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A refinement of Faulhaber's theorem concerning sums of powers of natural numbers

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ABSTRACT

In an attempt to present a refinement of Faulhaber's theorem concerning sums of powers of natural numbers, the authors investigate and derive all the possible decompositions of the polynomial $\mathcal{S}_{a,b}^k(x)$ which is given by

$$\mathcal{S}_{a,b}^k(x) = b^k + (a+b)^k + (2a+b)^k + \cdots + \big(a(x-1)+b\big)^k.$$

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1. Introduction, definitions and preliminaries

Throughout this work, we use the following standard notations:

$$\mathbb{N} := \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$$

and

$$\mathbb{Z}^- := \{-1, -2, -3, \ldots\} = \mathbb{Z}_0^- \setminus \{0\}.$$

Also, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. We denote by $\mathbb{C}[x]$ the ring of polynomials in the variable x with *complex* coefficients.

A decomposition of a polynomial $F(x) \in \mathbb{C}[x]$ is an equality of the following form:

$$F(x) = G_1\big(G_2(x)\big) \quad \big(G_1(x), G_2(x) \in \mathbb{C}[x]\big). \tag{1}$$

The decomposition in (1) is *nontrivial* if

$$\deg\{G_1(x)\} > 1 \quad \text{and} \quad \deg\{G_2(x)\} > 1.$$

Two decompositions

$$F(x) = G_1\big(G_2(x)\big) \quad \text{and} \quad F(x) = H_1\big(H_2(x)\big)$$

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are said to be equivalent if there exists a linear polynomial $\ell(x) \in \mathbb{C}[x]$ such that

$$G_1(x) = \ell\big(H_1(x)\big) \quad \text{and} \quad H_2(x) = \ell\big(G_2(x)\big).$$

The polynomial $F(x)$ is called *decomposable* if it has at least one nontrivial decomposition; otherwise, it is said to be *indecomposable*.

In his monumental book [1], Johann Faulhaber (1580–1635) discovered that the sums of the first n odd powers can be expressed as the polynomials of the simple sum N given by

$$N = 1 + 2 + 3 + \cdots + n = \frac{1}{2}\, n(n+1).$$

He also conjectured that similar representation exists for the sum of every odd power. The first correct proof of this conjecture was published by Jacobi [2] (see also [3]). The following sums of the powers of natural numbers:

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + (n-1)^k \quad (n \in \mathbb{N} \setminus \{1\};\ k \in \mathbb{N}_0) \tag{2}$$

are closely related to the classical Bernoulli polynomials $B_k(x)$, namely, for positive integers n we have

$$S_k(n) = \frac{1}{k+1}[B_{k+1}(n) - B_{k+1}] \quad (n \in \mathbb{N} \setminus \{1\};\ k \in \mathbb{N}_0), \tag{3}$$

where the classical Bernoulli polynomials $B_n(x)$ are usually defined by means of the following generating function (see, for details, [4, p. 59 *et seq.*]; see also [5,6] and the references cited in each of these recent investigations on the subject):

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} \quad (|z| < 2\pi) \tag{4}$$

with, of course, the classical Bernoulli numbers B_n given by

$$B_n := B_n(0) \quad (n \in \mathbb{N}_0).$$

By using the connection exhibited in (3), we can extend $S_k(n)$ appropriately to $\mathcal{S}_k(x)$ for every real value of $x \in \mathbb{R}$. We thus have

$$\mathcal{S}_k(x) := \frac{1}{k+1}[B_{k+1}(x) - B_{k+1}] \quad (x \in \mathbb{R} \setminus \{1\};\ k \in \mathbb{N}_0). \tag{5}$$

Such sums as $S_k(n)$ in (2) of powers of natural numbers, but with real or complex exponents, have also been investigated in the existing literature. For example, Srivastava et al. [7] made use of certain operators of fractional calculus to derive, among other results, the following summation identity:

$$\begin{aligned} Z(n, \lambda) &:= 1^\lambda + 2^\lambda + 3^\lambda + \cdots + n^\lambda \quad (\lambda \in \mathbb{C};\ ;\ n \in \mathbb{N};\ 1^\lambda := 1) \\ &= \frac{1}{\Gamma(-\lambda)} \sum_{k=1}^{n} \sum_{j=0}^{\infty} (j-\lambda)^{-1} L_j^{(-\lambda)}(k) \quad (\lambda \in \mathbb{C} \setminus \mathbb{N}_0), \end{aligned} \tag{6}$$

where $L_n^{(\alpha)}(z)$ denotes the classical Laguerre polynomial of order (or index) α and degree n in z, defined by (see [4, p. 55, Equation 1.4(72)])

