Curvature collineations in spray manifolds

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Abstract. Curvature collineations of a spray manifold induced by the Lie symmetries of the underlying spray are studied. The basic observation is that the Jacobi endomorphism and the Berwald curvature are invariant under these symmetries; this implies the invariance of the further curvature data. Our main technical tool is an appropriate Lie derivative operator along the tangent bundle projection.

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1 Introduction

Curvature collineations play an important role in the study of geometry and physics of classical space-times; for an excellent account on the subject we refer to G. S. Hall's book [5], especially its last chapter. On the other hand, Finsler geometry has appeared in a number of different physical contexts in recent years. Its increasing importance from the perspective of fundamental physical theories offers good reason for an elaboration of the theory of curvature collineation in this generality and in the language of present-day differential geometry. In the framework of the classical tensor calculus we have quite a large literature of curvature collineations in Finsler manifolds; a good survey of the state of art in 1993 is presented by R. B. Misra's paper [7]. The first steps toward a modern theory were taken in the papers of R. L. Lovas [6] and M. Crampin-D. J. Saunders [4]. The task is no easy one, since we have no unique distinguished covariant derivative operator in Finsler geometry. However, there is a unifying concept which makes a systematic and transparent treatment possible. This is the concept of a *spray*, which plays an essential role both in the foundations of (semi-) Riemannian geometry (see [1]) and Finsler geometry (see, e.g., [3], [9]). As a matter of fact, this is not a new discovery either. It was already known by Ludwig Berwald, as it turns out (among others) from his posthumous paper [2], that through the so-called *affine deviation tensor* (nowadays called also *Jacobi endomorphism*) of a spray a whole series of curvature quantities may be constructed. So it seems to be reasonable to start a systematic study of Finslerian curvature collineations in the generality of spray manifolds without any additional structure. Note that there is

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no complete consensus in the usage of the term 'spray' in differential geometry (and analysis). Our definition in section 4 will be in harmony with demands of Finsler geometry. In this spirit, the main scene of our considerations will be the pull-back bundle $(\mathring{T}M \times_M TM, \pi, \mathring{T}M)$ of the tangent bundle (TM, τ, M) over the projection of the deleted bundle $(\mathring{T}M, \mathring{\tau}, M)$. This setting is sketched in section 2.

A *curvature collineation* of a curvature tensor $\mathbf{C}$ of a spray manifold is a projectable vector field ξ on $\mathring{T}M$ such that the Lie derivative $\widetilde{\mathcal{L}}_\xi \mathbf{C}$ vanishes. Here $\mathbf{C}$ is a 'tensor along $\mathring{\tau}$', so we also need a concept of the Lie derivative of the sections of the pull-back bundle π (which is extended also to the whole tensor algebra of the module of sections of π). This generalization of the Lie derivative, together with some crucial commutation rules, is briefly discussed in section 3. For more details, we refer to [6] and [10].

In section 4, with the help of the affine deviation tensor, first we introduce the basic curvature data (affine and projective curvatures, Berwald curvature, Douglas curvature) of a spray manifold (M, S). Next we show that if a vector field X on M is a Lie symmetry of S, i.e., $\mathcal{L}_{X^c} S = [X^c, S] = 0$, then it is a curvature collineation for all the curvatures mentioned above. This is an expected result, however, to derive it, one has to overcome a lot of difficulties.

2 Preliminary constructions

Throughout the paper, M will be an n-dimensional smooth manifold ($n \geq 1$) whose underlying topological space is Hausdorff and second countable. The tangent bundle of M is the triplet (TM, τ, M), which will be denoted simply by its projection map τ. The deleted bundle for τ is $(\mathring{T}M, \mathring{\tau}, M)$, where $\mathring{T}M := \bigcup_{p\in M}(T_pM \setminus \{0_p\})$ (0_p is the zero vector in the tangent space T_pM) and $\mathring{\tau} := \tau \restriction \mathring{T}M$. The majority of our objects lives in the pull-back bundle $(\mathring{T}M \times_M TM, \pi, \mathring{T}M)$, $\pi := pr_1 \restriction \mathring{T}M \times_M TM$ of τ by $\mathring{\tau}$. The shorthand for this vector bundle will be π. The $C^\infty(\mathring{T}M)$-module of sections of π is denoted by $\mathrm{Sec}(\pi)$. (By a section we mean a smooth section.) A generic section of π is of the form $\widetilde{X}\colon u \in \mathring{T}M \mapsto (u, \underline{X}(u)) \in \mathring{T}M \times_M TM$, where $\underline{X}\colon \mathring{T}M \to TM$ is a smooth mapping such that $\tau \circ \underline{X} = \mathring{\tau}$, called the *principal part* of $\widetilde{X}$. If $X \in \mathfrak{X}(M)$, i.e., X is a vector field on M, then

$$\widehat{X}\colon u \in \mathring{T}M \mapsto (u, X \circ \mathring{\tau}(u)) \in \mathring{T}M \times_M TM$$

is a section of π, called a *basic section. Basic sections generate the module* $\mathrm{Sec}(\pi)$. We have a *canonical section* δ in $\mathrm{Sec}(\pi)$ whose principal part is $1_{\mathring{T}M}$. Thus for every $u \in \mathring{T}M$, $\delta(u) = (u, u)$. Most of our further canonical objects may be identified from the short exact sequence

$$0 \to \mathrm{Sec}(\pi) \xrightarrow{\mathbf{i}} \mathfrak{X}(\mathring{T}M) \xrightarrow{\mathbf{j}} \mathrm{Sec}(\pi) \to 0 \tag{2.1}$$

of $C^\infty(\mathring{T}M)$-homomorphisms defined by

$$\mathbf{i}(\widehat{X}) := X^{\mathsf{v}}; \quad \mathbf{j}(X^{\mathsf{v}}) := 0,\ \mathbf{j}(X^{\mathsf{c}}) := \widehat{X}, \tag{2.2}$$

where X^{v} and X^{c} are the vertical and complete lifts of $X \in \mathfrak{X}(M)$. Then $C := \mathbf{i}(\delta)$ is a canonical vertical vector field on $\mathring{T}M$, called the *Liouville vector field*, $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is a canonical type $(1,1)$ tensor field on $\mathring{T}M$, the *vertical endomorphism.*

