# SOME LINEAR PRESERVER PROBLEMS ON $B(H)$ CONCERNING RANK AND CORANK

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ABSTRACT. As a continuation of the work on linear maps between operator algebras which preserve certain subsets of operators with finite rank, or finite corank, here we consider the problem inbetween, that is, we treat the question of preserving operators with infinite rank and infinite corank. Since, as it turns out, in this generality our preservers cannot be written in a nice form what we have got used to when dealing with linear preserver problems, hence we restrict our attention to certain important classes of operators like idempotents, or projections, or partial isometries. We conclude the paper with a result on the form of linear maps which preserve the left ideals in $B(H)$.

## 1. INTRODUCTION

Linear preserver problems represent one of the most active research topics in matrix theory (see the survey paper [8]). In the last decade considerable attention has been also paid to similar questions in infinite dimension, that is, to linear preserver problems on operator algebras (see the survey paper [2]). In both cases, the problem is to characterize those linear maps on the algebra in question which leave invariant a given subset, or relation, or function. One of the most important such questions concerns the rank. This is because in many cases preserver problems can be reduced to the problem of rank preservers. Therefore, it is not surprising that a lot of work has been done on such preservers (see, for example, [1, 5] for the finite dimensional case and [7, 11] for the infinite dimensional case as well as the references therein). In our recent paper [6], we considered, among other things, the very similar problem of corank preservers which problem deserves attention, of course, only in the infinite dimensional case. If $H$ is a (complex) infinite dimensional Hilbert space, denote by $B(H)$ the algebra of all bounded linear operators acting on $H$. The result [6, Theorem 3] reads as follows. Let

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$\phi : B(H) \to B(H)$ be a bijective linear map which is weakly continuous on norm bounded sets. If $\phi$ preserves the corank-$k$ operators in both directions, then there exist invertible operators $A, B \in B(H)$ such that $\phi$ is of the form $\phi(T) = ATB$ $(T \in B(H))$.

Now, it seems to be a natural question to consider the problem of such preservers which are "inbetween" rank preservers and corank preservers, that is, to determine those linear maps which preserve the operators with infinite rank and infinite corank. We say that an operator $A \in B(H)$ has infinite rank and infinite corank if the (Hilbert space) dimensions of $\overline{\operatorname{rng} A}$ and $\overline{\operatorname{rng} A}^{\perp}$ are both infinite. Here, $\operatorname{rng} A$ stands for the range of $A$. We consider separable Hilbert spaces since in this case there is only one sort of infinite dimension. Unfortunately, the preservers above do not have such a nice form which we have got used to when dealing with linear preserver problems. Namely, there exist preservers of the above kind which cannot be expressed in terms of multiplications by fixed operators and, possibly, by transposition. To see this, let $\psi : B(H) \to B(H)$ be a linear map with norm less than 1 whose range consists of finite rank operators. Then it follows from a basic Banach algebra fact that the linear map $\phi$ defined by

$$\phi(T) = T - \psi(T) \qquad (T \in B(H))$$

is a bijection of $B(H)$ onto itself, and it is easy to check that $\phi$ preserves the operators with infinite rank and infinite corank in both directions (observe that this map preserves the Fredholm index as well which preserver problem might also seem to be natural after discussing corank preservers). So, in order to have one of the desired nice forms for our preservers we should somehow modify the problem by, for example, restricting the set of operators with infinite rank and infinite corank which we want preserve. This is exactly what we are doing here considering the important sets of idempotents, projections and partial isometries, respectively. In the last result of the paper we describe the linear bijections of $B(H)$ which preserve the left ideals in both directions. As it will be clear from the proof, this problem is also connected with the problem of rank preservers.

Let us fix the concepts and notation that we shall use throughout. By a projection we mean a self-adjoint idempotent in $B(H)$. An element $W \in B(H)$ is called a partial isometry if it is an isometry on a closed subspace of $H$ and 0 on its orthogonal complement. Algebraically, $W$ can be characterized by the equation $WW^*W = W$. We say that the operators $A, B \in B(H)$ are orthogonal to each other if $A^*B = AB^* = 0$. This means that the ranges of $A$ and $B$ as well as the orthogonal complements of their kernels are orthogonal to each other. If $x, y \in H$, then $x \otimes y$ denotes the operator defined by $(x \otimes y)z = \langle z, y\rangle x$ $(z \in H)$. In what follows $F(H)$ stands for the ideal of all finite rank operators in $B(H)$.

