ON EQUAL VALUES OF POWER SUMS OF ARITMETIC PROGRESSIONS

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ABSTRACT. In this paper, we consider the Diophantine equation

$$b^k + (a+b)^k + \cdots + (a(x-1)+b)^k =$$
$$= d^l + (c+d)^l + \cdots + (c(y-1)+d)^l,$$

where a, b, c, d, k, l are given integers with $\gcd(a, b) = \gcd(c, d) = 1$, $k \neq l$. We prove that, under some reasonable assumptions, the above equation has only finitely many solutions.

1. Introduction and the main result

For a positive integer $n \geq 2$, let

$$S_{a,b}^k(n) = b^k + (a+b)^k + \cdots + (a(n-1)+b)^k. \tag{1}$$

It is easy to see that the above power sum is related to the Bernoulli polynomials $B_k(x)$ in the following way:

$$S_{a,b}^k(n) = \frac{a^k}{k+1}\left(\left[B_{k+1}\left(n+\frac{b}{a}\right) - B_{k+1}\right] - \left[B_{k+1}\left(\frac{b}{a}\right) - B_{k+1}\right]\right), \tag{2}$$

where the polynomials $B_k(x)$ is defined by the generating series

$$\frac{t\exp(tx)}{\exp(t)-1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}$$

and $B_{k+1} = B_{k+1}(0)$. For the properties of Bernoulli polynomials which will be often used in this paper, sometimes without special reference,

2000 *Mathematics Subject Classification.* 11B68, 11D41.

Key words and phrases. Diophantine equations, exponential equations, Bernoulli polynomials.

we refer to [7, Chapters 1 and 2]. We can extend $S_{a,b}^{k}$ for every real value x as

$$S_{a,b}^{k}(x) = \frac{a^k}{k+1}\left(B_{k+1}\left(x+\frac{b}{a}\right) - B_{k+1}\left(\frac{b}{a}\right)\right). \tag{3}$$

We denote by $\mathbb{C}[x]$ the ring of polynomials in the variable x with complex coefficients. A decomposition of a polynomial $F(x) \in \mathbb{C}[x]$ is an equality of the following form

$$F(x) = G_1(G_2(x)) \quad (G_1(x), G_2(x) \in \mathbb{C}[x]),$$

which is nontrivial if

$$\deg G_1(x) > 1 \quad \text{and} \quad \deg G_2(x) > 1.$$

Two decompositions $F(x) = G_1(G_2(x))$ and $F(x) = H_1(H_2(x))$ are said to be equivalent if there exists a linear polynomial $\ell(x) \in \mathbb{C}[x]$ such that $G_1(x) = H_1(\ell(x))$ and $H_2(x) = \ell(G_2(x))$. The polynomial $F(x)$ is called decomposable if it has at least one nontrivial decomposition; otherwise it is said to be indecomposable.

In a recent paper, Bazsó, Pintér and Srivastava [1] proved the following theorem about the decomposition of the polynomial $S_{a,b}^{k}(x)$ defined above.

Theorem 1.1. *The polynomial $S_{a,b}^{k}(x)$ is indecomposable for even k. If $k = 2v-1$ is odd, then any nontrivial decomposition of $S_{a,b}^{k}(x)$ is equivalent to the following decomposition:*

$$S_{a,b}^{k}(x) = \widehat{S}_v\left(\left(x+\frac{b}{a}-\frac{1}{2}\right)^2\right). \tag{4}$$

Proof. This is Theorem 2 of [1]. □

Using Theorem 1.1 and the general finiteness criterion of Bilu and Tichy [2] for Diophantine equations of the form $f(x) = g(y)$, we prove the following result.

Theorem 1.2. *For $2 \leq k < l$, the equation*

$$S_{a,b}^{k}(x) = S_{c,d}^{l}(y) \tag{5}$$

has only finitely many solutions in integers x and y.

Since the finiteness criterion from [2] is based on the ineffective theorem of Siegel, our Theorem 1.2 is ineffective. We note that for $a = c = 1, b = d = 0$ our theorem gives the result of Bilu, Brindza, Kirschenhofer, Pintér and Tichy [3].

