THE EQUALITY PROBLEM IN THE CLASS OF CONJUGATE MEANS

BY PÁL BURAI AND JUDITA DASCĂL

ABSTRACT. Let $I \subset \mathbb{R}$ be a nonempty open interval and let $L : I^2 \to I$ be a fixed strict mean. A function $M : I^2 \to I$ is said to be an *L-conjugate mean on I* if there exist $p, q \in]0, 1]$ and a strictly monotone and continuous function φ such that

$$M(x,y) := \varphi^{-1}(p\varphi(x)+q\varphi(y)+(1-p-q)\varphi(L(x,y))) =: A_{\varphi}^{(p,q)}(x,y),$$

for all $x, y \in I$. Here $L(x, y)$ is a fixed quasi-arithmetic mean. We will solve the equality problem in this class of means.

1. INTRODUCTION

In [8] Daróczy and Páles introduced the class of conjugate means. There is a rich literature on the class of conjugate means in different cases (Bakula-Páles-Pečarić [2], Daróczy [3], Daróczy-Dascăl [4], [5], Daróczy-Páles [6], [7], [8]). When we deal with mean values, an important and natural question is the equality problem of means, which is a widely investigated area. For example in Jarczyk [10], Losonczi [12], [13], [14], [15], Losonczi-Páles [16], Makó-Páles [17], Matkowski [19], [20], [21], Páles [22] the authors deal with this problem for different types of means assuming necessarily smoothness conditions. In this paper we intend to solve the equality problem in the class of conjugate means derived from the quasi-arithmetic means. In order to define these means, we will need the following notation.

Let $I \subset \mathbb{R}$ be a nonempty open interval and let $\mathcal{CM}(I)$ denote the class of continuous and strictly monotone real valued functions defined on the interval I.

Date: September 19, 2011.

1991 *Mathematics Subject Classification.* Primary 39B22, Secondary 39B12, 26E60.

Key words and phrases. mean, functional equation, quasi-arithmetic mean, conjugate mean.

This research has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK 81402 (first and second author) and OTKA "Mobility" call HUMAN-MB08A-84581 (first author).

Assume that $L : I^2 \to I$ is a strict mean on I. It is well-known that for $p, q \in]0, 1]$

$$N(x, y) := px + qy + (1 - p - q)L(x, y) \quad (x, y \in I)$$

is a strict mean. ([8]). We can also say that N is a conjugate mean of order (p, q) derived from the mean L. If $\varphi \in \mathcal{CM}(I)$ then because of

$$min\{\varphi(x), \varphi(y)\} < p\varphi(x)+q\varphi(y)+(1-p-q)\varphi(L(x, y)) < max\{\varphi(x), \varphi(y)\}$$

for all $x, y \in I, x \neq y$, we get that

$$A_\varphi^{(p,q)}(x, y) := \varphi^{-1}(p\varphi(x) + q\varphi(y) + (1 - p - q)\varphi(L(x, y))) \quad (x, y \in I)$$

is also a strict mean on I.

Definition. Let $L : I^2 \to I$ be a fixed strict mean. A function $M : I^2 \to I$ is said to be an *L-conjugate mean on I* if there exist $p, q \in]0, 1]$ and $\varphi \in \mathcal{CM}(I)$ such that

$$M(x, y) = A_\varphi^{(p,q)}(x, y) \quad (x, y \in I).$$

The numbers p, q are said to be the *weights* and the function φ is called the *generating function* of the mean M.

If $p + q = 1$ it is obvious that

$$A_\varphi^{(p,1-p)}(x, y) = \varphi^{-1}(p\varphi(x) + (1 - p)\varphi(y)), \quad (x, y \in I),$$

thus this class of means (L-conjugate) includes the weighted quasi-arithmetic means.

Now let

$$L(x, y) := \frac{x + y}{2} \quad (x, y \in I)$$

be the arithmetic mean.

The question is for what weights $p, q, r, s \in]0, 1]$ and generating functions $\varphi, \psi \in \mathcal{CM}(I)$ will the $A_\varphi^{(p,q)}, A_\psi^{(r,s)} : I^2 \to I$ conjugate means be equal? This means we have to characterize the parameters $p, q, r, s \in]0, 1]$ and the functions φ, ψ such that

$$A_\varphi^{(p,q)}(x, y) = A_\psi^{(r,s)}(x, y) \tag{P}$$

holds for all $x, y \in I$. This question is the general equality problem:

$$\begin{aligned} &\varphi^{-1}\left(p\varphi(x) + q\varphi(y) + (1 - p - q)\varphi\left(\frac{x + y}{2}\right)\right) = \\ &= \psi^{-1}\left(r\psi(x) + s\psi(y) + (1 - r - s)\psi\left(\frac{x + y}{2}\right)\right) \ (x, y \in I). \end{aligned} \tag{1}$$

