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ON THE PROJECTIVE THEORY OF SPRAYS

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Dedicated to József Szilasi on the occasion of his 60th birthday.

Abstract. The paper aims to give a fairly self-contained survey on the fundamentals and the basic techniques of spray geometry, using a rigorously index-free formalism in the pull-back bundle framework, with applications to Finslerian sprays and metrizability problems. Thus we review a number of classically well-known facts from a modern viewpoint, and prove also known results using new ideas and tools. Among others, Laugwitz's metrization theorem (7.3) and the proof of the vanishing of the direction independent Landsberg and stretch tensor (6.6, 6.7) belong to this category. We present also some results we believe are new. We mention from this group the description of the projective factors which yield the invariance of the Berwald curvature under a projective change (5.3, 5.4) and the sufficient conditions of the Finsler metrizability of a spray in a broad sense deduced from the Rapcsák equations (7.7, 7.11).

1. Conventions and basic definitions

By a *manifold* we shall always mean a locally Euclidean, second countable, connected Hausdorff space with a smooth structure. If M and N are manifolds, $C^\infty(M,N)$ denotes the set of smooth maps from M to N; $C^\infty(M) := C^\infty(M,\mathbb{R})$. $\tau\colon TM \to M$ and $\tau_{TM}\colon TTM \to TM$ are the tangent bundles of the manifold M and the tangent manifold TM, respectively.
$\mathfrak{X}(M) := \{X \in C^\infty(M,TM) | \tau \circ X = 1_M\}$ is the $C^\infty(M)$-module of vector fields on M, its dual $\mathfrak{X}^*(M)$ is the module of 1-forms on M. If $X \in \mathfrak{X}(M)$, $\mathcal{L}_X$ denotes the *Lie derivative* with respect to X, and i_X is the *substitution operator* or *contraction* by X. d stands for the *exterior derivative* operator.

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If $o \in \mathfrak{X}(M)$ is the zero vector field, $\overset{\circ}{T}M := TM \setminus o(M)$, $\overset{\circ}{\tau} := \tau \upharpoonright \overset{\circ}{T}M$, then $\overset{\circ}{\tau}\colon \overset{\circ}{T}M \to M$ is said to be the *slit tangent bundle* of M. $\phi_* \in C^\infty(TM, TN)$ is the tangent linear map (or derivative) of $\phi \in C^\infty(M, N)$. If $I \subset \mathbb{R}$ is an open interval and $c\colon I \to M$ is a smooth curve then $\dot{c} := c_* \circ \frac{d}{du}$ is the velocity vector field of c. (Here $\frac{d}{du}$ is the canonical vector field on the real line.) The vertical lift of a function $f \in C^\infty(M)$ is $f^{\mathsf{v}} := f \circ \tau \in C^\infty(TM)$, the complete lift $f^{\mathsf{c}} \in C^\infty(TM)$ of f is defined by $f^{\mathsf{c}}(v) := v(f)$, $v \in TM$. For any vector field X on M there is a unique vector field $X^{\mathsf{c}} \in \mathfrak{X}(TM)$ such that $X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}$ for any function $f \in C^\infty(M)$. X^{c} is called the complete lift of X.

If K is a type $\binom{1}{1}$ tensor field on $\overset{\circ}{T}M$, i.e., an endomorphism of the $C^\infty(\overset{\circ}{T}M)$-module $\mathfrak{X}(\overset{\circ}{T}M)$ and $\eta \in \mathfrak{X}(\overset{\circ}{T}M)$, then we define the *Frölicher-Nijenhuis bracket* $[K, \eta]$ by

$$[K, \eta]\xi := [K\xi, \eta] - K[\xi, \eta] \ ; \ \xi \in \mathfrak{X}(\overset{\circ}{T}M).$$

Then $[K, \eta]$ is also a type $\binom{1}{1}$ tensor on $\overset{\circ}{T}M$; it is just the negative of the Lie derivative $\mathcal{L}_\eta K$. We also associate to K two graded derivations i_K and d_K of the Grassmann algebra of differential forms on $\overset{\circ}{T}M$, prescribing their operation on smooth functions and 1-forms by the following rules:

$$i_K F := 0 \ , \ i_K dF := dF \circ K \ ; \ F \in C^\infty(\overset{\circ}{T}M); \tag{1}$$

$$d_K := i_K \circ d - d \circ i_K. \tag{2}$$

Then the degree of i_K is 0, and the degree of d_K is 1. On functions d_K operates by

$$d_K F = i_K dF = dF \circ K,$$

so for any vector field ξ on $\overset{\circ}{T}M$ we have

$$d_K F(\xi) = dF(K(\xi)) = K(\xi)F.$$

The main scenes of our considerations will be the pull-back bundles of the tangent bundle $\tau\colon TM \to M$ over τ and $\overset{\circ}{\tau}$, i.e., the vector bundles

$$\pi\colon TM \times_M TM \to TM \text{ and } \overset{\circ}{\pi}\colon \overset{\circ}{T}M \times_M TM \to \overset{\circ}{T}M,$$

respectively. (Here $\times_M$ denotes fibre product.) The sections of π are smooth maps $\widetilde{X}\colon TM \to TM \times_M TM$ of the form

$$v \in TM \mapsto \widetilde{X}(v) = (v, \underline{X}(v)) \in TM \times_M TM,$$
$$\underline{X} \in C^\infty(TM, TM), \ \tau \circ \underline{X} = \tau.$$

We have a *canonical section*

$$\delta\colon v \in TM \mapsto \delta(v) := (v, v) \in TM \times_M TM,$$

and any vector field X on M induces a section

$$\widehat{X}\colon v \in TM \mapsto \widehat{X}(v) := (v, X(\tau(v))) \in TM \times_M TM,$$

called a *basic section* of π or a *basic vector field* along τ. The $C^\infty(TM)$-module $Sec(\pi)$ of sections of π is generated by the basic sections. If

$$\mathfrak{X}(\tau) := \{\underline{X} \in C^\infty(TM, TM) | \tau \circ \underline{X} = \tau\},$$

then $\mathfrak{X}(\tau)$ is naturally isomorphic to $Sec(\pi)$, so the two modules will be identified without any comment, whenever it is convenient. We get the $C^\infty(\overset{\circ}{T}M)$-modules $Sec(\overset{\circ}{\pi})$ and $\mathfrak{X}(\overset{\circ}{\tau})$ in the same way. The tensor algebras of $Sec(\pi) \cong \mathfrak{X}(\tau)$ and $Sec(\overset{\circ}{\pi}) \cong \mathfrak{X}(\overset{\circ}{\tau})$ will be denoted by $\mathcal{T}(\pi)$ and $\mathcal{T}(\overset{\circ}{\pi})$, respectively. The elements of these tensor algebras will frequently be mentioned as *tensors along τ or $\overset{\circ}{\tau}$*. Obviously, $\mathcal{T}(\pi)$ may be considered as a subalgebra of $\mathcal{T}(\overset{\circ}{\pi})$.

If $\mathbf{A}$ is a type $\binom{1}{3}$ tensor along $\overset{\circ}{\tau}$, then we define its *trace* $\mathrm{tr}\mathbf{A} \in \mathcal{T}^0_2(\overset{\circ}{\pi})$ by

$$(\mathrm{tr}\mathbf{A})(\widetilde{X}, \widetilde{Y}) := \mathrm{tr}\left(\widetilde{Z} \mapsto \mathbf{A}(\widetilde{Z}, \widetilde{X}, \widetilde{Y})\right) \ ; \widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\overset{\circ}{\pi}).$$

We have a canonical injective strong bundle map $\mathbf{i}\colon TM \times_M TM \to TTM$ given by

$$\mathbf{i}(v, w) := \dot{c}(0), \text{ if } c(t) := v + tw,$$

and a canonical surjective strong bundle map

$$\mathbf{j}\colon TTM \to TM \times_M TM,\ w \in T_vTM \mapsto \mathbf{j}(w) := (v, \tau_*(w))$$

such that the sequence

$$0 \longrightarrow TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \longrightarrow 0$$

is an exact sequence of vector bundle maps. $\mathbf{i}$ and $\mathbf{j}$ induce $C^\infty(TM)$-homomorphisms at the level of sections, which will be denoted by the same letters. So we also have the exact sequence

$$0 \longrightarrow \mathfrak{X}(\tau) \xrightarrow{\mathbf{i}} \mathfrak{X}(TM) \xrightarrow{\mathbf{j}} \mathfrak{X}(\tau) \longrightarrow 0$$

of module homomorphisms. $\mathfrak{X}^{\mathsf{v}}(TM) := \mathbf{i}\mathfrak{X}(\tau)$ is the module of *vertical vector fields* on TM, $X^{\mathsf{v}} := \mathbf{i}\widehat{X}$ is the *vertical lift* of $X \in \mathfrak{X}(M)$. If α is a 1-form on M, then there exists a unique 1-form α^{v} on TM such that

$$\alpha^{\mathsf{v}}(X^{\mathsf{v}}) = 0\ ,\ \alpha^{\mathsf{v}}(X^{\mathsf{c}}) = (\alpha(X))^{\mathsf{v}}$$

for all $X \in \mathfrak{X}(M)$. α^{v} is said to be the vertical lift of α.

$C := \mathbf{i}\delta$ is a canonical vertical vector field on TM, the *Liouville vector field.* For any vector field X on M we have

$$[C, X^{\mathsf{v}}] = -X^{\mathsf{v}}\ ,\ [C, X^{\mathsf{c}}] = 0. \tag{3}$$

$\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is a tensor field on TM of type $\binom{1}{1}$; it is called the *vertical endomorphism.* For all vector fields X on M we have

$$\mathbf{J}X^{\mathsf{v}} = 0 \ , \ \mathbf{J}X^{\mathsf{c}} = X^{\mathsf{v}};$$

therefore

$$\mathrm{Im}(\mathbf{J}) = \mathrm{Ker}(\mathbf{J}) = \mathfrak{X}^{\mathsf{v}}(TM) \ , \ \mathbf{J}^2 = 0.$$

The following useful relations may be verified immediately:

$$[\mathbf{J}, C] = \mathbf{J} \ ; \ [\mathbf{J}, X^{\mathsf{v}}] = [\mathbf{J}, X^{\mathsf{c}}] = 0 \ , \ X \in \mathfrak{X}(M). \tag{4}$$

We define the *vertical differential* $\nabla^{\mathsf{v}} F \in \mathcal{T}_1^0(\pi)$ of a function $F \in C^\infty(TM)$ by

$$\nabla^{\mathsf{v}} F(\widetilde{X}) := (\mathbf{i}\widetilde{X})F \ , \ \widetilde{X} \in \mathrm{Sec}(\pi). \tag{5}$$

We note that

$$\nabla^{\mathsf{v}} F \circ \mathbf{j} = d_{\mathbf{J}} F, \tag{6}$$

where $d_{\mathbf{J}}$ is the graded derivation associated to the vertical endomorphism by (1) and (2). The vertical differential of a section $\widetilde{Y} \in \mathrm{Sec}(\pi)$ is the type $\binom{1}{1}$ tensor $\nabla^{\mathsf{v}} \widetilde{Y} \in \mathcal{T}_1^1(\pi)$ given by

$$\nabla^{\mathsf{v}} \widetilde{Y}(\widetilde{X}) =: \nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \eta] \ , \ \widetilde{X} \in \mathrm{Sec}(\pi), \tag{7}$$

where $\eta \in \mathfrak{X}(TM)$ is such that $\mathbf{j}\eta = \widetilde{Y}$. (It is easy to check that the result does not depend on the choice of η.) Using the Leibnizian product rule as a guiding principle, the operators $\nabla^{\mathsf{v}}_{\widetilde{X}}$ may uniquely be extended to a tensor derivation of the tensor algebra of $\mathrm{Sec}(\pi)$. Forming the vertical differential of a tensor over $\mathrm{Sec}(\pi)$, we use the following convention: if, e.g., $\mathbf{A} \in \mathcal{T}_2^1(\pi)$, then $\nabla^{\mathsf{v}}(\mathbf{A}) \in \mathcal{T}_3^1(\pi)$, given by

$$\nabla^{\mathsf{v}} \mathbf{A}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) := (\nabla^{\mathsf{v}}_{\widetilde{X}} \mathbf{A})(\widetilde{Y}, \widetilde{Z}) = \nabla^{\mathsf{v}}_{\widetilde{X}} \mathbf{A}(\widetilde{Y}, \widetilde{Z}) - \mathbf{A}(\nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) - \mathbf{A}(\widetilde{Y}, \nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Z}).$$

A type $\binom{0}{s}$ or $\binom{1}{s}$ tensor $\mathbf{A}$ along $\overset{\circ}{\tau}$ is said to be *homogeneous of degree* k, where k is an integer, if

$$\nabla^{\mathsf{v}}_{\delta} \mathbf{A} = k\mathbf{A}.$$

2. Ehresmann connections and Berwald derivatives

By an *Ehresmann connection* over M we mean a map $\mathcal{H} \colon TM \times_M TM \to TTM$ satisfying the following conditions:

- (C$_1$) $\mathcal{H}$ is fibre preserving and fibrewise linear, i.e., for every $v \in TM$, $\mathcal{H}_v := \mathcal{H} \restriction \{v\} \times T_{\tau(v)}M$ is a linear map from $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$ into $T_v TM$.
- (C$_2$) $\mathbf{j} \circ \mathcal{H} = 1_{TM \times_M TM}$, i.e., "$\mathcal{H}$ splits".
- (C$_3$) $\mathcal{H}$ is smooth over $\overset{\circ}{T} M \times_M TM$.

