A q-Raabe formula and an integral of the fourth Jacobi theta function

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Abstract

We generalize the Raabe-formula to the q-loggamma function by giving an integral formula for $\log\Gamma_q$ when $q > 1$. As a consequence, we get that the integral of the logarithm of the fourth Jacobi theta function between its least imaginary zeros is connected to the partition function and the Riemann zeta function.

Key words: q-gamma function, q-loggamma function, Jacobi theta functions, hypergeometric function, Riemann zeta function, partition function, Raabe-formula
1991 MSC: 33E05, 33D05

1 Introduction

The *fourth Jacobi theta function* is defined by the infinite sums

$$\vartheta_4(x,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nix} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nx).$$

This is an entire function in the complex variable x for any fixed complex q for which $|q| < 1$. See [4,25] on the theta functions in general.

The *partition function* $P(n)$ gives the number of possible additive integer partitions of the natural number n. In other words, $P(n)$ is the number of ways writing n as a sum of positive integers [3,11].

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Preprint submitted to Journal of Number Theory *29 August 2012*

In the present paper we reveal an analytic connection between the fourth Jacobi function and the partition function.

It is easy to prove that for a fixed real $q \in]0,1[$ the function $\vartheta_4(ix,q)$ is real and strictly positive when $x \in]\log\sqrt{q}, -\log\sqrt{q}[$, and it is zero at these two endpoints (see the final remarks of Section 4). Our main result concerns on the area under the graph of the fourth Jacobi function on this interval (the graph can be found in the penultimate section).

Theorem 1 *For any real $q \in]0,1[$,*

$$\int_{-x^*}^{x^*} \log \vartheta_4(ix,q)dx = \zeta(2) - \log q \cdot \log\left(\sum_{n=0}^{\infty} P(n)q^{2n}\right).$$

Here $x^* = \log\sqrt{q}$, $P(n)$ is the partition function, and $i = \sqrt{-1}$.

(Note that x^* is negative and $-x^*$ is positive, so in the theorem the integration goes from "right to left" on the imaginary axis.)

To prove the theorem we need a q-analogue of Raabe's integral

$$\int_0^1 \log\Gamma(x)dx = \log\sqrt{2\pi}$$

for the Euler gamma function. This analogue uses the so-called q-gamma function and states:

$$\int_0^1 \log\Gamma_q(x)dx = \frac{\zeta(2)}{\log q} + \log\sqrt{\frac{q-1}{\sqrt[6]{q}}} + \log(q^{-1};q^{-1})_\infty \quad (q>1),$$

where $(q^{-1};q^{-1})_\infty = \prod_{n=0}^{\infty}(1-1/q^n)$.

For a more general statement, see Theorem 2. The definition of Γ_q is given in equation (2).

2 Preliminaries

In this section we set up all the necessary ingredients and preliminary propositions to prove our q-Raabe formula and the integral formula.

2.1 *Raabe's formula*

In 1840 J. L. Raabe [22] proved that for the Euler Γ function

$$\int_0^1 \log\Gamma(x+t)dx = \log\sqrt{2\pi} + t\log t - t \quad (t \geq 0).$$

This implies the special case (when we take the limit $t \to 0+$)

$$\int_0^1 \log\Gamma(x)dx = \log\sqrt{2\pi},$$

and an immediate consequence is that

$$\int_0^1 \log\Gamma(x)\Gamma(1-x)dx = \log 2\pi.$$

(See [1] for an elementary proof of this special case.) We shall prove the appropriate integral formula for the q-gamma function when $q > 1$ (Theorem 2.). Then we show that the Jacobi triple product identity connects the q-gamma function to ϑ_4 and our main formula (Theorem 1.) will follow.

To read more on the Raabe-formula and its extension to multidimensional case, the reader may consult [16,23].

2.2 *The q-gamma functions*

F. H. Jackson defined, for $0 < q < 1$, the q-analogue of the standard Euler $\Gamma(x)$ function for any $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ as [8,13,12]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} \quad (0 < q < 1) \tag{1}$$

with the so-called *q-Pochhammer symbol* $(x;q)_\infty = (1-x)(1-qx)(1-q^2x)\cdots$. This Γ_q function is called as *Jackson q-gamma function*. This plays an important role in the evaluation of basic hypergeometric series [8]. R. Askey also contributed to the Jackson q-gamma function in a profound way, see [5,6] for examples. On analytic properties of Γ_q (including information on poles, residues, infinite sum representations) one can turn to the book [24, Section 6.4].

