1} \alpha^{k_1-1-i_1} \varepsilon_{i_1,m_1} \Bigg) X_{k_1,m_1-1}, \\
&\qquad \Bigg(X_{k_2-1,m_2} - X_{k_2,m_2-1} - \sum_{i_2=-m_2+2}^{k_2-1} \alpha^{k_2-1-i_2} \varepsilon_{i_2,m_2} \Bigg) X_{k_2,m_2-1} \Bigg) \\
&= \sum_{k_1=-m_1+1}^{m_1} \sum_{k_2=-m_2+1}^{m_2} G^{(1)}_{k_1,m_1,k_2,m_2} - G^{(2)}_{k_1,m_1,k_2,m_2} - G^{(2)}_{k_2,m_2,k_1,m_1} + G^{(3)}_{k_1,m_1,k_2,m_2},
\end{aligned}$$

where

$$\begin{aligned}
G^{(1)}_{k_1,m_1,k_2,m_2} &:= \operatorname{Cov}\Big(\big(X_{k_1-1,m_1} - X_{k_1,m_1-1} \big) X_{k_1,m_1-1}, (X_{k_2-1,m_2} - X_{k_2,m_2-1}) X_{k_2,m_2-1} \Big), \\
G^{(2)}_{k_1,m_1,k_2,m_2} &:= \operatorname{Cov}\Bigg(\big(X_{k_1-1,m_1} - X_{k_1,m_1-1} \big) X_{k_1,m_1-1}, X_{k_2,m_2-1} \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i} \varepsilon_{i,m_2} \Bigg), \\
G^{(3)}_{k_1,m_1,k_2,m_2} &:= \operatorname{Cov}\Bigg(X_{k_1,m_1-1} \sum_{i_1=-m_1+2}^{k_1-1} \alpha^{k_1-1-i_1} \varepsilon_{i_1,m_1}, X_{k_2,m_2-1} \sum_{i_2=-m_2+2}^{k_2-1} \alpha^{k_2-1-i_2} \varepsilon_{i_2,m_2} \Bigg).
\end{aligned}$$

Thus, Lemma 2.4, representation (1.8) and independence of the error terms $\varepsilon_{i,j}$ imply

$$\begin{aligned}
G^{(2)}_{k_1,m_1,k_2,m_2} &= \operatorname{Cov}\Big(X_{k_1-1,m_1} - X_{k_1,m_1-1}, X_{k_2,m_2-1} \Big) \operatorname{Cov}\Bigg(X_{k_1,m_1-1}, \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i} \varepsilon_{i,m_2} \Bigg) \\
&\quad + \operatorname{Cov}\Big(X_{k_1,m_1-1}, X_{k_2,m_2-1} \Big) \operatorname{Cov}\Bigg(X_{k_1-1,m_1} - X_{k_1,m_1-1}, \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i} \varepsilon_{i,m_2} \Bigg), \\
G^{(3)}_{k_1,m_1,k_2,m_2} &= \operatorname{Cov}\Big(X_{k_1,m_1-1}, X_{k_2,m_2-1} \Big) \operatorname{Cov}\Bigg(\sum_{i_1=-m_1+2}^{k_1-1} \alpha^{k_1-1-i_1} \varepsilon_{i_1,m_1}, \sum_{i_2=-m_2+2}^{k_2-1} \alpha^{k_2-1-i_2} \varepsilon_{i_2,m_2} \Bigg) \\
&\quad + \operatorname{Cov}\Bigg(X_{k_1,m_1-1}, \sum_{i_2=-m_2+2}^{k_2-1} \alpha^{k_2-1-i_2} \varepsilon_{i_2,m_2} \Bigg) \operatorname{Cov}\Bigg(X_{k_2,m_2-1}, \sum_{i_1=-m_1+2}^{k_1-1} \alpha^{k_1-1-i_1} \varepsilon_{i_1,m_1} \Bigg).
\end{aligned}$$

