

INTEGRALS OF WEIGHTED MAXIMAL LOGARITHMIC KERNELS ON BOUNDED VILENKIN GROUPS

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ABSTRACT. The integrals of maximal Riesz- and Nörlund kernels are infinite, so we have to use some weight function to "pull them back" to the finite. In this paper we give necessary and sufficient conditions for weight function to get finite integral on bounded Vilenkin groups. For our motivation we refer the readers to [4, 5, 6].

1. INTRODUCTION

Kernel functions play an important role in harmonic analysis; this is the case in harmonic analysis on algebraic structures, too. The fundamental problem in Fourier theory is the convergence of the partial sums of Fourier series of functions in some spaces. According to the Banach–Steinhaus Theorem, this problem is approachable by the integrals of kernel functions as one can see in [3].

In this article we investigate integrals of maximal Riesz- and Nörlund kernels on bounded Vilenkin groups. Since, these integrals are infinite, we use some weight function to "pull them back" to the finite.

Let us introduce the standard notations [1, 8].

$$\mathbb{Z}_m := \{0, 1, \dots, m - 1\}$$

represents the m th cyclic group. Let us supply \mathbb{Z}_m with the discrete topology and with the measure μ_m which is defined by $\mu_m(\{x\}) = 1/m$ for each $x \in \mathbb{Z}_m$.

Let $(m_k : k \in \mathbb{N})$ be a sequence of natural numbers, which members are greater than 1. Let

$$G_m := \prod_{k=0}^{\infty} \mathbb{Z}_{m_k}.$$

This set is an abelian group with coordinate-wise addition modulo m_k . With product topology and product measure μ the group G_m is a compact topological space and probability space, called Vilenkin group. If $m_* := \sup_{k \in \mathbb{N}} m_k < \infty$, we say that G_m is bounded. In the present paper we deal with bounded Vilenkin groups, only. In special case $m_k = 2$ for all $k \in \mathbb{N}$, it is called Walsh group.

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The elements of G_m can be represented by a sequence $x = (x_i : i \in \mathbb{N})$, where $x_i \in \mathbb{Z}_{m_i}$ ($i \in \mathbb{N}$). The group operation on G_m (denoted by $+$) is the coordinate-wise addition modulo m_k .

A base for the neighbourhoods of G_m is given by

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_i = x_i \text{ if } i < n\}.$$

Let $0 := (0 : i \in \mathbb{N}) \in G_m$ denote the null element of G_m . By convenience, $I_n(0)$ is denoted by I_n .

Each natural number n can be expressed uniquely in the form

$$n = \sum_{i=0}^{\infty} n_i M_i,$$

where $n_i \in \{0, 1, \dots, m_i - 1\}$ and $M_0 := 1$, $M_{n+1} := m_n M_n$ ($n \in \mathbb{N}$). (For the Walsh group $M_n = 2^n$.) Let the order of $n > 0$ be defined by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$. That is, $M_{|n|} \leq n < M_{|n|+1}$ holds.

The generalised Rademacher functions are defined as

$$r_k(x) = \exp\left(2\pi i \frac{x_k}{m_k}\right) \quad (x \in G_m, n \in \mathbb{N}, i = \sqrt{-1}).$$

By these functions a complete orthonormal system is constructed on Vilenkin group, called Vilenkin system:

$$\psi_n(x) := \prod_{j=0}^{\infty} (r_j(x))^{n_j} \quad (x \in G_m).$$

Each ψ_n is a character of G_m , and all the characters of G_m are of this form [9]. Define the m -adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j \pmod{m_j}) M_j \quad (k, n \in \mathbb{N}).$$

Then, $\psi_{k \oplus n} = \psi_k \psi_n$, $\psi_n(x+y) = \psi_n(x) \psi_n(y)$, $\psi(-x) = \overline{\psi_n(x)}$, $|\psi_n| = 1$ ($k, n \in \mathbb{N}, x, y \in G_m$).

Now, we define the main notions of this paper.

The Fourier coefficients, Dirichlet kernels and the Fejér kernels are defined by

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi_n} d\mu,$$

$$D_n := \sum_{k=0}^{n-1} \psi_k, \quad K_n := \frac{1}{n} \sum_{k=0}^n D_k,$$

($n \in \mathbb{N}$) where $D_0 = K_0 = 0$. It is known [1] that

$$(1) \quad D_{M_n} = \begin{cases} M_n, & \text{if } x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

The general logarithmic kernels are

$$Q_n = \frac{1}{H_n} \sum_{k=1}^{n-1} q_k D_k \quad (n \in \mathbb{P}),$$

where $q_k := \frac{1}{k}$ or $\frac{1}{n-k}$ (where $Q_1 := 0$) and

$$H_n = \sum_{k=1}^{n-1} \frac{1}{k}$$

(set $H_1 := 1$) is the n th harmonic number (see below for more details).

