

Asymptotic inference for nearly unstable multidimensional AR processes

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Nearly unstable multidimensional AR models are studied where the coefficient matrices have some special form. Weak convergence of the sequence of the appropriately normalized LSE of the coefficient matrices is proved. A natural connection between the discrete and the corresponding continuous time models is presented.

1. Introduction

Consider the d -dimensional autoregressive model

$$\begin{cases} X_k = QX_{k-1} + \varepsilon_k, & k = 1, 2, \dots \\ X_0 = 0, \end{cases} \quad (1)$$

where the d -dimensional random column vector ε_k contains the (unobservable) random innovations (disturbances, noises) at time k , and the $d \times d$ matrix Q is the unknown parameter of the model. The least-squares estimator (LSE) of Q based on the observations X_1, \dots, X_n is given by

$$\hat{Q}_n = \left(\sum_{k=1}^n X_k X_{k-1}' \right) \left(\sum_{k=1}^n X_{k-1} X_{k-1}' \right)^{-1}. \quad (2)$$

Let $\varrho(Q)$ denote the spectral radius of the matrix Q , i. e., the maximum of the absolute value of the eigenvalues of the matrix Q .

When $\varrho(Q) < 1$, the model is said to be *asymptotically stationary*. Under the assumption that the ε_k 's are i.i.d. with $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k \varepsilon_k' = \Sigma$, the LSE of Q is asymptotically normal:

$$(\hat{Q}_n - Q) \left(\sum_{k=1}^n X_{k-1} X_{k-1}' \right)^{1/2} \xrightarrow{\mathcal{D}} \mathcal{N}_{d \times d}(0, I), \quad \text{as } n \rightarrow \infty, \quad (3)$$

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where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution and I is the unit matrix (see Mann and Wald [16] and Anderson [1]). By another normalization:

$$\sqrt{n}(\hat{Q}_n - Q) \xrightarrow{\mathcal{D}} \mathcal{N}_{d \times d}(0, F^{-1} \otimes \Sigma), \quad \text{as } n \rightarrow \infty,$$

where $F = \sum_{k=0}^{\infty} Q^k \Sigma (Q')^k$ is the covariance matrix of the stationary distribution (being the unique solution of the equation $\Sigma + QFQ' = F$).

When $\varrho(Q) = 1$, the model (1) is said to be *unstable*. It was shown by White [23] that in the case of the one-dimensional unstable AR(1) model $X_k = \beta X_{k-1} + \varepsilon_k$, $k \geq 1$, with $\beta = 1$, the variables $n(\hat{\beta}_n - \beta)$ converge in law to a random variable:

$$n(\hat{\beta}_n - \beta) \xrightarrow{\mathcal{D}} \frac{\int_0^1 W(t) dW(t)}{\int_0^1 W^2(t) dt}, \quad (4)$$

where $W(t)$, $t \geq 0$ is a standard Wiener process. In case of the one-dimensional unstable AR(p) model Chan and Wei [6] proved that with suitable normalizing matrices δ_n the sequence $\delta_n^{-1}(\hat{\beta}_n - \beta)$ converges in law and gave the representation of the limit distribution. This representation involves multiple stochastic integrals with respect to Wiener processes and has a very complicated form.

Fountis and Dickey [8] considered multidimensional unstable models with a matrix Q having one eigenvalue equal to 1 and the rest less than 1 in magnitude and obtained the limit distribution of the appropriately normalized LSE of the largest eigenvalue. The general multidimensional unstable models are studied in Sims, Stock and Watson [20], and in Tsay and Tiao [22]. Arató [3] has drawn the attention to the connection between discrete and continuous time multidimensional unstable models. This relationship was pointed out by Kormos [10], [11], and by Kormos and Piterbarg [13] in connection with hypothesis testing of nonstationarity for a (real-valued) Gaussian one-dimensional autoregressive time series.