Another q-gamma function can also be defined for $q > 1$. It is

$$\Gamma_q(x) = \frac{(q^{-1};q^{-1})_\infty}{(q^{-x};q^{-1})_\infty}(q-1)^{1-x}q^{\binom{x}{2}} \quad (q > 1). \tag{2}$$

This function was introduced by Jackson [12, p. 129], but he did not study its properties. There are two fundamental papers of D. S. Moak [20,21], in which he investigated its analytic properties (see also Exercise 1.23 of [8] and Exercise 14 on p. 546 in [24]). Therefore in considering this contribution of Moak, one might call the function in (2) the *Moak q-gamma function*.5

We emphasize that in the present paper we need and use only the q-gamma function when $q > 1$.

2.3 The zeta regularized product

We also need some recent results of Kurokawa and Wakayama. Let us consider a sequence $\mathbf{a} = (a_1, a_2, \dots)$. Its *zeta regularized product* is denoted and defined by [14,15]

$$\widehat{\prod_{n=1}^{\infty}} a_n = \exp\left(- \operatorname*{Res}_{s=0} \frac{\zeta_{\mathbf{a}}(s)}{s^2} \right). \tag{3}$$

Here

$$\zeta_{\mathbf{a}}(s) = \sum_{n=1}^{\infty} a_n^{-s}$$

is the zeta function associated with the sequence $\mathbf{a}$. It is assumed that $\zeta_{\mathbf{a}}(s)$ is meromorphic at $s = 0$ or at least it can be meromorphically continued to $s = 0$, and further, that around this point $\zeta_{\mathbf{a}}(s)$ has the Laurent expansion

$$\zeta_{\mathbf{a}}(s) = \sum_{m > m_0} c_m(\mathbf{a}) s^m$$

for some integer m_0. Thus the zeta regularized product equals to $\exp(-c_1(\mathbf{a}))$, as well.

See [19] for nice applications of regularized products.

M. Lerch's formula (5) implies that for $\mathbf{a} = (x, 1 + x, 2 + x, \dots)$

$$\widehat{\prod_{n=0}^{\infty}} (n + x) = \frac{\sqrt{2\pi}}{\Gamma(x)} \quad (x > 0). \tag{4}$$

The sequence $\mathbf{a}$ above has the associated zeta function

$$\zeta_{\mathbf{a}}(s) = \zeta(s, x) = \sum_{n=0}^{\infty} (n + x)^{-s} \quad (x > 0,\ \Re(s) > 1).$$

This is the well known *Hurwitz zeta function.* What Lerch proved is that [17]

$$\zeta'(0, x) = \log \frac{\Gamma(x)}{\sqrt{2\pi}} \quad (x > 0). \tag{5}$$

Then (4) easily follows. See [2, Theorem 1.3.4] for a proof of (5).

Now we step forward to the q-version of the above theorems. Let us introduce the short and standard notation

$$[n]_q = \frac{q^n - 1}{q - 1} \quad (q \neq 1).$$

With this abbreviation the zeta function associated with the sequence $\mathbf{a} = ([x], [1+x]_q, [2+x]_q, \dots)$ is the so-called *q-Hurwitz zeta function*:

$$\zeta_{\mathbf{a}}(s) = \zeta_q(s, x) = \sum_{n=0}^{\infty} [n + x]_q^{-s}.$$

There are different q-extensions of the ordinary Hurwitz zeta function, see [24] for different examples.

Since the analytical properties and convergence domains of this function are crucial in the present investigation, we formulate the next proposition.

Proposition 1 *For any fixed $q > 1$ the q-Hurwitz zeta function*

$$\zeta_q(s, x) = \sum_{n=0}^{\infty} [n + x]_q^{-s}$$

converges when $x > 0$ and $\Re(s) > 0$. At the point $s = 0$ this function has a simple pole with residue $1/\log(q)$.

The proof of this proposition can be found in the last section.

With respect to the q-gamma function, the parallel result of (4) is in the next proposition.