Namely, for $q_k = \frac{1}{k}$ we get $Q_n = R_n$ the n th Riesz kernels and for $q_k = \frac{1}{n-k}$ we have $Q_n = N_n$ the n th Nörlund kernels, respectively. That is,

$$R_n = \frac{1}{H_n} \sum_{k=1}^{n-1} \frac{D_k}{k}, \quad N_n = \frac{1}{H_n} \sum_{k=1}^{n-1} \frac{D_k}{n-k} \quad (n \in \mathbb{P}).$$

The most important distinction between these kernels is that in the Riesz case the rear members of the sum have greater weight than the fore ones. Nörlund kernels can be considered as a "reverse" of Riesz kernels.

The maximal function of the kernels Q_n 's is defined by

$$Q^*(x) := \sup_{n \in \mathbb{P}} |Q_n(x)| \quad (x \in G_m).$$

The partial sums of the Fourier series with respect to a function f are

$$S_n(f) = \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k.$$

It is easy to see that $S_n(f) = f * D_n$, where $*$ denotes the standard convolution.

As in general, ℓ^1 is the space which contains the sequences $(a_n : n \in \mathbb{N})$ for which $\sum_{k=0}^{\infty} |a_k| < \infty$.

Let $Q_n = R_n$ or N_n in any occurrences from this point. As we shall see that the problem is the next:

$$\|Q^*\|_1 := \int_{G_m} Q^* d\mu = +\infty$$

for any bounded Vilenkin group G_m . Thus, we would like to find some monotone increasing weight function $\alpha : [0, +\infty[\rightarrow [1, +\infty[$ such that the integral of weighted maximal kernel

$$Q_\alpha^*(x) = \sup_{n \in \mathbb{P}} \frac{|Q_n(x)|}{\alpha(n)} \quad (x \in G_m)$$

turn to be finite. That is, $\|Q_\alpha^*\|_1 \leq c < \infty$. This problem was raised by Gát [4] for the Walsh-Dirichlet and Walsh-Kaczmarz-Dirichlet kernels and he solved it. The second author discussed the behavior of maximal weighted kernels of Walsh-Fejér and Walsh-Kaczmarz-Fejér kernels [6].

For Vilenkin systems Simon and the first author [5] investigated the behavior of Vilenkin-Dirichlet and Vilenkin-Fejér kernels in this special point of view. We give more details later.

For Walsh group Gát used the map $\alpha \circ \log$, we denote it by $\tilde{\alpha}$.

2. MAIN RESULTS

Now, we formulate the results of this paper.

Theorem 1. *If G_m is a bounded Vilenkin group, then there exist constants $c, C > 0$ such that*

$$c \sum_{A=1}^{\infty} \frac{1}{A\alpha(M_A)} \leq \|R_{\alpha}^*\|_1 \leq C \sum_{A=1}^{\infty} \frac{1}{A\alpha(M_A)}.$$

We immediately have

Corollary 1. *For any bounded Vilenkin group $R_{\alpha}^* \in L^1$ if and only if $(1/A\alpha(M_A) : A \in \mathbb{P}) \in \ell^1$.*

Corollary 2. *In Walsh case*

$$c \sum_{A=1}^{\infty} \frac{1}{A\alpha(A)} \leq \|R_{\tilde{\alpha}}^*\|_1 \leq C \sum_{A=1}^{\infty} \frac{1}{A\alpha(A)}.$$

Theorem 2. *If G_m is a bounded Vilenkin group, then there exist constants $c, C > 0$ such that*

$$c \sum_{A=1}^{\infty} \frac{1}{\alpha(M_A)} \leq \|N_{\alpha}^*\|_1 \leq C \sum_{A=1}^{\infty} \frac{1}{\alpha(M_A)}.$$

We immediately have the following Corollaries:

Corollary 3. *For any bounded Vilenkin group $N_{\alpha}^* \in L^1$ if and only if $(1/\alpha(M_A) : A \in \mathbb{P}) \in \ell^1$.*

Corollary 4. *In Walsh case*

$$c \sum_{A=1}^{\infty} \frac{1}{\alpha(A)} \leq \|N_{\tilde{\alpha}}^*\|_1 \leq C \sum_{A=1}^{\infty} \frac{1}{\alpha(A)}.$$

This means that the behavior of the maximal function of weighted Nörlund logarithmic kernels is worse than the behavior of the maximal function of weighted Riesz logarithmic kernels. This means that the two summation method are far away from each other.