(C$_4$) If $o\colon M \to TM$ is the zero vector field, then $\mathcal{H}(o(p), v) = (o_*)_p(v)$, for all $p \in M$ and $v \in T_pM$.

We associate to an Ehresmann connection $\mathcal{H}$

the *horizontal projector* $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$, the *vertical projector* $\mathbf{v} := 1_{T\overset{\circ}{T}M} - \mathbf{h}$,
the *vertical map* $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v} : T\overset{\circ}{T}M \to \overset{\circ}{T}M \times_M TM$,
the *almost complex structure* $\mathbf{F} := \mathcal{H} \circ \mathcal{V} - \mathbf{i} \circ \mathbf{j} = \mathcal{H} \circ \mathcal{V} - \mathbf{J}$.

We have the following basic relations:

$$\mathbf{h}^2 = \mathbf{h},\ \mathbf{v}^2 = \mathbf{v};\ \mathbf{J} \circ \mathbf{h} = \mathbf{J},\ \mathbf{h} \circ \mathbf{J} = 0;\ \mathbf{J} \circ \mathbf{v} = 0,\ \mathbf{v} \circ \mathbf{J} = \mathbf{J};$$
$$\mathbf{F}^2 = -\mathbf{1},\ \mathbf{J} \circ \mathbf{F} = \mathbf{v},\ \mathbf{F} \circ \mathbf{J} = \mathbf{h};$$
$$\mathbf{F} \circ \mathbf{h} = -\mathbf{J},\ \mathbf{h} \circ \mathbf{F} = \mathbf{F} \circ \mathbf{v} = \mathbf{J} + \mathbf{F},\ \mathbf{v} \circ \mathbf{F} = -\mathbf{J}.$$

The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ (with respect to $\mathcal{H}$) is

$$X^{\mathsf{h}} := \mathcal{H} \circ \widehat{X} =: \mathcal{H}\widehat{X} = \mathbf{h}X^{\mathsf{c}}.$$

It may be shown (see e.g. [19]) that for all vector fields X, Y on M we have

$$\mathbf{J}[X^{\mathsf{h}}, Y^{\mathsf{h}}] = [X, Y]^{\mathsf{v}},\ \mathbf{h}[X^{\mathsf{h}}, Y^{\mathsf{h}}] = [X, Y]^{\mathsf{h}}. \tag{8}$$

By the *tension* of $\mathcal{H}$ we mean the type $\binom{1}{1}$ tensor field along $\overset{\circ}{\tau}$ given by

$$\mathbf{t}(\widetilde{X}) := \mathcal{V}[\mathcal{H}\widetilde{X}, C],\ \widetilde{X} \in \mathrm{Sec}(\overset{\circ}{\pi}).$$

Then

$$\mathbf{it}(\widehat{X}) = [X^{\mathsf{h}}, C],\ X \in \mathfrak{X}(M).$$

$\mathcal{H}$ is said to be *homogeneous* if its tension vanishes. We define the *torsion* and the *curvature* of $\mathcal{H}$ by

$$\mathbf{T}(\widetilde{X}, \widetilde{Y}) := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] - \mathcal{V}[\mathcal{H}\widetilde{Y}, \mathbf{i}\widetilde{X}] - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]$$

and

$$\mathbf{R}(\widetilde{X}, \widetilde{Y}) := -\mathcal{V}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]$$

($\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\overset{\circ}{\pi})$), respectively. Evaluating on basic vector fields, we obtain the more expressive relations

$$\mathbf{iT}(\widehat{X}, \widehat{Y}) := [X^{\mathsf{h}}, Y^{\mathsf{v}}] - [Y^{\mathsf{h}}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}}$$

and

$$\mathbf{iR}(\widehat{X}, \widehat{Y}) = -\mathbf{v}[X^{\mathsf{h}}, Y^{\mathsf{h}}],$$

where X and Y are vector fields on M.

Now we recall an elementary, but crucial construction of Ehresmann connections. To do this, we need the concept of a semispray and spray. By a *semispray* over a manifold M we mean a map $S\colon TM \to TTM$ satisfying the following conditions:

(S$_1$) $\tau_{TM} \circ S = 1_{TM}$;

(S_2) S is smooth over $\overset{\circ}{T}M$;
(S_3) $\mathbf{J}S = C$ (or, equivalently, $\mathbf{j}S = \delta$).

A semispray S is said to be a *spray*, if it satisfies the additional conditions

(S_4) S is of class C^1 over TM;
(S_5) $[C, S] = S$, i.e., S is positive-homogeneous of degree 2.

If a spray is of class C^2 (and hence smooth) over TM, then it is called an *affine spray*. Following S. Lang's terminology [9], we say that a *smooth* map $S\colon TM \to TTM$ is a *second-order vector field* over M, if it satisfies conditions (S_1) and (S_3). Notice, however, that by a 'spray' Lang means a second-order vector field satisfying the homogeneity condition (S_5), i.e., an 'affine spray' in our sense.

Given a semispray S over M, by a celebrated result of M. Crampin [6] and J. Grifone [7], there exists a unique Ehresmann connection $\mathcal{H}$ over M such that

$$\mathcal{H}(\widehat{X}) = \frac{1}{2}\left(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]\right) \tag{9}$$

for all vector fields X on M. $\mathcal{H}$ is said to be the *Ehresmann connection associated to* (or *generated by*) $\mathcal{H}$. The torsion of this Ehresmann connection vanishes. Furthermore, we have

$$\mathcal{H}(\delta) = \frac{1}{2}(S + [C, S]).$$

If, in particular, S is a spray, then $\mathcal{H}(\delta) = S$, and $\mathcal{H}$ is homogeneous, i.e., its tension also vanishes.

We define the *h-Berwald differentials* $\nabla^{\mathsf{h}} F \in \mathcal{T}_1^0(\overset{\circ}{\pi})$ $(F \in C^\infty(\overset{\circ}{T}M))$ and $\nabla^{\mathsf{h}}\widetilde{Y} \in \mathcal{T}_1^1(\overset{\circ}{\pi})$ $(\widetilde{Y} \in \operatorname{Sec}(\overset{\circ}{\pi}))$ by the following rules:

$$\nabla^{\mathsf{h}} F(\widetilde{X}) := (\mathcal{H}\widetilde{X})F,\ \widetilde{X} \in \operatorname{Sec}(\overset{\circ}{\pi}); \tag{10}$$

$$\nabla^{\mathsf{h}}\widetilde{Y}(\widetilde{X}) := \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}],\ \widetilde{X} \in \operatorname{Sec}(\overset{\circ}{\pi}). \tag{11}$$

The operators $\nabla^{\mathsf{h}}_{\widetilde{X}}$ $(\widetilde{X} \in \operatorname{Sec}(\overset{\circ}{\pi}))$ may also uniquely be extended to the whole tensor algebra of $\operatorname{Sec}(\overset{\circ}{\pi})$ as tensor derivations. Forming the h-Berwald differential of an arbitrary tensor, we adopt the same convention as in the vertical case. We note that the tension of $\mathcal{H}$ is just the h-covariant differential of the canonical section, i.e., $= \nabla^{\mathsf{h}}\delta$. So the homogeneity of $\mathcal{H}$ means that

$$\nabla^{\mathsf{h}}\delta = 0. \tag{12}$$

We may also consider the graded derivation $d_{\mathbf{h}}$ associated to the horizontal projector $\mathbf{h} = \mathcal{H} \circ \mathbf{j}$; then we have

$$\nabla^{\mathsf{h}} F \circ \mathbf{j} = d_{\mathbf{h}} F\ \ (F \in C^\infty(\overset{\circ}{T}M)). \tag{13}$$

From the operators ∇^{v} and ∇^{h} we build the *Berwald derivative*

$$\nabla\colon (\xi, \widetilde{Y}) \in \mathfrak{X}(\overset{\circ}{T}M) \times \operatorname{Sec}(\overset{\circ}{\pi}) \mapsto \nabla_\xi \widetilde{Y} := \nabla^{\mathsf{v}}_{\mathcal{V}\xi}\widetilde{Y} + \nabla^{\mathsf{h}}_{\mathbf{j}\xi}\widetilde{Y} \in \operatorname{Sec}(\overset{\circ}{\pi}).$$

Then, by (7) and (11),

$$\nabla_\xi \widetilde{Y} = \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] + \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}].$$

In particular,

$$\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y},\ \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} = \nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y};\ \widetilde{X}, \widetilde{Y} \in \operatorname{Sec}(\overset{\circ}{\pi});$$

$$\nabla_{X^{\mathsf{v}}}\widehat{Y} = 0,\ \mathbf{i}\nabla_{X^{\mathsf{h}}}\widehat{Y} = \left[X^{\mathsf{h}}, Y^{\mathsf{v}}\right];\ X, Y \in \mathfrak{X}(M). \tag{14}$$

Lemma 2.1 (hh-Ricci identity for functions)**.** *Let $\mathcal{H}$ be a torsion-free Ehresmann connection over M. If $f\colon \overset{\circ}{T}M \to \mathbb{R}$ is a smooth function, then for any sections $\widetilde{X}$, $\widetilde{Y}$ in* $\operatorname{Sec}(\overset{\circ}{\pi})$ *we have*

$$\nabla^{\mathsf{h}}\nabla^{\mathsf{h}} f(\widetilde{X}, \widetilde{Y}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{h}} f(\widetilde{Y}, \widetilde{X}) = -\mathbf{i}\mathbf{R}(\widetilde{X}, \widetilde{Y})f. \tag{15}$$

Proof. It is enough to show that formula (15) is true for basic vector fields $\widehat{X}, \widehat{Y} \in \operatorname{Sec}(\overset{\circ}{\pi})$. Then

$$\begin{aligned}(\nabla^{\mathsf{h}}\nabla^{\mathsf{h}} f)(\widehat{X}, \widehat{Y}) &= \left(\nabla_{X^{\mathsf{h}}}(\nabla^{\mathsf{h}} f)\right)(\widehat{Y}) = X^{\mathsf{h}}Y^{\mathsf{h}}f - \nabla^{\mathsf{h}} f(\nabla_{X^{\mathsf{h}}}\widehat{Y}) \\ &= X^{\mathsf{h}}Y^{\mathsf{h}}f - (\mathcal{H}\nabla_{X^{\mathsf{h}}}\widehat{Y})f = X^{\mathsf{h}}Y^{\mathsf{h}}f - \mathcal{H}\mathcal{V}[X^{\mathsf{h}}, Y^{\mathsf{v}}]f \\ &= X^{\mathsf{h}}Y^{\mathsf{h}}f - (\mathbf{F} + \mathbf{J})[X^{\mathsf{h}}, Y^{\mathsf{v}}]f = X^{\mathsf{h}}Y^{\mathsf{h}}f - \mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{v}}]f,\end{aligned}$$

and in the same way

$$(\nabla^{\mathsf{h}}\nabla^{\mathsf{h}} f)(\widehat{Y}, \widehat{X}) = Y^{\mathsf{h}}X^{\mathsf{h}}f - \mathbf{F}[Y^{\mathsf{h}}, X^{\mathsf{v}}]f.$$