The functional equation (1) was studied in the following particular cases:

(i) $p = q = r = s = \frac{1}{2}$ (see Aczél [1], Hardy et al. [9], Kuczma [11]),
(ii) $p = q = 1$ and $r = s = \frac{1}{2}$ by Daróczy and Páles [6], and by Daróczy and Dascăl [4],
(iii) $p = q \neq 1$ and $r = s = \frac{1}{2}$ by Daróczy and Dascăl [4],
(iv) $p = r$ and $q = s$ by Daróczy and Páles [8],
(v) $p \neq q$, $r \neq s$, $p + q = r + s = 1$ by Maksa and Páles [18],
(vi) $p \neq q$, $r \neq s$, $p + q \neq 1$, $r + s = 1$ by Daróczy and Dascăl [4],
(vii) $1 - p - q \geq 0$, $1 - r - s \geq 0$ by Makó and Páles [17].

The equality and comparison problem has been also studied by Daróczy [3] in the case when the mean $L(x, y) = min\{x, y\}$ and $p, q, r, s \in [0, 1]$. The first six cases were solved without assuming further regularity properties on the generating functions. In this paper, we investigate the general equality problem and we will solve the remaining cases. Due to the complexity of the problem, we will need at most fourth-order differentiability properties of the unknown functions φ and ψ.

In our main theorem we will not allow $p + q = 1$ and/or $r + s = 1$, although our proof would solve these cases too, but they have been already solved under weaker regularity assumptions. In order to give a complete solution to the equality problem of conjugate means in every possible case, in the next section we will cite the theorems which solve the cases we have excluded from our main theorem.

2. Preliminaries

Definition. Let $\varphi, \psi \in \mathcal{CM}(I)$. If there exist $a \neq 0$ and b such that

$$\psi(x) = a\varphi(x) + b \quad \text{if } x \in I$$

then we say that φ is equivalent to ψ on I and denote it by $\varphi(x) \sim \psi(x)$ if $x \in I$ or in short $\varphi \sim \psi$ on I.

In the next three theorems the conjugate mean is derived from a fixed quasi-arithmetic mean with the fixed generating function $\chi \in \mathcal{CM}(I)$. If we put the identity function instead of χ we get the case when $L(x, y)$ is the arithmetic mean.

The following two results treat the cases $p = q = 1$ and $p = q \neq 1$. Then the left hand side of equation (1) is a symmetric mean, hence, it is necessary that $r = s = \frac{1}{2}$.

Theorem (Daróczy-Dascăl, [4]). *Let $I \subset \mathbb{R}$ be a nonempty open interval and $\varphi, \psi \in \mathcal{CM}(I)$. The functional equation*

(2)
$$\varphi^{-1}\left(\varphi(x)+\varphi(y)-\varphi\left(\chi^{-1}\left(\frac{\chi(x)+\chi(y)}{2}\right)\right)\right)=\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)$$

holds for all $x, y \in I$ if and only if, with the notation $J := \chi(I)$, either

(i) $\varphi(x) \sim \chi(x)$ and $\psi(x) \sim \chi(x)$ if $x \in I$, or
(ii) there exists $t \in T_+(J) := \{t \in \mathbb{R} \mid J + t \subset \mathbb{R}_+\}$ such that $\varphi(x) \sim \log(\chi(x)+t)$ and $\psi(x) \sim \frac{1}{\chi(x)+t}$ if $x \in I$, or
(iii) there exists $t \in T_-(J) := \{t \in \mathbb{R} \mid -J + t \subset \mathbb{R}_+\}$ such that $\varphi(x) \sim \log(-\chi(x)+t)$ and $\psi(x) \sim \frac{1}{-\chi(x)+t}$ if $x \in I$.

If $T_+(J) \cup T_-(J) = \emptyset$ (i.e. $J = \mathbb{R}$) then only case (i) is possible.

Theorem (Daróczy-Dascăl, [4]). *Let $I \subset \mathbb{R}$ be a nonempty open interval, $p \in]0,1[$ and $\varphi, \psi \in \mathcal{CM}(I)$. The functional equation*

$$\varphi^{-1}\left(p\varphi(x)+p\varphi(y)+(1-2p)\varphi\left(\chi^{-1}\left(\frac{\chi(x)+\chi(y)}{2}\right)\right)\right)= \\ =\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right) \quad (x, y \in I) \tag{3}$$

holds if and only if either $\varphi(x) \sim \chi(x)$, $\psi(x) \sim \chi(x)$ if $x \in I$, or
if $p = \frac{1}{2}$ then $\varphi(x) \sim \psi(x)$ if $x \in I$, or
if $p = \frac{1}{4}$ then there exists a real number $\rho \neq 0$ such that $\varphi(x) \sim e^{\rho\chi(x)}$, $\psi(x) \sim e^{\frac{\rho}{2}\chi(x)}$ if $x \in I$.