Combining a result of Brindza [5] with recent theorems by Rakaczki [8] and Pintér and Rakaczki [6], for $k = 1$ and 3 we obtain effective statements.

Theorem 1.3. *For $k = 1$ and $l \notin \{1, 3, 5\}$, the equation*

$$S^1_{a,b}(x) = S^l_{c,d}(y) \tag{6}$$

implies $\max(|x|, |y|) < C_1$*, where C_1 is an effectively computable constant depending only on a, b, c, d and l.*

In the exceptional cases $l = 3, 5$ one can give some values for a, b, c, d such that the corresponding equations possess infinitely many solutions. For example, if $k = 1, a = 2, b = 1, l = 3$ or $l = 5, c = 1, d = 0$ we have

$$x^2 = 1 + 3 + \cdots + 2x - 1 = 1^3 + 2^3 + \cdots + (y - 1)^3$$

or

$$x^2 = 1 + 3 + \cdots + 2x - 1 = 1^5 + 2^5 + \cdots + (y - 1)^5,$$

respectively. These equations have infinitely many integer solutions, see [9].

Theorem 1.4. *For $k = 3$ and $l \notin \{1, 3, 5\}$, the equation*

$$S^3_{a,b}(x) = S^l_{c,d}(y) \tag{7}$$

implies $\max(|x|, |y|) < C_2$*, where C_2 is an effectively computable constant depending only on a, b, c, d and l.*

2. Auxiliary results

In this section, we collect some results needed to prove Theorem 1.2. First, we recall the finiteness criterion of Bilu and Tichy [2]. To do this, we need to define five kinds of so-called standard pairs of polynomials.

Let α, β be nonzero rational numbers, $\mu, \nu, q > 0$ and $\rho \geq 0$ be integers, and let $\nu(x) \in \mathbb{Q}[x]$ be a nonzero polynomial (which may be constant).

A *standard pair of the first kind* is $(x^q, \alpha x^\rho \nu(x)^q)$ or switched, $(\alpha x^\rho \nu(x)^q, x^q)$, where $0 \leq \rho < q, \gcd(\rho, q) = 1$ and $\rho + \deg \nu(x) > 0$.

A *standard pair of the second kind* is $(x^2, (\alpha x^2 + \beta)\nu(x)^2)$ or switched.

Denote by $D_\mu(x, \delta)$ the μ-th Dickson polynomial, defined by the functional equation $D_\mu(z + \delta/z, \delta) = z^\mu + (\delta/z)^\mu$ or by the explicit formula

$$D_\mu(x, \delta) = \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu-2i} \qquad \text{with} \qquad d_{\mu,i} = \frac{\mu}{\mu - i}\binom{\mu - i}{i}(-\delta)^i.$$

A *standard pair of the third kind* is $(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$, where $\gcd(\mu, \nu) = 1$.

A *standard pair of the fourth kind* is

$$(\alpha^{-\mu/2} D_\mu(x, \alpha), -\beta^{-\nu/2} D_\nu(x, \beta)),$$

where $\gcd(\mu, \nu) = 2$.

A *standard pair of the fifth kind* is $((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$ or switched.

The following theorem is the main result of [2].

Theorem 2.1. *Let $R(x), S(x) \in \mathbb{Q}[x]$ be nonconstant polynomials such that the equation $R(x) = S(y)$ has infinitely many solutions in rational integers x, y. Then $R = \varphi \circ f \circ \kappa$ and $S = \varphi \circ g \circ \lambda$, where $\kappa(x), \lambda(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $(f(x), g(x))$ is a standard pair.*

The following lemmas are the main ingredients for the proofs of Theorems 1. 3 and 1. 4.

Lemma 2.1. *For every $b \in \mathbb{Q}$ and rational integer $k \geq 3$ with $k \notin \{4, 6\}$ the polynomial $B_k(x) + b$ has at least three zeros of odd muliplicities.*

Proof of Lemma 2.1. For $b = 0$ and odd values of $k \geq 3$ this result is a consequence of a theorem by Brillhart [4, Corollary of Theorem 6]. For non-zero rational b and odd k with $k \geq 3$ and for even values of $k \geq 8$ our lemma follows from [6, Theorem] and [8, Theorem 2. 3], respectively. $\square$

Our next auxiliary result is an easy consequence of an effective theorem concerning the S-integer solutions of so-called hyperelliptic equations.