So we obtain

$$\begin{aligned}\nabla^{\mathsf{h}}\nabla^{\mathsf{h}} f(\widehat{X}, \widehat{Y}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{h}} f(\widehat{Y}, \widehat{X}) &= [X^{\mathsf{h}}, Y^{\mathsf{h}}]f - \mathbf{F}([X^{\mathsf{h}}, Y^{\mathsf{v}}] - [Y^{\mathsf{h}}, X^{\mathsf{v}}])f \\ &\overset{\mathbf{T}=0}{=} [X^{\mathsf{h}}, Y^{\mathsf{h}}]f - (\mathbf{F}[X, Y]^{\mathsf{v}})f = ([X^{\mathsf{h}}, Y^{\mathsf{h}}] - [X, Y]^{\mathsf{h}})f \\ &= ([X^{\mathsf{h}}, Y^{\mathsf{h}}] - \mathbf{h}[X^{\mathsf{h}}, Y^{\mathsf{h}}])f = \mathbf{v}[X^{\mathsf{h}}, Y^{\mathsf{h}}]f = -\mathbf{i}\mathbf{R}(\widehat{X}, \widehat{Y})f.\end{aligned}$$ □

3. The Berwald curvature of an Ehresmann connection

In this section we specify an Ehresmann connection $\mathcal{H}$ over M, and consider the Berwald derivative $\nabla = (\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ induced by $\mathcal{H}$. We denote by R^{∇} the usual curvature tensor of ∇. By the *Berwald curvature* of $\mathcal{H}$ we mean the type $\binom{1}{3}$ tensor field $\mathbf{B}$ along $\overset{\circ}{\tau}$ given by

$$\mathbf{B}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y})\widetilde{Z} = \nabla_{\mathbf{i}\widetilde{X}}\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{Z} - \nabla_{\mathcal{H}\widetilde{Y}}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Z} - \nabla_{[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]}\widetilde{Z}, \tag{16}$$

where $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ are vector fields along $\overset{\circ}{\tau}$.

Lemma 3.1. *For any vector field X, Y,Z on M we have*

$$\mathbf{B}(\widehat{X},\widehat{Y})\widehat{Z} = \mathbf{j}[X^{\mathsf{v}},\mathbf{F}[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = (\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{Z})(\widehat{X},\widehat{Y}), \tag{17}$$

or, equivalently,

$$\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} = [X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = [[X^{\mathsf{v}},Y^{\mathsf{h}}],Z^{\mathsf{v}}]. \tag{18}$$

Proof.

$$\begin{aligned}\mathbf{B}(\widehat{X},\widehat{Y})\widehat{Z} &:= R^{\nabla}(X^{\mathsf{v}},Y^{\mathsf{h}})\widehat{Z} = \nabla_{X^{\mathsf{v}}}\nabla_{Y^{\mathsf{h}}}\widehat{Z} - \nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{v}}}\widehat{Z} - \nabla_{[X^{\mathsf{v}},Y^{\mathsf{h}}]}\widehat{Z}\\ &= \nabla_{X^{\mathsf{v}}}(\mathcal{V}[Y^{\mathsf{h}},Z^{\mathsf{v}}]) = \mathbf{j}[X^{\mathsf{v}},\mathcal{H}\circ\mathcal{V}[Y^{\mathsf{h}},Z^{\mathsf{v}}]]\\ &= \mathbf{j}[X^{\mathsf{v}},(\mathbf{F}+\mathbf{J})[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = \mathbf{j}[X^{\mathsf{v}},\mathbf{F}[Y^{\mathsf{h}},Z^{\mathsf{v}}]].\end{aligned}$$

On the other hand,

$$\begin{aligned}(\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widehat{Z})(\widehat{X},\widehat{Y}) &= \nabla_{X^{\mathsf{v}}}(\nabla^{\mathsf{h}}\widehat{Z})(\widehat{Y}) = \nabla_{X^{\mathsf{v}}}\nabla_{Y^{\mathsf{h}}}\widehat{Z} = \mathbf{j}[X^{\mathsf{v}},\mathcal{H}\nabla_{Y^{\mathsf{h}}}\widehat{Z}]\\ &= \mathbf{j}[X^{\mathsf{v}},\mathcal{H}\circ\mathcal{V}[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = \mathbf{j}[X^{\mathsf{v}},\mathbf{F}[Y^{\mathsf{h}},Z^{\mathsf{v}}]],\end{aligned}$$

thus relations (17) hold. To prove the remainder, observe that

$$\begin{aligned}0 &= [\mathbf{J},X^{\mathsf{v}}]\mathbf{F}[Y^{\mathsf{h}},Z^{\mathsf{v}}] = [\mathbf{v}[Y^{\mathsf{h}},Z^{\mathsf{v}}],X^{\mathsf{v}}] - \mathbf{J}[\mathbf{F}[Y^{\mathsf{h}},Z^{\mathsf{v}}],X^{\mathsf{v}}]\\ &= -[X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] + \mathbf{J}[X^{\mathsf{v}},\mathbf{F}[Y^{\mathsf{h}},Z^{\mathsf{v}}]],\end{aligned}$$

and hence

$$\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} \overset{(17)}{=} \mathbf{J}[X^{\mathsf{v}},\mathbf{F}[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = [X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]].$$

Finally, using the Jacobi identity we obtain that

$$[X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = [[X^{\mathsf{v}},Y^{\mathsf{h}}],Z^{\mathsf{v}}].$$ □

Lemma 3.2. *The Berwald curvature of an Ehresmann connection is symmetric in its first and third variable. If the torsion of the Ehresmann connection vanishes, then the Berwald curvature is totally symmetric.*

Proof. Keeping the notation of the previous lemma, $\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} = [X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]]$. Since

$$\begin{aligned}0 &= [X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] + [Y^{\mathsf{h}},[Z^{\mathsf{v}},X^{\mathsf{v}}]] + [Z^{\mathsf{v}},[X^{\mathsf{v}},Y^{\mathsf{h}}]]\\ &= [X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] - [Z^{\mathsf{v}},[Y^{\mathsf{h}},X^{\mathsf{v}}]] = \mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} - \mathbf{iB}(\widehat{Z},\widehat{Y})\widehat{X},\end{aligned}$$

$\mathbf{B}$ is indeed symmetric in its first and third variable. If the Ehresmann connection has vanishing torsion, then

$$[X^{\mathsf{h}},Z^{\mathsf{v}}] - [Z^{\mathsf{h}},X^{\mathsf{v}}] - [X,Z]^{\mathsf{v}} = 0 \quad (X,Z\in\mathfrak{X}(M)),$$

and hence

$$\begin{aligned}\mathbf{iB}(\widehat{Y},\widehat{X})\widehat{Z} &= [Y^{\mathsf{v}},[X^{\mathsf{h}},Z^{\mathsf{v}}]] = [Y^{\mathsf{v}},[Z^{\mathsf{h}},X^{\mathsf{v}}]] + [Y^{\mathsf{v}},[X,Z]^{\mathsf{v}}]\\ &= [Y^{\mathsf{v}},[Z^{\mathsf{h}},X^{\mathsf{v}}]] = \mathbf{iB}(\widehat{Y},\widehat{Z})\widehat{X}.\end{aligned}$$

Thus, if the torsion vanishes,

$$\begin{aligned}\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} = \mathbf{iB}(\widehat{Z},\widehat{Y})\widehat{X} = \mathbf{iB}(\widehat{Z},\widehat{X})\widehat{Y} = \mathbf{iB}(\widehat{Y},\widehat{X})\widehat{Z} \\ = \mathbf{iB}(\widehat{Y},\widehat{Z})\widehat{X} = \mathbf{iB}(\widehat{X},\widehat{Z})\widehat{Y}.\end{aligned}$$ □

Lemma 3.3. *The tension and the Berwald curvature of an Ehresmann connection are related by*

$$\mathbf{B}(\widehat{X},\widehat{Y})\delta = \nabla^{\mathsf{v}}\mathbf{t}(\widehat{X},\widehat{Y});\ \ X,Y \in \mathfrak{X}(M). \tag{19}$$

Proof. By an important identity, due to J. Grifone, for any vector field ξ on TM we have

$$\mathbf{J}[\mathbf{J}\xi, S] = \mathbf{J}\xi, \tag{20}$$

where S is an arbitrary semispray over M ([7],[21]). Since $\mathcal{H}\circ\delta$ is a semispray over M, this implies that

$$\nabla_{\mathbf{i}\widetilde{X}}\delta = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\circ\delta] = \widetilde{X};\ \ \widetilde{X}\in\mathfrak{X}(\tau).$$

So we obtain

$$\begin{aligned}\mathbf{B}(\widehat{X},\widehat{Y})\delta &= R^{\nabla}(X^{\mathsf{v}},Y^{\mathsf{h}})\delta \\ &= \nabla_{X^{\mathsf{v}}}\nabla_{Y^{\mathsf{h}}}\delta - \nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{v}}}\delta - \nabla_{[X^{\mathsf{v}},Y^{\mathsf{h}}]}\delta \\ &= \nabla_{X^{\mathsf{v}}}(\mathbf{t}(\widehat{Y})) - \nabla_{Y^{\mathsf{h}}}\widehat{X} - \mathcal{V}[X^{\mathsf{v}},Y^{\mathsf{h}}] \\ &= \nabla_{X^{\mathsf{v}}}(\mathbf{t}(\widehat{Y})) - \mathcal{V}[Y^{\mathsf{h}},X^{\mathsf{v}}] + \mathcal{V}[Y^{\mathsf{h}},X^{\mathsf{v}}] = (\nabla_{X^{\mathsf{v}}}\mathbf{t})(\widehat{Y}) \\ &= (\nabla^{\mathsf{v}}\mathbf{t})(\widehat{X},\widehat{Y}).\end{aligned}$$ □

Corollary 3.4. *If the torsion and the vertical differential of the tension of an Ehresmann connection vanishes, then its Berwald curvature has the property*

$$\delta\in\left\{\widetilde{X},\widetilde{Y},\widetilde{Z}\right\} \Rightarrow \mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z} = 0. \tag{21}$$

Lemma 3.5. *The Berwald curvature of a homogeneous Ehresmann connection is homogeneous of degree -1, i.e.,*

$$\nabla^{\mathsf{v}}_{\delta}\mathbf{B} = \nabla_C\mathbf{B} = -\mathbf{B}.$$

Proof. Using the first relation in (4), the Jacobi identity (repeatedly) and the homogeneity of $\mathcal{H}$, for any vector fields X, Y, Z on M we get

$$\begin{aligned}&\mathbf{i}\,(\nabla_C\mathbf{B})\,(\widehat{X},\widehat{Y},\widehat{Z}) = \mathbf{i}\nabla_C(\mathbf{B}(\widehat{X},\widehat{Y})\widehat{Z}) \\ &\quad= \mathbf{J}[C,\mathcal{H}\mathbf{B}(\widehat{X},\widehat{Y})\widehat{Z}] = [\mathbf{J},C]\mathcal{H}\mathbf{B}(\widehat{X},\widehat{Y})\widehat{Z} - [\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z},C] \\ &\quad= \mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} - [[X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]],C] \\ &\quad= \mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} + [[[Y^{\mathsf{h}},Z^{\mathsf{v}}],C],X^{\mathsf{v}}] + [[C,X^{\mathsf{v}}],[Y^{\mathsf{h}},Z^{\mathsf{v}}]] \\ &\quad= [[[Y^{\mathsf{h}},Z^{\mathsf{v}}],C],X^{\mathsf{v}}] = -[[[Z^{\mathsf{v}},C],Y^{\mathsf{h}}],X^{\mathsf{v}}] + [[C,Y^{\mathsf{h}}],Z^{\mathsf{v}}],X^{\mathsf{v}}] \\ &\quad= -[[Z^{\mathsf{v}},Y^{\mathsf{h}}],X^{\mathsf{v}}] = -[X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = -\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z}.\end{aligned}$$ □

Lemma 3.6. (vh-Ricci formulae for functions and sections). *If F is a smooth function on $\overset{\circ}{T}M$ and $\widetilde{Z}$ is a section along $\overset{\circ}{\tau}$, then for any sections $\widetilde{X}$, $\widetilde{Y}$ in* $\operatorname{Sec}(\overset{\circ}{\pi})$ *we have*

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}F(\widetilde{X},\widetilde{Y}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}F(\widetilde{Y},\widetilde{X}); \tag{22}$$