Proposition 2 *For any real $q > 1$ and $x > 0$, there holds*

$$\widehat{\prod_{n=0}^{\infty}} [n + x]_q = \widehat{\prod_{n=1}^{\infty}} \frac{q^{n+x} - 1}{q - 1} = \frac{C_q}{\Gamma_q(x)}, \tag{6}$$

where

$$C_q = q^{-\frac{1}{12}} (q - 1)^{\frac{1}{2} - \frac{\log(q-1)}{2 \log q}} (q^{-1}; q^{-1})_\infty. \tag{7}$$

This is the second theorem of Kurokawa and Wakayama in [14] and this will be our main tool. (A more general form of this theorem is presented in [19].)

For practical reasons we rephrase this zeta regularization theorem in a more suitable form (employing (3) and (6)).

Proposition 3 *For any $q > 1$ and $x > 0$, there holds*

$$\log \Gamma_q(x) = \log C_q + \operatorname*{Res}_{s=0} \frac{\zeta_q(s,x)}{s^2},$$

where C_q is defined by (7).

The statement follows from Proposition 2 by using the regularization formula (3).

We split the proof of the main theorem to two sections. The next one contains the generalized Raabe's formula, the other contains the proof of the integral formula of Jacobi's function ϑ_4.

3 Integral of the q-loggamma function – the q-Raabe formula

The q-analogue of Raabe's theorem for $q > 1$ is given:

Theorem 2 *If $q > 1$ and $\Gamma_q(x)$ is defined by (2), then for any $t > 0$,*

$$\int_0^1 \log \Gamma_q(x+t)dx = \tag{8}$$

$$\log C_q - \frac{1}{2q^t \log q}\left[\frac{1-q^t}{1-q^{-t}}(2\operatorname{Li}_2(q^{-t}) + \log^2(1-q^{-t})) + \right.$$
$$\left. 2\frac{1-q^t}{1-q^{-t}} \log \frac{1-q}{1-q^t} \log(1-q^{-t}) - q^t \log^2 \frac{1-q}{1-q^t}\right].$$

In particular, if t tends to zero then

$$\int_0^1 \log \Gamma_q(x)dx = \frac{\zeta(2)}{\log q} + \log \sqrt{\frac{q-1}{\sqrt[6]{q}}} + \log(q^{-1};q^{-1})_\infty. \tag{9}$$

Here $\operatorname{Li}_2(z)$ is the *dilogarithm function* [18]:

$$\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

It is an interesting question that how such a theorem looks like when we use the Jackson q-gamma function (i.e., definition (1) and $0 < q < 1$). To look for a theorem of this flavour, our proof cannot be applied, since the two crucial points – the q-Hurwitz zeta and the Kurokawa-Wakayama theorem – work only when $q > 1$.

To prove Theorem 2, we need the following statement on the integral of the q-Hurwitz zeta function, which is interesting in itself.

Theorem 3 *If $q > 1$, $t > 0$ and $\Re(s) > 0$, then*

$$\int_0^1 \zeta_q(s, x+t)dx = \frac{(q-1)^s}{s \log q} \frac{(q^t-1)^{1-s}}{q^t} {}_2F_1(1, 1; s+1; q^{-t}).$$

Here

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

is a *hypergeometric function* and $(a)_n = a(a+1)\cdots(a+n-1)$ is the *Pochhammer symbol.* See a good introduction to hypergeometric functions in [10].

Proof of Theorem 3. To use the series representation of the q-Hurwitz zeta function, we have to assume that $t > 0$, $q > 1$ and $\Re(s) > 0$ (see Proposition 1). Then we have that

$$\int_0^1 \zeta_q(s, x+t)dx = \int_0^1 \sum_{n=0}^{\infty} [n+x+t]_q^{-s} dx =$$

$$(q-1)^s \sum_{n=0}^{\infty} \int_0^1 (q^{n+x+t}-1)^{-s} dx.$$

This latter integral can be computed if we determine the Taylor series of the integrand with respect to the variable x and then we integrate term by term. A lengthy computation finally shows that the integral can be expressed by hypergeometric functions:

$$\int_0^1 (q^{n+x+t}-1)^{-s} dx = \frac{1}{s \log q} \left[\frac{(q^{n+t}-1)^{1-s}}{q^{n+t}} {}_2F_1(1, 1; s+1; q^{-n-t}) - \right.$$

$$\left. \frac{(q^{n+t+1}-1)^{1-s}}{q^{n+t+1}} {}_2F_1(1, 1; s+1; q^{-n-t-1}) \right].$$