To gather informations on maximal functions of weighted kernels, we give a comprehensive tabular. For details, see [4, 5, 6].

Kernel type	Condition on α
Vilenkin-Dirichlet	$\frac{\ln(m_A)}{\alpha(M_A)} \in \ell^1 \Leftrightarrow D_\alpha^* \in L^1$
Vilenkin-Fejér	$\frac{\ln(m_A)}{\alpha(M_A)} \in \ell^1 \Leftrightarrow K_\alpha^* \in L^1$
Bounded Vilenkin-Riesz	$\frac{1}{A\alpha(M_A)} \in \ell^1 \Leftrightarrow R_\alpha^* \in L^1$
Bounded Vilenkin-Nörlund	$\frac{1}{\alpha(M_A)} \in \ell^1 \Leftrightarrow N_\alpha^* \in L^1$
Walsh-Kaczmarz-Dirichlet	$\frac{A}{\alpha(A)} \in \ell^1 \Leftrightarrow D_{\tilde{\alpha}}^{\kappa,*} \in L^1$
Walsh-Kaczmarz-Fejér	$\frac{1}{\alpha(A)} \in \ell^1 \Leftrightarrow K_{\tilde{\alpha}}^{\kappa,*} \in L^1$

As our theorems and earlier results show that, the best summability method have to be the Riesz's method.

We need some results on harmonic numbers and their sums. Here we recall some results described in [2]. First of all, the next estimation is valid:

$$\frac{1}{2n} + \ln(n-1) + \gamma < H_n < \frac{1}{2(n-1)} + \ln(n-1) + \gamma \quad (n \in \mathbb{N}),$$

where $\gamma = 0.5772\dots$ is the Euler-Mascheroni constant.

Second order harmonic numbers (also called hyperharmonic numbers) are well-known notions in number theory.

$$H_n^{(2)} = \sum_{k=2}^n H_k,$$

There is a useful connection between harmonic and hyperharmonic numbers:

$$H_n^{(2)} = n(H_{n+1} - 1).$$

We remark that one can define higher order harmonic numbers also. They appear in combinatoric number theory, too.

3. PROOFS

During the proofs the constants are not necessarily the same in several occurrences.

Proof of Theorem 1: First of all, we show that $\|R^*\|_1 = +\infty$. For every $x \in I_A \setminus I_{A+1}$ we have that $D_k(x) = k$ for $1 \leq k < M_{A+1}$ and

$$R_{M_A}(x) = \frac{1}{H_{M_A}} \sum_{k=1}^{M_A-1} \frac{D_k(x)}{k} = \frac{1}{H_{M_A}} (M_A - 1) \geq \frac{1}{2} \frac{M_A}{H_{M_A}}.$$

Now, we deduce the infiniteness of $\|R^*\|_1$. To do this we decompose the set G_m as a disjoint union

$$G_m = \bigcup_{A=0}^{\infty} (I_A \setminus I_{A+1})$$

and we write that

$$\begin{aligned}
\|R^*\|_1 &= \int_{G_m} \sup_{n \in \mathbb{P}} |R_n(x)| d\mu(x) = \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} \sup_{n \in \mathbb{P}} |R_n(x)| d\mu(x) \\
&\geq \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} |R_{M_A}(x)| d\mu(x) \geq \frac{1}{2} \sum_{A=0}^{\infty} \int_{I_A \setminus I_{A+1}} \frac{M_A}{H_{M_A}} d\mu \\
&\geq \frac{1}{2} \sum_{A=0}^{\infty} \frac{1}{M_{A+1}} \frac{M_A}{H_{M_A}} \geq \frac{1}{2m_*} \sum_{A=0}^{\infty} \frac{1}{H_{M_A}}.
\end{aligned}$$

Moreover, it is simple to show that

$$cA \leq c \ln 2^A \leq c \ln(M_A) \leq H_{M_A} \leq C \ln(M_A) \leq C \ln(m_*^A) \leq CA.$$

This immediately gives our statement.