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widetilde{Z}(\widetilde{X},\widetilde{Y}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\widetilde{Z}(\widetilde{Y},\widetilde{X}) = \mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z}. \tag{23}$$

Proof. The expression on the left-hand side of (22) is

$$\begin{aligned}\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}F(\widetilde{X},\widetilde{Y}) &= (\nabla_{\mathbf{i}\widetilde{X}}\nabla^{\mathsf{h}}F)(\widetilde{Y}) = (\mathbf{i}\widetilde{X})(\mathcal{H}\widetilde{Y})F - \nabla^{\mathsf{h}}F(\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y})\\ &= (\mathbf{i}\widetilde{X})(\mathcal{H}\widetilde{Y})F - (\mathcal{H}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y})F = (\mathbf{i}\widetilde{X})(\mathcal{H}\widetilde{Y})F - \mathbf{h}[\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y}]F\end{aligned}$$

The right-hand side of (22) can be written in the form

$$\begin{aligned}\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}F(\widetilde{Y},\widetilde{X}) &= (\nabla_{\mathcal{H}\widetilde{Y}}\nabla^{\mathsf{v}}F)(\widetilde{X}) = (\mathcal{H}\widetilde{Y})(\mathbf{i}\widetilde{X})F - \nabla^{\mathsf{v}}F(\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X})\\ &= (\mathcal{H}\widetilde{Y})(\mathbf{i}\widetilde{X})F - (\mathbf{i}\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X})F\\ &= (\mathcal{H}\widetilde{Y})(\mathbf{i}\widetilde{X})F - \mathbf{v}[\mathcal{H}\widetilde{Y},\mathbf{i}\widetilde{X}]F,\end{aligned}$$

so their difference is

$$[\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y}]F + \mathbf{v}[\mathcal{H}\widetilde{Y},\mathbf{i}\widetilde{X}]F - \mathbf{h}[\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y}]F = 0.$$

This proves relation (22). Relation (23) may be checked by a similar calculation. We have, on the one hand,

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\widetilde{Z}(\widetilde{X},\widetilde{Y}) = \nabla_{\mathbf{i}\widetilde{X}}\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{Z} - \nabla_{\mathcal{H}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y}}\widetilde{Z}.$$

On the other hand,

$$\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\widetilde{Z}(\widetilde{Y},\widetilde{X}) = \nabla_{\mathcal{H}\widetilde{Y}}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Z} - \nabla_{\mathbf{i}\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X}}\widetilde{Z}.$$

Since $\mathcal{H}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} - \mathbf{i}\nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X} = \mathbf{h}[\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y}] - \mathbf{v}[\mathcal{H}\widetilde{Y},\mathbf{i}\widetilde{X}] = [\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y}]$, it follows that the difference of the left-hand sides is indeed $\mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z}$. □

Lemma 3.7 (vh-Ricci formula for covariant tensors)**.** *Let $\mathbf{A} \in \mathcal{T}^0_s(\overset{\circ}{\pi})$, $s \geq 1$. For any sections $\widetilde{X},\widetilde{Y},\widetilde{Z_1},\dots\widetilde{Z_s}$ along $\overset{\circ}{\tau}$ we have*

$$\begin{aligned}&\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{A}(\widetilde{X},\widetilde{Y},\widetilde{Z_1},\dots\widetilde{Z_s}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{A}(\widetilde{Y},\widetilde{X},\widetilde{Z_1},\dots\widetilde{Z_s})\\ &= -\sum_{i=1}^{s}\mathbf{A}(\widetilde{Z_1},\dots,\mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z_i},\dots,\widetilde{Z_s}).\end{aligned} \tag{24}$$

Proof. For brevity, we sketch the argument only for a type $\binom{0}{2}$ tensor $\mathbf{A}$. It may easily be shown that the left-hand side of (24) is tensorial in its first two variables (actually, in all variables), so we may chose in the role of $\widetilde{X}$ and $\widetilde{Y}$ basic vector fields $\widehat{X}$, $\widehat{Y}$. Then

$$\begin{aligned}\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{A}(\widehat{X},\widehat{Y},\widetilde{Z_1},\widetilde{Z_2}) &= X^{\mathsf{v}}(Y^{\mathsf{h}}\mathbf{A}(\widetilde{Z_1},\widetilde{Z_2})) - X^{\mathsf{v}}\mathbf{A}(\nabla_{Y^{\mathsf{h}}}\widetilde{Z_1},\widetilde{Z_2})\\ &\quad - X^{\mathsf{v}}\mathbf{A}(\widetilde{Z_1},\nabla_{Y^{\mathsf{h}}}\widetilde{Z_2}) - Y^{\mathsf{h}}\mathbf{A}(\nabla_{X^{\mathsf{v}}}\widetilde{Z_1},\widetilde{Z_2}) + \mathbf{A}(\nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{v}}}\widetilde{Z_1},\widetilde{Z_2})\\ &\quad + \mathbf{A}(\nabla_{X^{\mathsf{v}}}\widetilde{Z_1},\nabla_{Y^{\mathsf{h}}}\widetilde{Z_2}) - Y^{\mathsf{h}}\mathbf{A}(\widetilde{Z_1},\nabla_{X^{\mathsf{v}}}\widetilde{Z_2}) + \mathbf{A}(\nabla_{Y^{\mathsf{h}}}\widetilde{Z_1},\nabla_{X^{\mathsf{v}}}\widetilde{Z_2})\\ &\quad + \mathbf{A}(\widetilde{Z_1},\nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{v}}}\widetilde{Z_2});\\ \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{A}(\widehat{Y},\widehat{X},\widetilde{Z_1},\widetilde{Z_2}) &= Y^{\mathsf{h}}(X^{\mathsf{v}}\mathbf{A}(\widetilde{Z_1},\widetilde{Z_2})) - Y^{\mathsf{h}}\mathbf{A}(\nabla_{X^{\mathsf{v}}}\widetilde{Z_1},\widetilde{Z_2})\\ &\quad - Y^{\mathsf{h}}\mathbf{A}(\widetilde{Z_1},\nabla_{X^{\mathsf{v}}}\widetilde{Z_2}) - [Y^{\mathsf{h}},X^{\mathsf{v}}]\mathbf{A}(\widetilde{Z_1},\widetilde{Z_2}) + \mathbf{A}(\nabla_{[Y^{\mathsf{h}},X^{\mathsf{v}}]}\widetilde{Z_1},\widetilde{Z_2})\\ &\quad + \mathbf{A}(\widetilde{Z_1},\nabla_{[Y^{\mathsf{h}},X^{\mathsf{v}}]}\widetilde{Z_2}) - X^{\mathsf{v}}\mathbf{A}(\nabla_{Y^{\mathsf{h}}}\widetilde{Z_1},\widetilde{Z_2}) + \mathbf{A}(\nabla_{X^{\mathsf{v}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z_1},\widetilde{Z_2})\\ &\quad + \mathbf{A}(\nabla_{Y^{\mathsf{h}}}\widetilde{Z_1},\nabla_{X^{\mathsf{v}}}\widetilde{Z_2}) - X^{\mathsf{v}}\mathbf{A}(\widetilde{Z_1},\nabla_{Y^{\mathsf{h}}}\widetilde{Z_2}) + \mathbf{A}(\nabla_{X^{\mathsf{v}}}\widetilde{Z_1},\nabla_{Y^{\mathsf{h}}}\widetilde{Z_2})\\ &\quad + \mathbf{A}(\widetilde{Z_1},\nabla_{X^{\mathsf{v}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z_2}),\end{aligned}$$

and after subtraction we get

$$\begin{aligned}&\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{A}(\widehat{X},\widehat{Y},\widetilde{Z_1},\widetilde{Z_2}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{A}(\widehat{Y},\widehat{X},\widetilde{Z_1},\widetilde{Z_2})\\ &\quad = \mathbf{A}(\nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{v}}}\widetilde{Z_1} - \nabla_{X^{\mathsf{v}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z_1} - \nabla_{[Y^{\mathsf{h}},X^{\mathsf{v}}]}\widetilde{Z_1},\widetilde{Z_2})\\ &\qquad + \mathbf{A}(\widetilde{Z_1},\nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{v}}}\widetilde{Z_2} - \nabla_{X^{\mathsf{v}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z_2} - \nabla_{[Y^{\mathsf{h}},X^{\mathsf{v}}]}\widetilde{Z_2}) =\\ &\qquad - \mathbf{A}(\mathbf{B}(\widehat{X},\widehat{Y})\widetilde{Z_1},\widetilde{Z_2}) - \mathbf{A}(\widetilde{Z_1},\mathbf{B}(\widehat{X},\widehat{Y})\widetilde{Z_2}).\end{aligned}$$

□

Proposition 3.8. *An Ehresmann connection $\mathcal{H}$ over M has vanishing Berwald curvature, if and only if, there exists a (necessarily unique) covariant derivative operator D on the base manifold M such that for any vector fields X, Y on M we have*

$$[X^{\mathsf{h}},Y^{\mathsf{v}}] = (D_XY)^{\mathsf{v}}. \tag{25}$$

Proof. The *sufficiency* of the condition is immediate: if there exists a covariant derivative operator D on M satisfying (25), then for all vector fields X, Y, Z on M we have

$$\mathbf{iB}(\widehat{X},\widehat{Y},\widehat{Z}) \overset{(18)}{=} [X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] \overset{(25)}{=} [X^{\mathsf{v}},(D_YZ)^{\mathsf{v}}] = 0,$$

since the Lie bracket of vertically lifted vector fields vanishes.

Conversely, if $\mathcal{H}$ has vanishing Berwald curvature, then for all vector fields X,Y,Z in $\mathfrak{X}(M)$,

$$[X^{\mathsf{v}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] = 0.$$

This implies that $[Y^{\mathsf{h}},Z^{\mathsf{v}}]$ is a vertical lift, so we may define a map

$$D\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M),\ (Y,Z)\mapsto D_YZ$$

by

$$(D_YZ)^{\mathsf{v}} := [Y^{\mathsf{h}},Z^{\mathsf{v}}].$$

It is easy to check, that D is a covariant derivative operator on M. For example, if $f \in C^\infty(M)$, then

$$\begin{aligned}(D_Y fZ)^{\mathsf{v}} &:= [Y^{\mathsf{h}}, (fZ)^{\mathsf{v}}] = [Y^{\mathsf{h}}, f^{\mathsf{v}} Z^{\mathsf{v}}] = (Y^{\mathsf{h}} f^{\mathsf{v}}) Z^{\mathsf{v}} + f^{\mathsf{v}} [Y^{\mathsf{h}}, Z^{\mathsf{v}}] \\ &= (Yf)^{\mathsf{v}} Z^{\mathsf{v}} + (f D_Y Z)^{\mathsf{v}} = ((Yf)Z + f D_Y Z)^{\mathsf{v}},\end{aligned}$$

hence

$$D_Y fZ = (Yf)Z + f D_Y Z.$$

Similarly,

$$\begin{aligned}(D_{fY} Z)^{\mathsf{v}} &:= \left[(fY)^{\mathsf{h}}, Z^{\mathsf{v}}\right] = [f^{\mathsf{v}} Y^{\mathsf{h}}, Z^{\mathsf{v}}] = -(Z^{\mathsf{v}} f^{\mathsf{v}}) Y^{\mathsf{h}} + f^{\mathsf{v}} [Y^{\mathsf{h}}, Z^{\mathsf{v}}] \\ &= f^{\mathsf{v}} [Y^{\mathsf{h}}, Z^{\mathsf{v}}] = (f D_Y Z)^{\mathsf{v}},\end{aligned}$$

which implies that

$$D_{fY} Z = f D_Y Z.$$

The other rules are immediate consequences of the definition of D. □

Relation (25) can also be written in the form

$$\nabla^{\mathsf{h}}_{\widehat{X}} \widehat{Y} = \widehat{D_X Y},$$

so it is reasonable to call an Ehresmann connection *h-basic* or briefly *basic*, if it has vanishing Berwald curvature: then the Christoffel symbols of the h-covariant derivative do not depend on the direction. More generally, we say that an Ehresmann connection is *weakly Berwald* if the trace of its Berwald curvature vanishes.

We shall use similar terminology for sprays. A spray will be called *Berwald*, if its associated Ehresmann connection has vanishing Berwald curvature, and will be called *weakly Berwald* if the Berwald curvature of its associated Ehresmann connection is traceless.