(It can be realized that the above subtraction comes from the Newton-Leibniz formula, so one can read out the primitive function and then check this integral by derivation, too.) Since

$${}_2F_1(1, 1; s+1; q^{-n-t}) = \sum_{k=0}^{\infty} \frac{k!}{(s+1)_k} \frac{1}{(q^{n+t})^k},$$

and because of $\Re(s) > 0$,

$$\sum_{n=0}^{\infty} \int_0^1 (q^{n+x+t}-1)^{-s} dx =$$

$$\frac{1}{s\log q}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{k!}{(s+1)_k}\left(\frac{(q^{n+t}-1)^{1-s}}{(q^{n+t})^{k+1}}-\frac{(q^{n+t+1}-1)^{1-s}}{(q^{n+t+1})^{k+1}}\right).$$

If we interchange the order of the summation – which can be done by absolute convergence –, we see that the sum over n is telescopic, so the only one term which not cancels belongs to $n=0$. Thus the above expression simplifies to

$$\frac{1}{s\log q}\sum_{k=0}^{\infty}\frac{k!}{(s+1)_k}\frac{(q^t-1)^{1-s}}{(q^t)^{k+1}}=\frac{(q^t-1)^{1-s}}{sq^t\log q}\sum_{k=0}^{\infty}\frac{k!}{(s+1)_k}\frac{1}{(q^t)^k}.$$

This latter sum is again hypergeometric with parameters $(1,1;s+1;q^{-t})$, hence we get our Theorem. □

Proof of Theorem 2. Proposition 3 gives that

$$\int_0^1\log\Gamma_q(x+t)dx=\log C_q+\int_0^1\operatorname*{Res}_{s=0}\frac{\zeta(s,x+t)}{s^2}dx.$$

Since the residue is taken with respect to s, we can carry out it before the integral. Hence, by Theorem 3,

$$\int_0^1\log\Gamma_q(x+t)dx=\log C_q+\operatorname*{Res}_{s=0}\frac{(q-1)^s}{s^3\log q}\frac{(q^t-1)^{1-s}}{q^t}{}_2F_1(1,1;s+1;q^{-t}).$$

The residue can be calculated as follows: we leave s^3 in the denominator, then we look for the coefficient of s^2 in the Taylor expansion of the remaining function. A lenghty calculation shows that the residue equals to

$$\frac{-1}{2q^t\log q}\left[(1-q^t)\left.\frac{\partial^2}{\partial s^2}{}_2F_1(1,1;s;q^{-t})\right|_{s=1}+\right. \tag{10}$$

$$\left.2(1-q^t)\log\frac{1-q}{1-q^t}\left.\frac{\partial}{\partial s}{}_2F_1(1,1;s;q^{-t})\right|_{s=1}-q^t\log^2\frac{1-q}{1-q^t}\right].$$

Now we deal with the partial derivatives. Symbolically,

$$\frac{\partial^n}{\partial s^n}{}_2F_1(1,1;s;z)=\sum_{n=0}^{\infty}(1)_n(1)_n\frac{\partial^n}{\partial s^n}\frac{1}{(s)_n}\frac{z^n}{n!}. \tag{11}$$

The Pochhammer symbol can be rewritten with the Γ function:

$$(s)_n=\frac{\Gamma(s+n)}{\Gamma(s)},$$

whence

$$\frac{\partial}{\partial s}\frac{1}{(s)_n}=\frac{-1}{(s)_n}(\psi(s+n)-\psi(s)), \tag{12}$$

and

$$\frac{\partial^2}{\partial s^2}\frac{1}{(s)_n}=\frac{(\psi(s+n)-\psi(s))^2}{(s)_n}-\frac{\psi'(s+n)-\psi'(s)}{(s)_n}.$$

Here

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

is the *digamma function* [5,9]. When n is a positive integer, then [2, p. 13]

$$\psi(n) = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} - \gamma = H_{n-1} - \gamma, \tag{13}$$

and

$$\psi'(n) = -\frac{1}{1^2} - \frac{1}{2^2} - \cdots - \frac{1}{(n-1)^2} + \zeta(2) = -H_{n-1,2} + \zeta(2).$$

