Pexiderized functional equations for vector products and quaternions

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Abstract

The purpose of the present paper is to solve the pexiderized versions of functional equations investigated by B. Nyul and G. Nyul [2], raised by the connection between products of quaternions and products of three-dimensional vectors.

1 Quaternionic products and vector products

Let $\mathbb{H} = \{r + x_1 i + x_2 j + x_3 k \mid r, x_1, x_2, x_3 \in \mathbb{R}\}$ be the skew field of quaternions with the basic relations $i^2 = j^2 = k^2 = ijk = -1$. Throughout this paper we use the following notions for a quaternion $h = r + x_1 i + x_2 j + x_3 k \in \mathbb{H}$: We say that h is purely imaginary if $r = 0$. The conjugate of h is $\overline{h} = r - x_1 i - x_2 j - x_3 k \in \mathbb{H}$, the absolute value of h is $|h| = \sqrt{r^2 + x_1^2 + x_2^2 + x_3^2}$, and the multiplicative inverse of h is $h^{-1} = \frac{1}{|h|^2}\overline{h}$ in case of $h \neq 0$. Quaternions $h_1 = r + x_1 i + x_2 j + x_3 k$, $h_2 = s + y_1 i + y_2 j + y_3 k \in \mathbb{H}$ are called orthogonal if $rs + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$, or equivalently if $h_1\overline{h_2} + h_2\overline{h_1} = 0$.

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If we identify purely imaginary quaternions $x_1 i + x_2 j + x_3 k$ with vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$, then the product of two purely imaginary quaternions is

$$\mathbf{xy} = -\langle \mathbf{x}, \mathbf{y} \rangle + \mathbf{x} \times \mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^3),$$

more generally the product of two arbitrary quaternions is

$$(r + \mathbf{x})(s + \mathbf{y}) = (rs - \langle \mathbf{x}, \mathbf{y} \rangle) + (s\mathbf{x} + r\mathbf{y} + \mathbf{x} \times \mathbf{y}) \quad (r, s \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^3),$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ and $\mathbf{x} \times \mathbf{y}$ denote the standard inner product (dot product) and the cross product of $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, respectively (see [1], [3], [4]).

Motivated by these connections between vector products and quaternions, B. Nyul and G. Nyul [2] solved functional equations

$$g(\mathbf{x})g(\mathbf{y}) = -\langle \mathbf{x}, \mathbf{y} \rangle + g(\mathbf{x} \times \mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^3) \tag{1}$$

and

$$f(r, \mathbf{x})f(s, \mathbf{y}) = -\langle \mathbf{x}, \mathbf{y} \rangle + f(rs, s\mathbf{x} + r\mathbf{y} + \mathbf{x} \times \mathbf{y}) \quad (r, s \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^3) \tag{2}$$

in functions $g : \mathbb{R}^3 \to \mathbb{H}$ and $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{H}$.

In the following theorems we determine all the solutions of the pexiderized versions of these equations, namely we completely solve functional equations

$$g_1(\mathbf{x})g_2(\mathbf{y}) = -\langle \mathbf{x}, \mathbf{y} \rangle + g_3(\mathbf{x} \times \mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^3) \tag{3}$$

and

$$f_1(r, \mathbf{x})f_2(s, \mathbf{y}) = -\langle \mathbf{x}, \mathbf{y} \rangle + f_3(rs, s\mathbf{x} + r\mathbf{y} + \mathbf{x} \times \mathbf{y}) \quad (r, s \in \mathbb{R}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^3) \tag{4}$$

in functions $g_1, g_2, g_3 : \mathbb{R}^3 \to \mathbb{H}$ and $f_1, f_2, f_3 : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{H}$.

Theorem 1. *The functions $g_1, g_2, g_3 : \mathbb{R}^3 \to \mathbb{H}$ satisfy (3) if and only if there exist pairwise orthogonal nonzero quaternions $h_1, h_2, h_3 \in \mathbb{H}$ with equal absolute values such that*

$$\begin{aligned}
g_1((x_1, x_2, x_3)) &= x_1 h_1 + x_2 h_2 + x_3 h_3, \\
g_2((x_1, x_2, x_3)) &= -x_1 h_1^{-1} - x_2 h_2^{-1} - x_3 h_3^{-1}, \\
g_3((x_1, x_2, x_3)) &= -x_1 h_2 h_3^{-1} - x_2 h_3 h_1^{-1} - x_3 h_1 h_2^{-1}.
\end{aligned}$$