Given a vector field η on $\mathring{T}M$, we define a (1,1) tensor field $[\mathbf{J},\eta]$ on $\mathring{T}M$ by

$$[\mathbf{J},\eta]\xi := [\mathbf{J}\xi,\eta] - \mathbf{J}[\xi,\eta], \quad \xi \in \mathfrak{X}(\mathring{T}M). \tag{2.3}$$

Then, in particular,

$$[\mathbf{J},X^{\mathsf{c}}] = [\mathbf{J},X^{\mathsf{v}}] = 0 \text{ for all } X \in \mathfrak{X}(M). \tag{2.4}$$

We denote by $\mathcal{T}^k_l(\pi)$ the $C^\infty(\mathring{T}M)$-module of type (k,l) tensors on the module $\mathrm{Sec}(\pi)$, and write $\mathcal{T}(\pi)$ for the full tensor algebra of $\mathrm{Sec}(\pi)$. Elements of $\mathcal{T}(\pi)$ are also mentioned as *tensors along* $\mathring{\tau}$. If $\mathbf{A} \in \mathcal{T}^1_{l+1}(\pi)$, we define its *trace* $\operatorname{tr}\mathbf{A} \in \mathcal{T}^0_l(\pi)$ by

$$(\operatorname{tr}\mathbf{A})(\widetilde{X}_1,\dots,\widetilde{X}_l) := \operatorname{tr}(\widetilde{Z} \mapsto \mathbf{A}(\widetilde{Z},\widetilde{X}_1,\dots,\widetilde{X}_l))$$

for $\widetilde{X}_1,\dots\widetilde{X}_l \in \mathrm{Sec}(\pi)$.

We recall that an $\mathbb{R}$-linear mapping $D\colon \mathcal{T}(\pi) \to \mathcal{T}(\pi)$ which preserves type and commutes with contractions is called a *derivation of* $\mathcal{T}(\pi)$ or a *derivation along* $\mathring{\tau}$, if it satisfies the *Leibniz rule* $D(\mathbf{A}\otimes\mathbf{B}) = (D\mathbf{A})\otimes\mathbf{B}+\mathbf{A}\otimes(D\mathbf{B})$; $\mathbf{A},\mathbf{B} \in \mathcal{T}(\pi)$. Willmore's classical theorem on tensor derivations (see, e.g., [8], p. 45–46) may immediately be translated in this context. So we have

Lemma 2.1. *Any derivation of* $\mathcal{T}(\pi)$ *is completely determined by its action over* $\mathcal{T}^0_0(\pi) := C^\infty(\mathring{T}M)$ *and* $\mathcal{T}^1_0(\pi) \cong \mathrm{Sec}(\pi)$*. Conversely, given a vector field* $\eta \in \mathfrak{X}(\mathring{T}M)$ *and an* $\mathbb{R}$*-linear mapping* $D_0\colon \mathrm{Sec}(\pi) \to \mathrm{Sec}(\pi)$ *such that*

$$D_0(f\widetilde{Y}) = (\eta f)\widetilde{Y} + fD_0\widetilde{Y} \text{ for all } f \in C^\infty(\mathring{T}M),\ \widetilde{Y} \in \mathrm{Sec}(\pi),$$

there exists a (necessarily unique) derivation D *along* $\mathring{\tau}$ *such that* $D \upharpoonright C^\infty(\mathring{T}M) = \eta$ *and* $D \upharpoonright \mathrm{Sec}(\pi) = D_0$.

By an *Ehresmann connection* in $\mathring{T}M$ we mean a right splitting of the exact sequence (2.1), i.e., a $C^\infty(\mathring{T}M)$-linear mapping $\mathcal{H}\colon \mathrm{Sec}(\pi) \to \mathfrak{X}(\mathring{T}M)$ such that $\mathbf{j}\circ\mathcal{H} = 1_{\mathrm{Sec}(\pi)}$. The *vertical mapping* associated to $\mathcal{H}$ is a left splitting of (2.1) satisfying $\mathrm{Ker}(\mathcal{V}) = \mathrm{Im}(\mathcal{H})$. So we get the 'double exact sequence'

$$0 \rightleftarrows \mathrm{Sec}(\pi) \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftarrows}} \mathfrak{X}(\mathring{T}M) \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftarrows}} \mathrm{Sec}(\pi) \rightleftarrows 0.$$

The mappings $\mathbf{h} := \mathcal{H}\circ\mathbf{j}$ and $\mathbf{v} := \mathbf{i}\circ\mathcal{V} = 1_{\mathfrak{X}(\mathring{T}M)} - \mathbf{h}$ are the *horizontal* and the *vertical projection* associated to $\mathcal{H}$, respectively. The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ (with respect to $\mathcal{H}$) is $X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \mathbf{h}(X^{\mathsf{c}})$. The Ehresmann connection $\mathcal{H}$ is said to be *homogeneous* if $[C,X^{\mathsf{h}}] = 0$ for all $X \in \mathfrak{X}(M)$.