(H_n and $H_{n,2}$ are the *harmonic-* and *second order harmonic numbers*, respectively. $H_0 = H_{0,2} = 0$.) Now (11), (12) and (13) gives that

$$\left.\frac{\partial}{\partial s}{}_2F_1(1,1;s;z)\right|_{s=1} = \sum_{n=0}^{\infty} n!n!\frac{-H_n}{n!}\frac{z^n}{n!} = -\sum_{n=0}^{\infty} H_n z^n = \frac{\log(1-z)}{1-z}.$$

The last equality is straightforward (see [10]). Similarly, for the second order derivative

$$\left.\frac{\partial^2}{\partial s^2}{}_2F_1(1,1;s;z)\right|_{s=1} = \sum_{n=0}^{\infty} n!n!\left(\frac{H_n^2}{n!} + \frac{H_{n,2}}{n!}\right)\frac{z^n}{n!} = \sum_{n=1}^{\infty} H_n^2 z^n + \sum_{n=1}^{\infty} H_{n,2} z^n. \tag{14}$$

By Cauchy's product, the latter sum is simply

$$\frac{1}{1-z}\sum_{n=1}^{\infty}\frac{z^n}{n^2} = \frac{\mathrm{Li}_2(z)}{1-z}.$$

The first sum can be determined easily. Note that

$$H_{n-1}^2 = \left(H_n - \frac{1}{n}\right)^2 = H_n^2 + \frac{1}{n^2} - 2\frac{H_n}{n},$$

whence

$$\sum_{n=1}^{\infty} H_{n-1}^2 z^n = \sum_{n=1}^{\infty} H_n^2 z^n + \sum_{n=1}^{\infty}\frac{z^n}{n^2} - 2\sum_{n=1}^{\infty}\frac{H_n}{n}z^n. \tag{15}$$

The last sum equals to [7]

$$\sum_{n=1}^{\infty}\frac{H_n}{n}z^n = \mathrm{Li}_2(z) + \frac{1}{2}\log^2(1-z). \tag{16}$$

If we temporarily introduce the function

$$f(z) = \sum_{n=1}^{\infty} H_n^2 z^n,$$

then (15) and (16) implies that

$$zf(z) = f(z) + \mathrm{Li}_2(z) - 2(\mathrm{Li}_2(z) + \frac{1}{2}\log^2(1-z)),$$

hence

$$f(z) = \sum_{n=1}^{\infty} H_n^2 z^n = \frac{\mathrm{Li}_2(z) + \log^2(1-z)}{1-z}.$$

Altogether, (14) becomes

$$\frac{\partial^2}{\partial s^2} {}_2F_1(1,1;s;z)\bigg|_{s=1} = \frac{2\,\mathrm{Li}_2(z) + \log^2(1-z)}{1-z}.$$

The partial derivatives in (10) are determined and the first part of the Theorem (i.e. formula (8)) is proved.

The second part, formula (9), can be proven if we take the limit $t \to 0$ in (8). Since this step is not trivial, we give the details:

$$\lim_{t\to 0} \frac{-1}{2q^t \log q}\left[\frac{1-q^t}{1-q^{-t}}(2\,\mathrm{Li}_2(q^{-t}) + \log^2(1-q^{-t})) + \right.$$

$$\left. 2\frac{1-q^t}{1-q^{-t}}\log\frac{1-q}{1-q^t}\log(1-q^{-t}) - q^t\log^2\frac{1-q}{1-q^t}\right] =$$

$$\frac{-1}{2\log q}\left[-2\,\mathrm{Li}_2(1) - \lim_{t\to 0}\left(+\log^2(1-q^{-t}) + 2\log\frac{1-q}{1-q^t}\log(1-q^{-t}) + \log^2\frac{1-q}{1-q^t}\right)\right] =$$

$$\frac{-1}{2\log q}\left[-2\,\mathrm{Li}_2(1) - \lim_{t\to 0}\left(\log(1-q^{-t}) + \log\frac{1-q}{1-q^t}\right)^2\right] =$$

$$\frac{-1}{2\log q}\left[-2\,\mathrm{Li}_2(1) - \lim_{t\to 0}\log^2\left((1-q)\frac{1-q^{-t}}{1-q^t}\right)\right] = \frac{1}{2\log q}(2\zeta(2) + \log^2(q-1)).$$