Theorem 2. *The functions $f_1, f_2, f_3 : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{H}$ satisfy (4) if and only if there exist pairwise orthogonal nonzero quaternions $h_1, h_2, h_3 \in \mathbb{H}$ with equal absolute values such that*

$$\begin{aligned}
f_1(r, (x_1, x_2, x_3)) &= r h_2 h_3^{-1} h_1 + x_1 h_1 + x_2 h_2 + x_3 h_3, \\
f_2(r, (x_1, x_2, x_3)) &= -r h_1^{-1} h_2 h_3^{-1} - x_1 h_1^{-1} - x_2 h_2^{-1} - x_3 h_3^{-1}, \\
f_3(r, (x_1, x_2, x_3)) &= r - x_1 h_2 h_3^{-1} - x_2 h_3 h_1^{-1} - x_3 h_1 h_2^{-1}.
\end{aligned}$$

2 Proofs

Proof of Theorem 1. We prove this theorem through seven claims.

Claim 1: $g_m(\mathbf{0}) = 0$ $(m = 1, 2, 3)$, $g_m(\mathbf{v}) = 0$ if only if $\mathbf{v} = \mathbf{0}$ $(m = 1, 2)$.

Substitute $(\mathbf{x}, \mathbf{y}) = (\mathbf{0}, \mathbf{v})$, $(\mathbf{v}, \mathbf{0})$ and $(\mathbf{v}, \mathbf{v})$ into (3) $(\mathbf{v} \in \mathbb{R}^3)$:

$$\begin{aligned} g_1(\mathbf{0})g_2(\mathbf{v}) &= g_3(\mathbf{0}), \\ g_1(\mathbf{v})g_2(\mathbf{0}) &= g_3(\mathbf{0}), \\ g_1(\mathbf{v})g_2(\mathbf{v}) &= -\langle \mathbf{v}, \mathbf{v} \rangle + g_3(\mathbf{0}). \end{aligned}$$

If $g_1(\mathbf{0}) \neq 0$, then the first equation gives $g_2(\mathbf{v}) = g_1(\mathbf{0})^{-1} g_3(\mathbf{0})$. When $g_3(\mathbf{0}) = 0$, we have $g_2(\mathbf{v}) = 0$. While in case of $g_3(\mathbf{0}) \neq 0$, it follows from the second equation that $g_1(\mathbf{v}) = g_1(\mathbf{0})$. In both cases the third equation becomes $0 = -\langle \mathbf{v}, \mathbf{v} \rangle$ for any $\mathbf{v} \in \mathbb{R}^3$, which is a contradiction. This means that $g_1(\mathbf{0}) = 0$.

Similarly, it can be shown that $g_2(\mathbf{0}) = 0$, and $g_3(\mathbf{0}) = 0$ follows from the first or the second equation. Then the third equation, $g_1(\mathbf{v})g_2(\mathbf{v}) = -\langle \mathbf{v}, \mathbf{v} \rangle$, implies that $g_1(\mathbf{v}) \neq 0$ and $g_2(\mathbf{v}) \neq 0$ for $\mathbf{v} \neq \mathbf{0}$.

In the rest of the proof, we shall use Claim 1 without referring to it.

Claim 2: g_1 and g_2 are homogeneous.

Substitute $(\mathbf{x}, \mathbf{y}) = (\mathbf{v}, \mathbf{v})$ and $(\lambda\mathbf{v}, \mathbf{v})$ into (3) $(\lambda \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^3)$:

$$\begin{aligned} g_1(\mathbf{v})g_2(\mathbf{v}) &= -\langle \mathbf{v}, \mathbf{v} \rangle, \\ g_1(\lambda\mathbf{v})g_2(\mathbf{v}) &= -\lambda\langle \mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

If we multiply the first equation by λ, we easily get $g_1(\lambda\mathbf{v}) = \lambda g_1(\mathbf{v})$ for $\mathbf{v} \neq \mathbf{0}$. Moreover, it is obviously true when $\mathbf{v} = \mathbf{0}$.

Homogeneity of g_2 can be proved similarly.

Claim 3: g_3 is homogeneous.