4. The affine curvature of an Ehresmann connection

We continue to assume that an Ehresmann connection $\mathcal{H}$ is specified over M, and consider the Berwald derivative $\nabla = (\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ determined by $\mathcal{H}$. By the *affine curvature* of $\mathcal{H}$ we mean the type $\binom{1}{3}$ tensor $\mathbf{H}$ along $\overset{\circ}{\tau}$ given by

$$\mathbf{H}(\widetilde{X}, \widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y})\widetilde{Z};\ \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \operatorname{Sec}(\overset{\circ}{\pi}).$$

This tensor was essentially introduced by L. Berwald ([5]) in terms of the classical tensor calculus and in the more specific context of an Ehresmann connection associated to a spray. So we think that it is appropriate to preserve his terminology. To indicate the meaning of the affine curvature, we remark, that if an Ehresmann connection is basic with base covariant derivative D on M, then its affine curvature may be identified with the curvature of D. More precisely, we have

$$\mathbf{i}\mathbf{H}(\widehat{X}, \widehat{Y})\widehat{Z} = (R^D(X, Y)Z)^{\mathsf{v}};\ X, Y, Z \in \mathfrak{X}(M).$$

Now we formulate and prove some basic relations found by Berwald in our setting. The first observation, roughly speaking, is that the affine curvature is just the vertical differential of the curvature of $\mathcal{H}$. The exact relation between $\mathbf{H}$ and $\mathbf{R}$ is formulated in

Lemma 4.1. *For all* $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \mathrm{Sec}(\overset{\circ}{\pi})$,

$$\mathbf{H}(\widetilde{X}, \widetilde{Y})\widetilde{Z} = \nabla^{\mathsf{v}}\mathbf{R}(\widetilde{Z}, \widetilde{X}, \widetilde{Y}). \tag{26}$$

Proof. It is enough to check that (26) is true for basic vector fields $\widehat{X}, \widehat{Y}, \widehat{Z}$ along $\overset{\circ}{\tau}$. Then, on the one hand,

$$\begin{aligned}\mathbf{H}(\widehat{X}, \widehat{Y})\widehat{Z} &:= \nabla_{X^{\mathsf{h}}}\nabla_{Y^{\mathsf{h}}}\widehat{Z} - \nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{h}}}\widehat{Z} - \nabla_{[X^{\mathsf{h}}, Y^{\mathsf{h}}]}\widehat{Z} \\ &\overset{(14)}{=} \nabla_{X^{\mathsf{h}}}\mathcal{V}[Y^{\mathsf{h}}, Z^{\mathsf{v}}] - \nabla_{Y^{\mathsf{h}}}\mathcal{V}[X^{\mathsf{h}}, Z^{\mathsf{v}}] - \nabla_{\mathbf{h}[X^{\mathsf{h}}, Y^{\mathsf{h}}]}\widehat{Z} \\ &\overset{(11),(8)}{=} \mathcal{V}[X^{\mathsf{h}}, [Y^{\mathsf{h}}, Z^{\mathsf{v}}]] - \mathcal{V}[Y^{\mathsf{h}}, [X^{\mathsf{h}}, Z^{\mathsf{v}}]] - \mathcal{V}[[X, Y]^{\mathsf{h}}, Z^{\mathsf{v}}] \\ &= \mathcal{V}([X^{\mathsf{h}}, [Y^{\mathsf{h}}, Z^{\mathsf{v}}]] + [Y^{\mathsf{h}}, [Z^{\mathsf{v}}, X^{\mathsf{h}}]] + [Z^{\mathsf{v}}, [X, Y]^{\mathsf{h}}]) \\ &= \mathcal{V}([-Z^{\mathsf{v}}, [X^{\mathsf{h}}, Y^{\mathsf{h}}]] + [Z^{\mathsf{v}}, [X, Y]^{\mathsf{h}}]) \\ &= \mathcal{V}[Z^{\mathsf{v}}, [X, Y]^{\mathsf{h}} - [X^{\mathsf{h}}, Y^{\mathsf{h}}]] = \mathcal{V}[Z^{\mathsf{v}}, \mathbf{i}\mathbf{R}(\widehat{X}, \widehat{Y})].\end{aligned}$$

On the other hand,

$$\begin{aligned}\nabla^{\mathsf{v}}\mathbf{R}(\widehat{Z}, \widehat{X}, \widehat{Y}) &= (\nabla_{Z^{\mathsf{v}}}\mathbf{R})\,(\widehat{X}, \widehat{Y}) \\ &\overset{(14)}{=} \nabla_{Z^{\mathsf{v}}}(\mathbf{R}(\widehat{X}, \widehat{Y})) \overset{(7)}{=} \mathbf{j}[Z^{\mathsf{v}}, \mathcal{H}\mathbf{R}(\widehat{X}, \widehat{Y})] = -\mathbf{j}[Z^{\mathsf{v}}, \mathcal{H}\mathcal{V}[X^{\mathsf{h}}, Y^{\mathsf{h}}]] \\ &= -\mathbf{j}[Z^{\mathsf{v}}, \mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}] + \mathbf{J}[X^{\mathsf{h}}, Y^{\mathsf{h}}]] = -\mathbf{j}[Z^{\mathsf{v}}, \mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}]].\end{aligned}$$

Now, taking into account the second relation in (4),

$$\begin{aligned}0 &= [\mathbf{J}, Z^{\mathsf{v}}]\mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}] = [\mathbf{J}\mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}], Z^{\mathsf{v}}] - \mathbf{J}[\mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}], Z^{\mathsf{v}}] \\ &= [\mathbf{v}[X^{\mathsf{h}}, Y^{\mathsf{h}}], Z^{\mathsf{v}}] - \mathbf{J}[\mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}], Z^{\mathsf{v}}] \\ &= [Z^{\mathsf{v}}, \mathbf{i}\mathbf{R}(\widehat{X}, \widehat{Y})] + \mathbf{J}[Z^{\mathsf{v}}, \mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}]] \\ &= \mathbf{i}\mathcal{V}[Z^{\mathsf{v}}, \mathbf{i}\mathbf{R}(\widehat{X}, \widehat{Y})] + \mathbf{J}[Z^{\mathsf{v}}, \mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}]],\end{aligned}$$

hence

$$\nabla^{\mathsf{v}}\mathbf{R}(\widehat{Z}, \widehat{X}, \widehat{Y}) = -\mathbf{j}[Z^{\mathsf{v}}, \mathbf{F}[X^{\mathsf{h}}, Y^{\mathsf{h}}]] = \mathcal{V}[Z^{\mathsf{v}}, \mathbf{i}\mathbf{R}(\widehat{X}, \widehat{Y})] = \mathbf{H}(\widehat{X}, \widehat{Y})\widehat{Z}. \qquad \square$$

Lemma 4.2. *If $\mathcal{H}$ is a homogeneous Ehresmann connection, then the curvature of $\mathcal{H}$ may be reproduced from the affine curvature, namely, we have*

$$\mathbf{R}(\widetilde{X}, \widetilde{Y}) = \mathbf{H}(\widetilde{X}, \widetilde{Y})\delta\ ;\ \widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\overset{\circ}{\pi}). \tag{27}$$

Proof.

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\delta := \nabla_{\mathcal{H}\widetilde{X}}\nabla_{\mathcal{H}\widetilde{Y}}\delta - \nabla_{\mathcal{H}\widetilde{Y}}\nabla_{\mathcal{H}\widetilde{X}}\delta - \nabla_{[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}]}\delta \overset{(12)}{=} -\nabla_{\mathbf{v}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}]}\delta = -\mathbf{j}[\mathbf{v}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}],\mathcal{H}\circ\delta].$$

Since $\mathcal{H}\circ\delta = S$ is a spray, we obtain that

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\delta = -\mathbf{i}^{-1}\mathbf{J}[\mathbf{JF}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}],S] \overset{(20)}{=} -\mathbf{i}^{-1}\mathbf{JF}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}] = -\mathcal{V}[\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}] = \mathbf{R}(\widetilde{X},\widetilde{Y}),$$

as we claimed. □

Corollary 4.3. *If an Ehresmann connection is homogeneous, then its curvature* $\mathbf{R}$ *is homogeneous of degree 1, i.e.,* $\nabla_C\mathbf{R} = \mathbf{R}$.

Proof. For any vector fields X, Y on M,

$$(\nabla_C\mathbf{R})(\widehat{X},\widehat{Y}) = \nabla^{\mathsf{v}}\mathbf{R}(\delta,\widehat{X},\widehat{Y}) \overset{(26)}{=} \mathbf{H}(\widehat{X},\widehat{Y})\delta \overset{(27)}{=} \mathbf{R}(\widehat{X},\widehat{Y}).$$ □

Lemma 4.4. *The affine curvature of a homogeneous Ehresmann connection is homogeneous of degree zero, i.e.,* $\nabla_C\mathbf{H} = 0$.

Proof. By a similar technique as above, we have for any vector fields X, Y, Z on M:

$$\begin{aligned}\mathbf{i}(\nabla_C\mathbf{H})(\widehat{X},\widehat{Y},\widehat{Z}) &= \mathbf{i}\nabla_C(\mathbf{H}(\widehat{X},\widehat{Y})\widehat{Z})\\ &\overset{(26)}{=} \mathbf{i}\nabla_C\nabla_{Z^{\mathsf{v}}}(\mathbf{R}(\widehat{X},\widehat{Y})) = \mathbf{i}\nabla_C\mathbf{j}[Z^{\mathsf{v}},\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y})]\\ &= \mathbf{J}[C,\mathbf{h}[Z^{\mathsf{v}},\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y})]] = \mathbf{J}[C,[Z^{\mathsf{v}},\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y})]]\\ &= -\mathbf{J}([Z^{\mathsf{v}},[\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y}),C]] + [\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y}),[C,Z^{\mathsf{v}}]])\\ &= -\mathbf{J}([\mathbf{h}[C,\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y})],Z^{\mathsf{v}}] - [\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y}),Z^{\mathsf{v}}]) = 0,\end{aligned}$$

since

$$\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y}) \overset{(4.3)}{=} \mathcal{H}\nabla_C(\mathbf{R}(\widehat{X},\widehat{Y})) = \mathcal{H}\mathbf{j}[C,\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y})] = \mathbf{h}[C,\mathcal{H}\mathbf{R}(\widehat{X},\widehat{Y})].$$ □

Lemma 4.5. *If* $\mathcal{H}$ *is a torsion-free Ehresmann connection, then its affine curvature satisfies the Bianchi identity*

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\widetilde{Z} + \mathbf{H}(\widetilde{Y},\widetilde{Z})\widetilde{X} + \mathbf{H}(\widetilde{Z},\widetilde{X})\widetilde{Y} = 0 \quad (\widetilde{X},\widetilde{Y},\widetilde{Z} \in \mathrm{Sec}(\mathring{\pi})). \tag{28}$$

Proof. In our calculations we may use basic vector fields $\widehat{X},\widehat{Y},\widehat{Z}$ again. Then, taking into account the first partial result in the proof of 4.1, we get

$$\begin{aligned}\mathbf{iH}(\widehat{X},\widehat{Y})\widehat{Z} &= [Z^{\mathsf{v}},\mathbf{iR}(\widehat{X},\widehat{Y})]\\ &= [Z^{\mathsf{v}},-\mathbf{v}[X^{\mathsf{h}},Y^{\mathsf{h}}]] = [Z^{\mathsf{v}},[X,Y]^{\mathsf{h}}] - [Z^{\mathsf{v}},[X^{\mathsf{h}},Y^{\mathsf{h}}]]\\ &= [Z^{\mathsf{v}},[X,Y]^{\mathsf{h}}] + [X^{\mathsf{h}},[Y^{\mathsf{h}},Z^{\mathsf{v}}]] + [Y^{\mathsf{h}},[Z^{\mathsf{v}},X^{\mathsf{h}}]].\end{aligned}$$

In the same way,

$$\mathbf{iH}(\widehat{Y},\widehat{Z})\widehat{X} = [X^{\mathsf{v}},[Y,Z]^{\mathsf{h}}] + [Y^{\mathsf{h}},[Z^{\mathsf{h}},X^{\mathsf{v}}]] + [Z^{\mathsf{h}},[X^{\mathsf{v}},Y^{\mathsf{h}}]],$$
$$\mathbf{iH}(\widehat{Z},\widehat{X})\widehat{Y} = [Y^{\mathsf{v}},[Z,X]^{\mathsf{h}}] + [Z^{\mathsf{h}},[X^{\mathsf{h}},Y^{\mathsf{v}}]] + [X^{\mathsf{h}},[Y^{\mathsf{v}},Z^{\mathsf{h}}]].$$