The result (4) led to the study of the following so-called nearly nonstationary one-dimensional AR(1) model (better to call it nearly unstable):

$$\begin{cases} X_{n,k} = \beta_n X_{n,k-1} + \varepsilon_{n,k}, & k = 1, 2, \dots, n \\ X_{n,0} = 0, \end{cases} \quad (5)$$

where $\beta_n = 1 + h/n$. It was shown by Chan and Wei [4], [5] that

$$\left(\sum_{k=1}^n X_{n,k-1}^2 \right)^{1/2} (\hat{\beta}_n - \beta_n) \xrightarrow{\mathcal{D}} \frac{\int_0^1 Y(t) dW(t)}{\left(\int_0^1 Y^2(t) dt \right)^{1/2}}, \quad (6)$$

where $Y(t)$, $t \in [0, 1]$ is an Ornstein-Uhlenbeck process defined as the solution of the stochastic differential equation

$$dY(t) = hY(t) dt + dW(t), \quad Y(0) = 0.$$

By another normalization

$$n(\hat{\beta}_n - \beta_n) \xrightarrow{\mathcal{D}} \frac{\int_0^1 Y(t) dW(t)}{\int_0^1 Y^2(t) dt}, \quad (7)$$

see, for example, Phillips [18], Jeganathan [9], Dzhaparidze, Kormos, van der Meer and van Zuijlen [7]). (The above model is called also near integrated and is applied often in economic theory; see Phillips [18].)

Recently, Jeganathan [9] has considered nearly nonstationary one-dimensional AR(p) models, i. e., AR(p) models near to an unstable model. He proved that the appropriately normalized LSE of the coefficients converges in law and gave a very complicated representation for the limiting distribution in terms of multiple stochastic integrals with respect to Wiener processes. In the forthcoming paper van der Meer, Pap and van Zuijlen [17] a simpler form and explanation is given for the asymptotic behaviour of the least-squares estimators in the nearly nonstationary AR(p) model and the relation between discrete and continuous time models is clarified.

Nearly unstable multidimensional AR processes are generated according to the scheme

$$\begin{cases} X_{n,k} = Q_n X_{n,k-1} + \varepsilon_{n,k}, & k = 1, 2, \dots, n \\ X_{n,0} = 0, \end{cases} \quad (8)$$

where $\{\varepsilon_{n,k}\}$ is an array of d -dimensional random vectors and Q_n , $n \geq 1$, is a sequence of $d \times d$ matrices such that $Q_n \rightarrow Q$, where Q is a matrix with $\varrho(Q) = 1$. Phillips [18] treated the case where $Q_n = e^{A/n}$, $n \geq 1$, where A is a fixed $d \times d$ matrix. Kormos and Pap [12] investigated the case where $Q_n = e^{(\gamma I + A)/n}$, $n \geq 1$, where $\gamma \in \mathbb{R}$ and A is a skew-symmetric matrix and obtained the weak convergence of the log-likelihood ratio under the assumption that ε_k 's are i.i.d. normal. The limit distribution turned out to be the Radon-Nikodym derivative of some continuous time multidimensional Ornstein-Uhlenbeck process. This indicates a natural connection between discrete and continuous time models. Stockmarr and Jacobsen [21] studied essentially the case where $Q_n = I + n^{-1}A$. We remark that these authors considered in fact an additional matrix in their model. However, this situation will also be covered in our model below, since one can easily include this extra matrix in the noise process.

The aim of the present paper is to investigate nearly unstable models (8) where the coefficient matrices have some special form which includes the situations considered by the authors mentioned above. First we give a necessary and sufficient condition on the innovations $\{\varepsilon_{n,k}\}$ for convergence (in the Skorokhod space $D([0, 1] \rightarrow \mathbb{R}^d)$) of the *rotated* sequence

$$\frac{1}{\sqrt{n}} e^{-[nt]B} X_{n,[nt]}, \quad t \in [0, 1], \quad n = 1, 2, \dots$$

to a d -dimensional continuous time AR(1) process, where B is a suitable $d \times d$ matrix. Then we shall prove weak convergence of the sequence of appropriately normalized LSE's

and finally we give a natural connection between the discrete and the related continuous time models. It should be remarked that our results essentially include the earlier work of the above mentioned authors and our method is completely different.

2. Convergence results

For every $n = 1, 2, \dots$ consider the d -dimensional AR(1) model

$$\begin{cases} X_{n,k} = Q_n e^B X_{n,k-1} + \varepsilon_{n,k}, & k = 1, 2, \dots, n \\ X_{n,0} = 0, \end{cases} \quad (9)$$

where $\{\varepsilon_{n,k}\}$, is an array of d -dimensional random vectors and $Q_n = e^{A_n/n}$, $n \geq 1$ are $d \times d$ matrices such that $A_n \rightarrow A$, B is a known skew-symmetric $d \times d$ matrix, and $A_n B = B A_n$, $n \geq 1$. (We remark that for each orthogonal matrix C there exists uniquely a matrix B such that $C = e^B$ and B is a skew-symmetric matrix, i. e. $B' = -B$.) The model (9) is nearly unstable, since $Q_n e^B \rightarrow e^B$ and e^B is an orthogonal matrix.