Thus (8) tends to the simple expression

$$\int_0^1 \log\Gamma_q(x)dx = \log C_q + \frac{1}{2\log q}(2\zeta(2) + \log^2(q-1)).$$

The definition (7) of C_q enables us to get a more simple identity. Since

$$\log C_q = -\frac{1}{12}\log q + \frac{1}{2}\log(q-1) - \frac{\log^2(q-1)}{2\log q} + \log(q^{-1};q^{-1})_\infty,$$

the term $\frac{\log^2(q-1)}{2\log q}$ cancels and a trivial modification gives the second formula (9) of our Theorem 2. □

4 The proof of Theorem 1.

The (2) definition of the q-gamma function and some reduction gives that for any $q > 1$ and $y > 0$

$$\frac{1}{\Gamma_{q^2}\left(\frac{1}{2}\log_q\left(\frac{q}{y}\right)\right)\Gamma_{q^2}\left(\frac{1}{2}\log_q(qy)\right)} = \tag{17}$$

$$\frac{(q^{\frac{1}{2}})^{1-\log_q^2 y}}{(q^{-2};q^{-2})_\infty^3(q^2-1)}(q^{-2};q^{-2})_\infty(y/q;q^{-2})_\infty(1/(yq);q^{-2})_\infty$$

This product can be rewritten by Jacobi's triple product identity [8, p. 15]:

$$(q^{-2};q^{-2})_\infty(y/q;q^{-2})_\infty(1/(qy);q^{-2})_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{-n^2}y^n \quad (q > 1, y \neq 0).$$

In (17) we choose $y = q^{1-2x}$. Then $\frac{1}{2}\log_q\left(\frac{q}{y}\right) = x$ and $\frac{1}{2}\log_q(qy) = 1-x$, so Jacobi's triple product identity yields

$$\frac{1}{\Gamma_{q^2}(x)\Gamma_{q^2}(1-x)} = \frac{(q^{\frac{1}{2}})^{1-(1-2x)^2}}{(q^{-2};q^{-2})_\infty^3(q^2-1)}\sum_{n=-\infty}^{\infty}(-1)^n q^{-n^2}(q^{1-2x})^n.$$

Next consider the definition of the Jacobi function ϑ_4 on the first page. It is not hard to see that we arrive at the next formula:

$$\frac{1}{\Gamma_{q^2}(x)\Gamma_{q^2}(1-x)} = \frac{q^{2x(1-x)}}{(q^{-2};q^{-2})_\infty^3(q^2-1)}\vartheta_4\left(\frac{1}{2i}(1-2x)\log q, \frac{1}{q}\right).$$

In the next step we take logarithm of both sides and integrate on $[0,1]$.

$$\int_0^1 \log\Gamma_{q^2}(x)\Gamma_{q^2}(1-x)dx =$$

$$\log(q^{-2};q^{-2})_\infty^3(q^2-1) - \log q\int_0^1 2x(1-x)dx - \int_0^1 \log\vartheta_4\left(\frac{1}{2i}(1-2x)\log q, \frac{1}{q}\right)dx.$$

Using Theorem 2, the right hand side must be equal to

$$\frac{2\zeta(2)}{\log q^2} + \log\frac{q^2-1}{\sqrt[6]{q^2}} + \log(q^{-2};q^{-2})_\infty^2.$$

An elementary simplification implies that

$$\int_0^1 \log\vartheta_4\left(\frac{1}{2i}(1-2x)\log q, \frac{1}{q}\right)dx = \log(q^{-2};q^{-2})_\infty - \frac{\zeta(2)}{\log q}.$$

We transform the integral:

$$\int_0^1 \log\vartheta_4\left(\frac{1}{2i}(1-2x)\log q, \frac{1}{q}\right)dx = \frac{-i}{\log q}\int_{-\frac{1}{2}i\log q}^{\frac{1}{2}i\log q}\log\vartheta_4\left(x, \frac{1}{q}\right)dx.$$

Therefore

$$\int_{-\frac{1}{2}i\log q}^{\frac{1}{2}i\log q} \log \vartheta_4\left(x, \frac{1}{q}\right) dx = \frac{1}{i}\left[\zeta(2) - \log q \log(q^{-2}; q^{-2})_\infty\right]. \tag{18}$$