Moreover, using again the independence of the error terms $\varepsilon_{i,j}$ one can easily see that $G^{(2)}_{k_1,m_1,k_2,m_2}=0$ if $m_2>m_1$ and $G^{(3)}_{k_1,m_1,k_2,m_2}=0$ if $m_2\neq m_1$. In this way

$$\begin{aligned}\sum_{m_1=1}^{n}\sum_{m_2=1}^{n}\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2} G^{(3)}_{k_1,m_1,k_2,m_2} \qquad (4.7)\\ =\sum_{m=1}^{n}\sum_{k_1=-m+1}^{m}\sum_{k_2=-m+1}^{m}\operatorname{Cov}\big(X_{k_1,m-1},X_{k_2,m-1}\big)\alpha^{|k_1-k_2|}\sum_{i=0}^{m+k_1\wedge k_2-3}\alpha^{2i}\\ \leq \frac{C_\alpha}{1-\alpha^2}\sum_{m=1}^{n}\sum_{k_1=0}^{2m-1}\sum_{k_2=0}^{2m-1}(k_1+k_2)^{1/2}\leq\frac{3C_\alpha}{1-\alpha^2}(n+1)^{7/2},\end{aligned}$$

where the first inequality is a consequence of Lemma 2.1 and the empty sum is defined to be zero.

Further, let

$$\begin{aligned}B^{(1)}_{k_1,m_1,k_2,m_2}&:=\operatorname{Cov}\Bigg(X_{k_1,m_1-1},\sum_{i=-m_2+2}^{k_2-1}\alpha^{k_2-1-i}\varepsilon_{i,m_2}\Bigg),\\ B^{(2)}_{k_1,m_1,k_2,m_2}&:=\operatorname{Cov}\Bigg(X_{k_1-1,m_1}-X_{k_1,m_1-1},\sum_{i=-m_2+2}^{k_2-1}\alpha^{k_2-1-i}\varepsilon_{i,m_2}\Bigg).\end{aligned}$$

Assuming $m_2<m_1$, with the help of representation (1.9) we obtain

$$\begin{aligned}B^{(1)}_{k_1,m_1,k_2,m_2}&=\sum_{i=-m_2+2}^{k_1\wedge k_2-1}\mathsf{P}\big(S^{(\alpha)}_{k_1+m_1-m_2-1-i}=k_1-i\big)\alpha^{k_2-1-i}+\alpha^{k_2-k_1-1}(1-\alpha)^{m_1-m_2-1}\mathbb{1}_{\{k_1\leq k_2-1\}},\\ B^{(2)}_{k_1,m_1,k_2,m_2}&=\sum_{i=-m_2+2}^{k_1\wedge k_2-1}\Big(\mathsf{P}\big(S^{(\alpha)}_{k_1+m_1-m_2-1-i}=k_1-1-i\big)-\mathsf{P}\big(S^{(\alpha)}_{k_1+m_1-m_2-1-i}=k_1-i\big)\Big)\alpha^{k_2-1-i}\\ &\quad-\alpha^{k_2-k_1-1}(1-\alpha)^{m_1-m_2-1}\mathbb{1}_{\{k_1\leq k_2-1\}}\end{aligned}$$

for $k_1+m_1\geq 3$, otherwise the above quantities are equal to zero. Hence, using (2.2) one can easily show that for $k_1 \le k_2 - 1$

$$\begin{aligned}
\big|B^{(1)}_{k_1,m_1,k_2,m_2}\big| &\le \alpha^{k_2-k_1-1}(1-\alpha)^{m_1-m_2-1} + \alpha^{k_2-k_1} \sum_{i=-m_2+2}^{k_1-1} \frac{D_{\alpha,\alpha}}{(k_1+m_1-m_2-1-i)^{1/2}} \\
&\le H_\alpha \alpha^{k_2-k_1}(k_1+m_1)^{1/2}, \\
\big|B^{(2)}_{k_1,m_1,k_2,m_2}\big| &\le \alpha^{k_2-k_1-1}(1-\alpha)^{m_1-m_2-1} + \alpha^{k_2-k_1} \sum_{i=-m_2+2}^{k_1-1} \frac{D_{\alpha,\alpha}}{k_1+m_1-m_2-1-i} \\
&\le H_\alpha \alpha^{k_2-k_1} \log(k_1+m_1)
\end{aligned}$$