## 2. Results

We begin with the description of all linear bijections $\phi$ of $B(H)$ which preserve the partial isometries of infinite rank and infinite corank in both directions (this means that $W$ is a partial isometry with infinite rank and infinite corank if and only if so is $\phi(A)$).

**Theorem 1.** *Let $H$ be a separable infinite dimensional Hilbert space. Let $\phi : B(H) \to B(H)$ be a linear bijection which preserves the partial isometries of infinite rank and infinite corank in both directions. Then there exist unitary operators $U, V \in B(H)$ such that $\phi$ is either of the form*

$$\phi(T) = UTV \qquad (T \in B(H))$$

*or of the form*

$$\phi(T) = UT^{tr}V \qquad (T \in B(H))$$

*where $^{tr}$ denotes the transpose with respect to an arbitrary but fixed complete orthonormal sequence in $H$.*

In the proof we shall use the following two auxiliary results.

**Lemma 1.** *Let $T, S \in B(H)$ be partial isometries with $S = ST^*S$. Then we have $TT^*S = S$ and $ST^*T = S$.*

*Proof.* Denote $Q = TS^*$. Since $SS^*$ and $T^*T$ are projections, we compute

$$SS^* = ST^*SS^*TS^* = Q^*(SS^*)Q \leq Q^*Q = S(T^*T)S^* \leq SS^*.$$

This implies $Q^*Q = SS^*$. In particular, we obtain $\|Q\| \leq 1$ (in fact, the norm of $Q$ is either 0 or 1). But $Q$ is an idempotent. Indeed, we have

$$Q^2 = TS^*TS^* = T(ST^*S)^* = TS^* = Q.$$

So, $Q$ is a contractive idempotent. It is easy to see that this implies that $Q$ is a self-adjoint idempotent, that is, a projection. To verify this, pick arbitrary elements $x \in \ker Q$ and $y \in \operatorname{rng} Q$. Then we have

$$\|y\|^2 \leq \|\mu x + y\|^2 \qquad (\mu \in \mathbb{C}).$$

An elementary argument shows that this implies that $x \perp y$. Hence the kernel and the range of $Q$ are orthogonal to each other and this verifies that $Q$ is a projection. Now, from $Q^*Q = SS^*$ we obtain $Q = SS^*$. Therefore, $TS^* = SS^*$ and, as $SS^*$ is the projection onto $\operatorname{rng} S$, it follows that the range of $S$ is included in that of $T$. Since $TT^*$ is the projection onto $\operatorname{rng} T$, we have $TT^*S = S$. Similarly, from the equality $S^* = S^*TS^*$ one can deduce $T^*TS^* = S^*$ which is equivalent to $ST^*T = S$. $\square$

**Lemma 2.** *Suppose that $T, S \in B(H)$ are partial isometries. The operator $T + \lambda S$ is a partial isometry for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ if and only if $T$ and $S$ are orthogonal to each other.*

*Proof.* Suppose first that

$$(T+\lambda S)(T+\lambda S)^*(T+\lambda S)=T+\lambda S$$

holds for every $\lambda \in \mathbb{C}$ with $|\lambda|=1$. Using the fact that $T, S$ are partial isometries, one can conclude that

$$0=\lambda^2 ST^*S+\lambda(TT^*S+ST^*T)+\bar{\lambda}TS^*T+SS^*T+TS^*S.$$

Since this is valid for every $\lambda \in \mathbb{C}$ of modulus 1, choosing the particular values $\lambda = 1, -1, i, -i$, it is easy to deduce that

$$ST^*S=0 \tag{1}$$

$$TT^*S+ST^*T=0 \tag{2}$$

$$TS^*T=0 \tag{3}$$

$$SS^*T+TS^*S=0. \tag{4}$$

Multiplying (2) by $T^*$ from the left and taking (3) into account, we obtain $T^*S=0$. Similarly, multiplying (4) by $S^*$ from the right and taking (1) into account, we have $TS^*=0$. So, $T$ and $S$ are orthogonal. As for the reverse implication, if $T, S$ are mutually orthogonal partial isometries, then it is just a simple calculation that $T+\lambda S$ is a partial isometry for every $\lambda \in \mathbb{C}$ of modulus 1. □