Let $\mathbf{v} \in \mathbb{R}^3$. Choose $\mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{w} \neq \mathbf{0}$ and $\mathbf{v}$, $\mathbf{w}$ being orthogonal. Then substitute $(\mathbf{x}, \mathbf{y}) = (\frac{1}{\|\mathbf{w}\|^2}(\mathbf{w} \times \mathbf{v}), \mathbf{w})$ and $(\frac{\lambda}{\|\mathbf{w}\|^2}(\mathbf{w} \times \mathbf{v}), \mathbf{w})$ into (3) $(\lambda \in \mathbb{R})$:

$$\begin{aligned} g_1(\frac{1}{\|\mathbf{w}\|^2}(\mathbf{w} \times \mathbf{v}))g_2(\mathbf{w}) &= g_3(\mathbf{v}), \\ g_1(\frac{\lambda}{\|\mathbf{w}\|^2}(\mathbf{w} \times \mathbf{v}))g_2(\mathbf{w}) &= g_3(\lambda\mathbf{v}). \end{aligned}$$

Since g_1 is homogeneous by Claim 2, it follows that g_3 is also homogeneous.

Claim 4: g_1 and g_2 are additive.

Substitute $(\mathbf{x}, \mathbf{y}) = (\mathbf{v} + \mathbf{w}, \mathbf{w})$, $(\mathbf{v}, \mathbf{w})$ and $(\mathbf{w}, \mathbf{w})$ into (3) ($\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$):

$$\begin{aligned} g_1(\mathbf{v} + \mathbf{w}) g_2(\mathbf{w}) &= -\langle \mathbf{v} + \mathbf{w}, \mathbf{w} \rangle + g_3(\mathbf{v} \times \mathbf{w}), \\ g_1(\mathbf{v}) g_2(\mathbf{w}) &= -\langle \mathbf{v}, \mathbf{w} \rangle + g_3(\mathbf{v} \times \mathbf{w}), \\ g_1(\mathbf{w}) g_2(\mathbf{w}) &= -\langle \mathbf{w}, \mathbf{w} \rangle. \end{aligned}$$

After adding the second and the third equations, together with the first one they give that $g_1(\mathbf{v} + \mathbf{w}) = g_1(\mathbf{v}) + g_1(\mathbf{w})$ if $\mathbf{w} \neq \mathbf{0}$. This relation also holds for $\mathbf{w} = \mathbf{0}$.

It can be deduced similarly that g_2 is additive.

Claim 5: g_3 is additive.

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ be arbitrary vectors. Choose $\mathbf{u} \in \mathbb{R}^3$ such that $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u}$ is orthogonal to $\mathbf{v}$, $\mathbf{w}$ (any nonzero vector from the orthogonal complement of the subspace generated by $\mathbf{v}$ and $\mathbf{w}$). Now substitute $(\mathbf{x}, \mathbf{y}) = (\frac{1}{\|\mathbf{u}\|^2}(\mathbf{u} \times (\mathbf{v} + \mathbf{w})), \mathbf{u})$, $(\frac{1}{\|\mathbf{u}\|^2}(\mathbf{u} \times \mathbf{v}), \mathbf{u})$ and $(\frac{1}{\|\mathbf{u}\|^2}(\mathbf{u} \times \mathbf{w}), \mathbf{u})$ into (3):

$$\begin{aligned} g_1(\frac{1}{\|\mathbf{u}\|^2}(\mathbf{u} \times (\mathbf{v} + \mathbf{w}))) g_2(\mathbf{u}) &= g_3(\mathbf{v} + \mathbf{w}), \\ g_1(\frac{1}{\|\mathbf{u}\|^2}(\mathbf{u} \times \mathbf{v})) g_2(\mathbf{u}) &= g_3(\mathbf{v}), \\ g_1(\frac{1}{\|\mathbf{u}\|^2}(\mathbf{u} \times \mathbf{w})) g_2(\mathbf{u}) &= g_3(\mathbf{w}). \end{aligned}$$

Then the additivity of g_3 is an immediate consequence of the same property of g_1 by Claim 4.

Claim 6: Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$. Furthermore, let $g_1(\mathbf{e}_m) = h_m \in \mathbb{H} \setminus \{0\}$ $(m = 1,2,3)$. Then $g_2(\mathbf{e}_m) = -h_m^{-1}$, $g_3(\mathbf{e}_m) = -h_{m+1} h_{m+2}^{-1}$ (through Claim 6, subscripts have to be understood modulo 3), and h_1, h_2, h_3 are pairwise orthogonal quaternions with equal absolute values.

If we substitute $(\mathbf{x}, \mathbf{y}) = (\mathbf{e}_m, \mathbf{e}_m)$ into (3), we get $g_1(\mathbf{e}_m) g_2(\mathbf{e}_m) = -1$, hence $g_2(\mathbf{e}_m) = -h_m^{-1}$. In addition, substituting $(\mathbf{x}, \mathbf{y}) = (\mathbf{e}_{m+1}, \mathbf{e}_{m+2})$ into (3), we obtain $g_3(\mathbf{e}_m) = g_1(\mathbf{e}_{m+1}) g_2(\mathbf{e}_{m+2}) = -h_{m+1} h_{m+2}^{-1}$.

