Characterization of Rössler and Duffing maps with Rényi entropy and generalized complexity measures

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Abstract.
  Rényi entropy and generalized complexity measures are used to describe the chaotic behaviour of dynamical systems. These measures are found to be sensitive to the fine details of the Rössler and the Duffing maps. They are good descriptors of chaotic behaviour. Periodic windows and the fractal character of the chaotic dynamics are nicely detected.

1. Introduction
There exist several quantities to study the chaotic behaviour of dynamical systems. Complexity measures have proved to be especially efficient in this respect. One of these measures is the LMC (Lopez-Ruiz - Mancini - Calbet) statistical complexity [1]. A couple of years ago, a one- and a two-parameter extension [2] of this measure were put forward. These generalizations are based on the Rényi entropy. First, some simple quantum systems (H-atom, harmonic oscillator and square well) were studied with these measures. Recently, it has been demonstrated [3] that these generalized complexity measures are suitable to describe chaotic behavior. The logistic and Tinkerbell maps were analyzed.

In this work the Rössler and the Duffing maps are studied with the Rényi entropy and the generalized complexity measures.

2. Rényi Entropy and Generalized Statistical Complexity Measures
Consider a set of discrete probabilites $p_1, ..., p_N$ with $\sum_{i=1}^{N} p_i = 1$. The Rényi entropy of order α has the form

$$R^{(\alpha)} = \frac{1}{1-\alpha} \ln \sum p_i^\alpha, \quad 0 < \alpha < \infty, \ \alpha \neq 1. \tag{1}$$

The limit $\alpha \to 1$ gives the Shannon entropy:

$$S = -\sum p_i \ln p_i. \tag{2}$$

The LMC complexity was defined as the product of two important information-theoretical quantities: $C = HQ$, where $H = e^S$ is the Shannon entropy power, while $Q = e^{-D} = e^{-R^{(2)}}$ is the logarithm of the Rényi entropy or order 2. The disequilibrium D quantifies the deviation of the probability distribution from uniformity. The Shannon entropy S, on the other hand, is a measure of uncertainty. A one-parameter extension of the generalized statistical measure of complexity [2] is $C^{(\alpha)} = e^{R^{(\alpha)} - R^{(\beta=2)}}$. If $\alpha \to 1$ we obtain the LMC complexity.
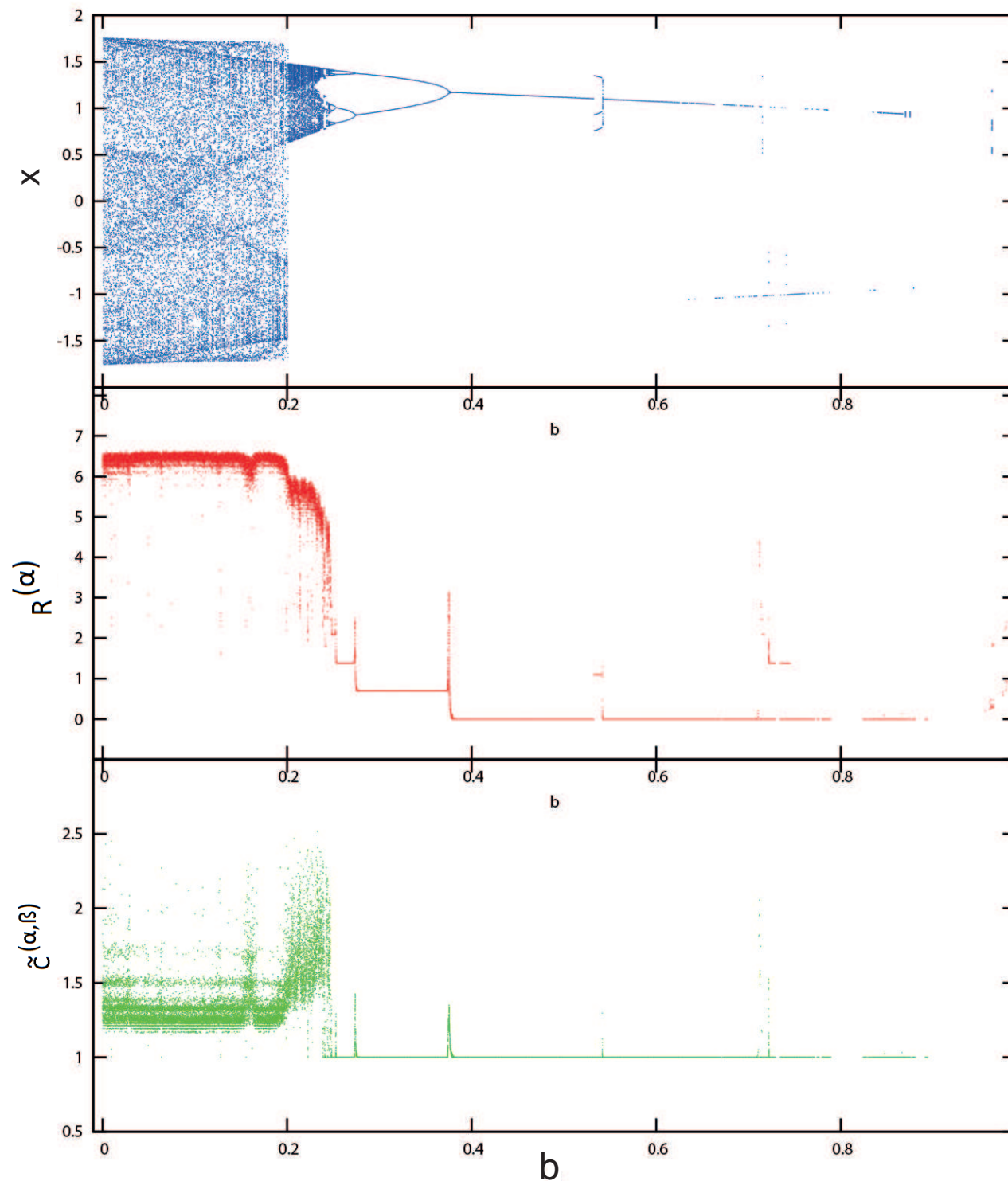
Figure 1. Bifurcation diagram, Rényi entropy($\alpha = 6$) and generalized complexity($\alpha = 3$, $\beta = 6$) for the Duffing map.

In the two-parameter extension, on the other hand, the generalized statistical measure of complexity [2] has the form

$$\tilde{C}^{(\alpha,\beta)} = e^{R^{(\alpha)} - R^{(\beta)}}, \qquad 0 < \alpha, \beta < \infty. \tag{3}$$