*Proof of Theorem 1.* Let $\{x_1, \ldots, x_k\} \subseteq H$ and $\{y_1, \ldots, y_k\} \subseteq H$ be two systems of pairwise orthogonal unit vectors. We claim that the image of the finite rank partial isometry $R=\sum_{j=1}^k x_j \otimes y_j$ under $\phi$ is also a finite rank partial isometry. Let $(e_n)$ be an orthonormal sequence in the orthogonal complement of $\{x_1, \ldots, x_k\}$ which generates a closed subspace of infinite codimension. Similarly, let $(f_n)$ be an orthonormal sequence in $\{y_1, \ldots, y_k\}^\perp$. Denote $U=\sum_n e_n \otimes f_n$ and let $V=U+R$. Clearly, $U$ and $V$ are partial isometries of infinite rank and infinite corank. Moreover, for every $\lambda \in \mathbb{C}$ of modulus 1, the operator $R+\lambda U=(V-U)+\lambda U$ is also a partial isometry of infinite rank and infinite corank. Therefore, $\phi(V)+(\lambda-1)\phi(U)$ is a partial isometry for every $\lambda \in \mathbb{C}$ with $|\lambda|=1$. This means that with the notation $V'=\phi(V), U'=\phi(U)$ we have

$$(V'+(\lambda-1)U')(V'+(\lambda-1)U')^*(V'+(\lambda-1)U')=(V'+(\lambda-1)U')$$

for every $\lambda \in \mathbb{C}$ of modulus 1. Performing the above operations we obtain a polynomial in $\lambda, \bar{\lambda}$ with operator coefficients which equals 0 on the perimeter of the unit disc in the complex plane. Just as in the proof of Lemma 2, choosing the particular values $\lambda = 1, -1, i, -i$ we find that the coefficients of the polynomial in question are all 0. Therefore, we have

$$-U'+U'V'^*U'=0, \tag{5}$$

$$2U'+V'V'^*U'+U'V'^*V'-U'U'^*V'-2U'V'^*U'-V'U'^*U'=0, \tag{6}$$

$$U' + V'U'^*V' - U'U'^*V' - V'U'^*U' = 0, \tag{7}$$

and

$$\begin{aligned} -2U' - V'V'^*U' - V'U'^*V' - U'V'^*V' + \\ 2U'U'^*V' + U'V'^*U' + 2V'U'^*U' = 0, \end{aligned} \tag{8}$$

where the left hand sides of (5), (6), (7), (8) are the coefficients of $\lambda^2$, $\lambda$, $\bar{\lambda}$ and 1, respectively. From (5) and (6) we deduce

$$V'V'^*U' + U'V'^*V' = U'U'^*V' + V'U'^*U'. \tag{9}$$

We prove that $\phi(R) = V' - U'$ is a partial isometry. Indeed, we compute

$$(V' - U')(V' - U')^*(V' - U') = \tag{10}$$

$$V' - U' - V'V'^*U' - V'U'^*V' - U'V'^*V' + U'U'^*V' + U'V'^*U' + V'U'^*U'.$$

From (9) we know that

$$-V'V'^*U' - U'V'^*V' + U'U'^*V' + V'U'^*U' = 0.$$

So, we have to show that $V' - U' - V'U'^*V' + U'V'^*U' = V' - U'$. By (5) we have $U'V'^*U' = U'$. It remains to verify that $V'U'^*V' = U'$. From (7) and (9) we infer that

$$U' + V'U'^*V' - V'V'^*U' - U'V'^*V' = 0. \tag{11}$$

By Lemma 1 it follows that $V'V'^*U' = U'$ and $U'V'^*V' = U'$. Now, (11) gives $V'U'^*V' = U'$. Consequently, the right hand side of the equation (10) is equal to $V' - U'$ which verifies that $\phi(R)$ is a partial isometry.