Since the matrices e^{A_n} and e^B also commute, the *rotated* observations

$$Z_{n,k} = e^{-kB} X_{n,k}, \quad k = 0, 1, \dots, n; \quad n = 1, 2, \dots$$

form again a nearly unstable d -dimensional model

$$\begin{cases} Z_{n,k} = e^{A_n/n} Z_{n,k-1} + \zeta_{n,k}, & k = 1, 2, \dots, n \\ Z_{n,0} = 0, \end{cases} \quad (10)$$

where $\{\zeta_{nk}\} = \{e^{-kB} \varepsilon_{n,k}\}$ is the rotated array of the random disturbances.

The random step functions

$$\begin{aligned} Y_n(t) &= \frac{1}{\sqrt{n}} e^{-[nt]B} X_{n,[nt]}, & t \in [0, 1] \\ M_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} e^{-kB} \varepsilon_{n,k}, & t \in [0, 1] \end{aligned}$$

can be considered as random elements in the Skorokhod space $D([0, 1] \rightarrow \mathbb{R}^d)$. We denote by $C([0, 1] \rightarrow \mathbb{R}^d)$ the space of \mathbb{R}^d -valued continuous functions endowed with the supremum norm. The supremum norm and the Skorokhod metric on the space $D([0, 1] \rightarrow \mathbb{R}^d)$ will be denoted by $\|\cdot\|_\infty$ and ρ , respectively. We shall need the following simple lemma, which is based on the continuous mapping theorem and the Skorokhod-construction.

Lemma 1. *Let $\Phi, \Phi_n : D([0, 1] \rightarrow \mathbb{R}^k) \rightarrow D([0, 1] \rightarrow \mathbb{R}^\ell)$, $n = 1, 2, \dots$ be measurable mappings such that $\|\Phi_n(x_n) - \Phi(x)\|_\infty \rightarrow 0$ for all $x, x_n \in D([0, 1] \rightarrow \mathbb{R}^k)$ with*

$\|x_n - x\|_\infty \rightarrow 0$. Let $Z, Z_n, n = 1, 2, \dots$ be stochastic processes with values in $D([0, 1] \rightarrow \mathbb{R}^k)$ such that $Z_n \xrightarrow{\mathcal{D}} Z$ in $D([0, 1] \rightarrow \mathbb{R}^k)$ and almost all trajectories of Z are continuous. Then $\Phi_n(Z_n) \xrightarrow{\mathcal{D}} \Phi(Z)$ in $D([0, 1] \rightarrow \mathbb{R}^\ell)$.

Proof. Due to the Skorokhod-construction we can find processes \tilde{Z}_n and a process \tilde{Z} , such that $\tilde{Z}_n \stackrel{\mathcal{D}}{=} Z_n$, $\tilde{Z} \stackrel{\mathcal{D}}{=} Z$ and

$$\rho(\tilde{Z}_n, \tilde{Z}) \rightarrow 0 \quad \text{a.s.}$$

Using the fact that \tilde{Z} has continuous trajectories a. s., we conclude that

$$\|\tilde{Z}_n - \tilde{Z}\|_\infty \rightarrow 0 \quad \text{a.s.}$$

Thus we have

$$\|\Phi_n(\tilde{Z}_n) - \Phi(\tilde{Z})\|_\infty \rightarrow 0 \quad \text{a.s.}$$

and hence

$$\Phi_n(\tilde{Z}_n) \xrightarrow{\mathcal{D}} \Phi(\tilde{Z}) \quad \text{in } D([0, 1] \rightarrow \mathbb{R}^\ell).$$

The last relation implies the desired result. \square

Let $M(t), t \in [0, 1]$, be a continuous semimartingale with values in \mathbb{R}^d . Consider the continuous time autoregressive process $Y(t), t \in [0, 1]$, defined as the solution of the stochastic differential equation

$$dY(t) = AY(t) dt + dM(t), \quad Y(0) = 0. \quad (11)$$

Our first theorem gives a necessary and sufficient condition for the weak convergence $Y_n \xrightarrow{\mathcal{D}} Y$ in $D([0, 1] \rightarrow \mathbb{R}^d)$.

Theorem 1. *The following statements are equivalent:*

- (i) $M_n \xrightarrow{\mathcal{D}} M$ in $D([0, 1] \rightarrow \mathbb{R}^d)$
- (ii) $Y_n \xrightarrow{\mathcal{D}} Y$ in $D([0, 1] \rightarrow \mathbb{R}^d)$
- (iii) $(M_n, Y_n) \xrightarrow{\mathcal{D}} (M, Y)$ in $D([0, 1] \rightarrow \mathbb{R}^{2d})$.