We define the *vertical differential* of a function $F \in C^\infty(\mathring{T}M)$ and a section $\widetilde{Y} \in \mathrm{Sec}(\pi)$ as the 1-form $\nabla^{\mathsf{v}}F \in \mathcal{T}^0_1(\pi)$ and the $(1,1)$-tensor $\nabla^{\mathsf{v}}\widetilde{Y} \in \mathcal{T}^1_1(\pi)$ given by

$$\nabla^{\mathsf{v}}F(\widetilde{X}) := (\mathbf{i}\widetilde{X})F, \quad \widetilde{X} \in \mathrm{Sec}(\pi) \tag{2.5}$$

and

$$\nabla^{\mathsf{v}}\widetilde{Y}(\widetilde{X}) := \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X},\eta];\ \widetilde{X} \in \mathrm{Sec}(\pi),\ \eta \in \mathfrak{X}(\mathring{T}M),\ \mathbf{j}\eta = \widetilde{Y}. \tag{2.6}$$

respectively. With the help of an 'auxiliary' Ehresmann connection $\mathcal{H}$, (2.6) may be written in the more convenient form

$$\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]. \tag{2.7}$$

Nevertheless, ∇^{v} is a canonical operator which does not depend on any extra structure in π. Notice that basic sections have vanishing vertical derivatives:

$$\nabla^{\mathsf{v}}_{\widetilde{X}}\widehat{Y} = 0 \text{ for all } \widetilde{X} \in \mathrm{Sec}(\pi),\ Y \in \mathfrak{X}(M). \tag{2.8}$$

Indeed $\nabla^{\mathsf{v}}_{\widetilde{X}}\widehat{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, Y^{\mathsf{h}}] = 0$, since $[\mathbf{i}\widetilde{X}, Y^{\mathsf{h}}]$ is a vertical vector field, and hence belongs to $\mathrm{Ker}\,(\mathbf{j})$.

By Lemma 2.1, the operators $\nabla^{\mathsf{v}}_{\widetilde{X}}$ ($\widetilde{X} \in \mathrm{Sec}(\pi)$) can be uniquely extended to derivations of the full tensor algebra $\mathcal{T}(\pi)$. So we may also form the vertical differential of any tensor along $\mathring{\tau}$. If, for example, $\mathbf{A} \in \mathcal{T}^1_k(\pi)$, then $\nabla^{\mathsf{v}}\mathbf{A} \in \mathcal{T}^1_{k+1}(\pi)$ is given by

$$\nabla^{\mathsf{v}}\mathbf{A}(\widetilde{X}, \widetilde{Y}_1, \dots, \widetilde{Y}_k) := (\nabla^{\mathsf{v}}_{\widetilde{X}}\mathbf{A})(\widetilde{Y}_1, \dots \widetilde{Y}_k); \quad \widetilde{X}, \widetilde{Y}_i \in \mathrm{Sec}(\pi),\ i \in \{1, \dots, k\}.$$

If an Ehresmann connection $\mathcal{H}$ is specified in $\mathring{T}M$, we define the *h-Berwald differential* ∇^{h} in the same way as the vertical differential, but with the starting steps

$$\nabla^{\mathsf{h}}F(\widetilde{X}) := (\mathcal{H}\widetilde{X})F; \quad F \in C^\infty(\mathring{T}M),\ \widetilde{X} \in \mathrm{Sec}(\pi); \tag{2.9}$$

$$\nabla^{\mathsf{h}}\widetilde{Y}(\widetilde{X}) := \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]; \quad \widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\pi). \tag{2.10}$$

3 Lie derivative along $\mathring{\tau}$

We recall that a vector field ξ on $\mathring{T}M$ is said to be *projectable* if there is a vector field X on M such that $\mathring{\tau}_* \circ \xi = X \circ \mathring{\tau}$.

Lemma 3.1. *Let a projectable vector field $\xi \in \mathfrak{X}(\mathring{T}M)$ be given. There exists a unique derivation $\widetilde{\mathcal{L}}_\xi$ along $\mathring{\tau}$ such that*

$$\widetilde{\mathcal{L}}_\xi F := \xi F \quad \text{if } F \in C^\infty(\mathring{T}M), \tag{3.1}$$

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}] \quad \text{if } \widetilde{Y} \in \mathrm{Sec}(\pi). \tag{3.2}$$

Proof. Since ξ is a projectable and $\mathbf{i}\widetilde{Y}$ is a vertical vector field on $\mathring{T}M$, their Lie bracket is also vertical, so the right-hand side of (3.2) yields a well-defined section of π. For any function $F \in C^\infty(\mathring{T}M)$ we have

$$\widetilde{\mathcal{L}}_\xi F\widetilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}(F\widetilde{Y})] = \mathbf{i}^{-1}[\xi, F(\mathbf{i}\widetilde{Y})] = \mathbf{i}^{-1}(F[\xi, \mathbf{i}\widetilde{Y}] + (\xi F)\mathbf{i}\widetilde{Y}) = (\xi F)\widetilde{Y} + F\widetilde{\mathcal{L}}_\xi\widetilde{Y},$$

so our assertion follows by Lemma 2.1. □

The derivation $\widetilde{\mathcal{L}}_\xi$ is called the *Lie derivative along* $\mathring{\tau}$ with respect to ξ.

If $\mathcal{H}$ is an arbitrarily chosen Ehresmann connection in $\mathring{T}M$, then $[\xi, \mathbf{i}\widetilde{Y}] = \mathbf{v}[\xi, \mathbf{i}\widetilde{Y}] = \mathbf{i} \circ \mathcal{V}[\xi, \mathbf{i}\widetilde{Y}]$, so (3.2) may also be written in the form

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y} = \mathcal{V}[\xi, \mathbf{i}\widetilde{Y}]. \tag{3.3}$$

(However, $\widetilde{\mathcal{L}}_\xi$ is a natural operator!) We mention some frequently used special cases.

Lemma 3.2. *For any vector field X on M,*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y} = \widehat{[X,Y]} = \widehat{\mathcal{L}_X Y}, \quad Y \in \mathfrak{X}(M); \tag{3.4}$$

$$\widetilde{\mathcal{L}}_{X^{\mathsf{v}}}\widetilde{Y} = \nabla^{\mathsf{v}}_{\widehat{X}}\widetilde{Y}, \quad \widetilde{Y} \in \operatorname{Sec}(\pi). \tag{3.5}$$

If an Ehresmann connection $\mathcal{H}$ is specified in $\mathring{T}M$, then

$$\widetilde{\mathcal{L}}_{X^{\mathsf{h}}}\widetilde{Y} = \nabla^{\mathsf{h}}_{\widehat{X}}\widetilde{Y}. \tag{3.6}$$

Thus $\widetilde{\mathcal{L}}_{X^{\mathsf{v}}} = \nabla^{\mathsf{v}}_{\widehat{X}}, \quad \widetilde{\mathcal{L}}_{X^{\mathsf{h}}} = \nabla^{\mathsf{h}}_{\widehat{X}}$.