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} \quad (z, \alpha \in \mathbb{C}; \quad n \in \mathbb{N}_0). \tag{7}$$

On the other hand, the following functional relation for the sum $Z(n, m)$ defined by (6) with $\lambda = m \ \ (m \in \mathbb{N}_0)$ was proven by Nishimoto and Srivastava [8, p. 130, Equation 2.3]:

$$Z(n, m) = m! \sum_{j=0}^{m} (-1)^j \binom{m+\alpha}{m-j} \sum_{k=0}^{j} \frac{(-1)^k}{k!} \binom{j+\alpha}{j-k} Z(n, k) \quad (m \in \mathbb{N}_0;\ n \in \mathbb{N}). \tag{8}$$

Rakaczki [9] proved that the polynomial $\mathcal{S}_k(x)$, defined by (5), is indecomposable for even values of $k \in \mathbb{N}_0$. Furthermore, for odd $k \in \mathbb{N}_0$, Rakaczki [9] observed that all the decompositions of $S_k(x)$ are equivalent to the following decomposition:

$$\mathcal{S}_k(x) = \tilde{S}_k\left(\left(x - \frac{1}{2}\right)^2\right). \tag{9}$$

His result is a consequence of Theorem 1 below, which is due to Bilu et al. [10].

Theorem 1. *The polynomial $B_n(x)$ is indecomposable for odd $n \in \mathbb{N}_0$. If $n = 2m$ ($m, n \in \mathbb{N}_0$ is even), then any nontrivial decomposition of $B_n(x)$ is equivalent to the following decomposition:*

$$B_n(x) = \tilde{B}_m\left(\left(x - \frac{1}{2}\right)^2\right).$$

In particular, the polynomial $\tilde{B}_m(x)$ is indecomposable for any $m \in \mathbb{N}_0$.

Chen *et al.* [11] formulated a generalization of the classical Faulhaber theorem. For a given arithmetic progression

$$a + b, 2a + b, \ldots, a(n-1) + b,$$

Faulhaber's result implies that odd power sums of this arithmetic progression are polynomials in

$$(n-1)b + \frac{1}{2}\, n(n-1)a.$$

For example,

$$1^{2m-1} + 3^{2m-1} + 5^{2m-1} + \cdots + (2n-1)^{2m-1}$$

is a polynomial in n^2. Furthermore,

$$1^{2m-1} + 4^{2m-1} + 7^{2m-1} + \cdots + (3n-2)^{2m-1}$$

is a polynomial in the pentagonal number $\frac{1}{2}\, n(3n-1)$.

For a positive integer $n \in \mathbb{N} \setminus \{1\}$, let

$$S^k_{a,b}(n) := b^k + (a+b)^k + (2a+b)^k + \cdots + \big(a(n-1)+b\big)^k. \tag{10}$$

It is easy to see that

$$S^k_{a,b}(n) = \frac{a^k}{k+1}\left(\left[B_{k+1}\left(n + \frac{b}{a}\right) - B_{k+1}\right] - \left[B_{k+1}\left(\frac{b}{a}\right) - B_{k+1}\right]\right). \tag{11}$$

We can thus extend the definition (10) to hold true for every real value of $x \in \mathbb{R}$ as follows:

$$\mathcal{S}^k_{a,b}(x) := \frac{a^k}{k+1}\left[B_{k+1}\left(x + \frac{b}{a}\right) - B_{k+1}\left(\frac{b}{a}\right)\right]. \tag{12}$$

The main object of this work is to prove a generalization and refinement of the above-cited results of Rakaczki [9] and Chen *et al.* [11].

2. The main result

In this section, we apply Theorem 1 in order to derive the following generalization and refinement of the results of Rakaczki [9] and Chen *et al.* [11], which we referred to in the preceding section.