Remark. The statement (i) is in other words: the functional central limit theorem holds for the rotated triangular array $\{e^{-kB} \varepsilon_{n,k}\}_{k=1, \dots, n; n \geq 1}$.

Proof. (i) \implies (iii). Using Itô's formula we obtain

$$Y(t) = M(t) + A \int_0^t e^{(t-s)A} M(s) ds, \quad t \in [0, 1].$$

A similar formula holds for the random step functions $Y_n(t), t \in [0, 1]$:

$$Y_n(t) = M_n(t) + A_n \int_0^{\lceil nt \rceil / n} e^{\gamma_n(\lceil nt \rceil / n - s)} M_n(s) ds, \quad (12)$$

since (10) implies

$$Y_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} e^{([nt]-k)A_n/n} \zeta_{n,k},$$

and

$$\begin{aligned} A_n \int_0^{[nt]/n} e^{-sA_n} M_n(s) ds &= A_n \sum_{j=1}^{[nt]} \int_{(j-1)/n}^{j/n} e^{-sA_n} ds M_n((j-1)/n) \\ &= \sum_{j=1}^{[nt]} \left(e^{-(j-1)A_n/n} - e^{-jA_n/n} \right) M_n((j-1)/n) \\ &= \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \sum_{k=1}^{j-1} \left(e^{-(j-1)A_n/n} - e^{-jA_n/n} \right) \zeta_{n,k} \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]-1} \sum_{j=k+1}^{[nt]} \left(e^{-(j-1)A_n/n} - e^{-jA_n/n} \right) \zeta_{n,k} \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]-1} \left(e^{-kA_n/n} - e^{-[nt]A_n/n} \right) \zeta_{n,k} \\ &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \left(e^{-kA_n/n} - e^{-[nt]A_n/n} \right) \zeta_{n,k} \\ &= e^{-[nt]A_n/n} (Y_n(t) - M_n(t)). \end{aligned}$$

Hence the processes (M, Y) and (M_n, Y_n) can be expressed as

$$(M, Y) = \Phi(M), \quad (M_n, Y_n) = \Phi_n(M_n), \quad n = 1, 2, \dots,$$

where the measurable mappings $\Phi, \Phi_n : D([0, 1] \rightarrow \mathbb{R}^d) \rightarrow D([0, 1] \rightarrow \mathbb{R}^{2d})$, $n = 1, 2, \dots$ are defined as

$$\begin{aligned} \Phi(x)(t) &= \left(x(t), x(t) + A \int_0^t e^{(t-s)A} x(s) ds \right), \\ \Phi_n(x)(t) &= \left(x(t), x(t) + A_n \int_0^{[nt]/n} e^{([nt]/n-s)A_n} x(s) ds \right). \end{aligned}$$

Applying Lemma 1 we obtain $(M_n, Y_n) \xrightarrow{\mathcal{D}} (M, Y)$ in $D([0, 1] \rightarrow \mathbb{R}^{2d})$.

(iii) \implies (i) and (iii) \implies (ii) are trivial.

(ii) \implies (iii). Since the process $Y(t)$, $t \in [0, 1]$, is a solution of the stochastic differential equation (11) we have

$$Y(t) = A \int_0^t Y(s) ds + M(t),$$

thus

$$M(t) = Y(t) - A \int_0^t Y(s) ds.$$

A similar formula holds for the partial sum process $M_n(t)$, $t \in [0, 1]$:

$$\begin{aligned} M_n(t) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Z_{n,k} - e^{A_n/n} Z_{n,k-1}) \\ &= \frac{1}{\sqrt{n}} Z_{n,[nt]} - \frac{1}{\sqrt{n}} (e^{A_n/n} - I) \sum_{k=1}^{[nt]-1} Z_{n,k} \\ &= Y_n(t) - n(e^{A_n/n} - I) \int_0^{[nt]/n} Y_n(s) ds. \end{aligned}$$