Certainly, the special case $\alpha \to 1$ and $\beta = 2$ gives back the LMC complexity. Important properties of the generalized complexity are detailed in [2]. It has been shown that the generalized complexity extends the complexity measure to any kind of well behaved distribution.
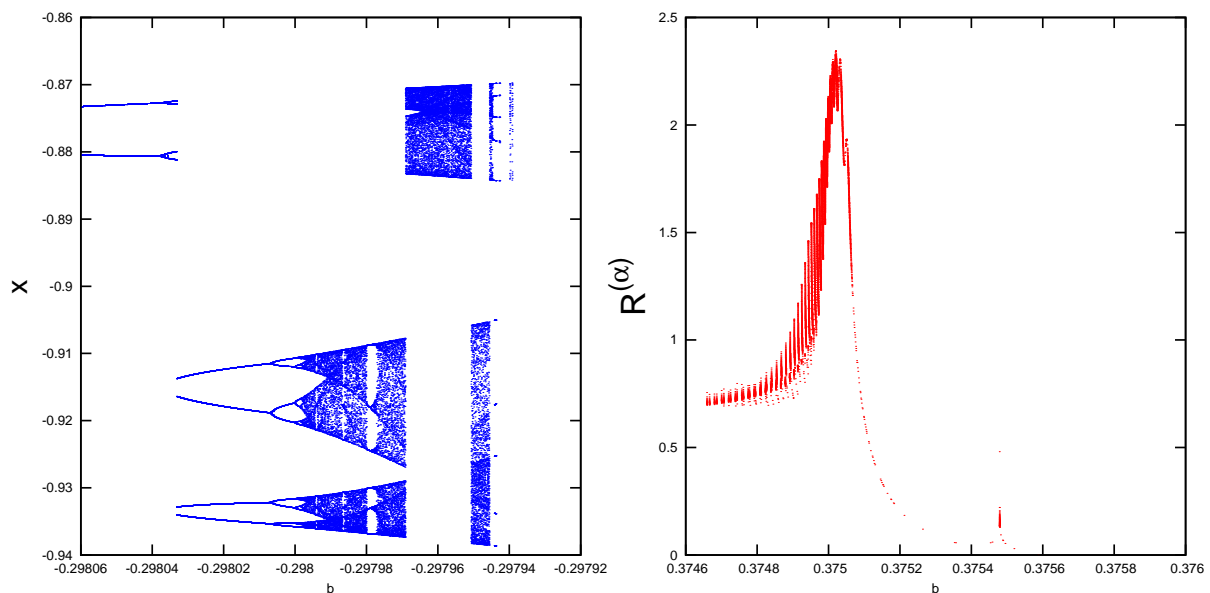

Figure 2. Enlarged bifurcation diagram of the Duffing map for $-0.2981 < b < -0.2979$ and the Rényi entropy in the vicinity of a bifurcation point.

3. Application: Duffing and Rössler maps

Now, we apply the generalized complexity measure to characterize the Duffing and Rössler maps. The Duffing map has the form:

$$x_{n+1} = y_n, \qquad y_{n+1} = -bx_n + ay_n - y_n^3. \tag{4}$$

The parameter a is taken as $a = 2.75$ and the parameter b is selected as a control parameter. The initial coordinates were: $x = 0.1$ and $y = 0.1$. Fig.2 shows the x coordinate. (y behaves similarly.) The probabilities p_i were determined [4] by subdividing the interval $[-2, 2]$ into 10000 equal bins. The number of iterates falling within a bin divided by the total number of iterations (10^4) gives the probability. For an n-periodic dynamics there are only n probabilities that are not zero. As these probabilities are all equal, the Rényi entropy is $\ln n$, independent from the parameter α, therefore the complexity is 1. From the definition (3) follows that $\tilde{C}^{(\alpha,\beta)} \geq 1$ if $\alpha < \beta$ and $\tilde{C}^{(\alpha,\beta)} \leq 1$ if $\alpha > \beta$. As one expects that complexity is larger for a more complex behaviour, the case $\alpha < \beta$ is selected.