We next prove that $\phi(R)$ has finite rank. By the preserver property of $\phi$ it is sufficient to prove that $\phi(R)$ has infinite corank. We have seen that for every $\lambda \in \mathbb{C}$ of modulus 1, the operator $R + \lambda U$ is a partial isometry of infinite rank and infinite corank. This implies that for $R' = \phi(R)$, the operator $R' + \lambda U'$ is a partial isometry for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. According to Lemma 2, we obtain that $R'$ and $U'$ are orthogonal to each other. Since the range of $U'$ is infinite dimensional, it follows that $R'$ is of infinite corank which implies that $R'$ is a finite rank partial isometry.

We next prove that $\phi$ preserves the partial isometries in general. To see this, let $W$ be a partial isometry. If it is of finite rank, then there is now nothing to prove. So, let $W$ be of infinite rank. In that case we have an orthogonal sequence $(W_n)$ of partial isometries of infinite rank and infinite corank whose sum is $W$. By the preserver property of $\phi$ it follows that the operators $A_n = \phi(W) - \sum_{k=1}^{n+1} \phi(W_k) = \phi(W - \sum_{k=1}^{n+1} W_k)$ and $B_n = \sum_{k=1}^{n} \phi(W_k) = \phi(\sum_{k=1}^{n} W_k)$ are partial isometries. Because of the same reason, $A_n + \lambda B_n$ is a partial isometry for every $\lambda \in \mathbb{C}$ of modulus 1. By Lemma 2 this implies that $A_n$ and $B_n$ are orthogonal to each other. The statement [10, Lemma 1.3] tells us that the series of pairwise orthogonal partial isometries is convergent in the strong operator topology and its sum is also a partial isometry. Consider the operators $A = \phi(W) - \sum_n \phi(W_n)$

and $B = \sum_n \phi(W_n)$. By the just mentioned result $\sum \phi(W_n)^*$ is strongly convergent as well, and $\sum_n \phi(W_n)^* = B^*$. We then also have $\phi(W)^* - \sum_n \phi(W_n)^* = A^*$. Since $A_n$ and $B_n$ are orthogonal for every $n \in \mathbb{N}$, it is now easy to verify that $A$ is orthogonal to $B$. The operator $B$ is a partial isometry. As for $A$, we know that $(A_n)$ strongly converges to $A$ and, as we have seen, $(A_n^*)$ strongly converges to $A^*$. It is well-known that the multiplication is strongly countinuous on the norm-bounded subsets of $B(H)$. Consequently, we infer that $(A_n A_n^*)$ strongly converges to $AA^*$ and then that $(A_n A_n^* A_n)$ strongly converges to $AA^*A$. Since $A_n$ is a partial isometry for every $n$, we obtain that $A$ is also a partial isometry. Now, since $\phi(W)$ is the sum of the mutually orthogonal partial isometries $A$ and $B$, it follows that $\phi(W)$ is a partial isometry as well. We have assumed that $\phi^{-1}$ has the same preserver properties as $\phi$. Therefore, $\phi$ preserves the partial isometries in both directions. Suppose that $W$ is a maximal partial isometry, that is, suppose that $W$ is a partial isometry and there is no nonzero partial isometry which is orthogonal to $W$. If $V \in B(H)$ is a nonzero partial isometry which is orthogonal to $\phi(W)$, then $V + \lambda\phi(W)$ is a partial isometry for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. This gives us that $\phi^{-1}(V) + \lambda W$ is also a partial isometry for every $\lambda \in \mathbb{C}$ of modulus 1. By Lemma 2 this results in the orthogonality of $\phi^{-1}(V)$ and $W$ which is a contradiction. Consequently, we obtain that $\phi$ preserves the maximal partial isometries which are precisely the isometries and the coisometries. It is well-known that the set of all extreme points of the unit ball of $B(H)$ consists of these operators exactly. So, $\phi$ is a linear map on $B(H)$ which preserves the extreme points of the unit ball. The form of all linear maps with this property acting on a von Neumann factor was determined in [9]. The result [9, Theorem 1] says that there is a unitary operator $U \in B(H)$ such that either there exists a *-homomorphism $\psi : B(H) \to B(H)$ such that

$$\phi(T) = U\psi(T) \qquad (T \in B(H))$$

or there exists a *-antihomomorphism $\psi' : B(H) \to B(H)$ such that

$$\phi(T) = U\psi'(T) \qquad (T \in B(H)).$$

Since our map $\phi$ is bijective, the same must hold for the corresponding morphism $\psi$ or $\psi'$ above. Now, referring to folk results on the form of *-automorphisms and *-antiautomorphisms of $B(H)$, we conclude the proof. $\square$

We continue with a result of the same spirit on idempotent preservers.