Hence the processes (M, Y) and (M_n, Y_n) can be expressed as

$$(M, Y) = \Psi(Y), \quad (M_n, Y_n) = \Psi_n(Y_n), \quad n = 1, 2, \dots,$$

where the measurable mappings $\Psi, \Psi_n : D([0, 1] \rightarrow \mathbb{R}^d) \rightarrow D([0, 1] \rightarrow \mathbb{R}^{2d})$, $n = 1, 2, \dots$ are defined as follows

$$\begin{aligned} \Psi(x)(t) &= \left(x(t) - A \int_0^t x(s) ds, x(t) \right), \\ \Psi_n(x)(t) &= \left(x(t) - n(e^{A_n/n} - I) \int_0^{[nt]} x(s) ds, x(t) \right). \end{aligned}$$

Applying Lemma 1 we obtain $(M_n, Y_n) \xrightarrow{\mathcal{D}} (M, Y)$ in $D([0, 1] \rightarrow \mathbb{R}^{2d})$. \square

3. Least-squares estimators

The least-squares estimator of the matrix Q_n in the model (9) can be obtained minimizing the sum of squares

$$\sum_{k=1}^n \|X_{n,k} - Q_n e^B X_{n,k-1}\|^2.$$

The orthogonality of e^B implies that the above sum is equal to

$$\sum_{k=1}^n \|Z_{n,k} - Q_n Z_{n,k-1}\|^2,$$

consequently the LSE is

$$\hat{Q}_n = \left(\sum_{k=1}^n Z_{n,k} Z'_{n,k-1} \right) \left(\sum_{k=1}^n Z_{n,k-1} Z'_{n,k-1} \right)^{-1}. \quad (13)$$

From now on we shall put the following condition on the random disturbances $\{\varepsilon_{n,k}\}$.

- (C) $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$ is a triangular array of d -dimensional square integrable martingale differences with respect to the filtrations $(\mathcal{F}_{nk})_{k=0,1,\dots,n;n \geq 1}$ such that for all $t \in [0, 1]$

$$\frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E}(\varepsilon_{n,k} \varepsilon'_{n,k} | \mathcal{F}_{n,k-1}) \xrightarrow{\mathbb{P}} I, \quad \text{as } n \rightarrow \infty$$

and

$$\forall \alpha > 0 \quad \frac{1}{n} \sum_{k=1}^{[nt]} \mathbb{E}(\|\varepsilon_{n,k}\|^2 \chi_{\{\|\varepsilon_{n,k}\| > \alpha \sqrt{n}\}} | \mathcal{F}_{n,k-1}) \xrightarrow{\mathbb{P}} 0, \quad \text{as } n \rightarrow \infty.$$

Theorem 2. Suppose that the array $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, satisfies the condition (C). Then

$$n(\widehat{Q}_n - Q_n) \xrightarrow{\mathcal{D}} \int_0^1 (dW(t)) Y'(t) \left(\int_0^1 Y(t) Y'(t) dt \right)^{-1},$$

where the process $Y(t)$, $t \in [0, 1]$, is given by

$$dY(t) = AY(t)dt + dW(t), \quad Y(0) = 0,$$

and $W(t)$, $t \in [0, 1]$, is a standard d -dimensional Wiener process.

By another normalization

$$(\widehat{Q}_n - Q_n) \left(\sum_{k=1}^n X_{k-1} X'_{k-1} \right)^{1/2} \xrightarrow{\mathcal{D}} \int_0^1 (dW(t)) Y'(t) \left(\int_0^1 Y(t) Y'(t) dt \right)^{-1/2}.$$

Proof. Orthogonality of the matrix e^B implies that the rotated array $\zeta_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, also satisfies the condition (C), so using a version of the functional central limit theorem on the space $D([0, 1] \rightarrow \mathbb{R}^d)$ (Theorem 7.11 in Liptser and Shirayev [15]) we obtain $M_n \xrightarrow{\mathcal{D}} W$ in $D([0, 1] \rightarrow \mathbb{R}^d)$. Applying Theorem 1 we can conclude $(M_n, Y_n) \xrightarrow{\mathcal{D}} (W, Y)$ in $D([0, 1] \rightarrow \mathbb{R}^{2d})$. This implies weak convergence of the stochastic integrals

$$\left(\int_0^1 Y_n(t) Y'_n(t) dt, \int_0^1 (dM_n(t)) Y'_n(t) \right) \xrightarrow{\mathcal{D}} \left(\int_0^1 Y(t) Y'(t) dt, \int_0^1 (dW(t)) Y'(t) \right)$$