with some constant $H_\alpha > 0$, while for $k_1 > k_2 - 1$ we have

$$\begin{aligned}
\big|B^{(1)}_{k_1,m_1,k_2,m_2}\big| &\le \frac{D_{\alpha,\alpha}}{(k_1-k_2+m_1-m_2)^{1/2}} \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i} \le \frac{H_\alpha}{(k_1-k_2+m_1-m_2)^{1/2}}, \\
\big|B^{(2)}_{k_1,m_1,k_2,m_2}\big| &\le \frac{D_{\alpha,\alpha}}{k_1-k_2+m_1-m_2} \sum_{i=-m_2+2}^{k_2-1} \alpha^{k_2-1-i} \le \frac{H_\alpha}{k_1-k_2+m_1-m_2}.
\end{aligned}$$

Obviously, if $m_1 = m_2$ then

$$B^{(1)}_{k_1,m_1,k_2,m_2} = 0 \quad \text{and} \quad B^{(2)}_{k_1,m_1,k_2,m_2} = \sum_{i=-m_1+2}^{k_1 \wedge k_2 - 1} \mathsf{P}\big(S^{(\alpha)}_{k_1-1-i} = k_1 - 1 - i\big)\alpha^{k_2-1-i} \le \frac{\alpha^{|k_1-k_2|}}{1-\alpha^2}.$$

In this way, by Lemma 2.1 and Proposition 2.3,

$$\begin{aligned}
\sum_{m_1=1}^{n}\sum_{m_2=1}^{n}\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2}\big|G^{(2)}_{k_1,m_1,k_2,m_2}\big| &\le \frac{C_\alpha}{1-\alpha^2}\sum_{m=1}^{n}\sum_{k_1=-m+3}^{m}\sum_{k_2=-m+3}^{m}(k_1+k_2+2m)^{1/2}\alpha^{|k_1-k_2|} \\
&+H_\alpha \sum_{m_1=2}^{n}\sum_{m_2=1}^{m_1-1}\sum_{k_2=-m_2+1}^{m_2}\sum_{k_1=-m_1+3}^{k_2-1}\alpha^{k_2-k_1}(k_2+m_1)^{1/2}\Big(K_\alpha + 2C_\alpha\log(k_2+m_1)\Big) \qquad (4.8) \\
&+ H_\alpha \sum_{m_1=2}^{n}\sum_{m_2=1}^{m_1-1}\sum_{k_1=-m_1+3}^{m_1}\sum_{k_2=-m_2+1}^{k_1}\bigg(\frac{K_\alpha}{(k_1-k_2+m_1-m_2)^{1/2}} + \frac{C_\alpha(k_1+k_2+m_1+m_2)^{1/2}}{k_1-k_2+m_1-m_2}\bigg) \\
&\le \frac{8C_\alpha}{(1-\alpha)(1-\alpha^2)}\sum_{m=1}^{n} m^{3/2} + \frac{H_\alpha}{1-\alpha}\sum_{m=2}^{n}\sum_{k=-m+1}^{m} m(k+m)^{1/2}\Big(K_\alpha + 2C_\alpha\log(k+m)\Big) \\
&+ 2H_\alpha\sum_{m=2}^{n}\sum_{k=-m+3}^{m} m(k+m)^{1/2}\Big(K_\alpha+2C_\alpha\log(k+m)\Big) \le Q_\alpha(n+1)^{7/2}\log(n+1)
\end{aligned}$$

with some constant $Q_\alpha > 0$. Inequalities (4.7) and (4.8) imply

$$\sum_{m_1=1}^{n}\sum_{m_2=1}^{n}\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2} G_{m_1,m_2,k_1,m_1,k_2,m_2} = \sum_{m_1=1}^{n}\sum_{m_2=1}^{n}\sum_{k_1=-m_1+1}^{m_1}\sum_{k_2=-m_2+1}^{m_2} G^{(1)}_{k_1,m_1,k_2,m_2} + \mathcal{O}\big(n^{7/2}\log(n)\big),$$