**Theorem 2.** *Let $H$ be a separable infinite dimensional Hilbert space. Suppose that $\phi : B(H) \to B(H)$ is a linear bijection which preserves the idempotents of infinite rank and infinite corank in both directions. Then there is an invertible operator $A \in B(H)$ such that $\phi$ is either of the form*

$$\phi(T) = ATA^{-1} \qquad (T \in B(H))$$

*or of the form*

$$\phi(T) = AT^{tr}A^{-1} \qquad (T \in B(H)).$$

In the proof we shall use the following lemma which is certainly well-known and is included here only for the sake of completeness.

**Lemma 3.** *If $P, Q \in B(H)$ are idempotents, then*

(*i*) *$P + Q$ is an idempotent if and only if $PQ = QP = 0$;*
(*ii*) *$P - Q$ is an idempotent if and only if $PQ = QP = Q$.*

*Proof.* It follows from elementary algebraic computations. □

*Proof of Theorem 2.* If $P, Q \in B(H)$ are idempotents, then we write $P \leq Q$ if $PQ = QP = P$. Clearly, this is equivalent to the condition that $\operatorname{rng} P \subseteq \operatorname{rng} Q$ and $\ker Q \subseteq \ker P$. Let us say that an idempotent $P \in B(H)$ is regular if it has infinite rank and infinite corank. We prove that for any two regular idempotents $P, Q$ we have $P \leq Q$ if and only if for every regular idempotent $R \in B(H)$, if $Q+R$ is a regular idempotent, then so is $P+R$. The necessity is almost evident. To the sufficiency suppose first that $\operatorname{rng} P \not\subseteq \operatorname{rng} Q$. Let $x \in H$ be such that $Px = x$ and $Qx \neq x$. Choose a regular idempotent $R \leq I - Q$ for which $Q + R$ is a regular idempotent and $Rx \neq 0$ (observe that $(I - Q)x \neq 0$). Since $P + R$ is an idempotent, we have $PR = RP = 0$. It follows that $0 = RPx = Rx$ which is a contradiction. Hence, we have $\operatorname{rng} P \subseteq \operatorname{rng} Q$. The relation $\ker Q \subseteq \ker P$ can be proved in a similar manner. Using the above characterization and the preserver property of $\phi$, we obtain that $\phi$ preserves the relation $\leq$ between regular idempotents. Now, if $R$ is a finite rank idempotent, then $R$ can be written in the form $R = Q - P$ with some regular idempotents $P \leq Q$. Since $\phi(P) \leq \phi(Q)$, it follows that $\phi(R) = \phi(Q) - \phi(P)$ is also an idempotent. We prove that $\phi(R)$ is of finite rank. Choosing a regular idempotent $P$ with $R \leq P$, it follows that $P - R$ is a regular idempotent and hence $\phi(P) - \phi(R)$ is also an idempotent. By Lemma 3 (*ii*) this implies that $\phi(R) \leq \phi(P)$. If $\phi(R)$ is not of finite rank, then it is regular which implies that $R$ is also regular and this is a contradition. Therefore, using the preserver properties of $\phi$ and $\phi^{-1}$ we obtain that $\phi$ preserves the finite rank idempotents in both directions. It is now easy to see that $\phi$ is a linear bijection of $F(H)$ onto itself. By Lemma 3 (*i*), for any idempotents $R, R' \in F(H)$ we have $RR' = R'R = 0$ if and only if $\phi(R)\phi(R') = \phi(R')\phi(R) = 0$. Using this property it is easy to verify that $\phi$ preserves the rank-one idempotents in both directions. By [11, Theorem 4.4] we infer that there is an invertible bounded linear operator $A \in B(H)$ such that $\phi$ is either of the form