Now we add these three relations, and apply the vanishing of the torsion repeatedly:

$$\begin{aligned}\mathbf{i}(\mathbf{H}(\widehat{X},\widehat{Y})\widehat{Z} + \mathbf{H}(\widehat{Y},\widehat{Z})\widehat{X} + \mathbf{H}(\widehat{Z},\widehat{X})\widehat{Y}) &= [X^{\mathsf{h}},[Y^{\mathsf{h}},Z^{\mathsf{v}}] - [Z^{\mathsf{h}},Y^{\mathsf{v}}]] \\ &+ [Y^{\mathsf{h}},[Z^{\mathsf{v}},X^{\mathsf{h}}] - [X^{\mathsf{v}},Z^{\mathsf{h}}]] + [Z^{\mathsf{h}},[X^{\mathsf{h}},Y^{\mathsf{v}}] - [Y^{\mathsf{h}},X^{\mathsf{v}}]] + [X^{\mathsf{v}},[Y,Z]^{\mathsf{h}}] \\ &+ [Y^{\mathsf{v}},[Z,X]^{\mathsf{h}}] + [Z^{\mathsf{v}},[X,Y]^{\mathsf{h}}] \\ &= [X^{\mathsf{h}},[Y,Z]^{\mathsf{v}}] - [[Y,Z]^{\mathsf{h}},X^{\mathsf{v}}] + [Y^{\mathsf{h}},[Z,X]^{\mathsf{v}}] - [[Z,X]^{\mathsf{h}},Y^{\mathsf{v}}] + [Z^{\mathsf{h}},[X,Y]^{\mathsf{v}}] \\ &- [[X,Y]^{\mathsf{h}},Z^{\mathsf{v}}] = ([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]])^{\mathsf{v}} = 0. \qquad \square\end{aligned}$$

Now we suppose that the Ehresmann connection $\mathcal{H}$ is associated to a spray S. Then, as we have already mentioned, $\mathcal{H}$ is homogeneous and torsion-free. The type $\binom{1}{1}$ tensor field $\mathbf{K}$ along $\overset{\circ}{\tau}$ defined by

$$\text{(29)} \qquad \mathbf{K}(\widetilde{X}) := \mathcal{V}[S,\mathcal{H}\widetilde{X}], \ \ \widetilde{X} \in \operatorname{Sec}(\overset{\circ}{\pi})$$

is said to be the *affine deviation tensor* (L. Berwald [5]) or the *Jacobi endomorphism* (W. Sarlet et al. [12]) of the spray S, or of the Ehresmann connection associated to S. The homogeneity of S implies that $S = \mathcal{H}\delta$, so it follows that

$$\text{(30)} \qquad \mathbf{K}(\widetilde{X}) = \mathcal{V}[\mathcal{H}\delta,\mathcal{H}\widetilde{X}] = -\mathbf{R}(\delta,\widetilde{X}) = \mathbf{R}(\widetilde{X},\delta).$$

This means that the affine deviation can immediately be obtained from the curvature of the Ehresmann connection. The converse is also true:

Proposition 4.6. *Let S be a spray over M, and let $\mathcal{H}$ be the Ehresmann connection associated to S. Then the curvature and the affine deviation of $\mathcal{H}$ are related by*

$$\mathbf{R}(\widetilde{X},\widetilde{Y}) = \frac{1}{3}(\nabla^{\mathsf{v}}\mathbf{K}(\widetilde{Y},\widetilde{X}) - \nabla^{\mathsf{v}}\mathbf{K}(\widetilde{X},\widetilde{Y})) \ ; \ \widetilde{X},\widetilde{Y} \in \operatorname{Sec}(\overset{\circ}{\pi}).$$

Proof. Let X, Y be vector fields on M. Then

$$\begin{aligned}\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{X}) &= (\nabla_{Y^{\mathsf{v}}}\mathbf{K})(\widehat{X}) = \nabla_{Y^{\mathsf{v}}}(\mathbf{K}(\widehat{X})) \\ &\overset{(30)}{=} \nabla_{Y^{\mathsf{v}}}(\mathbf{R}(\widehat{X},\delta)) = (\nabla_{Y^{\mathsf{v}}}\mathbf{R})(\widehat{X},\delta) + \mathbf{R}\left(\nabla_{Y^{\mathsf{v}}}\widehat{X},\delta\right) + \mathbf{R}\left(\widehat{X},\nabla_{Y^{\mathsf{v}}}\delta\right) \\ &= \nabla^{\mathsf{v}}\mathbf{R}(\widehat{Y},\widehat{X},\delta) + \mathbf{R}(\widehat{X},\widehat{Y}).\end{aligned}$$

Similarly,

$$\nabla^{\mathsf{v}}\mathbf{K}(\widehat{X},\widehat{Y}) = \nabla^{\mathsf{v}}\mathbf{R}(\widehat{X},\widehat{Y},\delta) + \mathbf{R}(\widehat{Y},\widehat{X}),$$

therefore

$$\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{X}) - \nabla^{\mathsf{v}}\mathbf{K}(\widehat{X},\widehat{Y}) = \nabla^{\mathsf{v}}\mathbf{R}(\widehat{Y},\widehat{X},\delta) - \nabla^{\mathsf{v}}\mathbf{R}(\widehat{X},\widehat{Y},\delta) + 2\mathbf{R}(\widehat{X},\widehat{Y})$$
$$\overset{(26)}{=} \mathbf{H}(\widehat{X},\delta)\widehat{Y} - \mathbf{H}(\widehat{Y},\delta)\widehat{X} + 2\mathbf{R}(\widehat{X},\widehat{Y}) = \mathbf{H}(\widehat{X},\delta)\widehat{Y} + \mathbf{H}(\delta,\widehat{Y})\widehat{X} + 2\mathbf{R}(\widehat{X},\widehat{Y})$$
$$\overset{4.5}{=} -\mathbf{H}(\widehat{Y},\widehat{X})\delta + 2\mathbf{R}(\widehat{X},\widehat{Y}) = \mathbf{H}(\widehat{X},\widehat{Y})\delta + 2\mathbf{R}(\widehat{X},\widehat{Y}) \overset{4.2}{=} 3\mathbf{R}(\widehat{X},\widehat{Y}). \qquad \square$$

Remark. Historically, Berwald's starting point in his above mentioned, famous posthumous paper was a SODE of form

$$(x^i)'' + 2G^i(x,x') = 0,\ i \in \{1,\dots,n\}, \tag{31}$$

where

$$G^i \in C^1(\tau^{-1}(\mathcal{U})) \cap C^\infty(\overset{\circ}{\tau}{}^{-1}(\mathcal{U})),\ CG^i = 2G^i$$

($\mathcal{U} \subset M$ is a coordinate neighborhood).

As a first step, Berwald deduces the following 'equation of affine deviation':

$$\frac{D^2\xi^i}{ds} + K^i_j\left(x, \frac{dx}{ds}\right)\xi^j = 0,\ i \in \{1,\dots,n\},$$

where

$$\frac{D\xi^i}{ds} := \frac{d\xi^i}{ds} + G^i_r\xi^r,\ G^i_r := \frac{\partial G^i}{\partial y^r},$$

and the functions

$$K^i_j := 2\frac{\partial G^i}{\partial x^j} - \frac{\partial G^i_j}{\partial x^r}y^r + 2G^i_{jr}G^r - G^i_rG^r_j,\ (i,j \in \{1,\dots,n\})$$

are the components of a type $\binom{1}{1}$ tensor, called *affinen Abweichungstensor* by Berwald. It may easily be checked that our tensor $\mathbf{K}$ is indeed an intrinsic form of the tensor obtained by him.

In the second step, Berwald introduces the 'Grundtensor der affinen Krmmung', giving its components by

$$K^i_{jk} := \frac{1}{3}\left(\frac{\partial K^i_k}{\partial y^j} - \frac{\partial K^i_j}{\partial y^k}\right).$$

Proposition 4.6 shows that this is just the curvature of the Ehresmann connection which may be associated to the SODE (31).

Finally, in the third step, Berwald defines the 'affine Krmmungstensor' by its components

$$K^i_{hjk} := \frac{\partial K^i_{jk}}{\partial y^h}.$$

In view of Lemma 4.1, this is just our tensor $\mathbf{H}$.

Lemma 4.7 (hh-Ricci formulae for sections and 1-forms)**.** *If $\mathcal{H}$ is a torsion-free Ehresmann connection, then for any section $\widetilde{Z} \in \mathrm{Sec}(\mathring{\pi})$ and 1-form $\widetilde{\alpha} \in \mathcal{T}_1^0(\mathring{\pi})$ we have*

$$\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{Z}(\widehat{X},\widehat{Y}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{Z}(\widehat{Y},\widehat{X}) = \mathbf{H}(\widehat{X},\widehat{Y})\widetilde{Z} - \nabla^{\mathsf{v}}\widetilde{Z}(\mathbf{R}(\widehat{X},\widehat{Y})), \tag{32}$$

$$\begin{aligned}&\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{\alpha}(\widehat{X},\widehat{Y},\widehat{Z}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{\alpha}(\widehat{Y},\widehat{X},\widehat{Z})\\ &\quad = -\widetilde{\alpha}(\mathbf{H}(\widehat{X},\widehat{Y})\widehat{Z}) - \nabla^{\mathsf{v}}\widetilde{\alpha}(\mathbf{R}(\widehat{X},\widehat{Y}),\widehat{Z}),\ (X,Y,Z \in \mathfrak{X}(M)).\end{aligned} \tag{33}$$

Proof.

$$\begin{aligned}\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{Z}(\widehat{X},\widehat{Y}) &= \nabla_{X^{\mathsf{h}}}(\nabla^{\mathsf{h}}\widetilde{Z})(\widehat{Y}) = \nabla_{X^{\mathsf{h}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z} - \nabla^{\mathsf{h}}\widetilde{Z}\left(\nabla_{X^{\mathsf{h}}}\widehat{Y}\right)\\ &= \nabla_{X^{\mathsf{h}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z} - \nabla_{\mathcal{H}\nabla_{X^{\mathsf{h}}}\widehat{Y}}\widetilde{Z} = \nabla_{X^{\mathsf{h}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z} - \nabla_{\mathcal{H}\mathcal{V}[X^{\mathsf{h}},Y^{\mathsf{v}}]}\widetilde{Z},\end{aligned}$$

and, similarly,

$$\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{Z}(\widehat{Y},\widehat{X}) = \nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{h}}}\widetilde{Z} - \nabla_{\mathcal{H}\mathcal{V}[Y^{\mathsf{h}},X^{\mathsf{v}}]}\widetilde{Z}.$$