Proof. Relation (3.4) is a direct consequence of the definition (3.2), or of (3.3). To check (3.5), observe that from (2.4)

$$0 = [\mathbf{J}, X^{\mathsf{v}}]\mathcal{H}\widetilde{Y} \overset{(2.3)}{=} [\mathbf{J}\mathcal{H}\widetilde{Y}, X^{\mathsf{v}}] - \mathbf{J}[\mathcal{H}\widetilde{Y}, X^{\mathsf{v}}] = [\mathbf{i}\widetilde{Y}, X^{\mathsf{v}}] - \mathbf{J}[\mathcal{H}\widetilde{Y}, X^{\mathsf{v}}],$$

hence

$$\mathbf{J}[X^{\mathsf{v}}, \mathcal{H}\widetilde{Y}] = [X^{\mathsf{v}}, \mathbf{i}\widetilde{Y}], \tag{3.7}$$

therefore $\widetilde{\mathcal{L}}_{X^{\mathsf{v}}}\widetilde{Y} \overset{(3.3)}{=} \mathcal{V}[X^{\mathsf{v}}, \mathbf{i}\widetilde{Y}] \overset{(3.7)}{=} \mathcal{V} \circ \mathbf{i} \circ \mathbf{j}[X^{\mathsf{v}}, \mathcal{H}\widetilde{Y}] \overset{(2.7)}{=} \nabla^{\mathsf{v}}_{\widehat{X}}\widetilde{Y}$. Finally, (3.6) is immediate from (3.3) and (2.10). □

Proposition 3.3 (Commutation rules). *Let X and Y be vector fields on M. Then*

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{j} = \mathbf{j} \circ \mathcal{L}_{X^{\mathsf{c}}}, \tag{3.8}$$

where $\mathcal{L}_{X^{\mathsf{c}}}$ is the usual Lie derivative on $\mathring{T}M$ with respect to X^{c};

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{v}}_{\widehat{Y}} - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}. \tag{3.9}$$

If an Ehresmann connection $\mathcal{H}$ is also specified in $\mathring{T}M$ and ∇^h is the h-Berwald differential arising from $\mathcal{H}$, then

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{h}}_{\widehat{Y}} - \nabla^{\mathsf{h}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} = \widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Y^{\mathsf{h}}]}. \tag{3.10}$$

Proof. Since for any vector field η on $\mathring{T}M$, $0 \overset{(2.4)}{=} [\mathbf{J}, X^{\mathsf{c}}]\eta = [\mathbf{J}\eta, X^{\mathsf{c}}] - \mathbf{J}[\eta, X^{\mathsf{c}}]$, we have

$$\mathbf{J}[X^{\mathsf{c}}, \eta] = [X^{\mathsf{c}}, \mathbf{J}\eta]. \tag{3.11}$$

Thus $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}\eta = \mathcal{V}[X^{\mathsf{c}}, \mathbf{J}\eta] \overset{(3.11)}{=} \mathcal{V}\mathbf{J}[X^{\mathsf{c}}, \eta] = \mathbf{j}[X^{\mathsf{c}}, \eta] = \mathbf{j}\mathcal{L}_{X^{\mathsf{c}}}\eta$, which proves (3.8).

Next we show that the left-hand sides and the right-hand sides of (3.9) and (3.10) act in the same way on $C^\infty(\mathring{T}M)$ and $\operatorname{Sec}(\pi)$; then our assertions come from Lemma 2.1. Let $F \in C^\infty(\mathring{T}M)$, $\widetilde{Z} \in \operatorname{Sec}(\pi)$. Then, on the one hand,

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \nabla^{\mathsf{v}}_{\widehat{Y}} - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}})F = X^{\mathsf{c}}(Y^{\mathsf{v}}F) - Y^{\mathsf{v}}(X^{\mathsf{c}}F) = [X^{\mathsf{c}}, Y^{\mathsf{v}}]F = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}F.$$

On the other hand,

$$\mathbf{i}\circ(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\circ\nabla^{\mathsf{v}}_{\widehat{Y}}-\nabla^{\mathsf{v}}_{\widehat{Y}}\circ\widetilde{\mathcal{L}}_{X^{\mathsf{c}}})(\widetilde{Z})=\mathbf{i}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}[Y^{\mathsf{v}},\mathcal{H}\widetilde{Z}])-\mathbf{i}(\nabla^{\mathsf{v}}_{\widehat{Y}}\mathbf{i}^{-1}[X^{\mathsf{c}},\mathbf{i}\widetilde{Z}])$$
$$\overset{(3.8),(2.6)}{=}\mathbf{J}[X^{\mathsf{c}},[Y^{\mathsf{v}},\mathcal{H}\widetilde{Z}]]-\mathbf{J}[Y^{\mathsf{v}},\mathcal{H}\circ\mathbf{i}^{-1}[X^{\mathsf{c}},\mathbf{i}\widetilde{Z}]]\overset{(3.11),(3.7)}{=}[X^{\mathsf{c}},[Y^{\mathsf{v}},\mathbf{i}\widetilde{Z}]]$$
$$-[Y^{\mathsf{v}},[X^{\mathsf{c}},\mathbf{i}\widetilde{Z}]]=[X^{\mathsf{c}},[Y^{\mathsf{v}},\mathbf{i}\widetilde{Z}]]+[Y^{\mathsf{v}},[\mathbf{i}\widetilde{Z},X^{\mathsf{c}}]]\overset{\text{Jacobi}}{=}-[\mathbf{i}\widetilde{Z},[X^{\mathsf{c}},Y^{\mathsf{v}}]]$$
$$=[[X,Y]^{\mathsf{v}},\mathbf{i}\widetilde{Z}],$$

hence

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\circ\nabla^{\mathsf{v}}_{\widehat{Y}}-\nabla^{\mathsf{v}}_{\widehat{Y}}\circ\widetilde{\mathcal{L}}_{X^{\mathsf{c}}})(\widetilde{Z})=\mathbf{i}^{-1}[[X,Y]^{\mathsf{v}},\mathbf{i}\widetilde{Z}]=\widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}\widetilde{Z},$$

thus proving (3.9).