Let us consider the left endpoint of the integration:

$$\vartheta_4\left(-\frac{1}{2}i\log q, \frac{1}{q}\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n-n^2}.$$

It is straightforward to see that all terms cancel, so

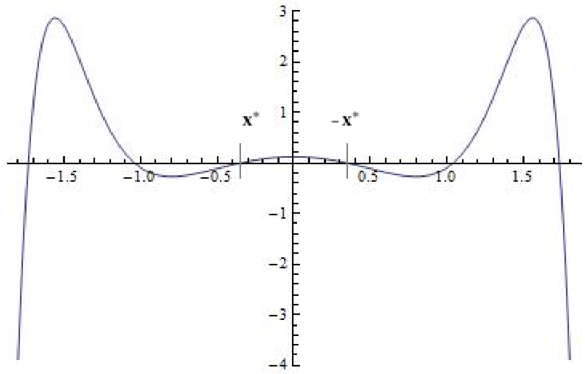
$$\vartheta_4\left(-\frac{1}{2}i\log q, \frac{1}{q}\right) = 0.$$

Similarly,

$$\vartheta_4\left(\frac{1}{2}i\log q, \frac{1}{q}\right) = 0.$$

Note that if $q > 1$, as we suppose, then the second argument of the theta function, $1/q$, is between 0 and 1. So, for the sake of simplicity, from now on we change to $q = \frac{1}{q}$ and we restrict q to the interval $]0, 1[$.

To visualize the roots, we draw $\vartheta_4(ix, 1/2)$



From the graph it is obvious that there are no roots of $\vartheta_4(ix, q)$ between $]i\log\sqrt{q}, -i\log\sqrt{q}[$ and that this function is positive on this interval. Thus the first claim of the introduction is strenghtened.

In addition, changing our variable q as we did above the graph, (18) also modifies:

$$\int_{i\log\sqrt{q}}^{-i\log\sqrt{q}} \log \vartheta_4(x, q)\, dx = \frac{1}{i}\left[\zeta(2) + \log q \log(q^2; q^2)_\infty\right] \quad (0 < q < 1).$$

Interchanging the limits of the integration, substituting ix in place of x and using the well known generating function [3]

$$\sum_{n=0}^{\infty} P(n)q^n = (q;q)_\infty^{-1},$$

we are done. $\square$

5 Proof of Proposition 1

To put the paper in a logically closed form, there is one more statement left to prove. We have to justify that the series

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} [n+x]_q^{-s}$$

converges when x is a positive real number, $q > 1$ and $\Re(s) > 0$. Having these assumptions, for the general term the next estimation is valid:

$$0 < \frac{1}{[n+x]_q} = \frac{q-1}{q^{n+x}-1} < \frac{q-1}{q^{n+x}-q} = \frac{1}{q}\frac{q-1}{q^{n-1+x}-1} = \frac{1}{[n-1+x]_q}.$$

Therefore, by induction,

$$\frac{1}{[n+x]_q} < \frac{1}{q^n}\frac{1}{[0+x]_q} = \frac{1}{q^n}\frac{q-1}{q^x-1}.$$

This shows that the general term exponentially decreases for any fixed, positive real x. (Note that for complex x there are additional singularities of this function and the present argument cannot be applied.) Now we have that

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} [n+x]_q^{-s} < \frac{q-1}{q^x-1}\sum_{n=0}^{\infty}\frac{1}{(q^s)^n}.$$

If $\Re(s) > 0$ then $1/q^s$ has absolute value less than one, so the sum on the right converges.

However, when $s = 0$, $\zeta_q(s,x)$ obviously diverges. Kurokawa and Wakayama [14, p. 297] proved the next asymptotic estimation around $s = 0$ of the q-Hurwitz function:

$$\zeta_q(s,x) = \frac{1}{\log(q)}\frac{1}{s} + \frac{1}{2} - x + \frac{\log(q-1)}{\log(q)} + O(s).$$

This shows that at $s = 0$ this function has a simple pole with residue $1/\log(q)$, as we stated in our proposition.

Acknowledgement

The author wishes to thank to the unknown referee for his/her suggestions which highly improved the quality of the presentation.

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