The upper panel of Fig. 1 presents the bifurcation diagram. (The values of x are plotted against the parameter b.) Fig. 1 also shows the Rényi entropy for $\alpha = 6$ (middle panel) and the
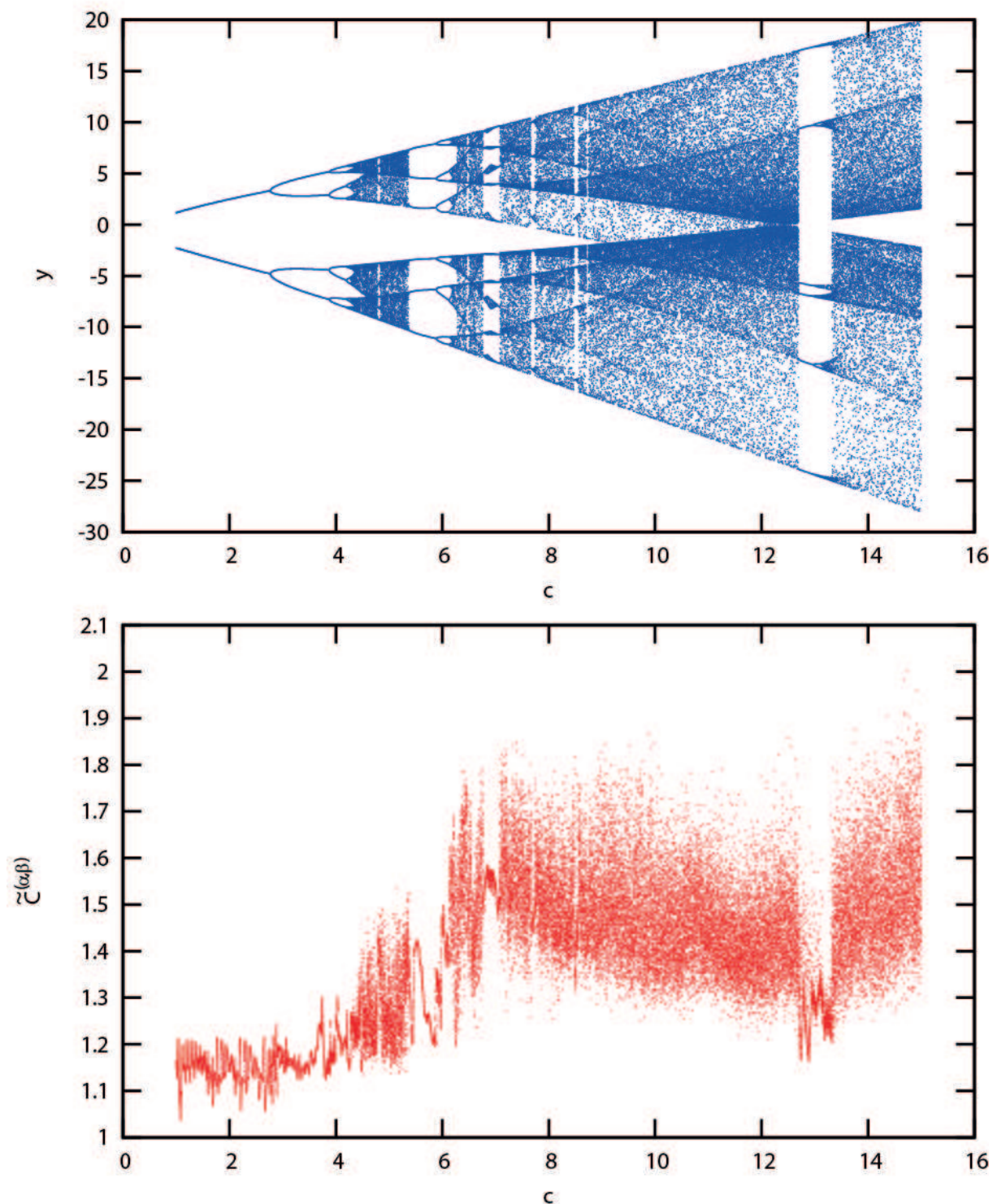
Figure 3. Rössler bifurcation diagram and generalized complexity($\alpha = 3$, $\beta = 6$).