It is clear that the left-hand side and the right-hand side of (3.10) act in the same way on $C^\infty(\mathring{T}M)$. As to their actions on $\mathrm{Sec}(\pi)$, we find

$$\mathbf{i}\circ(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\circ\nabla^{\mathsf{h}}_{\widehat{Y}}-\nabla^{\mathsf{h}}_{\widehat{Y}}\circ\widetilde{\mathcal{L}}_{X^{\mathsf{c}}})(\widetilde{Z})=\mathbf{i}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathcal{V}[Y^{\mathsf{h}},\mathbf{i}\widetilde{Z}])-\mathbf{i}\nabla^{\mathsf{h}}_{\widehat{Y}}\mathbf{i}^{-1}[X^{\mathsf{c}},\mathbf{i}\widetilde{Z}]$$
$$=[X^{\mathsf{c}},\mathbf{i}\mathcal{V}[Y^{\mathsf{h}},\mathbf{i}\widetilde{Z}]]-\mathbf{i}\mathcal{V}[Y^{\mathsf{h}},[X^{\mathsf{c}},\mathbf{i}\widetilde{Z}]]=[X^{\mathsf{c}},\mathbf{v}[Y^{\mathsf{h}},\mathbf{i}\widetilde{Z}]]-\mathbf{v}[Y^{\mathsf{h}},[X^{\mathsf{c}},\mathbf{i}\widetilde{Z}]]$$
$$\overset{(*)}{=}[X^{\mathsf{c}},[Y^{\mathsf{h}},\mathbf{i}\widetilde{Z}]]+[Y^{\mathsf{h}},[\mathbf{i}\widetilde{Z},X^{\mathsf{c}}]]=-[\mathbf{i}\widetilde{Z},[X^{\mathsf{c}},Y^{\mathsf{h}}]]=[[X^{\mathsf{c}},Y^{\mathsf{h}}],\mathbf{i}\widetilde{Z}],$$

whence our claim. (At step $(*)$ we used the fact that $\mathbf{v}$ is acting on vertical vector fields, which belong to its fixed submodule.) □

Notice that in formula (3.9) the first term on the left-hand side and the term on the right-hand side annulate the basic sections by (2.8) and (3.5), so it follows that

$$\nabla^{\mathsf{v}}_{\widehat{Y}}\circ\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\widehat{Z})=0;\quad X,Y,Z\in\mathfrak{X}(M). \tag{3.12}$$

Of course, this can also be checked by an easy direct calculation.

4 Curvature collineations of spray manifolds

A C^1 vector field $S\colon TM\to TTM$ is said to be a *spray for* M if it is smooth on $\mathring{T}M$ and satisfies the conditions

$$\tau_*\circ S=1_{TM}\quad\text{and}\quad[C,S]=S.$$

The first condition expresses that S defines a second-order differential equation on M; it may equivalently be written in the form $\mathbf{J}S=C$. Condition $[C,S]=S$ requires positive homogeneity of degree 2. A manifold equipped with a spray is called a *spray manifold*.

Any spray S for M induces a homogeneous Ehresmann connection $\mathcal{H}$ in $\mathring{T}M$ such that for every vector field X on M

$$X^{\mathsf{h}}=\mathcal{H}(\widehat{X})=\frac{1}{2}(X^{\mathsf{c}}+[X^{\mathsf{v}},S]). \tag{4.1}$$

We say that the so specified Ehresmann connection is the *Berwald connection* for the spray manifold (M,S).

In terms of the classical tensor calculus, the basic curvature data of a spray manifold were introduced in a very lucid manner by L. Berwald in his posthumous paper [2]. We follow his treatment as close as we can, but using an index-free technique. In this spirit, we start with the *Jacobi endomorphism* $\mathbf{K} \in \mathcal{T}^1_1(\pi)$ (called *affine deviation* by Berwald) given by

$$\mathbf{K}(\widetilde{X}) := \mathcal{V}[S, \mathcal{H}(\widetilde{X})], \quad \widetilde{X} \in \operatorname{Sec}(\pi). \tag{4.2}$$

Next, with the help of $\mathbf{K}$, we define the *fundamental affine curvature* $\mathbf{R} \in \mathcal{T}^1_2(\pi)$ and the *affine curvature* $\mathbf{H} \in \mathcal{T}^1_3(\pi)$ by the formulae

$$\mathbf{R}(\widetilde{X}, \widetilde{Y}) := \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{K}(\widetilde{Y}, \widetilde{X}) - \nabla^{\mathsf{v}}\mathbf{K}(\widetilde{X}, \widetilde{Y})) \tag{4.3}$$

and

$$\mathbf{H}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := \nabla^{\mathsf{v}}\mathbf{R}(\widetilde{Z}, \widetilde{X}, \widetilde{Y}) \tag{4.4}$$

($\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \operatorname{Sec}(\pi)$). From the Jacobi endomorphism one can construct a further important $(1,1)$ tensor, which is unchanged if the spray S is replaced by

$$\overline{S} = S - 2\varphi C, \qquad \varphi \in C^\infty(\mathring{T}M). \tag{4.5}$$

