

SQUARE PRODUCT OF THREE INTEGERS IN SHORT INTERVALS

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ABSTRACT. In this paper we list all the integer triplets taken from an interval of length ≤ 12 , whose products are perfect squares.

1. INTRODUCTION

Let f and k be positive integers with $f \leq k$. The sets of distinct integers $n_1, \dots, n_f \in [n+1, \dots, n+k]$ with the property that there is a nontrivial way to multiply them to obtain a perfect power, was investigated by Erdős and Turk [ET]. This question is related to the Erdős-Selfridge theorem (see [ES]), which states that the product of two or more consecutive integers is never a perfect power, that is, if $f = k \geq 2$, then

$$(1) \quad \prod_{i=1}^f n_i = x^m \quad (x \in \mathbb{N}, m \geq 2)$$

has no solutions. Moreover, Erdős and Turk conjectured (cf. [ET]) that (1) has no solutions with $(k, f, m) = (4, 3, 2)$. This conjecture was verified by Tzanakis [T].

In this paper we list all the integer triplets ($f = 3$) taken from a short interval ($k \leq 12$), whose products are perfect squares.

2. RESULT

Now we formulate our result.

Theorem. *Let $(a, b, c) \in \mathbb{Z}^3$ with $a < b < c$ such that $c - a = k - 1 < 12$. If $abc \neq 0$ is a perfect square, then the triplet (a, b, c) is one of the following:*

- $k = 5$: $(-2, -1, 2), (2, 3, 6),$
- $k = 6$: $(-4, -1, 1), (3, 6, 8), (5, 8, 10), (240, 243, 245),$
- $k = 7$: $(-4, -2, 2), (-3, -1, 3), (2, 4, 8), (6, 8, 12), (48, 50, 54),$
- $k = 8$: $(-4, -3, 3), (1, 2, 8), (2, 8, 9), (7, 8, 14), (21, 27, 28),$
- $k = 9$: $(-6, -3, 2), (-4, -1, 4), (1, 4, 9), (2, 5, 10), (12, 15, 20), (24, 27, 32),$
 $(242, 245, 250),$

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$k = 10$: $(-8, -2, 1)$, $(-6, -2, 3)$, $(-3, -2, 6)$, $(3, 4, 12)$, $(3, 9, 12)$, $(6, 10, 15)$,
 $(18, 24, 27)$,

$k = 11$: $(-9, -4, 1)$, $(-9, -1, 1)$, $(-8, -4, 2)$, $(-8, -1, 2)$, $(-5, -4, 5)$, $(-5, -1, 5)$,
 $(-2, -1, 8)$, $(2, 6, 12)$, $(5, 12, 15)$, $(8, 9, 18)$, $(8, 16, 18)$, $(10, 18, 20)$, $(14, 21, 24)$,
 $(20, 24, 30)$, $(40, 45, 50)$, $(2880, 2888, 2890)$, $(10082, 10086, 10092)$,

$k = 12$: $(-9, -8, 2)$, $(-9, -2, 2)$, $(-8, -6, 3)$, $(1, 3, 12)$, $(7, 14, 18)$, $(11, 18, 22)$,
 $(22, 24, 33)$, $(44, 45, 55)$, $(88, 98, 99)$, $(693, 700, 704)$.

As a consequence of the theorem we obtain that the interval $[44, 45, \dots, 55]$ is the smallest one which contains two disjoint triplets of positive integers with the relevant property: $\{44, 45, 55\}$ and $\{48, 50, 54\}$.

3. PROOF

To prove our theorem, we will reduce equation (1) to several elliptic equations. Recently, Gebel, Pethő and Zimmer [GPZ], and independently Stroeker and Tzanakis [ST] have developed an algorithm for resolving elliptic equations. Their method is based on the approach of Zagier [Z], and on the recent estimates of linear forms in elliptic logarithms, due to David [D]. The algorithm outlined in [GPZ] has been implemented by Gebel in the program package SIMATH (cf. [SIM]), and we use this program package to resolve our elliptic equations.

Proof of the Theorem. Let $(a, b, c) \in \mathbb{Z}^3$ be a triplet with the desired property, and put $x = a$, $u = b - a$ and $v = c - a$. To prove the Theorem we have to resolve the system of elliptic equations

$$x(x - u)(x - v) = y^2$$

with $0 < u < v < 12$ in integers x, y . Using the results of Erdős and Selfridge [ES], and Tzanakis [T], we may suppose that $v \geq 4$, and we obtain 52 equations. By a simple substitution we transform these elliptic equations into Weierstrass normal form, and we can resolve them by the SIMATH. We obtained just the solutions listed in our Theorem. \square

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