generalized complexity for $\alpha = 3$ and $\beta = 6$ (lower panel) for the interval $0 < b < 1$. Periodic and chaotic behaviour can be seen in the bifurcation diagram, and can also be detected by the
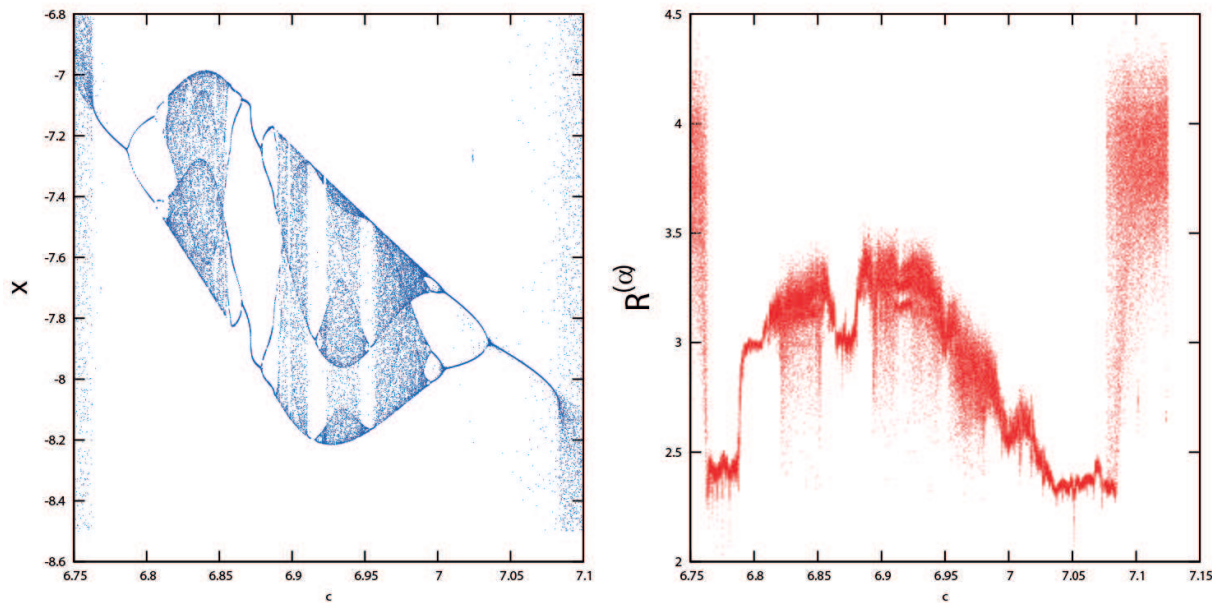
Figure 4. Enlarged Rössler bifurcation diagram and Rényi entropy.

Rényi entropy and the generalized complexity. In the bifurcation points both the Rényi entropy and the generalized complexity icreases abruptly. Fig. 2b enlarges the Rényi entropy in the vicinity of a bifurcation point. Fig. 2a shows an enlargement of the bifurcation diagram: a very interesting behaviour in the intervals $-0.86 < x < -0.94$ and $-0.2981 < b < -0.2979$. At $b = -0.298075$ the diagram is shifted, at $b = -0.29801$ it goes back to the original position. There is another shift in the interval $-0.29799 < b < -0.297985$. A similar behaviour can be observed for other values of x. These shifts can not be detected in the Rényi entropy and the generalized complexity, because the values of the probabilities do not change.

The Rössler model is given by

$$\frac{dx}{dt} = -y - z, \qquad \frac{dy}{dt} = x + ay, \qquad \frac{dz}{dt} = b + z(x - c). \qquad (5)$$

The parameters a and b were taken as $a = 0.2$, $b = 0.2$ and c is the control parameter. The initial coordinates were: $x = 0$, $y = -5$ and $z = 0$. The differential equations were solved numerically by the Runge-Kutta (second order). Poincaré sections were taken at $x = 0$ and the figures show the coordinate y. Fig. 3 presents the bifurcation diagram and the generalized complexity($\alpha = 3$, $\beta = 6$) for $1 < c < 15$. Fig. 4 shows the enlarged bifurcation diagram and the Rényi entropy for $6.75 < c < 7.1$. It is a very rich structure, the bifurcation diagram and the Rényi entropy reflects different aspects. The regular and chaotic parts can be clearly distinguished. When periodic windows appear, the Rényi entropy decreases. Further enlargements (not presented here) would reveal additional fine details and the fractal character of the chaotic dynamics.

In summary, we used the Rényi entropy and the generalized complexity measures to describe Rössler and the Duffing maps. These measures nicely show the regular and the chaotic behaviour of dynamical systems. Periodic windows and the fractal character of the chaotic dynamics are clearly detected.

Acknowledgments

References

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