(see, for example, Proposition 6 in Jeganathan [9]). Moreover,

$$\begin{aligned} \widehat{Q}_n &= \left(\sum_{k=1}^n (Q_n Z_{n,k-1} + \zeta_{n,k}) Z'_{n,k-1} \right) \left(\sum_{k=1}^n Z_{n,k-1} Z'_{n,k-1} \right)^{-1} \\ &= Q_n + \frac{1}{n} \left(\int_0^1 (dM_n(t)) Y'_n(t) \right) \left(\int_0^1 Y_n(t) Y'_n(t) dt \right)^{-1}. \end{aligned}$$

Since the matrix $\int_0^1 Y(t) Y'(t) dt$ is invertible with probability 1 we can apply the continuous mapping theorem. \square

4. Connection with continuous time AR processes

Let $W(t)$, $t \in [0, 1]$, be a standard d -dimensional Wiener process. Consider the continuous time autoregressive process given by

$$dY(t) = AY(t)dt + dW(t), \quad Y(0) = 0. \quad (14)$$

Consider the measures \mathbb{P}_W and \mathbb{P}_Y on $C([0, 1] \rightarrow \mathbb{R}^d)$ generated by the processes W and Y , respectively. It is known that \mathbb{P}_Y is absolutely continuous with respect to the measure \mathbb{P}_W and the Radon-Nikodym derivative has the form

$$\frac{d\mathbb{P}_Y}{d\mathbb{P}_W}(Y) = \exp \left\{ -\frac{1}{2} \int_0^1 \|AY(t)\|^2 dt + \int_0^1 \langle AY(t), dY(t) \rangle \right\}. \quad (15)$$

Consequently the maximum likelihood estimator (MLE) of the matrix A based on the observations $Y(t)$, $t \in [0, 1]$, in the model (14) is given by

$$\hat{A} = \int_0^1 (dY(t))Y'(t) \left(\int_0^1 Y(t)Y'(t) dt \right)^{-1}.$$

(See, for example, Le Breton [14], Arató [2]).

Corollary 1. *Suppose that the array $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, satisfies the condition (C). Then*

$$n(\hat{Q}_n - I) \xrightarrow{\mathcal{D}} \hat{A},$$

where \hat{A} is the maximum likelihood estimator of the matrix A based on the observations $Y(t)$, $t \in [0, 1]$, in the model (14).

Proof. Using Itô's formula we obtain

$$\hat{A} = A + \int_0^1 (dW(t))Y'(t) \left(\int_0^1 Y(t)Y'(t) dt \right)^{-1}.$$

Theorem 2 implies

$$n(\hat{Q}_n - I) = n(\hat{Q}_n - Q_n) + n(e^{A_n/n} - I) \xrightarrow{\mathcal{D}} \hat{A}. \quad \square$$

Consider now the quadratic form

$$L_n(X_{n,1}, \dots, X_{n,n}) = -\frac{1}{2} \left(\sum_{k=1}^n \|X_{n,k} - Q_n e^B X_{n,k-1}\|^2 - \sum_{k=1}^n \|X_{n,k} - e^B X_{n,k-1}\|^2 \right),$$

connected with the LSE \hat{Q}_n . We remark that if the innovations $\{\varepsilon_{n,n}\}$ are normal then $L_n(X_{n,1}, \dots, X_{n,n})$ is just the log-likelihood ratio. Denote the log-likelihood ratio of the process $Y(t)$, $t \in [0, 1]$ by

$$L(Y) = \log \frac{d\mathbb{P}_Y}{d\mathbb{P}_W}(Y).$$

Corollary 2. *Suppose that the array $\varepsilon_{n,k}$, $k = 1, \dots, n$, $n \geq 1$, satisfies the condition (C). Then*

$$L_n(X_{n,1}, \dots, X_{n,n}) \xrightarrow{\mathcal{D}} L(Y)$$

in \mathbb{R} .

Proof. Using the same arguments as in the proof of Theorem 1 we obtain

$$\begin{aligned}
& -2L_n(X_{n,1}, \dots, X_{n,n}) = \\
& = \sum_{k=1}^n \|(Q_n - I)Z_{n,k-1}\|^2 - 2 \sum_{k=1}^n \langle Z_{n,k} - Z_{n,k-1}, (Q_n - I)Z_{n,k-1} \rangle \\
& = \int_0^1 \|n(Q_n - I)Y_n(t)\|^2 dt - 2 \int_0^1 \langle dY_n(t), n(Q_n - I)Y_n(t) \rangle \\
& \xrightarrow{\mathcal{D}} \int_0^1 \|AY(t)\|^2 dt - 2 \int_0^1 \langle dY(t), AY(t) \rangle.
\end{aligned}$$

The assertion is proved. □

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