When on the right-hand side there is a weighted quasi-arithmetic mean, the solution is given by the next theorem.

Theorem (Daróczy-Dascăl, [4]). *Let $I \subset \mathbb{R}$ be a nonempty open interval, $p, q \in]0,1]$, $p \neq q$, $r \in]0,1[$, $r \neq \frac{1}{2}$. If the functions $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of the functional equation*

$$\varphi^{-1}\left(p\varphi(x)+q\varphi(y)+(1-p-q)\varphi\left(\chi^{-1}\left(\frac{\chi(x)+\chi(y)}{2}\right)\right)\right)= \\ =\psi^{-1}(r\psi(x)+(1-r)\psi(y)) \ (x, y \in I)$$

then the following cases are possible:

(i) if $p+q=1$ then $p=r$ and $\varphi(x) \sim \psi(x)$ if $x \in I$;
(ii) if $p+q \neq 1$ then $r = \frac{p-q+1}{2}$ and either
$\varphi(x) \sim \chi(x)$, $\psi(x) \sim \chi(x)$ if $x \in I$, or
if $(1-p-q)^2 = 4pq$ then there exists a real number $\rho \neq 0$ such that $\varphi(x) \sim e^{\rho\chi(x)}$, $\psi(x) \sim e^{\frac{\rho}{2}\chi(x)}$ if $x \in I$.

Conversely, the functions given in the above cases are solutions of the functional equation.

In order to solve the equality problem in general, we will also use the generalization of the quasi-arithmetic means considered by Makó and Páles in [17], which comprises the conjugate means derived from the arithmetic mean and we will also use the notation of [17].

For $k \geq 1$, let $\mathcal{C}^k(I)$ denote the class of all those (φ, ψ) k-times continuously differentiable functions defined on I such that $\varphi'(x)\psi'(x) \neq 0$ for $x \in I$.

Furthermore, let μ be a signed measure on the Borel sets of $[0,1]$. We define the kth moment and the kth centralized moment of μ by

$$\widehat{\mu}_k := \int_0^1 t^k d\mu(t) \quad \text{and } \mu_k = \int_0^1 (t - \widehat{\mu}_1)^k d\mu(t) \quad (k \in \mathbb{N} \cup \{0\}).$$

Obviously, $\widehat{\mu}_0 = \mu_0 = 1$ and $\mu_1 = 0$. In the sequel, δ_τ will denote the Dirac measure concentrated at the point $\tau \in [0,1]$. Let $\varphi \in \mathcal{CM}(I)$ and $p, q \in]0,1]$. Then the conjugate mean $A_\varphi^{(p,q)}$ defined in the previous section, derived from the arithmetic mean, can be written as

$$A_\varphi^{(p,q)}(x,y) = \varphi^{-1}\left(\int_0^1 \varphi(tx + (1-t)y)d\mu(t)\right) =: M_{\varphi,\mu}(x,y) \quad (x, y \in I),$$

where the measure μ is

$$\mu = q\delta_0 + p\delta_1 + (1-p-q)\delta_{\frac{1}{2}}.$$

If $p = q = \frac{1}{2}$, i.e. $\mu = \frac{\delta_0 + \delta_1}{2}$, then $M_{\varphi,\mu}$ is a quasi-arithmetic mean.

We can easily compute the kth centralized moment of μ:

$$\mu_k = (-1)^k q \left(\frac{1+p-q}{2}\right)^k + p\left(\frac{1-p+q}{2}\right)^k + (1-p-q)\left(\frac{q-p}{2}\right)^k,$$

$k \in \mathbb{N}$. For instance,

$$\mu_2 = \frac{1}{4}(p + q - p^2 - q^2 + 2pq).$$

According to the above notation, our main question can be rephrased as: characterize those pairs (φ, μ) and (ψ, ν) such that

$$\varphi^{-1}\left(\int_0^1 \varphi(tx + (1-t)y)d\mu(t)\right) = \psi^{-1}\left(\int_0^1 \psi(tx + (1-t)y)d\nu(t)\right)$$

holds for all $x, y \in I$, where $\mu = q\delta_0 + p\delta_1 + (1-p-q)\delta_{\frac{1}{2}}$, $\nu = s\delta_0 + r\delta_1 + (1-r-s)\delta_{\frac{1}{2}}$ and $p, q, r, s \in]0,1]$. In our investigation, we will often make use of this formulation of the equality problem, too.