This new ingredient is the *projective deviation tensor*

$$\mathbf{W}^\circ := \mathbf{K} - \frac{1}{n-1}(\operatorname{tr}\mathbf{K})\mathbf{1}_{\operatorname{Sec}(\pi)} + \frac{3}{n+1}(\operatorname{tr}\mathbf{R}) \otimes \delta + \frac{2-n}{n^2-1}(\nabla^{\mathsf{v}} \operatorname{tr}\mathbf{K}) \otimes \delta, \tag{4.6}$$

baptized so by Berwald. On the analogy of (4.3) and (4.4), we define the *fundamental projective curvature* $\mathbf{W} \in \mathcal{T}^1_2(\pi)$ and the *projective curvature* $\mathbf{W}^* \in \mathcal{T}^1_3(\pi)$ by

$$\mathbf{W}(\widetilde{X}, \widetilde{Y}) := \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{W}^\circ(\widetilde{Y}, \widetilde{X}) - \nabla^{\mathsf{v}}\mathbf{W}^\circ(\widetilde{X}, \widetilde{Y})) \tag{4.7}$$

and

$$\mathbf{W}^*(\widetilde{X}, \widetilde{Y})\widetilde{Z} := \nabla^{\mathsf{v}}\mathbf{W}(\widetilde{Z}, \widetilde{X}, \widetilde{Y}), \tag{4.8}$$

respectively ($\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \operatorname{Sec}(\pi)$). The tensors $\mathbf{W}$ and $\mathbf{W}^*$ are also invariant under 'the projective change' (4.5).

The tensors just defined have a counterpart in (semi-) Riemannian geometry. The next two curvature tensors are the prototypes of non-Riemannian data in spray geometry. The *Berwald curvature* $\mathbf{B}$ and the *Douglas curvature* $\mathbf{D}$ of (M, S) are given by

$$\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} := (\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{Z})(\widehat{X}, \widehat{Y}); \quad X, Y, Z \in \mathfrak{X}(M) \tag{4.9}$$

and

$$\mathbf{D} := \mathbf{B} - \frac{1}{n+1}((\operatorname{tr}\mathbf{B}) \odot \mathbf{1}_{\operatorname{Sec}(\pi)} + (\nabla^{\mathsf{v}} \operatorname{tr}\mathbf{B}) \otimes \delta), \tag{4.10}$$

where the symbol $\odot$ stands for symmetric product without numerical factor. The importance of the Douglas curvature lies in the fact that it is also invariant under the projective changes of the spray.

We recall that a vector field X on M is said to be a *Lie symmetry* of S or an *affine vector field* on M (with respect to S) if

$$\mathcal{L}_{X^{\mathsf{c}}}S = [X^{\mathsf{c}}, S] = 0. \tag{4.11}$$

In our earlier paper [10] we have shown that this property (among others) is equivalent to the relations

$$[X^{\mathsf{c}}, Y^{\mathsf{h}}] = [X, Y]^{\mathsf{h}}, \quad Y \in \mathfrak{X}(M) \tag{4.12}$$

and

$$[\mathbf{v}, X^{\mathsf{c}}] = 0. \tag{4.13}$$

(Of course, the horizontal lift is formed in (4.12) with respect to the Berwald connection of (M, S), and $\mathbf{v}$ is the vertical projection associated to the Berwald connection. The bracket $[\mathbf{v}, \eta]$, $\eta \in \mathfrak{X}(\mathring{T}M)$, is defined by the same rule as $[\mathbf{J}, \eta]$ in (2.3).)

A projectable vector field ξ on $\mathring{T}M$ is said to be a *curvature collineation of a curvature tensor* $\mathbf{C} \in \mathcal{T}^1_k(\pi)$ of a spray manifold ($k \in \{1, 2, 3\}$; affine and projective deviation are included), if $\widetilde{\mathcal{L}}_\xi \mathbf{C} = 0$. Then we also say that ξ is a *curvature collineation of the spray manifold.*

Theorem 4.1. *Let (M, S) be a spray manifold. If X is an affine vector field on M, then X^{c} is a curvature collineation for the Jacobi endomorphism $\mathbf{K}$, and hence for the fundamental affine curvature $\mathbf{R}$ and the affine curvature $\mathbf{H}$.*

Proof. First we show that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0$. For any vector field Y on M,

$$\begin{aligned}
&(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K})(\widehat{Y}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y})) - \mathbf{K}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathcal{V}[S, Y^{\mathsf{h}}]) - \mathcal{V}[S, \mathcal{H}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})] \\
&= \mathbf{i}^{-1}[X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] - \mathcal{V}[S, \mathcal{H} \circ \mathbf{i}^{-1}[X^{\mathsf{c}}, Y^{\mathsf{v}}]] = \mathbf{i}^{-1}([X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] \\
&-\mathbf{v}[S, [X, Y]^{\mathsf{h}}]) \overset{(4.12)}{=} \mathbf{i}^{-1}([X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] - \mathbf{v}[S, [X^{\mathsf{c}}, Y^{\mathsf{h}}]]) \overset{\text{Jacobi} + (4.11)}{=} \\
&\mathbf{i}^{-1}([X^{\mathsf{c}}, \mathbf{v}[S, Y^{\mathsf{h}}]] - \mathbf{v}[X^{\mathsf{c}}, [S, Y^{\mathsf{h}}]]) = -\mathbf{i}^{-1}([\mathbf{v}, X^{\mathsf{c}}][S, Y^{\mathsf{h}}]) \overset{(4.13)}{=} 0,
\end{aligned}$$

as was to be proven. Notice that relation $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} = 0$ is equivalent to

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathbf{K} = \mathbf{K} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \tag{4.14}$$

over $\operatorname{Sec}(\pi)$. From this, taking into account (3.9), we obtain that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Z}, \widehat{Y})) = \nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y})) + \widetilde{\mathcal{L}}_{[X,Z]^{\mathsf{v}}}(\mathbf{K}(\widehat{Y})) \tag{4.15}$$

for any vector fields Y, Z on M.