To get the lower estimation for the weighted Riesz kernel, let us follow the steps in the proof of infiniteness. We get

$$\begin{aligned}
\|R_\alpha^*\|_1 &= \int_{G_m} \sup_{n \in \mathbb{P}} \frac{|R_n(x)|}{\alpha(n)} d\mu(x) \geq \sum_{A=1}^{\infty} \int_{I_A \setminus I_{A+1}} \frac{|R_{M_A}(x)|}{\alpha(M_A)} d\mu(x) \\
&\geq \frac{1}{2} \sum_{A=1}^{\infty} \frac{1}{M_{A+1}} \frac{M_A}{H_{M_A} \alpha(M_A)} \geq \frac{1}{2m_*} \sum_{A=1}^{\infty} \frac{1}{H_{M_A} \alpha(M_A)} \\
&\geq c \sum_{A=1}^{\infty} \frac{1}{A \alpha(M_A)}.
\end{aligned}$$

The upper bound estimation:

$$\begin{aligned}
R_\alpha^* &= \sup_{n \in \mathbb{P}} \frac{|R_n|}{\alpha(n)} \leq \sup_{n \in \mathbb{P}} \frac{1}{H_n \alpha(n)} \left| \sum_{k=1}^{M_{|n|}-1} \frac{D_k}{k} \right| + \sup_{n \in \mathbb{P}} \frac{1}{H_n \alpha(n)} \left| \sum_{k=M_{|n|}}^{n_{|n|} M_{|n|}-1} \frac{D_k}{k} \right| \\
&\quad + \sup_{n \in \mathbb{P}} \frac{1}{H_n \alpha(n)} \left| \sum_{k=n_{|n|} M_{|n|}}^{n-1} \frac{D_k}{k} \right| =: S^1 + S^2 + S^3.
\end{aligned}$$

It is well-known [1] that

$$(2) \quad D_{n_A M_A + j} = D_{n_A M_A} + \psi_{n_A M_A} D_j, \quad 0 \leq j < M_A, 0 \leq n_A < m_A.$$

First, we discuss S^1 (S^2 goes analogously).

$$\begin{aligned}
 S^1 &\leq \sup_{n \in \mathbb{P}} \frac{1}{H_n \alpha(n)} \sum_{A=0}^{|n|-1} \left| \sum_{k=M_A}^{M_{A+1}-1} \frac{D_k}{k} \right| \\
 &\leq \sup_{n \in \mathbb{P}} \sum_{A=0}^{|n|-1} \frac{1}{H_{M_A} \alpha(M_A)} \sum_{B=1}^{m_A-1} \left| \sum_{k=BM_A}^{(B+1)M_A-1} \frac{D_k}{k} \right| \\
 &\leq \sup_{n \in \mathbb{P}} \sum_{A=0}^{|n|-1} \frac{1}{H_{M_A} \alpha(M_A)} \sum_{B=1}^{m_A-1} \left| \sum_{l=0}^{M_A-1} \frac{D_{BM_A+l}}{BM_A+l} \right| \\
 &\leq \sum_{A=0}^{\infty} \frac{1}{H_{M_A} \alpha(M_A)} \sum_{B=1}^{m_A-1} \left| \sum_{l=0}^{M_A-1} \frac{D_{BM_A}}{BM_A+l} \right| \\
 &\quad + \sum_{A=0}^{\infty} \frac{1}{H_{M_A} \alpha(M_A)} \sum_{B=1}^{m_A-1} \left| \sum_{l=0}^{M_A-1} \frac{D_l}{BM_A+l} \right| \\
 &=: S^{1,1} + S^{1,2}.
 \end{aligned}$$

From equation (1) we have that $\|D_{BM_A}\|_1 \leq c$ and

$$\begin{aligned}
 \|S^{1,1}\|_1 &\leq \sum_{A=0}^{\infty} \frac{c}{H_{M_A} \alpha(M_A)} \sum_{B=1}^{m_A-1} \sum_{l=0}^{M_A-1} \frac{1}{BM_A+l} \\
 &\leq c \sum_{A=0}^{\infty} \frac{(H_{m_A M_A} - H_{M_A-1})}{H_{M_A} \alpha(M_A)} \\
 &\leq c \sum_{A=0}^{\infty} \frac{\ln m_A}{H_{M_A} \alpha(M_A)} \leq c \sum_{A=1}^{\infty} \frac{1}{A \alpha(M_A)}.
 \end{aligned}$$

To discuss $S^{1,2}$ we use Abel's transformation.

$$\begin{aligned}
 \left| \sum_{l=0}^{M_A-1} \frac{D_l}{BM_A+l} \right| &\leq \left| \sum_{l=1}^{M_A-2} \left(\frac{1}{BM_A+l} - \frac{1}{BM_A+l+1} \right) l K_l \right| \\
 &\quad + \frac{M_A-1}{(B+1)M_A-1} |K_{M_A-1}|.
 \end{aligned}$$