We will also use a result of Makó and Páles [17, Theorem 5, p. 415], which, assuming $\mathcal{C}^{n+1}$, gives further conditions necessary for the equality. Although in [17], the authors define μ as a Borel probability measure, the proof of the theorem for signed measures remains word for word the same. Thus, we will omit the proof, but for the reader's convenience we cite the theorem:

Theorem 1. *Assume $\mathcal{C}^{n+1}(I)$ for some $n \in \mathbb{N}$ and $\mu_1 = \nu_1$. Then, in order that $M_{\varphi,\mu} = M_{\psi,\nu}$ be valid, it is necessary that*

$$\sum_{i=0}^{n} \binom{n}{i} (\mu_{i+1}\nu_{n-i} - \mu_i \nu_{n+1-i}) \frac{\varphi^{(i+1)}}{\varphi'} \cdot \frac{\psi^{(n+1-i)}}{\psi'} = 0. \tag{4}$$

Conversely, if φ, ψ are analytic functions and (4) *holds for all $n \in \mathbb{N}$, then $M_{\varphi,\mu} = M_{\psi,\nu}$ is satisfied.*

We draw the reader's attention to the fact that in our problem we have $\mu_1 = \nu_1 = 0$.

3. Main result

It is enough to solve the functional equation (1) up to the equivalence of the functions φ and ψ. Thus we may assume that $\varphi'(x) > 0$ and $\psi'(x) > 0$ if $x \in I$.

In the following lemma we give the first necessary condition for the equality of the conjugate means.

Lemma 1. *Let (φ, ψ) be a pair of class $\mathcal{C}^1(I)$ that satisfies the functional equation (1) and $p, q, r, s \in]0, 1]$. Then*

$$p - q = r - s. \tag{5}$$

Proof. Differentiating (1) with respect to the first variable we have

$$\begin{aligned} &\frac{p\varphi'(x) + \frac{1-p-q}{2}\varphi'(\frac{x+y}{2})}{\varphi'(\varphi^{-1}\left(p\varphi(x) + q\varphi(y) + (1-p-q)\varphi\left(\frac{x+y}{2}\right)\right))} = \\ &= \frac{r\psi'(x) + \frac{1-r-s}{2}\psi'(\frac{x+y}{2})}{\psi'(\psi^{-1}\left(r\psi(x) + s\psi(y) + (1-r-s)\psi\left(\frac{x+y}{2}\right)\right))}, \end{aligned} \tag{6}$$

and then substituting $y := x$, we get $p - q = r - s$. This condition is equivalent with the equality of the first moments: $\hat{\mu}_1 = \hat{\nu}_1$. □

Corollary 1. *Let (φ, ψ) be a pair of class $\mathcal{C}^2(I)$ that satisfies the functional equation (1) and $p, q, r, s \in]0, 1]$. Then there exists a positive real constant c such that*

$$\varphi'(x) = c\psi'(x)^{\frac{\nu_2}{\mu_2}} \text{ for all } x \in I. \tag{7}$$

Proof. By differentiating (1) with respect to the second variable and then substituting $y := x$ we get

$$(-p-q+(p-q)^2)\frac{\varphi''}{\varphi'}(x) = (-r-s+(r-s)^2)\frac{\psi''}{\psi'}(x)$$

for all $x, y \in I$.

$-p-q+(p-q)^2 \neq 0$ and $-r-s+(r-s)^2 \neq 0$ since p, q, r, s are from the interval $]0,1]$.

Integrating the equation above we get that there exists a positive real number $c \neq 0$ such that $\varphi'(x) = c\psi'(x)^{\frac{-r-s+(r-s)^2}{-p-q+(p-q)^2}}$ for all $x \in I$, but this is equivalent to $\varphi'(x) = c\psi'(x)^{\frac{\nu_2}{\mu_2}}$. □

From Theorem 1 we can derive further necessary conditions for the equality (1). The proof is analogous to the proof given in [17, Theorem 10, p. 420].

Theorem 2. *Assume* $\mathcal{C}^2$ *and assume that equality* $M_{\varphi,\mu} = M_{\psi,\nu}$ *holds. Then,*

$$\mu_2\frac{\varphi''}{\varphi'} = \nu_2\frac{\psi''}{\psi'} =: \phi. \tag{8}$$

If $\mathcal{C}^3$ *is valid then the function* $\phi : I \to \mathbb{R}$ *introduced in* (8) *satisfies the differential equation*

$$\left(\frac{\mu_3}{\mu_2} - \frac{\nu_3}{\nu_2}\right)\phi' + \left(\frac{\mu_3}{\mu_2^2} - \frac{\nu_3}{\nu_2^2}\right)\phi^2 = 0. \tag{9}$$

If $\mathcal{C}^4$ *is also valid, then* ϕ *satisfies the differential equation*

$$\left(\frac{\mu_4}{\mu_2} - \frac{\nu_4}{\nu_2}\right)\phi'' + \left(\frac{3\mu_4}{\mu_2^2} - \frac{3\nu_4}{\nu_2^2}\right)\phi\phi' + \left(\frac{\mu_4 - 3\mu_2^2}{\mu_2^3} - \frac{\nu_4 - 3\nu_2^2}{\nu_2^3}\right)\phi^3 = 0. \tag{10}$$