Hence

$$\begin{aligned}&\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{Z}(\widehat{X},\widehat{Y}) - \nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\widetilde{Z}(\widehat{Y},\widehat{X})\\ &= \nabla_{X^{\mathsf{h}}}\nabla_{Y^{\mathsf{h}}}\widetilde{Z} - \nabla_{Y^{\mathsf{h}}}\nabla_{X^{\mathsf{h}}}\widetilde{Z} + \nabla_{\mathcal{H}\mathcal{V}([Y^{\mathsf{h}},X^{\mathsf{v}}]-[X^{\mathsf{h}},Y^{\mathsf{v}}])}\widetilde{Z}\\ &= \mathbf{H}(\widehat{X},\widehat{Y})\widetilde{Z} + \nabla_{[X^{\mathsf{h}},Y^{\mathsf{h}}]-(\mathbf{F}+\mathbf{J})[X,Y]^{\mathsf{v}}}\widetilde{Z}\\ &= \mathbf{H}(\widehat{X},\widehat{Y})\widetilde{Z} + \nabla_{[X^{\mathsf{h}},Y^{\mathsf{h}}]-[X,Y]^{\mathsf{h}}}\widetilde{Z} = \mathbf{H}(\widehat{X},\widehat{Y})\widetilde{Z} - \nabla_{\mathbf{iR}(\widehat{X},\widehat{Y})}\widetilde{Z}\\ &= \mathbf{H}(\widehat{X},\widehat{Y})\widetilde{Z} - \nabla^{\mathsf{v}}\widetilde{Z}(\mathbf{R}(\widehat{X},\widehat{Y})),\end{aligned}$$

which proves relation (32). Relation (33) can be checked in the same way. □

5. Projectively related sprays

We recall (for details, see [16], [23], [24], [21]) that two sprays S and $\overline{S}$ over M are said to be (pointwise) *projectively related*, if there exists a function $P\colon TM \to \mathbb{R}$, C^1 on TM, smooth on $\mathring{T}M$, such that

$$\overline{S} = S - 2PC. \tag{34}$$

The *projective factor* P in (34) is necessarily positive-homogeneous of degree 1, i.e., $CP = P$. If $\mathbf{A}$ is a geometric object associated to S, then we denote by $\overline{\mathbf{A}}$ the corresponding geometric object determined by $\overline{S}$. The following relations are well-known ([21]), and may easily be checked. If $\mathcal{H}$ is the Ehresmann connection associated to S, then

$$\overline{\mathcal{H}} = \mathcal{H} - P\mathbf{i} - \nabla^{\mathsf{v}}P \otimes C, \tag{35}$$

$$\overline{\mathbf{h}} = \mathbf{h} - P\mathbf{J} - (\nabla^{\mathsf{v}}P \circ \mathbf{j}) \otimes C = \mathbf{h} - P\mathbf{J} - d_{\mathbf{J}}P \otimes C, \tag{36}$$

$$X^{\bar{\mathsf{h}}} = X^{\mathsf{h}} - PX^{\mathsf{v}} - (X^{\mathsf{v}}P)C \quad (X \in \mathfrak{X}(M)), \tag{37}$$

$$\overline{\mathcal{V}} = \mathcal{V} + P\mathbf{j} + (\nabla^{\mathsf{v}}P \circ \mathbf{j}) \otimes \delta = \mathcal{V} + P\mathbf{j} - d_{\mathbf{J}}P \otimes \delta. \tag{38}$$

We also have the less immediate

Lemma 5.1.

$$\overline{\nabla}^{\bar{\mathsf{h}}} = \nabla^{\mathsf{h}} - P\nabla^{\mathsf{v}} - \nabla^{\mathsf{v}}P \otimes \nabla_C + \nabla^{\mathsf{v}}P \odot \mathbf{1} + \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P \otimes \delta, \tag{39}$$

where the symbol $\odot$ denotes symmetric product (without any numerical factor), and $\mathbf{1} \in \mathcal{T}_1^1(\pi)$ is the unit tensor.

Proof. Let $\widetilde{X}$ and $\widetilde{Y}$ be vector fields along $\overset{\circ}{\tau}$. Then

$$\begin{aligned}
\overline{\nabla}_{\overline{\mathcal{H}}\widetilde{X}}\widetilde{Y} &= \overline{\mathcal{V}}[\overline{\mathcal{H}}\widetilde{X}, \mathbf{i}\widetilde{Y}] \overset{(35)}{=} \overline{\mathcal{V}}[\mathcal{H}\widetilde{X} - P\mathbf{i}\widetilde{X} - \nabla^{\mathsf{v}}P(\widetilde{X})C, \mathbf{i}\widetilde{Y}] \\
&= \overline{\mathcal{V}}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] + \overline{\mathcal{V}}\left(\mathbf{i}\widetilde{Y}(P)\mathbf{i}\widetilde{X} - P[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}]\right) \\
&\quad + \overline{\mathcal{V}}\left(\mathbf{i}\widetilde{Y}(\mathbf{i}\widetilde{X}P)C - \mathbf{i}\widetilde{X}(P)[C, \mathbf{i}\widetilde{Y}]\right) \\
&\overset{(38)}{=} \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}] - P\mathbf{j}[\mathbf{i}\widetilde{Y}, \mathcal{H}\widetilde{X}] - (\mathbf{J}[\mathbf{i}\widetilde{Y}, \mathcal{H}\widetilde{X}]P)\delta + \mathbf{i}\widetilde{Y}(P)\widetilde{X} \\
&\quad - P\mathbf{i}^{-1}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}] + \mathbf{i}\widetilde{Y}(\mathbf{i}\widetilde{X}P)\delta - \mathbf{i}\widetilde{X}(P)\mathbf{i}^{-1}[\mathbf{i}\delta, \mathbf{i}\widetilde{Y}].
\end{aligned}$$

An easy calculation shows that

$$\begin{aligned}
\mathbf{i}^{-1}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}] &= \nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} - \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X}, \\
\mathbf{i}^{-1}[\mathbf{i}\delta, \mathbf{i}\widetilde{Y}] &= \nabla_C\widetilde{Y} - \nabla_{\mathbf{i}\widetilde{Y}}\delta = \nabla_C\widetilde{Y} - \widetilde{Y},
\end{aligned}$$

so we obtain

$$\begin{aligned}
\overline{\nabla}^{\bar{\mathsf{h}}}(\widetilde{X}, \widetilde{Y}) &= \overline{\nabla}_{\overline{\mathcal{H}}\widetilde{X}}\widetilde{Y} = \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - P\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} - (\mathbf{i}\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X})P\delta + \mathbf{i}\widetilde{Y}(P)\widetilde{X} \\
&\quad - P\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y} + P\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} + (\mathbf{i}\widetilde{Y}(\mathbf{i}\widetilde{X})P)\delta - \mathbf{i}\widetilde{X}(P)\nabla_C\widetilde{Y} + \mathbf{i}\widetilde{X}(P)\widetilde{Y} \\
&= \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - P\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} - \nabla^{\mathsf{v}}P(\widetilde{X})\nabla_C\widetilde{Y} + \nabla^{\mathsf{v}}P(\widetilde{X})\widetilde{Y} + \nabla^{\mathsf{v}}P(\widetilde{Y})\widetilde{X} \\
&\quad - \left(\mathbf{i}\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X}\right)P\delta + \left(\mathbf{i}\nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathbf{i}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y}\right)P\delta + (\mathbf{i}\widetilde{X}(\mathbf{i}\widetilde{Y})P)\delta \\
&= (\nabla^{\mathsf{h}} - P\nabla^{\mathsf{v}} - \nabla^{\mathsf{v}}P \otimes \nabla_C + \nabla^{\mathsf{v}}P \odot \mathbf{1} + \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}P \otimes \delta)(\widetilde{X}, \widetilde{Y}). \quad \square
\end{aligned}$$

Corollary 5.2. *For all vector fields X, Y on M,*

$$\overline{\nabla}_{X^{\bar{\mathsf{h}}}}\widehat{Y} = \nabla_{X^{\mathsf{h}}}\widehat{Y} + (X^{\mathsf{v}}P)\widehat{Y} + (Y^{\mathsf{v}}P)\widehat{X} + X^{\mathsf{v}}(Y^{\mathsf{v}}P)\delta, \tag{40}$$

$$\overline{\nabla}_{\overline{S}}\widehat{Y} = \nabla_S\widehat{Y} + P\widehat{Y} + (Y^{\mathsf{v}}P)\delta. \tag{41}$$

Proof. Since

$$\begin{aligned}
\nabla^{\mathsf{v}}(\widehat{X}, \widehat{Y}) &= \nabla_{X^{\mathsf{v}}}\widehat{Y} = 0, \\
\nabla_C\widehat{Y} &= \mathbf{j}[C, Y^{\mathsf{v}}] = -\mathbf{j}Y^{\mathsf{v}} = 0,
\end{aligned}$$

for basic vector fields $\widehat{X}, \widehat{Y}$ relation (39) leads to (40). As to the second relation, observe that

$$\overline{\mathbf{h}}\overline{S} = (\mathbf{h} - P\mathbf{J} - (\nabla^{\mathsf{v}} P \circ \mathbf{j})C)(S - 2PC) = S - PC - (CP)C = S - 2PC = \overline{S}.$$

Hence

$$\begin{aligned}\overline{\nabla}_{\overline{S}}\widehat{Y} = \overline{\nabla}_{\overline{\mathbf{h}}\,\overline{S}}\widehat{Y} = \overline{\nabla}^{\overline{\mathsf{h}}}_{\delta}\widehat{Y} &\overset{(39)}{=} \nabla^{\mathsf{h}}_{\delta}\widehat{Y} - P\nabla^{\mathsf{v}}_{\delta}\widehat{Y} - \nabla^{\mathsf{v}} P(\delta)\nabla_C\widehat{Y} \\ &+ \nabla^{\mathsf{v}} P(\delta)\widehat{Y} + \nabla^{\mathsf{v}} P(\widehat{Y})\delta + \nabla^{\mathsf{v}}\nabla^{\mathsf{v}} P(\delta, \widehat{Y})\widehat{Y} = \nabla_S\widehat{Y} + P\widehat{Y} + (Y^{\mathsf{v}} P)\delta,\end{aligned}$$

since

$$\begin{aligned}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} P(\delta, \widehat{Y}) &= \nabla_C(\nabla^{\mathsf{v}} P)(\widehat{Y}) = C(Y^{\mathsf{v}} P) = [C, Y^{\mathsf{v}}]P + Y^{\mathsf{v}}(CP) \\ &= -Y^{\mathsf{v}} P + Y^{\mathsf{v}} P = 0.\end{aligned}$$

□

It was shown in [21], that the Berwald curvatures of S and $\overline{S}$ and their traces are related by

$$\overline{\mathbf{B}} = \mathbf{B} - \nabla^{\mathsf{v}}\nabla^{\mathsf{v}} P \odot \mathbf{1} - \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} P \otimes \delta \tag{42}$$

and

$$\mathrm{tr}\overline{\mathbf{B}} = \mathrm{tr}\mathbf{B} - (n+1)\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} P, \tag{43}$$

respectively.

From relation (43) it follows at once that the trace of the Berwald curvature is a projective invariant, if and only if, the projective factor satisfies the PDE

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} P = 0. \tag{44}$$

However, *this relation gives also the criterion of the projective invariance of the Berwald curvature.*

Indeed, if P satisfies (44), then (42) yields $\overline{\mathbf{B}} = \mathbf{B}$. Conversely, if $\overline{\mathbf{B}} = \mathbf{B}$, then $\mathrm{tr}\overline{\mathbf{B}} = \mathrm{tr}\mathbf{B}$, and (43) implies that $\nabla^{\mathsf{v}}\nabla^{\mathsf{v}} P = 0$.

The solutions of this PDE may be described easily, so we obtain

Proposition 5.3. *The Berwald curvature of a spray S is invariant under a projective change $\overline{S} = S - 2PC$, if and only if, the projective factor is of form*

$$P = i_\xi \alpha^{\mathsf{v}} =: \overline{\alpha},\ \alpha \in \mathfrak{X}^*(M), \tag{45}$$

where ξ is an arbitrary second-order vector field over M.

Proof. First we check that the functions given by (45) solve (44). To do this, observe that for any vector field Y on M,

$$Y^{\mathsf{v}}\overline{\alpha} = (\alpha(Y))^{\mathsf{v}}. \tag{46}$$

Indeed,

$$Y^{\mathsf{v}}\overline{\alpha} = Y^{\mathsf{v}} i_\xi \alpha^{\mathsf{v}} = \mathcal{L}_{Y^{\mathsf{v}}} i_\xi \alpha^{\mathsf{v}} = i_\xi \mathcal{L}_{Y^{\mathsf{v}}}\alpha^{\mathsf{v}} + i_{[Y^{\mathsf{v}}, \xi]}\alpha^{\mathsf{v}}.$$

The first term on the right-hand side vanishes, since for any vector field X on M we have

$$(\mathcal{L}_{Y^{\mathsf{v}}}\alpha^{\mathsf{v}})(X^{\mathsf{v}}) = Y^{\mathsf{v}}(\alpha^{\mathsf{v}}(X^{\mathsf{v}})) - \alpha^{\mathsf{v}}([Y^{\mathsf{v}}, X^{\mathsf{v}}]) = 0,$$

$$(\mathcal{L}_{Y^\mathsf{v}}\alpha^\mathsf{v})(X^\mathsf{c}) = Y^\mathsf{v}(\alpha^\mathsf{v}(X^\mathsf{c})) - \alpha^\mathsf{v}([Y^\mathsf{v}, X^\mathsf{c}]) = Y^\mathsf{v}(\alpha(X))^\mathsf{v} - \alpha^\mathsf{v}([Y,X]^\mathsf{v}) = 0.$$

As for the second term, it is known (see e.g. [19], 3.2, Corollary) that

$$[Y^\mathsf{v}, \xi] = Y^\mathsf{c} + \eta,\ \ \eta \in \mathfrak{X}^\mathsf{v}(TM),$$

therefore

$$Y^\mathsf{v}\overline{\alpha} = \alpha^\mathsf{v}\left([Y^\mathsf{v}, \xi]\right) = \alpha^\mathsf{v}(Y^\mathsf{c}) + \alpha^\mathsf{v}(\eta) = (\alpha(Y))^\mathsf{v},$$

as we claimed.