$$\phi(T) = ATA^{-1} \qquad (T \in F(H))$$

or of the form

$$\phi(T) = AT^{tr}A^{-1} \qquad (T \in F(H)).$$

Without loss of generality we may assume that $\phi$ is of the first form and then that $A = I$. We intend to show that $\phi(T) = T$ $(T \in B(H))$. Let $P \in B(H)$ be a regular idempotent. If $R$ is any finite rank idempotent with $R \leq P$, then just as above, we obtain $R = \phi(R) \leq \phi(P)$. Since $R \leq P$ was arbitrary, it now follows that $P \leq \phi(P)$. Since $\phi^{-1}$ has the same preserver property as $\phi$, it follows that $P \leq \phi^{-1}(P)$. But $\phi$ preserves the order between the regular idempotents. Hence, we have $\phi(P) \leq P$. Therefore, $\phi(P) = P$ for every regular idempotent $P$. Since every idempotent of finite corank is the sum of two regular idempotents, we obtain that $\phi(P) = P$ holds for every idempotent $P \in B(H)$. Since every element of $B(H)$ is a finite linear combination of projections [4, Theorem 2], we conclude that $\phi(T) = T$ is valid for every $T \in B(H)$. This completes the proof. $\square$

In a similar fashion one can verify the following result concerning projection preservers.

**Theorem 3.** *Let $H$ be a separable infinite dimensional Hilbert space. Suppose that $\phi : B(H) \to B(H)$ is a linear bijection which preserves the projections of infinite rank and infinite corank in both directions. Then there is a unitary operator $U \in B(H)$ such that $\phi$ is either of the form*

$$\phi(T) = UTU^* \qquad (T \in B(H))$$

*or of the form*

$$\phi(T) = UT^{tr}U^* \qquad (T \in B(H)).$$

Our final result describes the linear bijections $\phi$ of $B(H)$ which preserve the left ideals in both directions (this means that $\mathcal{L} \subseteq B(H)$ is a left ideal if and only if $\phi(\mathcal{L})$ is a left ideal). As it will be clear from the proof, this problem is also connected with the problem of rank preservers.

**Theorem 4.** *Let $H$ be a Hilbert space. Suppose that $\phi : B(H) \to B(H)$ is a linear bijection preserving the left ideals of $B(H)$ in both directions. Then there are invertible operators $A, B \in B(H)$ such that $\phi$ is of the form*

$$\phi(T) = ATB \qquad (T \in B(H)).$$

*Proof.* The minimal left ideals of $B(H)$ are precisely the sets $\{x \otimes y \, : \, x \in H\}$ for nonzero $y \in H$. Since $\phi$ clearly preserves the minimal left ideals of $B(H)$ in both directions, we easily deduce that $\phi$ is a linear bijection of $F(H)$ onto itself which preserves the rank-one operators. By [11, Theorem 3.3] (see also [7]) it follows that there are linear bijections $A, B : H \to H$ such that $\phi$ is either of the form

$$\phi(x \otimes y) = Ax \otimes By \qquad (x, y \in H) \tag{12}$$

or of the form

$$\phi(x \otimes y) = Ay \otimes Bx \qquad (x, y \in H).$$

Since $\phi$ is left ideal preserving, the second possibility above obviously cannot occur.

We prove that $\phi(I)$ is invertible. First we note the following. It is true in any algebra with unit that an element fails to have a left inverse if and only if this element is included in a maximal left ideal (recall that every proper left ideal is included in a maximal left ideal). Therefore, $\phi$ preserves the left invertible elements of $B(H)$ in both directions. We recall that an operator $S$ in $B(H)$ is left invertible if and only if $S$ is injective and $S$ has closed range. Now, let $x, y \in H$ be arbitrary nonzero vectors. Let $\lambda \in \mathbb{C}$. By Fredholm alternative $x \otimes y - \lambda I$ is injective if and only if it is surjective. This gives us that $x \otimes y - \lambda I$ is left invertible if and only if it is invertible. Since the spectrum of any element in $B(H)$ is nonempty, we infer that there is a $\lambda \in \mathbb{C}$ for which $x \otimes y - \lambda I$ is not left invertible. Suppose that $x \otimes y$ is not quasinilpotent, that is, $\langle x, y\rangle \neq 0$. Then the scalar $\lambda$ above can be chosen to be nonzero. It follows that $Ax \otimes By - \lambda\phi(I)$ is not left invertible. On the other hand, $\phi(I)$ is left invertible and hence it is a left Fredholm operator (see [3, 2.3. Definition, p. 356]). But any compact perturbation of a left Fredholm operator has closed range [3, 2.5. Theorem, p. 356]. So, the operator $Ax \otimes By - \lambda\phi(I)$ is not left invertible but it has closed range. Therefore, this operator is not injective, that is, there exists a nonzero vector $z \in H$ such that $\lambda\phi(I)z = \langle z, By\rangle Ax$. Clearly, this implies that $Ax \in \operatorname{rng}\phi(I)$. Since $x \in H$ was arbitrary, we conclude that $H = \operatorname{rng} A \subseteq \phi(I)$ which means that $\phi(I)$ is surjective. This gives us that $\phi(I)$ is invertible.