Our main result is the following

Theorem 3. *Let* $I \subset \mathbb{R}$ *be a nonempty open interval,* $p, q, r, s \in]0,1]$, $p+q \neq 1$, $r+s \neq 1$. *If the functions* $\varphi, \psi \in \mathcal{CM}(I)$ *are solutions of the functional equation*

$$\varphi^{-1}\left(p\varphi(x) + q\varphi(y) + (1-p-q)\varphi\left(\frac{x+y}{2}\right)\right) =$$
$$= \psi^{-1}\left(r\psi(x) + s\psi(y) + (1-r-s)\psi\left(\frac{x+y}{2}\right)\right) \quad (x, y \in I) \tag{11}$$

then $p - q = r - s$ *and the following cases are possible:*

(i) if $p = r$ *then* $q = s$ *and* $\varphi(x) \sim \psi(x)$ *for all* $x \in I$;

(ii) *if* $p \neq r$, $p \neq q$, $r \neq s$ *and* $\mathcal{C}^3$ *is valid, then* $\varphi(x) \sim x$, $\psi(x) \sim x$ *for all* $x \in I$,

(iii) *if* $p \neq r$, $p = q$ *and* $\mathcal{C}^4$ *is valid, then* $r = s$ *and* $\varphi(x) \sim x$, $\psi(x) \sim x$ *for all* $x \in I$.

Conversely, the functions given in the above cases are solutions of (11).

Remark. In the proof, several moment conditions were factorized by using the Maple 14 symbolic package.

Proof. From Lemma 1 we have that $p - q = r - s$.

The case (i) has been solved by Daróczy-Páles in [8].

In case (ii), we know from Theorem 2 that equation (9) holds. To solve the differential equation (9) we distinguish three cases.

Case 1: $\phi = 0$, which is trivially a solution of (9). Then $\varphi'' = 0$, whence φ' and ψ' are constant functions. Therefore, $\varphi(x) \sim x$, $\psi(x) \sim x$ for all $x \in I$. By short computation it is easy to verify that the identity function is always a solution of the functional equation (1).

In the rest of the proof we may assume that ϕ is not identically zero. Denote by J a maximal subinterval of I where ϕ does not vanish. Clearly, J is open and nonempty. Then we can rewrite the differential equation (9) as

$$\frac{\phi'(x)}{\phi^2(x)} = \frac{\mu_2^2\nu_3 - \mu_3\nu_2^2}{\mu_2\nu_2(\mu_3\nu_2 - \mu_2\nu_3)} \quad (x \in J). \tag{12}$$

Now we consider the cases when the right-hand side is zero and when it is different from zero. The denominator $\mu_2\nu_2(\mu_3\nu_2 - \mu_2\nu_3)$ is different from zero as it is equivalent to

$$\mu_2\nu_2(\mu_3\nu_2 - \mu_2\nu_3) =$$

$$= \frac{1}{256}(p+q-(p-q)^2)(r+s-(r-s)^2)(p-q)(-1+p-q)(1+p-q)(q-s) \neq 0$$

Case 2: $\frac{\mu_2^2\nu_3 - \mu_3\nu_2^2}{\mu_2\nu_2(\mu_3\nu_2 - \mu_2\nu_3)} = 0$, i.e. $\phi'(x) = 0$ on J. Then, from $\mu_2^2\nu_3 - \mu_3\nu_2^2 = 0$ we have that

$$\frac{1}{64}(q-s)(p-q) \cdot$$
$$(-4qp - 2s - 2p - q^2 + 5p^2 - 4p^3 + 8qp^2 + 6sp - 4pq^2 +$$
$$+6sq - 4sq^2 - 4sp^2 - 4pq^3 - 4qp^3 + 6p^2q^2 + q^4 + p^4 + 8sqp) = 0.$$

But we know that $q = s$ gives $p = r$, i.e. case (i). Since $p \neq q$, only the third factor can be zero:

$$\begin{aligned} &-4qp - 2s - 2p - q^2 + 5p^2 - 4p^3 + 8qp^2 + 6sp - 4pq^2 - 4sq^2 + \\ &+6sq - 4sp^2 - 4pq^3 - 4qp^3 + 6p^2q^2 + q^4 + p^4 + 8sqp = 0. \end{aligned} \tag{13}$$