Now we turn to prove that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R} = 0$. Since $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R}$ is a tensor along $\mathring{\tau}$, it is enough to check that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R}(\widehat{Y}, \widehat{Z}) = 0$ for all $Y, Z \in \mathfrak{X}(M)$. By the product rule,

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R})(\widehat{Y}, \widehat{Z}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{R}(\widehat{Y}, \widehat{Z})) - \mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}, \widehat{Z}) - \mathbf{R}(\widehat{Y}, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}). \tag{4.16}$$

At the right-hand side

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{R}(\widehat{Y},\widehat{Z})) \overset{(4.3)}{=} \frac{1}{3}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Z},\widehat{Y})) - \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{Z}))) \overset{(4.15)}{=}$$
$$\frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y})) + \widetilde{\mathcal{L}}_{[X,Z]^{\mathsf{v}}}(\mathbf{K}(\widehat{Y})) - \nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Z})) - \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}(\mathbf{K}(\widehat{Z})));$$
$$\mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) = \frac{1}{3}((\nabla^{\mathsf{v}}\mathbf{K})(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}) - (\nabla^{\mathsf{v}}\mathbf{K})(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z})) = \frac{1}{3}((\nabla^{\mathsf{v}}_{\widehat{Z}}\mathbf{K})(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})$$
$$-(\nabla^{\mathsf{v}}_{\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y}}\mathbf{K})(\widehat{Z})) \overset{(3.12),(2.8)}{=} \frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{Z}}(\mathbf{K}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y})) - \nabla^{\mathsf{v}}_{\mathbf{i}^{-1}[X^{\mathsf{c}},Y^{\mathsf{v}}]}(\mathbf{K}(\widehat{Z})))$$
$$\overset{(4.14)}{=} \frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y})) - \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}(\mathbf{K}(\widehat{Z}))).$$

Interchanging Y and Z, it follows that

$$-\mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) - \mathbf{R}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}) = \mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{Y}) - \mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) = \frac{1}{3}(\nabla^{\mathsf{v}}_{\widehat{Y}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Z}))$$
$$-\nabla^{\mathsf{v}}_{\widehat{[X,Z]}}(\mathbf{K}(\widehat{Y})) - \nabla^{\mathsf{v}}_{\widehat{Z}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{K}(\widehat{Y})) + \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}(\mathbf{K}(\widehat{Z}))).$$

Thus we find that the right-hand side of (4.16) is

$$\frac{1}{3}(\widetilde{\mathcal{L}}_{[X,Z]^{\mathsf{v}}}(\mathbf{K}(\widehat{Y})) - \nabla^{\mathsf{v}}_{\widehat{[X,Z]}}(\mathbf{K}(\widehat{Y})) - \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}(\mathbf{K}(\widehat{Z})) + \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}(\mathbf{K}(\widehat{Z}))) \overset{(3.5)}{=} 0,$$

which proves the desired relation $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R} = 0$. Note that this is equivalent to

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{R}(\widehat{Y},\widehat{Z})) = \mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) + \mathbf{R}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}); \quad Y,Z \in \mathfrak{X}(M). \tag{4.17}$$

We show finally that X is a curvature collineation also for the affine curvature $\mathbf{H}$. For any vector fields Y, Z, U on M,

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{H})(\widehat{Y},\widehat{Z},\widehat{U}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{H}(\widehat{Y},\widehat{Z})\widehat{U}) - \mathbf{H}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z})\widehat{U} - \mathbf{H}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z})\widehat{U}$$
$$-\mathbf{H}(\widehat{Y},\widehat{Z})\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U} = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla^{\mathsf{v}}\mathbf{R})(\widehat{U},\widehat{Y},\widehat{Z})) - (\nabla^{\mathsf{v}}\mathbf{R})(\widehat{U},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z})$$
$$-(\nabla^{\mathsf{v}}\mathbf{R})(\widehat{U},\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}) - (\nabla^{\mathsf{v}}\mathbf{R})(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U},\widehat{Y},\widehat{Z}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}_{\widehat{U}}(\mathbf{R}(\widehat{Y},\widehat{Z})))$$
$$-(\nabla^{\mathsf{v}}_{\widehat{U}}\mathbf{R})(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}) - (\nabla^{\mathsf{v}}_{\widehat{U}}\mathbf{R})(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z}) - (\nabla^{\mathsf{v}}_{\widehat{[X,U]}}\mathbf{R})(\widehat{Y},\widehat{Z})$$
$$\overset{(3.9),(3.12)}{=} \nabla^{\mathsf{v}}_{\widehat{U}} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{R}(\widehat{Y},\widehat{Z})) + \widetilde{\mathcal{L}}_{[X,U]^{\mathsf{v}}}(\mathbf{R}(\widehat{Y},\widehat{Z})) - \nabla^{\mathsf{v}}_{\widehat{U}}(\mathbf{R}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z}))$$
$$-\nabla^{\mathsf{v}}_{\widehat{U}}(\mathbf{R}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z})) - \nabla^{\mathsf{v}}_{\widehat{[X,U]}}(\mathbf{R}(\widehat{Y},\widehat{Z})) \overset{(4.17)}{=} \widetilde{\mathcal{L}}_{[X,U]^{\mathsf{v}}}(\mathbf{R}(\widehat{Y},\widehat{Z}))$$
$$-\nabla^{\mathsf{v}}_{\widehat{[X,U]}}\mathbf{R}(\widehat{Y},\widehat{Z}) \overset{(3.5)}{=} 0,$$

which completes the proof. □

Corollary 4.2. *Any affine vector field is a curvature collineation for the projective deviation, the fundamental projective curvature and the projective curvature.*

Proof. Suppose $X \in \mathfrak{X}(M)$ is an affine vector field of the spray manifold (M,S). Then, taking into account that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{1}_{\mathrm{Sec}(\pi)} = 0$, $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\delta = \mathcal{V}[X^{\mathsf{c}},C] = 0$ and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathrm{tr} = \mathrm{tr} \circ \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}$, it follows that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W}^{\circ} \overset{(4.6)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K} - \frac{1}{n-1}\,\mathrm{tr}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K})\mathbf{1}_{\mathrm{Sec}(\pi)} + \frac{3}{n+1}(\mathrm{tr}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R})) \otimes \delta$$

$$+\frac{2-n}{n^2-1}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{K}))\otimes\delta \overset{\text{Th. 4.1}}{=} \frac{2-n}{n^2-1}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{K}))\otimes\delta.$$