and the same can be proved for the remaining three terms of (4.6). Hence

$$\operatorname{Var}\bigg(\sum_{m=1}^{n} U_m\bigg) = \sum_{m_1=1}^{n}\sum_{(k_1,\ell_1)\in T_{m_1}\setminus T_{m_1-1}}\sum_{m_2=1}^{n}\sum_{(k_2,\ell_2)\in T_{m_2}\setminus T_{m_2-1}} G^{(1)}_{k_1,\ell_1,k_2,\ell_2} = \sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n} G^{(1)}_{k_1,\ell_1,k_2,\ell_2} + \mathcal{R}_n, \tag{4.9}$$

and $\mathcal{R}_n = \mathcal{O}\big(n^{7/2}\log(n)\big)$. Further, Lemma (2.4) implies

$$G^{(1)}_{k_1,\ell_1,k_2,\ell_2} = \mathcal{L}^{(1)}_{k_1,\ell_1,k_2,\ell_2} + \mathcal{L}^{(2)}_{k_1,\ell_1,k_2,\ell_2},$$

where

$$\begin{aligned}\mathcal{L}^{(1)}_{k_1,m_1,k_2,m_2} :=& \operatorname{Cov}\big(X_{k_1-1,m_1} - X_{k_1,m_1-1}, X_{k_2-1,m_2} - X_{k_2,m_2-1}\big)\operatorname{Cov}\big(X_{k_1,m_1-1}, X_{k_2,m_2-1}\big)\\ &+ \operatorname{Cov}\big(X_{k_1-1,m_1} - X_{k_1,m_1-1}, X_{k_2,m_2-1}\big)\operatorname{Cov}\big(X_{k_2-1,m_2} - X_{k_2,m_2-1}, X_{k_1,m_1-1}\big),\end{aligned}$$

and using the same ideas as in the proof of (3.4) on can show

$$\lim_{n\to\infty} n^{-9/2}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n}\mathcal{L}^{(2)}_{k_1,\ell_1,k_2,\ell_2} = 0. \tag{4.10}$$

Finally,

$$\begin{aligned} n^{-9/2}&\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n}\mathcal{L}^{(1)}_{k_1,\ell_1,k_2,\ell_2}\\ &= \iint_T\iint_T \Big(\sqrt{n}\operatorname{Cov}\big(Z^{(n)}_{0,1}(s_1,t_1) - Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{0,1}(s_2,t_2) - Z^{(n)}_{1,0}(s_2,t_2)\big) \qquad (4.11)\\ &\qquad \times \operatorname{Cov}\big(Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2)\big) + \operatorname{Cov}\big(Z^{(n)}_{0,1}(s_1,t_1) - Z^{(n)}_{1,0}(s_1,t_1), Z^{(n)}_{1,0}(s_2,t_2)\big)\\ &\qquad \times \sqrt{n}\operatorname{Cov}\big(Z^{(n)}_{0,1}(s_2,t_2) - Z^{(n)}_{1,0}(s_2,t_2), Z^{(n)}_{1,0}(s_1,t_1)\big)\Big)\mathrm{d}s_1\,\mathrm{d}t_1\,\mathrm{d}s_2\,\mathrm{d}t_2.\end{aligned}$$

With the help of Lemma 2.1 and Proposition 2.3 one can easily show that the integrand on the right hand side of (4.11) can be dominated by $K_\alpha\big(C_\alpha\sqrt{s_1+t_1+s_2+t_2+1}+K_\alpha\big)$, which has a finite integral on $T\times T$. Hence, by Proposition 2.2 and DCT

$$\lim_{n\to\infty} n^{-9/2}\sum_{(k_1,\ell_1)\in T_n}\sum_{(k_2,\ell_2)\in T_n}\mathcal{L}^{(1)}_{k_1,\ell_1,k_2,\ell_2}=0,$$

which together with (4.9) and (4.10) implies (4.5). □

Proof of Proposition 4.2. We have

$$\mathbb{1}_{\{\|\widetilde{d}_{T_m}-\widetilde{d}_{T_{m-1}}\|\geq\delta\}}\leq\delta^{-2}\|\widetilde{d}_{T_m}-\widetilde{d}_{T_{m-1}}\|^2,$$