In this case, from the differential equation, we know that $\phi'(x) = 0$ on J. Thus, there exists a nonzero constant α such that $\phi = \mu_2\alpha$ on J. If J were a proper subinterval of I, then one of the endpoints, say a, would be contained in I. By continuity, we have $\phi(a) = \mu_2\alpha \neq 0$, which means that J is not maximal. So $J = I$. Using the definition of ϕ, we get that

$$\frac{\varphi''}{\varphi'} = \alpha.$$

Integrating this equality, we can find a constant c such that $\varphi' = \alpha(\varphi - c)$. This is a first-order linear differential equation for φ, whose general solution is of the form $\varphi(x) = de^{\alpha x} + c$ for some constant d. Obviously, $\alpha d \neq 0$ cannot be zero, since φ is strictly monotone. Using Corollary 1, it follows that $\psi \sim e^{\beta x}$, where $\beta = \frac{\mu_2}{\nu_2}\alpha$. The functions φ and ψ are obviously analytic, hence, Theorem 1 can be applied. We put in (4) in Theorem 1 $\varphi = e^{c\frac{\nu_2}{\mu_2}x}$ and $\psi = e^{cx}$ for some constant c. From (13) we can express s. Using this form of s, for $n = 1$ and $n = 2$ equation (4) holds. For $n = 3$, from equation (4) we get

$$\begin{gathered} (p^2 - 2p + 1 - 2pq - 2q + q^2)(p^2 - 2pq - 3p - 3q + q^2 + 2)\cdot \\ (p - q)^2(-1 + p - q)^2(-1 + p + q)(1 + p - q)^2 = 0 \end{gathered}$$

and this is possible if either

$$(p^2 - 2pq - 3p - 3q + q^2 + 2) = 0,$$

or

$$(p^2 - 2p + 1 - 2pq - 2q + q^2) = 0.$$

From the first equation we get $p = q + \frac{3}{2} - \frac{1}{2}\sqrt{24q + 1}$, but this gives $s = q$, i.e. case (i).

From the second equation we get $p = q + 1 - 2\sqrt{q}$, which means that $s = \sqrt{q}$ and by Lemma 1 $r + s = 1$, but this contradicts with the initial condition on the parameters.

Case 3: Let $c := \frac{\mu_2^2\nu_3 - \mu_3\nu_2^2}{\mu_2\nu_2(\mu_3\nu_2 - \mu_2\nu_3)} \neq 0$. Then, with the notation $\alpha := 1 + \frac{1}{\mu_2 c} \neq 1$, the functional equation in ϕ can be rewritten as follows:

$$\frac{\phi'(x)}{\phi^2(x)} = \frac{1}{\mu_2(\alpha - 1)} \qquad (x \in J).$$

Integrating this equality, it follows for some x_0, that

$$\frac{1}{\phi(x)} = \frac{x - x_0}{\mu_2(\alpha - 1)} \qquad (x \in J). \tag{14}$$

Hence x_0 cannot be in J. If J were a proper subinterval of I, then one of the endpoints, say a, would be contained in I. By taking the limit $x \to a$ in the above equation, it follows that ϕ has a nonzero finite limit at a. By continuity, we have that $\phi(a) = \frac{\mu_2(\alpha-1)}{a-x_0} \neq 0$, showing that J is not maximal, so the contradiction proves that $J = I$. Applying (14) and the definition of the function ϕ, we get

$$\frac{\varphi''(x)}{\varphi'(x)} = \frac{\phi(x)}{\mu_2} = \frac{\alpha - 1}{x - x_0} \qquad (x \in J).$$

Integrating this equation, we get that

$$\varphi'(x) = \begin{cases} k|x - x_0|^{\alpha-1} & \text{if } \alpha \neq 0 \\ k|x - x_0|^{-1} & \text{if } \alpha = 0 \end{cases}$$

for some constant k. After integration this yields that φ is of the form

$$\varphi(x) \sim \begin{cases} |x - x_0|^{\alpha} & \text{if } \alpha \neq 0 \\ \ln|x - x_0| & \text{if } \alpha = 0 \end{cases},$$

and using Corollary 1, we get that

$$\psi(x) \sim \begin{cases} |x - x_0|^{\beta} & \text{if } \beta \neq 0 \\ \ln|x - x_0| & \text{if } \beta = 0 \end{cases},$$

with $\beta := 1 + (\mu_2/\nu_2)(\alpha - 1) \neq 1$. Obviously, $(\alpha - 1)(\beta - 1) = (\mu_2/\nu_2)(\alpha - 1)^2 > 0$.