Lemma 2.2. *Let $f(x)$ be a polynomial with rational coefficients and with at least three zeros of odd multiplicities. Further, let u be a fixed positive integer. If x and y are integer solutions of the equation*

$$f\left(\frac{x}{u}\right) = y^2,$$

then we have $\max(|x|, |y|) < C_3$, where C_3 is an effectively computable constant depending only on u and the parameters of f.

Proof of Lemma 2.2. This is a special case of the main result of [5]. □

Let $c_1, e_1 \in \mathbb{Q}^*$ and $c_0, e_0 \in \mathbb{Q}$.

Lemma 2.3. *The polynomial $S^k_{a,b}(c_1 x + c_0)$ is not of the form $e_1 x^q + e_0$ with $q \geq 3$.*

Lemma 2.4. *The polynomial $S^k_{a,b}(c_1 x + c_0)$ is not of the form*

$$e_1 D_\nu(x, \delta) + e_0,$$

where $D_\nu(x, \delta)$ is the ν-th Dickson polynomial with $\nu > 4, \delta \in \mathbb{Q}^*$.

Before proving the above lemmas, we introduce the following notation. Put

$$S^k_{a,b}(c_1 x + c_0) = s_{k+1} x^{k+1} + s_k x^k + \cdots + s_0,$$

and

$$c_0' = \frac{b}{a} + c_0.$$

We have

$$s_{k+1} = \frac{a^k c_1^{k+1}}{k+1}, \tag{8}$$

$$s_k = \frac{a^k c_1^k}{2}(2c_0' - 1), \tag{9}$$

$$s_{k-1} = \frac{a^k c_1^{k-1}}{12} k(6c_0'^2 - 6c_0' + 1), k \geq 2, \tag{10}$$

and for $k \geq 4$,

$$s_{k-3} = \frac{a^k c_1^{k-3}}{720} k(k-1)(k-2)(30c_0'^4 - 60c_0'^3 + 30c_0'^2 - 1). \tag{11}$$

Proof of Lemma 2.3. Suppose that $S_{a,b}^k(c_1x+c_0) = e_1x^q + e_0$, where we have $q = k+1 \geq 3$. It follows that $s_{k-1} = 0$, so $6c_0'^2 - 6c_0' + 1 = 0$. Hence, $c_0' \notin \mathbb{Q}$, which is a contradiction. □

Proof of Lemma 2.4. Suppose that $S_{a,b}^k(c_1x+c_0) = e_1D_\nu(x,\delta)+e_0$ with $\nu > 4$. Then

$$s_{k+1} = e_1, \tag{12}$$

$$s_k = 0, \tag{13}$$

$$s_{k-1} = -e_1\nu\delta, \tag{14}$$

$$s_{k-3} = \frac{e_1(\nu-3)\nu\delta^2}{2}. \tag{15}$$

From (8), (12) and (9), (13), respectively, it follows that

$$e_1 = \frac{a^{\nu-1}c_1^\nu}{\nu} \quad \text{and} \quad c_0' = \frac{1}{2}. \tag{16}$$

In view of (10), substituting (16) together with $k = \nu - 1$ into (14), we obtain

$$-\frac{a^{\nu-1}c_1^{\nu-2}(\nu-1)}{24} = -\frac{a^{\nu-1}c_1^\nu\nu\delta}{\nu}, \tag{17}$$

which implies

$$c_1^2 = \frac{\nu-1}{24\delta}. \tag{18}$$

Similarly, comparing the forms (11) and (15) of s_{k-3} with the substitutions $k = \nu - 1$ and (16), we obtain

$$\frac{7a^{\nu-1}c_1^{\nu-4}(\nu-1)(\nu-2)(\nu-3)}{5760} = \frac{a^{\nu-1}c_1^\nu(\nu-3)\nu\delta^2}{2\nu}, \tag{19}$$

which implies

$$c_1^4 = \frac{7(\nu-1)(\nu-2)}{2880\,\delta^2}. \tag{20}$$

After substituting (18) into (20), we obtain $7(\nu-2) = 5(\nu-1)$, which implies $\nu = 9/2$, a contradiction. □