hence to prove the proposition it suffices to show

$$\sum_{m=1}^{n}\mathsf{E}\Big(\big\|\widetilde{d}_{T_m}-\widetilde{d}_{T_{m-1}}\big\|^4\,\Big|\,\mathcal{F}_{m-1}\Big)\stackrel{\mathsf{P}}{\longrightarrow}0\qquad\text{as}\ \ n\to\infty,$$

which is a direct consequence of

$$n^{-4}\sum_{m=1}^{n}\mathsf{E}\Big(\big|d^{(1)}_m-d^{(1)}_{m-1}\big|^4\,\Big|\,\mathcal{F}_{m-1}\Big)\stackrel{\mathsf{P}}{\longrightarrow}0\qquad\text{and}\qquad n^{-5}\sum_{m=1}^{n}\mathsf{E}\Big(\big|d^{(2)}_m-d^{(2)}_{m-1}\big|^4\,\Big|\,\mathcal{F}_{m-1}\Big)\stackrel{\mathsf{P}}{\longrightarrow}0.$$

However, these statements have already been proved, see Baran et al. [2005, Section 6, pp. 47-48] and Baran et al. [2005, Section 4, pp. 31-32], respectively. □

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References

Baran, S., Pap, G. (2012). Parameter estimation in a spatial unit root autoregressive model. *J. Multivariate Anal.* 107:282–305.

Baran, S., Pap, G. and Zuijlen, M.v. (2004). Asymptotic inference for an unstable spatial AR model. *Statistics* 38:465–482.

Baran, S., Pap, G. and Zuijlen, M.v. (2005). Asymptotic inference for unit roots in spatial triangular autoregression. Department of Mathematics, Radboud University Nijmegen, The Netherlands, *Report No.* 0506 (April 2005). Url: www.inf.unideb.hu/~barans/prepr.html.

Baran, S., Pap, G. and Zuijlen, M.v. (2007). Asymptotic inference for unit roots in spatial triangular autoregression. *Acta Appl. Math.* 96:17–42.

Basu, S. and Reinsel, G.C. (1993). Properties of the spatial unilateral first-order ARMA model. *Adv. in Appl. Probab.* 25:631–648.

Basu, S. and Reinsel, G.C. (1994). Regression models with spatially correlated errors. *J. Amer. Statist. Assoc.* 89:88–99.

Besag, J.E. (1972). On the correlation structure of some two dimensional stationary processes. *Biometrika* 59:43–48.

Bustos, O.H., Ruiz, M., Ojeda, S., Vallejos, R. and Alejandro, A.C. (2009). Asymptotic behavior of RA-estimates in autoregressive 2D processes. *J. Statist. Plann. Inference* 139:3649–3664.

Cullis, B.R. and Gleeson, A.C. (1991). Spatial analysis of field experiments — an extension to two dimensions. *Biometrics* 47:1449–1460.

Davydov, Y. and Paulauskas, V. (2008) On estimation of parameters for spatial autoregressive model *Stat. Inference Stoch. Process.* 11:237–247.

Hamilton, J.D. (1994). *Time Series Analysis.* Princeton:Princeton University Press.

Jacod, J. and Shiryaev, A.N. (1987). *Limit Theorems for Stochastic Processes.* Berlin:Springer–Verlag.

Martin, R.J. (1990). The use of time-series models and methods in the analysis of agricultural field trials. *Comm. Statist. -Theory Methods* 19:55–81.

Paulauskas, V. (2007). On unit roots for spatial autoregressive models. *J. Multivariate Anal.* 98:209–226.

Tory, E.M. and Pickard, G.K. (1992). Unilateral Gaussian fields. *Adv. Appl. Prob*. 24:95–112.

Sims, J.D., Stock, J.H. and Watson, M.W. (1990). Inference in linear time series models with some unit roots. *Econometrica* 58:113–144.

Vallejos, R., Ojeda, S. (2012). Image segmentation and time series clustering based on spatial and temporal ARMA processes. In Pei-Gee Peter Ho (Ed.), *Advances in Image Segmentation*. Rijeka:InTech.

Whittle, P. (1954). On stationary processes in the plane. *Biometrika* 41:434–449.