Theorem 2. *The polynomial $\mathcal{S}^k_{a,b}(x)$ is indecomposable for even $k \in \mathbb{N}_0$. If $k = 2v - 1$ is odd, then any nontrivial decomposition of $\mathcal{S}^k_{a,b}(x)$ is equivalent to the following decomposition:*

$$\mathcal{S}^k_{a,b}(x) = \hat{S}_v\left(\left(x + \frac{b}{a} - \frac{1}{2}\right)^2\right) \quad (k = 2v - 1). \tag{13}$$

Proof. We prove Theorem 2 by suitably applying Theorem 1. First of all, we let $k \in \mathbb{N}_0$ be an even integer,

$$t = x + \frac{b}{a}$$

and suppose that there exist polynomials f_1 and f_2 such that

$$\deg\{f_1(t)\} > 1 \quad \text{and} \quad \deg\{f_2(t)\} > 1 \tag{14}$$

and

$$\frac{a^k}{k+1}\left[B_{k+1}(t) - B_{k+1}\left(\frac{b}{a}\right)\right] = f_1\big(f_2(t)\big). \tag{15}$$

From this last expression, Eq. (15), we have a decomposition for the $(k+1)$th Bernoulli polynomial, which is a contradiction.

We now let $k \in \mathbb{N} \setminus \{1\}$ be an odd integer. Then we have

$$B_{k+1}(t) = \frac{k+1}{a^k} f_1\bigl(f_2(t)\bigr) + B_{k+1}\left(\frac{b}{a}\right)$$

and upon choosing

$$f(t) = \frac{k+1}{a^k} f_1(t) + B_{k+1}\left(\frac{b}{a}\right),$$

we obtain

$$B_{k+1}(t) = f\bigl(f_2(t)\bigr).$$

We thus find from Theorem 1 that

$$f_2(t) = \left(t - \frac{1}{2}\right)^2 = \left(x + \frac{b}{a} - \frac{1}{2}\right)^2. \tag{16}$$

This evidently completes our proof of Theorem 2. □

3. Remarks and observations

By considering the following power sum:

$$\sum_{j=0}^{n-1} (5j+3)^3 = \frac{125}{4}n^4 + \frac{25}{2}n^3 - \frac{55}{4}n^2 - 3n, \tag{17}$$

we can easily observe that the polynomial in (17) is given by

$$\sum_{j=0}^{n-1} (5j+3)^3 = \frac{125}{4}\left(n + \frac{1}{10}\right)^4 - \frac{125}{8}\left(n + \frac{1}{10}\right)^2 + \frac{49}{320}. \tag{18}$$

We conclude our investigation by remarking that the decomposition properties of a polynomial with rational coefficients play an important rôle in the theory of such separable diophantine equations as follows (see [12]):

$$f(x) = g(y).$$

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References

[1] J. Faulhaber, Academia Algebræ, Darinnen die miraculosische Inventiones zu den höchsten Cossen weiters continuirt und profitiert werden, Johann Ulrich Schönigs, Augspurg, 1631.
[2] C.G.J. Jacobi, De usu legitimo formulae summatoriae Maclaurinianae, J. Reine Angew. Math. 12 (1834) 263–272.
[3] D.E. Knuth, Johann Faulhaber and sums of powers, Math. Comput. 61 (1993) 277–294.
[4] H.M. Srivastava, J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 2001.
[5] H.M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, Math. Proc. Cambridge Philos. Soc. 129 (2000) 77–84.
[6] H.M. Srivastava, Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, Appl. Math. Lett. 17 (2004) 375–380.
[7] H.M. Srivastava, J.M.C. Joshi, C.S. Bisht, Fractional calculus and the sum of powers of natural numbers, Stud. Appl. Math. 85 (1991) 183–193.
[8] K. Nishimoto, H.M. Srivastava, Evaluation of the sum of powers of natural numbers by means of fractional calculus, J. Coll. Engrg. Nihon Univ. Ser. B 32 (1991) 127–132.
[9] Cs. Rakaczki, On the Diophantine equation $S_m(x) = g(y)$, Publ. Math. Debrecen 65 (2004) 439–460.
[10] Yu.F. Bilu, B. Brindza, P. Kirschenhofer, Á. Pintér, R.F. Tichy, Diophantine equations and Bernoulli polynomials (with an appendix by A. Schinzel), Compositio Math. 131 (2002) 173–188.
[11] W.Y.C. Chen, A.M. Fu, I.F. Zhang, Faulhaber's theorem on power sums, Discrete Math. 309 (2009) 2974–2981.
[12] Yu.F. Bilu, R.F. Tichy, The Diophantine equation $f(x) = g(y)$, Acta Arith. 95 (2000) 261–288.