Substitution $(\mathbf{x}, \mathbf{y}) = (\mathbf{e}_{m+2}, \mathbf{e}_{m+1})$ into (3) gives $g_1(\mathbf{e}_{m+2}) g_2(\mathbf{e}_{m+1}) = g_3(-\mathbf{e}_m)$. Using Claim 3 and the previously proved parts of Claim 6, this yields $-h_{m+2} h_{m+1}^{-1} = h_{m+1} h_{m+2}^{-1}$. Taking absolute values, $\frac{|h_{m+2}|}{|h_{m+1}|} = \frac{|h_{m+1}|}{|h_{m+2}|}$, that is $|h_{m+1}| = |h_{m+2}|$. Then $-h_{m+2} h_{m+1}^{-1} = h_{m+1} h_{m+2}^{-1}$ is equivalent to

$-h_{m+2}\overline{h_{m+1}} = h_{m+1}\overline{h_{m+2}}$, which means that h_{m+1} and h_{m+2} are orthogonal quaternions.

Claim 7: The solutions of (3) are exactly the functions given in the theorem.

By linearity of g_m, $g_m((x_1, x_2, x_3)) = x_1 g_m(\mathbf{e}_1) + x_2 g_m(\mathbf{e}_2) + x_3 g_m(\mathbf{e}_3)$ $(m = 1, 2, 3)$. From Claim 6 we arrive at the desired formulas. Direct calculation shows that these are indeed solutions of (3). $\square$

Proof of Theorem 2. As we shall see, Theorem 1 will be an important tool in solving (4).

Claim 1: Let $g_m(\mathbf{x}) = f_m(0, \mathbf{x})$ $(\mathbf{x} \in \mathbb{R}^3)$ $(m = 1, 2, 3)$. Then the statements of Theorem 1 and the claims in its proof hold for g_1, g_2, g_3.

Substituting $(r, s) = (0, 0)$ into (4), we get

$$f_1(0, \mathbf{x}) f_2(0, \mathbf{y}) = -\langle \mathbf{x}, \mathbf{y} \rangle + f_3(0, \mathbf{x} \times \mathbf{y}),$$

which means that g_1, g_2, g_3 satisfy (3).

Claim 2: $f_1(\lambda, \mathbf{0}) = \lambda h_2 h_3^{-1} h_1$, $f_2(\lambda, \mathbf{0}) = -\lambda h_1^{-1} h_2 h_3^{-1}$ and $f_3(\lambda, \mathbf{0}) = \lambda$ $(\lambda \in \mathbb{R})$.

Substitute $(r, \mathbf{x}, s, \mathbf{y}) = (\lambda, \mathbf{0}, 0, \mathbf{e}_1)$, $(0, \mathbf{e}_1, \lambda, \mathbf{0})$ and $(\lambda, \mathbf{0}, 1, \mathbf{0})$ into (4):

$$\begin{aligned} f_1(\lambda, \mathbf{0}) f_2(0, \mathbf{e}_1) &= f_3(0, \lambda \mathbf{e}_1), \\ f_1(0, \mathbf{e}_1) f_2(\lambda, \mathbf{0}) &= f_3(0, \lambda \mathbf{e}_1), \\ f_1(\lambda, \mathbf{0}) f_2(1, \mathbf{0}) &= f_3(\lambda, \mathbf{0}). \end{aligned}$$

From the first equation, using homogeneity of g_3 and Claim 6 of Theorem 1, it follows that $-f_1(\lambda, \mathbf{0}) h_1^{-1} = -\lambda h_2 h_3^{-1}$, thus $f_1(\lambda, \mathbf{0}) = \lambda h_2 h_3^{-1} h_1$.

In a similar way, from the second equation we get $h_1 f_2(\lambda, \mathbf{0}) = -\lambda h_2 h_3^{-1}$, hence $f_2(\lambda, \mathbf{0}) = -\lambda h_1^{-1} h_2 h_3^{-1}$.

Finally, the third equation gives $f_3(\lambda, \mathbf{0}) = (\lambda h_2 h_3^{-1} h_1)(-h_1^{-1} h_2 h_3^{-1}) = \lambda h_2 h_3^{-1} h_3 h_2^{-1} = \lambda$.

Claim 3: $f_m(\lambda, \mathbf{v}) = f_m(\lambda, \mathbf{0}) + f_m(0, \mathbf{v})$ $(\lambda \in \mathbb{R},\ \mathbf{v} \in \mathbb{R}^3)$ $(m = 1, 2)$.