Now assume that $x_0 < \inf I$ (the case $x_0 \geq \inf I$ is analogous). We can assume $\alpha \neq \beta$, which is equivalent to $q \neq s$. The functions φ and ψ are analytic and we have

$$\frac{\varphi^{(j)}(x)}{\varphi'(x)} = (j-1)!\binom{\alpha - 1}{j - 1}(x - x_0)^{1-j},$$

$$\frac{\psi^{(j)}(x)}{\psi'(x)} = (j-1)!\binom{\beta - 1}{j - 1}(x - x_0)^{1-j}, \quad (x \in J, j \in \mathbb{N}).$$

Replacing these forms of φ and ψ into equation (4) we get by Theorem 1, that the equality of the means $A_\varphi^{(p,q)}$ and $A_\psi^{(r,s)}$ is equivalent to the validity of the following condition for all $n \in \mathbb{N}$

$$\sum_{i=0}^{n} \binom{\alpha - 1}{i}\binom{\beta - 1}{n - i}(\mu_{i+1}\nu_{n-i} - \mu_i\nu_{n+1-i}) = 0. \tag{15}$$

In the case $n = 1$, from (15) we can express for instance α:

$$\alpha = \frac{(2s - 2\beta s + \beta q^2 + \beta q - \beta p - 2\beta qp + \beta p^2 - 2q)}{(-q - 2qp + q^2 - p + p^2)},$$

then using this form of α, in the case $n = 2$, equation (15) gives either

$$\beta = 1,$$

which is impossible, or

$$\begin{gathered}\beta = (-2s - q - 3p^3 + q^3 - 3p + 8sqp - 6qp + 6p^2 + \\ +6sp + 6sq - 4sp^2 - 4sq^2 + 7qp^2 - 5pq^2)/ \\ (-4qp - 2s - 2p - q^2 + 5p^2 + 8sqp + 8qp^2 + 6sp - 4pq^2 + \\ +6sq - 4p^3 - 4sq^2 - 4sp^2 + q^4 + p^4 + 6p^2q^2 - 4pq^3 - 4qp^3).\end{gathered}$$

But we know that initially $\beta = 1 + (\mu_2/\nu_2)(\alpha - 1)$. Hence, replacing the initial form of β in the above equation we get the following equation only in p, q

$$(p + 1 - q)(p - 1 - q)(p^2 - p - 2qp + q^2 - q) = 0,$$

which is a contradiction. So the only possible solutions are $\varphi(x) \sim x$, $\psi(x) \sim x$ for all $x \in I$.

In case (iii), from $p = q \neq r$ we have $r = s$ by Lemma 1, and the functional equation to solve is

$$\begin{aligned}\varphi^{-1}\left(p\varphi(x) + p\varphi(y) + (1 - 2p)\varphi\left(\frac{x+y}{2}\right)\right) = \\ = \psi^{-1}\left(r\psi(x) + r\psi(y) + (1 - 2r)\psi\left(\frac{x+y}{2}\right)\right) \quad (x, y \in I).\end{aligned}$$

Then $\mu_3 = \nu_3 = 0$. Hence in this case (9) doesn't gives us any information. Thus, to solve case (iii) we will need to solve the differential equation (10). With $p = q$ and $r = s$ we have $\mu_2 = \frac{p}{2}$, $\mu_4 = \frac{p}{8}$, $\nu_2 = \frac{r}{2}$ and $\nu_4 = \frac{r}{8}$ and the differential equation (10) has the form

$$\phi\left[\frac{3}{2}\left(\frac{1}{p} - \frac{1}{r}\right)\phi' + \frac{(6pr - p - r)(p - r)}{p^2r^2}\phi^2\right] = 0.$$

Obviously, $\phi = 0$ is trivially a solution of the above differential equation. Therefore, $\varphi(x) \sim x$, $\psi(x) \sim x$ for all $x \in I$.

In the rest of the proof we may assume that ϕ is not identically zero. Denote by J a maximal subinterval of I where ϕ does not vanish. Clearly, J is open and nonempty. Then we can rewrite the differential equation (10) as

$$\frac{\phi'(x)}{\phi^2(x)} = \frac{2}{3}\cdot\frac{6pr-p-r}{pr} \quad (x \in J). \tag{16}$$

This differential equation can be solved similarly as equation (12) in case (ii), thus we will give only the main steps, omitting the computational details.

If the right-hand side of equation (16) is zero, i.e. $6pr-p-r=0$, from the differential equation we have that $\phi' = 0$ on J. Using the definition of ϕ, after some computation we get that there exists a nonzero constant λ such that $\varphi \sim e^{\lambda\frac{r}{p}x}$ and $\psi \sim e^{\lambda x}$ on I. The functions φ and ψ are obviously analytic, hence we can apply Theorem 1 and for $n=5$, from equation (4) and using $6pr-p-r=0$, we get that either $r=\frac{1}{6}$, or $r=\frac{1}{3}$, or $r=\frac{1}{2}$, or $r=\frac{1}{4}$. The first two cases imply $p=r$, which is not possible, because here $p \neq r$. The cases $r=\frac{1}{2}$, $r=\frac{1}{4}$ imply $r+s=1$ and $p+q=1$, respectively, which is again impossible.