We next show that the linear operators $A, B$ in (12) are bounded. Let $x, y \in H$. We have seen above that $x \otimes y - \lambda I$ is not left invertible if and only if $\lambda \in \sigma(x \otimes y)$, where $\sigma(.)$ denotes the spectrum. Similarly, by Fredholm alternative again, $\phi(I)^{-1}(Ax \otimes By - \lambda\phi(I))$ is not left invertible if and only if $\lambda \in \sigma(\phi(I)^{-1}Ax \otimes By)$. Since $\phi$ preserves the left invertible operators in both directions, we obtain

$$\sigma(x \otimes y) = \sigma(\phi(I)^{-1}Ax \otimes By).$$

By the spectral radius formula we have

$$|\langle x, y\rangle| = |\langle \phi(I)^{-1}Ax, By\rangle| \qquad (x, y \in H).$$

Now, an easy application of the closed graph theorem shows that $A, B$ are continuous.

Evidently, we may suppose without any loss of generality that $A = B = I$. Let $S \in B(H)$ be invertible and write $C = \phi(S)$. We claim that $C = S$. Let $x \in H$ be an arbitrary unit vector. Then $S(I - \lambda x \otimes x)$ has a left inverse if and only if $\lambda \neq 1$. Consequently, the operator $C - \lambda Sx \otimes x$ is injective for every $\lambda \neq 1$ and for every unit vector $x \in H$. Let $z \in H$ be a nonzero vector. Let $y = S^{-1}Cz$ which is also nonzero since $C = \phi(S)$ is left invertible. If $\langle z, y\rangle \neq 0$, then choosing $\lambda = \|y\|^2/\langle z, y\rangle$ we see that

$$Cz - \lambda(1/\|y\|^2 Sy \otimes y)(z) = Sy - Sy = 0.$$

Since $z$ is nonzero, we deduce $\lambda = 1$ which means $\|y\|^2 = \langle z, y\rangle$. Therefore, for every nonzero vector $z \in H$ we have two possibilities. Either

$\langle z, S^{-1}Cz\rangle = 0$ or $\langle S^{-1}Cz, S^{-1}Cz\rangle = \langle z, S^{-1}Cz\rangle$. Clearly, the set of all nonzero vectors satisfying the first equality as well as the set of those ones which satisfy the second equality are both closed in $H \setminus \{0\}$. Since $H \setminus \{0\}$ is a connected set, we infer that either

$$\langle z, S^{-1}Cz\rangle = 0 \qquad (z \in H)$$

or

$$\langle S^{-1}Cz, S^{-1}Cz\rangle = \langle z, S^{-1}Cz\rangle \qquad (z \in H).$$

The first possibility would imply that $S^{-1}C = 0$. Therefore, we have the second one which can be reformulated as $(S^{-1}C)^*(S^{-1}C) = S^{-1}C$. This shows that $S^{-1}C$ is a projection. But, on the other hand, it is left invertible, and hence we have $S^{-1}C = I$ which results in $\phi(S) = C = S$. Since $B(H)$ is linearly generated by the set of all invertible operators, it follows that $\phi$ is the identity on $B(H)$. This completes the proof. □

It is easy to see that in the proof above we used only the preservation of two extreme kinds of left ideals, namely, that of the minimal ones and that of the maximal ones. Preserving minimal left ideals is connected with the problem of rank preservers. On the other hand, preserving maximal left ideals is connected with the problem of left invertibility preservers. Because of the great interest in linear maps preserving invertibility in one direction, or in both directions, it might be interesting to consider the "one-sided" analogues of those problems.

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