Substitute $(r, \mathbf{x}, s, \mathbf{y}) = (\lambda, \mathbf{v}, 0, \mathbf{v})$, $(\lambda, \mathbf{0}, 0, \mathbf{v})$ and $(0, \mathbf{v}, 0, \mathbf{v})$ into (4):

$$\begin{aligned} f_1(\lambda, \mathbf{v}) f_2(0, \mathbf{v}) &= -\langle \mathbf{v}, \mathbf{v} \rangle + f_3(0, \lambda \mathbf{v}), \\ f_1(\lambda, \mathbf{0}) f_2(0, \mathbf{v}) &= f_3(0, \lambda \mathbf{v}), \\ f_1(0, \mathbf{v}) f_2(0, \mathbf{v}) &= -\langle \mathbf{v}, \mathbf{v} \rangle. \end{aligned}$$

Comparing the first equation with the sum of the other two equations, we get $f_1(\lambda, \mathbf{v}) = f_1(\lambda, \mathbf{0}) + f_1(0, \mathbf{v})$ if $\mathbf{v} \neq \mathbf{0}$. But this is clearly true for $\mathbf{v} = \mathbf{0}$, too.

Similarly, this claim can be shown for f_2.

Claim 4: $f_3(\lambda,\mathbf{v}) = f_3(\lambda,\mathbf{0}) + f_3(0,\mathbf{v})$ ($\lambda \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^3$).

Substitute $(r,\mathbf{x},s,\mathbf{y}) = (\lambda,\mathbf{v},1,\mathbf{0})$, $(\lambda,\mathbf{0},1,\mathbf{0})$ and $(0,\mathbf{v},1,\mathbf{0})$ into (4):

$$\begin{aligned} f_1(\lambda,\mathbf{v})f_2(1,\mathbf{0}) &= f_3(\lambda,\mathbf{v}),\\ f_1(\lambda,\mathbf{0})f_2(1,\mathbf{0}) &= f_3(\lambda,\mathbf{0}),\\ f_1(0,\mathbf{v})f_2(1,\mathbf{0}) &= f_3(0,\mathbf{v}). \end{aligned}$$

Now it is straightforward that Claim 3 implies our statement for f_3, because $f_2(1,\mathbf{0}) = -h_1^{-1}h_2h_3^{-1} \neq 0$ by Claim 2.

Claim 5: The solutions of (4) are precisely the functions given in the theorem.

This is an easy consequence of Claims 1, 2, 3, 4, and a direct verification that these functions are indeed solutions of (4). □

3 Special cases

As a special case, if we are interested in solving the partially pexiderized version of (3) with $g_1 = g_2$, that is the functional equation

$$g_1(\mathbf{x})g_1(\mathbf{y}) = -\langle \mathbf{x},\mathbf{y}\rangle + g_3(\mathbf{x}\times\mathbf{y}) \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}^3) \tag{5}$$

in functions $g_1, g_3 : \mathbb{R}^3 \to \mathbb{H}$, we can easily describe its solutions.

Corollary. *The functions $g_1, g_3 : \mathbb{R}^3 \to \mathbb{H}$ satisfy (5) if and only if there exist pairwise orthogonal purely imaginary quaternions $h_1, h_2, h_3 \in \mathbb{H}$ with absolute values 1 such that*

$$\begin{aligned} g_1((x_1,x_2,x_3)) &= x_1h_1 + x_2h_2 + x_3h_3,\\ g_3((x_1,x_2,x_3)) &= x_1h_2h_3 + x_2h_3h_1 + x_3h_1h_2. \end{aligned}$$

Proof of Corollary. For $g_1 = g_2$ in Theorem 1, we need to have $h_m = -h_m^{-1}$ ($m = 1,2,3$). Taking absolute values, we get $|h_m| = \frac{1}{|h_m|}$, hence $|h_m| = 1$. Then $h_m = -h_m^{-1}$ is equivalent to $h_m = -\overline{h_m}$, in other words h_m is purely imaginary.

On the other hand, it can be checked that the given functions are solutions of (5). □

Remark. As a consequence, we obtain that $g : \mathbb{R}^3 \to \mathbb{H}$ is a solution of functional equation (1) if and only if there exist orthogonal purely imaginary quaternions $h_1, h_2 \in \mathbb{H}$ with absolute values 1 such that $g((x_1, x_2, x_3)) = x_1 h_1 + x_2 h_2 + x_3 h_1 h_2$, which is a reformulation of Theorem 1 in [2].

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