There exists a $c > 0$ such that $\|K_j\|_1 \leq c$ for all $j \in \mathbb{P}$ (see [7]). Thus,

$$\begin{aligned}
 \|S^{1,2}\|_1 &\leq \sum_{A=0}^{\infty} \frac{c}{H_{M_A} \alpha(M_A)} \sum_{B=1}^{m_A-1} \left(\sum_{l=1}^{M_A-2} \frac{1}{BM_A+l} + \frac{M_A-1}{(B+1)M_A-1} \right) \\
 &\leq \sum_{A=0}^{\infty} \frac{c(H_{m_A M_A} - H_{M_A} + m_A)}{H_{M_A} \alpha(M_A)} \leq \sum_{A=1}^{\infty} \frac{c}{A \alpha(M_A)}.
 \end{aligned}$$

At last, we discuss S^3 . By the help of equation (2) we write that

$$\begin{aligned}
S^3 &= \sup_{n \in \mathbb{P}} \frac{1}{H_n \alpha(n)} \left| \sum_{k=n_{|n|} M_{|n|}}^{n-1} \frac{D_k}{k} \right| \\
&\leq \sum_{A=0}^{\infty} \frac{1}{H_{M_A} \alpha(M_A)} \sup_{|n|=A} \left| \sum_{k=n_A M_A}^{n-1} \frac{D_k}{k} \right| \\
&\leq \sum_{A=0}^{\infty} \frac{1}{H_{M_A} \alpha(M_A)} \sup_{|n|=A} \left| \sum_{l=0}^{n-n_A M_A-1} \frac{D_{n_A M_A+l}}{n_A M_A+l} \right| \\
&\leq \sum_{A=0}^{\infty} \frac{|D_{n_A M_A}|}{H_{M_A} \alpha(M_A)} \sup_{|n|=A} \left| \sum_{l=0}^{n-n_A M_A-1} \frac{1}{n_A M_A+l} \right| \\
&\quad + \sum_{A=0}^{\infty} \frac{1}{H_{M_A} \alpha(M_A)} \sup_{|n|=A} \left| \sum_{l=0}^{n-n_A M_A-1} \frac{D_l}{n_A M_A+l} \right| \\
&=: S^{3,1} + S^{3,2}.
\end{aligned}$$

From the above written

$$\|S^{3,1}\|_1 \leq c \sum_{A=0}^{\infty} \frac{(H_{m_A M_A} - H_{M_A})}{H_{M_A} \alpha(M_A)} \leq c \sum_{A=1}^{\infty} \frac{1}{A \alpha(M_A)}.$$

Now, we use Abel's transformation again

$$\begin{aligned}
\left| \sum_{l=0}^{n-n_A M_A-1} \frac{D_l}{n_A M_A+l} \right| &\leq \sum_{l=1}^{n-n_A M_A-2} \frac{1}{n_A M_A+l} |K_l| \\
&\quad + \frac{n-n_A M_A-1}{n-1} |K_{n-n_A M_A-1}| \\
&\leq \sum_{l=1}^{M_A-1} \frac{1}{n_A M_A+l} |K_l| + \frac{n-n_A M_A-1}{n-1} |K_{n-n_A M_A-1}|.
\end{aligned}$$

This yields (by $\|K_n\|_1 \leq c$ for all n)

$$\|S^{3,2}\|_1 \leq \sum_{A=1}^{\infty} \frac{c(H_{m_A M_A} - H_{M_A}) + \left\| \sup_{|n|=A} \frac{n-n_A M_A-1}{n-1} |K_{n-n_A M_A-1}| \right\|_1}{A \alpha(M_A)}.$$

We will show that there exists a $c > 0$ such that $\left\| \sup_{|n|=A} \frac{n-n_A M_A-1}{n-1} |K_{n-n_A M_A-1}| \right\|_1 \leq c$, which immediately gives that

$$\|S^{3,2}\|_1 \leq c \sum_{A=1}^{\infty} \frac{1}{A \alpha(M_A)}.$$

Let $n \in \mathbb{N}$ then (see [7])

$$n|K_n| \leq c \sum_{j=0}^{|n|} M_j \sum_{i=j}^{|n|} \sum_{l=0}^{m_j-1} D_{M_i}(\cdot + l e_j)$$