One can see that the condition $\nu > 4$ is necessary. Indeed,

$$S_{2,1}^2(x) = \frac{4}{3}x^3 - \frac{1}{3}x = \frac{4}{3}D_3\left(x, \frac{1}{12}\right),$$

and

$$S_{2,1}^3(x) = 2x^4 - x^2 = 2D_4\left(x, \frac{1}{8}\right) - \frac{1}{16}.$$

3. Proofs of the Theorems

Proof of Theorem 1.3. Using (3), one can rewrite equation (6) as

$$\frac{c^l}{l+1}\left(B_{l+1}\left(y + \frac{d}{c}\right) - B_{l+1}\left(\frac{d}{c}\right)\right) = \frac{1}{2}ax^2 + \left(b - \frac{a}{2}\right)x$$

or

$$\begin{aligned}\frac{8ac^l}{l+1}\left(B_{l+1}\left(y + \frac{d}{c}\right) - B_{l+1}\left(\frac{d}{c}\right)\right) &= 4a^2x^2 + 8a\left(b - \frac{a}{2}\right)x \\ &= (2ax + 2b - a)^2 - (2b - a)^2.\end{aligned}$$

Then our result is a simple consequence of Lemmas 2.1 and 2.2.

Proof of Theorem 1.4. Following Theorem 1.1, we have

$$\begin{aligned}S_{a,b}^3(x) &= \frac{a^3}{4}\left(x + \frac{b}{a} - \frac{1}{2}\right)^4 - \frac{a^3}{8}\left(x + \frac{b}{a} - \frac{1}{2}\right)^2 \\ &+ \frac{a^4 - 16a^2b^2 + 32ab^3 - 16b^4}{64a}.\end{aligned}$$

Using the above representation, we rewrite equation (7) as

$$64aS_{c,d}^l(y) = (2ax+2b-a)^4 - 4a^2(2ax+2b-a)^2 + a^4 - 16a^2b^2 + 32ab^3 - 16b^4$$

or

$$64aS_{c,d}^l(y) + 3a^4 + 16a^2b^2 - 32ab^3 - 16b^4 = (X - 2a^2)^2,$$

where $X = (2ax + 2b - a)^2$. As in the previous case, Lemmas 2.1 and 2.2 complete the proof.

Proof of Theorem 1.2. If the equation (5) has infinitely many integer solutions, then by Theorem 2.1 it follows that $S_{a,b}^k(a_1x + a_0) = \phi(f(x))$ and $S_{c,d}^l(b_1x + b_0) = \phi(g(x))$, where (f, g) is a standard pair over $\mathbb{Q}$, a_0, a_1, b_0, b_1 are rationals with $a_1b_1 \neq 0$ and $\phi(x)$ is a polynomial with rational coefficients.

Assume that $h = \deg \phi > 1$. Then Theorem 1.1 implies

$$0 < \deg f, \deg g \leq 2,$$

and since $k < l$, we have $\deg f = 1, \deg g = 2$. In particular, $k + 1 = h$ and $l + 1 = 2h$, so $l = 2k + 1$. Therefore, if $l \neq k + 1$, we then must have $h = \deg \phi = 1$ and $l = 2k + 1$.

Condition $k \neq 1$ implies $k \geq 2$ and since $l = 2k + 1$, it follows that $l \geq 5$. Since $\deg f = 1$, there exist $f_1, f_0 \in \mathbb{Q}$, $f_1 \neq 0$, such that $S^k_{a,b}(f_1 x + f_0) = \phi(x)$, so

$$S^k_{a,b}(f_1 g(x) + f_0) = \phi(g(x)) = S^l_{c,d}(b_1 x + b_0).$$

As $g(x)$ is quadratic, by making the substitution $x \mapsto (x - b_0)/b_1$, we obtain that there are $c_2, c_1, c_0 \in \mathbb{Q}$, $c_2 \neq 0$, such that

$$S^k_{a,b}(c_2 x^2 + c_1 x + c_0) = S^l_{c,d}(x).$$

Since $\deg S^k_{a,b}(x) = k + 1 \geq 2$ and $c_2 \neq 0$, we have a decomposition of $S^l_{c,d}(x)$ which is equivalent to $S((x + b/a - 1/2)^2)$ for some $S \in \mathbb{Q}[x]$ with $\deg S = k + 1$, according to Theorem 1.1. Therefore, there exists a linear polynomial $l(x)$ in $\mathbb{C}[x]$ such that