Observe that for every function $F \in C^\infty(\mathring{T}M)$ and vector field Y on M we have

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{v}}F)(\widehat{Y}) = Y^{\mathsf{v}}(X^{\mathsf{c}}F).$$

Indeed, $(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}F)(\widehat{Y})) = X^{\mathsf{c}}(Y^{\mathsf{v}}F) - \nabla^{\mathsf{v}}F(\widehat{[X,Y]}) = [X^{\mathsf{c}}, Y^{\mathsf{v}}]F + Y^{\mathsf{v}}(X^{\mathsf{c}}F)$ $- [X,Y]^{\mathsf{v}}F = Y^{\mathsf{v}}(X^{\mathsf{c}}F)$.

Now, with the choice $F := \operatorname{tr}\mathbf{K}$, we find that

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{v}}(\operatorname{tr}\mathbf{K}))(\widehat{Y}) = Y^{\mathsf{v}}(X^{\mathsf{c}}\operatorname{tr}\mathbf{K}) = Y^{\mathsf{v}}(\operatorname{tr}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{K}) = 0,$$

thus proving that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W}^\circ = 0$. Having this result, relations $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W} = 0$ and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{W}^* = 0$ may be shown in the same way as $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{R} = 0$ and $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{H} = 0$ in the preceding proof. □

Theorem 4.3. *The affine vector fields of a spray manifold are curvature collineations for the Berwald curvature.*

Proof. Let (M,S) be a spray manifold and $X \in \mathfrak{X}(M)$ an affine vector field on M with respect to S. For any vector fields Y, Z, U on M,

$$\begin{aligned}
&(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{B})(\widehat{Y},\widehat{Z},\widehat{U}) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathbf{B}(\widehat{Y},\widehat{Z})\widehat{U}) - \mathbf{B}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Y},\widehat{Z})\widehat{U} - \mathbf{B}(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z})\widehat{U}\\
&-\mathbf{B}(\widehat{Y},\widehat{Z})\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U} \overset{(4.9),\,(3.4)}{=} \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{U})(\widehat{Y},\widehat{Z})) - ((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{U})(\widehat{[X,Y]},\widehat{Z}))\\
&-((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{U})(\widehat{Y},\widehat{[X,Z]})) - ((\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U})(\widehat{Y},\widehat{Z})) = \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U}) - \nabla^{\mathsf{v}}_{\widehat{[X,Y]}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U}\\
&-\nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{[X,Z]}}\widehat{U} - \nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U} \overset{(3.9),\,(3.10)}{=} \nabla^{\mathsf{v}}_{\widehat{Y}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U}) + \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{v}}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U}\\
&-\nabla^{\mathsf{v}}_{\widehat{[X,Y]}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U} - \nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{[X,Z]}}\widehat{U} - \nabla^{\mathsf{v}}_{\widehat{Y}}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla^{\mathsf{h}}_{\widehat{Z}}\widehat{U} + \nabla^{\mathsf{v}}_{\widehat{Y}}\widetilde{\mathcal{L}}_{[X^{\mathsf{c}},Z^{\mathsf{h}}]}\widehat{U}\\
&\overset{(3.5),\,(4.12)}{=} -\nabla^{\mathsf{v}}_{\widehat{Y}}\nabla^{\mathsf{h}}_{\widehat{[X,Z]}}\widehat{U} + \nabla^{\mathsf{v}}_{\widehat{Y}}\widetilde{\mathcal{L}}_{[X,Z]^{\mathsf{h}}}\widehat{U} \overset{(3.6)}{=} 0,
\end{aligned}$$

which proves the theorem. □

Corollary 4.4. *Any affine vector field is a curvature collineation for the Douglas curvature.*

Proof. Let $X \in \mathfrak{X}(M)$ be an affine vector field of the spray manifold (M,S). Then, by the preceding theorem, $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}$ kills the first two members on the left-hand side of (4.10), so it remains to show that $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{B}) = 0$. For any vector fields Y, Z, U on M,

$$\begin{aligned}
&(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{B}))(\widehat{Y},\widehat{Z},\widehat{U}) = X^{\mathsf{c}}((\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{B})(\widehat{Y},\widehat{Z},\widehat{U})) - (\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{B})(\widehat{[X,Y]},\widehat{Z},\widehat{U})\\
&-(\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{B})(\widehat{Y},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) - (\nabla^{\mathsf{v}}\operatorname{tr}\mathbf{B})(\widehat{Y},\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}) = X^{\mathsf{c}}Y^{\mathsf{v}}(\operatorname{tr}\mathbf{B}(\widehat{Z},\widehat{U}))\\
&-[X,Y]^{\mathsf{v}}(\operatorname{tr}\mathbf{B}(\widehat{Z},\widehat{U})) - Y^{\mathsf{v}}(\operatorname{tr}\mathbf{B}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U})) - Y^{\mathsf{v}}(\operatorname{tr}\mathbf{B}(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}))\\
&= Y^{\mathsf{v}}(X^{\mathsf{c}}(\operatorname{tr}\mathbf{B})(\widehat{Z},\widehat{U}) - \operatorname{tr}\mathbf{B}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{Z},\widehat{U}) - \operatorname{tr}\mathbf{B}(\widehat{Z},\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widehat{U}))\\
&= Y^{\mathsf{v}}(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\operatorname{tr}\mathbf{B}(\widehat{Z},\widehat{U})) = Y^{\mathsf{v}}((\operatorname{tr}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{B})(\widehat{Z},\widehat{U})) \overset{\text{Th. 4.3}}{=} 0,
\end{aligned}$$

as was to be shown. □

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