Thus,

$$\begin{aligned} \sup_{|n|=A} \frac{n - n_A M_A - 1}{n - 1} |K_{n - n_A M_A - 1}| &\leq \\ &\leq \sup_{|n|=A} \frac{c}{n - 1} \sum_{j=0}^{A-1} M_j \sum_{i=j}^{A-1} \sum_{l=0}^{m_j-1} D_{M_i}(\cdot + l e_j) \\ &\leq \frac{c}{M_A} \sum_{j=0}^{A-1} M_j \sum_{i=j}^{A-1} \sum_{l=0}^{m_j-1} D_{M_i}(\cdot + l e_j). \end{aligned}$$

From this and equality (1) we have

$$\left\| \sup_{|n|=A} \frac{n - n_A M_A - 1}{n - 1} |K_{n - n_A M_A - 1}| \right\|_1 \leq \frac{c}{M_A} \sum_{j=0}^{A-1} M_{j+1} (A - j) \leq c.$$

For more details see [7].

This completes the proof of Theorem 1. \square

Proof of Theorem 2: The main approach is same as in the previous case, but the individual steps are slightly different.

First, we discuss the sum $\sum_{k=1}^{n-1} \frac{k}{n-k}$. Set $s = n - k$, then

$$\sum_{k=1}^{n-1} \frac{k}{n-k} = \sum_{s=1}^{n-1} \frac{n-s}{s}.$$

Moreover,

$$\begin{aligned} H_n^{(2)} &= H_2 + H_3 + \cdots + H_n = \\ &= \frac{1}{1} + \left(\frac{1}{1} + \frac{1}{2} \right) + \cdots + \left(\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n-1} \right) = \sum_{k=1}^{n-1} \frac{n-k}{k}. \end{aligned}$$

That is,

$$\sum_{k=1}^{n-1} \frac{k}{n-k} = H_n^{(2)}.$$

For all $x \in I_A \setminus I_{A+1}$ we have that $D_k(x) = k$ for $1 \leq k < M_{A+1}$ and

$$N_{M_A}(x) = \frac{1}{H_{M_A}} \sum_{k=1}^{M_A-1} \frac{D_k(x)}{n-k} = \frac{1}{H_{M_A}} \sum_{k=1}^{M_A-1} \frac{k}{n-k} = \frac{H_{M_A}^{(2)}}{H_{M_A}}.$$

$H_n^{(2)} = n(H_{n+1} - 1)$ (see above and [2]) yields that

$$N_{M_A}(x) \geq \frac{M_A(H_{M_{A+1}} - 1)}{H_{M_A}}$$

for all $x \in I_A \setminus I_{A+1}$.

It means that we are totally in the same situation as in the Riesz case. Namely,

$$\|N^*\|_1 = +\infty$$

for all bounded Vilenkin group. But,

$$\begin{aligned} \|N_\alpha^*\|_1 &\geq \sum_{A=1}^{\infty} \int_{I_A \setminus I_{A+1}} N_\alpha^* d\mu \geq \sum_{A=1}^{\infty} \int_{I_A \setminus I_{A+1}} \frac{N_{M_A}}{\alpha(M_A)} d\mu \\ &\geq c \sum_{A=1}^{\infty} \frac{M_A(H_{M_{A+1}} - 1)}{H_{M_A} \alpha(M_A)} \frac{1}{M_A} \geq c \sum_{A=1}^{\infty} \frac{1}{\alpha(M_A)}. \end{aligned}$$

The upper bound estimation:

$$\begin{aligned} N_\alpha^* &= \sup_{n \in \mathbb{P}} \frac{|N_n|}{\alpha(n)} = \sup_{n \in \mathbb{P}} \frac{\frac{1}{H_n} \left| \sum_{k=1}^{n-1} \frac{D_k}{n-k} \right|}{\alpha(n)} \leq \sup_{n \in \mathbb{P}} \frac{\frac{1}{H_n} \sum_{k=1}^{n-1} \frac{|D_k|}{n-k}}{\alpha(n)} \leq \\ &\leq \sup_{n \in \mathbb{P}} \frac{1}{H_n} \sum_{k=1}^{n-1} \frac{|D_k|}{\alpha(k)(n-k)} \leq \sup_{n \in \mathbb{P}} \frac{1}{H_n} \sum_{k=1}^{n-1} \frac{|D_\alpha^*|}{n-k} = D_\alpha^*. \end{aligned}$$

Recalling the bounded version of the theorem in [5] on maximal function of weighted Vilenkin-Dirichlet kernels, the result comes immediately. \square

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