$$c_2 x^2 + c_1 x + c_0 = l((x + b/a - 1/2)^2)$$

and $S(x) = S^k_{a,b}(l(x))$. Hence, there are $A, B \in \mathbb{C}$, $A \neq 0$, such that $c_2 x^2 + c_1 x + c_0 = A(x + b/a - 1/2)^2 + B$. Clearly, this implies that $A, B \in \mathbb{Q}$ and

$$S^k_{a,b}\left(A(x + b/a - 1/2)^2 + B\right) = S^{2k+1}_{c,d}(x).$$

By the linear substitution $x \mapsto x - b/a + 1/2$, we obtain

$$S^k_{a,b}(Ax^2 + B) = S^{2k+1}_{c,d}(x - b/a + 1/2). \tag{21}$$

Thus, we have an equality of polynomials of degree $2k + 2 \geq 6$. We calculate and compare coefficients of the first few highest monomials participating in the above polynomials. The coefficients of the polynomial in the right–hand side above are easily deduced by setting $c_1 = 1, c_0 = -b/a + 1/2$ in (8), (9), (10) and (11). Therefore, if we denote

$$S^{2k+1}_{c,d}(x - b/a + 1/2) = r_{2k+2} x^{2k+2} + \cdots + r_1 x + r_0,$$

and

$$c_0' = \frac{d}{c} - \frac{b}{a} + \frac{1}{2},$$

then the coefficients are:

$$\begin{aligned}
r_{2k+2} &= \frac{c^{2k+1}}{2k+2}, \\
r_{2k+1} &= \frac{c^{2k+1}}{2}(2c_0' - 1), \\
r_{2k} &= \frac{c^{2k+1}(2k+1)}{12}(6c_0'^2 - 6c_0' + 1), \\
r_{2k-2} &= \frac{c^{2k+1}(2k+1)2k(2k-1)}{720}(30c_0'^4 - 60c_0'^3 + 30c_0'^2 - 1).
\end{aligned}$$

On the other hand, the coefficients $s_{k+1}, s_k, \dots s_0$ for the polynomial $S_{a,b}^k(x)$ can be found by setting $c_1 = 1, c_0 = 0$ in (8), (9), (10) and (11). Since

$$S_{a,b}^k(Ax^2 + B) = \sum_{m=0}^{k+1} s_m \sum_{i=0}^{m} \binom{m}{i} (Ax^2)^i B^{m-i},$$

it follows that if we put

$$S_{a,b}^k(Ax^2 + B) = t_{2k+2}x^{2k+2} + \cdots + t_1 x + t_0,$$

then

$$\begin{aligned}
t_{2k+2} &= \frac{a^k A^{k+1}}{k+1}, \\
t_{2k+1} &= 0, \\
t_{2k} &= a^k A^k B + \frac{a^k A^k}{2}\left(2\left(\frac{b}{a}\right) - 1\right), \\
t_{2k-1} &= 0, \\
t_{2k-2} &= \frac{a^k k}{2} A^{k-1} B^2 + \frac{a^k k}{2} A^{k-1} B \left(2\left(\frac{b}{a}\right) - 1\right) \\
&+ \frac{a^k k}{12} A^{k-1} \left(6\left(\frac{b}{a}\right)^2 - 6\left(\frac{b}{a}\right) + 1\right).
\end{aligned}$$