If in (16), $\frac{2}{3}\cdot\frac{6pr-p-r}{pr} \neq 0$, applying the same method as in case (ii) and from (15) for $n=1,2,3$, we get again that the possible solutions are only $\varphi(x) \sim x$, $\psi(x) \sim x$ for all $x \in I$. $\square$

4. Application

Applying the main result of the previous section, we can answer the equality problem of conjugate means derived from a fixed quasi-arithmetic mean. Now in problem (P) let

$$L(x,y) := \chi^{-1}\left(\frac{\chi(x)+\chi(y)}{2}\right),$$

be a fixed quasi-arithmetic mean with the fixed generating function $\chi \in \mathcal{CM}(I)$. In this case, the answer to the equality problem is given by the following

Theorem 4. *Let $I \subset \mathbb{R}$ be a nonempty open interval, $p,q,r,s \in]0,1]$, $p+q \neq 1$, $r+s \neq 1$. If the functions $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of the functional equation*

$$\begin{aligned} \varphi^{-1}\left(p\varphi(x)+q\varphi(y)+(1-p-q)\varphi\left(L(x,y)\right)\right) = \\ = \psi^{-1}\left(r\psi(x)+s\psi(y)+(1-r-s)\psi\left(L(x,y)\right)\right) \ (x,y \in I) \end{aligned} \tag{17}$$

then $p-q=r-s$ and the following cases are possible:

(i) if $p=r$ then $q=s$ and $\varphi(x) \sim \psi(x)$ for all $x \in I$;
(ii) if $p \neq r$, $p \neq q$, $r \neq s$ and $\mathcal{C}^3$ is valid, then $\varphi(x) \sim \chi(x)$, $\psi(x) \sim \chi(x)$ for all $x \in I$,

(iii) if $p \neq r$, $p = q$ and $\mathcal{C}^4$ is valid, then $r = s$ and $\varphi(x) \sim \chi(x)$, $\psi(x) \sim \chi(x)$ for all $x \in I$.

Conversely, the functions given in the above cases are solutions of (11).

Proof. Let $u := \chi(x)$, $v := \chi(y)$, $K := \chi(I)$. Then equation (17) is equivalent to

$$\chi \circ \varphi^{-1}\left(p\varphi \circ \chi^{-1}(u) + q\varphi \circ \chi^{-1}(v) + (1-p-q)\varphi \circ \chi^{-1}\left(\frac{u+v}{2}\right)\right) =$$

$$= \chi \circ \psi^{-1}\left(r\psi \circ \chi^{-1}(u) + s\psi \circ \chi^{-1}(v) + (1-r-s)\psi \circ \chi^{-1}\left(\frac{u+v}{2}\right)\right)$$

for all $u, v \in K$. With the notation $f := \varphi \circ \chi^{-1}$, $g := \psi \circ \chi^{-1}$, from the above equation we have

$$f^{-1}\left(pf(u) + qf(v) + (1-p-q)f\left(\frac{u+v}{2}\right)\right) =$$
$$= g^{-1}\left(rg(u) + sg(v) + (1-r-s)g\left(\frac{u+v}{2}\right)\right) \quad (u, v \in K),$$

but this is equivalent with the equality problem of conjugate means derived from the arithmetic mean, therefore we can apply Theorem 3 for the unknown continuous, strictly monotone functions f and g, and we get the statement of our theorem. $\square$

Remark. It is worthy of note that the equality problem of conjugate means includes a well-known problem, the original Matkowski-Sutô problem and some of its generalizations (see Daróczy-Páles [7]). For example, by putting $p = q = 1$ and $r = s = \frac{1}{2}$ in equation (1), we get

$$\varphi^{-1}\left(\varphi(x) + \varphi(y) - \varphi\left(\frac{x+y}{2}\right)\right) = \psi^{-1}\left(\frac{\psi(x) + \psi(y)}{2}\right)$$

for all $x, y \in I$. Let $u := \varphi(x)$, $v := \varphi(y)$ $(u, v \in K := \varphi(I))$ be arbitrary, then from the above equation we get that

$$\varphi \circ \psi^{-1}\left(\frac{\psi \circ \varphi^{-1}(u) + \psi \circ \varphi^{-1}(v)}{2}\right) + \varphi\left(\frac{\varphi^{-1}(u) + \varphi^{-1}(v)}{2}\right) = u + v$$

for all $u, v \in K$. This equation is the original Matkowski-Sutô problem. Similarly, we can derive some generalizations of this problem from other special cases of the functional equation (1).

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Pál Burai
University of Debrecen, Faculty of Informatics
H-4010 Debrecen, Pf. 12, Hungary
E-mail address: burai.pal@inf.unideb.hu

and
TU Berlin, Department of Mathematics
June 17th street, 136, 10623 Berlin, Germany
E-mail address: burai@math.tu-berlin.de

Judita Dascăl
University of Luxembourg, Campus Kirchberg
Mathematics Research Unit, BLG
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg, Grand Duchy of Luxembourg
E-mail address: judita.dascal@uni.lu