Now we compare the coefficients. Comparing the leading coefficients yields

$$\frac{a^k A^{k+1}}{k+1} = \frac{c^{2k+1}}{2k+2}, \qquad \text{so} \qquad 2a^k A^{k+1} = c^{2k+1}, \tag{22}$$

and

$$\frac{2c}{a} = \frac{c^{2k+2}}{a^{k+1}A^{k+1}}.$$

Therefore,

$$\sqrt[k+1]{\frac{2c}{a}} \in \mathbb{Q}.$$

If a and c do not fulfill the above condition, we are through, otherwise we proceed. Comparing the coefficients of index $2k+1$, we get

$$\frac{c^{2k+1}}{2}(2c_0' - 1) = 0,$$

so $c_0' = 1/2$, which implies

$$\frac{d}{c} = \frac{b}{a}.$$

If the coefficients a, b, c and d do not satisfy the last property above, then we eliminate the possibility $\deg \phi > 1$. Therefore, we proceed with the case where a, b, c and d do satisfy this property. Comparing the next coefficients and using (22), we obtain

$$\frac{b}{a} - \frac{1}{2} = -\frac{1}{12}A(2k+1) - B. \tag{23}$$

Comparing the coefficients of index $2k-2$ and using $c_0' = 1/2$, we get

$$\begin{aligned}\frac{a^k k}{2}A^{k-1}B^2 &+ \frac{a^k k}{2}A^{k-1}B\left(2\left(\frac{b}{a}\right) - 1\right)\\ &+ \frac{a^k k}{12}A^{k-1}\left(6\left(\frac{b}{a}\right)^2 - 6\left(\frac{b}{a}\right) + 1\right)\\ &= \frac{7}{8}\cdot\frac{c^{2k+1}(2k+1)2k(2k-1)}{720}.\end{aligned}$$

By using also (22) and simplifying, we obtain

$$\frac{B^2}{2} + \frac{B}{2}\left(2\left(\frac{b}{a}\right) - 1\right) + \frac{1}{12}\left(6\left(\frac{b}{a}\right)^2 - 6\left(\frac{b}{a}\right) + 1\right) = \frac{7(4k^2-1)A^2}{1440}.$$

By using also (23), the last relation above can be transformed into

$$\begin{aligned}\frac{B^2}{2} + B\left(-\frac{1}{12}A(2k+1) - B\right) &+ \frac{1}{2}\left(-\frac{1}{12}A(2k+1) - B\right)^2 - \frac{1}{24}\\ &= \frac{7A^2(4k^2-1)}{1440}.\end{aligned}$$

After simplification, we obtain

$$A^2(k-3)(-2k-1) = 15.$$

For $k \geq 3$, the expression in the left–hand side above is negative or zero, which is a contradiction. If $k = 2$, then $A^2 = 3$, which contradicts the fact that $A \in \mathbb{Q}$. Therefore there are no rational coefficients a, b, c, d, A and B such that (21) is fulfilled, which implies that $\deg \phi = 1$.

Now, we have

$$S^k_{a,b}(a_1x + a_0) = e_1 f(x) + e_0 \qquad \text{and} \qquad S^l_{c,d}(b_1x + b_0) = e_1 g(x) + e_0,$$

where $0 \neq e_1, e_0 \in \mathbb{Q}$. Further, we have $\deg f = k+1$ and $\deg g = l+1$.

In view of the assumptions on k and l, it follows that the standard pair (f, g) cannot be of the second kind, and with the exception of the case $(k, l) = (3, 5)$, of the fifth kind either.

If it is of the first kind, then one of the polynomials $S^k_{a,b}(a_1x + a_0)$ and $S^l_{c,d}(b_1x+b_0)$ is of the form $e_1x^q+e_0$ with $q \geq 3$. This is impossible by Lemma 2.3.

If (f, g) is a standard pair of the third or fourth kind, we then have $S^l_{c,d}(b_1x + b_0) = e_1D_\nu(x, \delta) + e_0$ with $\nu = l + 1 \geq 5$ and $\delta \in \mathbb{Q}^*$, which contradicts Lemma 2.4 or $k = 2, l = 3$. In this case Theorem 1.4 gives an effective finiteness result.

Now returning to the special case $(k, l) = (3, 5)$, by using formula (10) for $k = 3$ it is easy to see that $S^3_{a,b}(c_1c + c_0) = e_1(3x^4 - 4x^3) + e_0$ is impossible, see the proof of Lemma 2.3. □

Acknowledgements. The authors are grateful to the referee for her/his careful reading and helful remarks.

The research was supported in part by the Hungarian Academy of Sciences, by the OTKA grant K75566, and by the TÁMOP 4.2.1./B-09/1/KONV-2010-0007 project implemented through the New Hungary Development Plan cofinanced by the European Social Fund and the European Regional Development Fund.

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