Now, for any vector fields X, Y on M,

$$\nabla^\mathsf{v}\nabla^\mathsf{v}\overline{\alpha}(\widehat{X}, \widehat{Y}) = \left(\nabla_{X^\mathsf{v}}\nabla^\mathsf{v}\overline{\alpha}\right)(\widehat{Y}) = X^\mathsf{v}(\nabla^\mathsf{v}\overline{\alpha}(\widehat{Y})) = X^\mathsf{v}(Y^\mathsf{v}\overline{\alpha}) \overset{(46)}{=} X^\mathsf{v}(\alpha(Y))^\mathsf{v} = 0,$$

therefore the functions $\overline{\alpha}, \alpha \in \mathfrak{X}^*(M)$ solve our PDE (44). We show that these solutions satisfy the homogeneity condition $C\overline{\alpha} = \overline{\alpha}$.

We may suppose that ξ is homogeneous of degree two, i.e., $[C, \xi] = \xi$, since the definition of $\overline{\alpha}$ does not depend on the choice of ξ. Then we obtain

$$C\overline{\alpha} = \mathcal{L}_C i_\xi \alpha^\mathsf{v} = i_\xi \mathcal{L}_C \alpha^\mathsf{v} - i_{[\xi, C]}\alpha^\mathsf{v} = i_{[C,\xi]}\alpha^\mathsf{v} = i_\xi\alpha^\mathsf{v} = \overline{\alpha},$$

since $\mathcal{L}_C\alpha^\mathsf{v} = 0$.

Conversely, if $\nabla^\mathsf{v}\nabla^\mathsf{v}P = 0$, then for all $X, Y \in \mathfrak{X}(M)$,

$$0 = \nabla^\mathsf{v}\nabla^\mathsf{v}P(\widehat{X}, \widehat{Y}) = (\nabla_{X^\mathsf{v}}(\nabla^\mathsf{v}P))(\widehat{Y}) = X^\mathsf{v}(Y^\mathsf{v}P).$$

Then $Y^\mathsf{v}P$ is the vertical lift of a smooth function on M, therefore it is of form

$$Y^\mathsf{v}P = (\alpha(Y))^\mathsf{v} \overset{(46)}{=} Y^\mathsf{v}\overline{\alpha},\ \ \alpha \in \mathfrak{X}^*(M).$$

Since Y is arbitrary, this implies that

$$P = \overline{\alpha} + f^\mathsf{v},\ \ f \in C^\infty(M).$$

However, the homogeneity condition $CP = P$ forces that $f = 0$, since $C\overline{\alpha} = \overline{\alpha}$ and $Cf^\mathsf{v} = 0$. □

Corollary 5.4. *A Berwald or a weakly Berwald spray remains of that type under a projective change, if and only if, the projective factor is given by (45).*

6. Basic objects associated to a Finsler function

By a *Finsler function* over M we mean a function $F\colon TM \to \mathbb{R}$ satisfying the following conditions:

- (F_1) F is smooth on $\overset{\circ}{T}M$.
- (F_2) *F is positive-homogeneous of degree 1* in the sense that for each non-negative real number λ and each vector $v \in TM$, we have

$$F(\lambda v) = \lambda F(v).$$

(F_3) The *metric tensor*

$$g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 \in \mathcal{T}_2^0(\mathring{\pi})$$

is (fibrewise) non-degenerate.

A *Finsler manifold* is a pair (M, F) consisting of a manifold M and a Finsler function on its tangent manifold.

By (F_1) and (F_2), F is continuous on TM and identically zero on $\sigma(M)$. The function $E := \frac{1}{2}F^2$ is called the *energy function* of the Finsler manifold (M, F). It is continuous on TM, smooth on $\mathring{T}M$, and also identically zero on $\sigma(M)$. By (F_2), E satisfies

$$E(\lambda v) = \lambda^2 E(v)$$

for all $v \in TM$ and non-negative $\lambda \in \mathbb{R}$, i.e., E is positive-homogeneous of degree 2. Over $\mathring{T}M$ this holds, if and only if, $CE = 2E$. It may be shown (see e.g. [25]) that, actually, *E is of class C^1 on TM.*

For any vector fields X, Y on M we have

$$g(\widehat{X}, \widehat{Y}) = X^{\mathsf{v}}(Y^{\mathsf{v}}E), \tag{47}$$

from which it follows immediately that g is *symmetric*. More generally, if $\widetilde{X}$ and $\widetilde{Y}$ are in $\mathrm{Sec}(\mathring{\pi})$, then

$$g(\widetilde{X}, \widetilde{Y}) = (\mathbf{i}\widetilde{X})(\mathbf{i}\widetilde{Y})E - (\mathbf{i}\nabla_{\mathbf{i}\widetilde{X}}\widetilde{Y})E = (\mathbf{i}\widetilde{X})(\mathbf{i}\widetilde{Y})E - \mathbf{J}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]E, \tag{48}$$

where $\mathcal{H}$ is an arbitrary Ehresmann connection over M. In particular, we get

$$g(\delta, \delta) = 2E. \tag{49}$$

A further elementary property of the metric tensor is that it is homogeneous of degree 0, i.e.,

$$\nabla^{\mathsf{v}}_{\delta}g = \nabla_C g = 0. \tag{50}$$

Indeed, for any vector fields X, Y on M,

$$\begin{aligned}(\nabla_C g)(\widehat{X}, \widehat{Y}) &= Cg(\widehat{X}, \widehat{Y}) \overset{(47)}{=} C(X^{\mathsf{v}}(Y^{\mathsf{v}}E)) = [C, X^{\mathsf{v}}](Y^{\mathsf{v}}E) + X^{\mathsf{v}}(C(Y^{\mathsf{v}}E)) \\ &= -X^{\mathsf{v}}Y^{\mathsf{v}}E + X^{\mathsf{v}}([C, Y^{\mathsf{v}}]E + Y^{\mathsf{v}}(CE)) \\ &= -2X^{\mathsf{v}}(Y^{\mathsf{v}}E) + 2X^{\mathsf{v}}(Y^{\mathsf{v}}E) = 0.\end{aligned}$$

A Finsler manifold (M, F) is said to be *positive definite* if the condition

(F_4) $F(v) > 0$ whenever $v \in \mathring{T}M$

is also satisfied. It may be shown ([11]) that in this case the metric tensor is (fibrewise) positive definite.

The type $\binom{0}{3}$ tensor

$$\mathcal{C}_\flat := \frac{1}{2}\nabla^{\mathsf{v}}g = \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E \tag{51}$$

is said to be the *Cartan tensor* of the Finsler manifold (M, F). It may easily be seen that $\mathcal{C}_\flat$ is *totally symmetric*: for any sections $\widetilde{X}_1$, $\widetilde{X}_2$, $\widetilde{X}_3$ in $\mathrm{Sec}(\overset{\circ}{\pi})$ and any permutation $\sigma\colon \{1,2,3\} \to \{1,2,3\}$ we have

$$\mathcal{C}_\flat(\widetilde{X}_{\sigma(1)}, \widetilde{X}_{\sigma(2)}, \widetilde{X}_{\sigma(3)}) = \mathcal{C}_\flat(\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3).$$

Since g is homogeneous of degree 0, it follows that $\mathcal{C}_\flat$ is homogeneous of degree -1, i.e.,

$$\nabla^{\mathsf{v}}_{\delta}\mathcal{C}_\flat = \nabla_C \mathcal{C}_\flat = -\mathcal{C}_\flat. \tag{52}$$

As a consequence of the total symmetry of $\mathcal{C}_\flat$ and the 0-homogeneity of g we get

$$\delta \in \left\{\widetilde{X}, \widetilde{Y}, \widetilde{Z}\right\} \;\Rightarrow\; \mathcal{C}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = 0. \tag{53}$$

Indeed,

$$2\mathcal{C}_\flat(\delta, \widetilde{Y}, \widetilde{Z}) = \nabla^{\mathsf{v}} g(\delta, \widetilde{Y}, \widetilde{Z}) = (\nabla_C g)\,(\widetilde{Y}, \widetilde{Z}) = 0.$$

It is also known (see e.g. [25] again) that the following assertions are equivalent for a positive definite Finsler manifold (M, F):

(i) *The energy function E of (M, F) is of class C^2 (and hence smooth) on TM.*

(ii) *E is the norm associated to a Riemannian metric on M.*

(iii) *There exists a Riemannian metric γ on M, such that*

$$g(\widehat{X}, \widehat{Y}) = \gamma(X, Y) \circ \tau,$$

for all vector fields $X, Y \in \mathfrak{X}(M)$.

(iv) *The Cartan tensor of (M, F) vanishes.*

The 1-form

$$\theta\colon \widetilde{X} \in \mathrm{Sec}(\overset{\circ}{\pi}) \mapsto \theta(\widetilde{X}) := g(\widetilde{X}, \delta) \in C^\infty(\overset{\circ}{T}M)$$

is said to be the *canonical 1-form* or *Hilbert 1-form* of (M, F). It may be seen immediately that

$$\theta = F\nabla^{\mathsf{v}} F = \nabla^{\mathsf{v}} E. \tag{54}$$

The section

$$\ell := \frac{1}{F}\delta$$

is traditionally called the *normalized support element field* of (M, F). Its dual form is

$$\ell_\flat := \frac{1}{F}\theta = \nabla^{\mathsf{v}} F \tag{55}$$

since

$$\ell_\flat(\ell) = \frac{1}{F^2}\theta(\delta) = \frac{1}{F^2} g(\delta, \delta) \overset{(49)}{=} \frac{1}{2E} \cdot 2E = 1.$$

By the *angular metric tensor* of (M, F) we mean the type $\binom{0}{2}$ tensor

$$\eta := g - \ell_\flat \otimes \ell_\flat = g - \nabla^{\mathsf{v}} F \otimes \nabla^{\mathsf{v}} F \tag{56}$$

along $\overset{\circ}{\tau}$. We obtain:

$$\frac{1}{F}\eta = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F. \tag{57}$$

Indeed, for any vector fields X, Y, Z on M we have

$$\begin{aligned}\frac{1}{F}\eta(\widehat{X},\widehat{Y}) &= \frac{1}{F}\left(g(\widehat{X},\widehat{Y}) - \nabla^{\mathsf{v}}F(\widehat{X})\nabla^{\mathsf{v}}F(\widehat{Y})\right)\\ &\overset{(47)}{=} \frac{1}{F}\left(\frac{1}{2}X^{\mathsf{v}}(Y^{\mathsf{v}}F^2) - (X^{\mathsf{v}}F)(Y^{\mathsf{v}}F)\right)\\ &= \frac{1}{F}\left(X^{\mathsf{v}}((Y^{\mathsf{v}}F)F) - (X^{\mathsf{v}}F)(Y^{\mathsf{v}}F)\right) = X^{\mathsf{v}}(Y^{\mathsf{v}}F) = (\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F)(\widehat{X},\widehat{Y}).\end{aligned}$$

We note that (55) and (57) imply

$$\nabla^{\mathsf{v}}\ell_\flat = \frac{1}{F}\eta. \tag{58}$$

Lemma 6.1. *If (M,F) is a Finsler manifold, then for any vector fields X, Y, Z on M we have*

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F(\widehat{X},\widehat{Y},\widehat{Z}) = \frac{2}{F}\mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) - \frac{1}{F^2}\underset{(\widehat{X},\widehat{Y},\widehat{Z})}{\mathfrak{S}}\ell_\flat\otimes\eta(\widehat{X},\widehat{Y},\widehat{Z}) \tag{59}$$

where the symbol $\underset{(\widehat{X},\widehat{Y},\widehat{Z})}{\mathfrak{S}}$ *means cyclic sum over* $\widehat{X},\widehat{Y},\widehat{Z}$*.*

Proof.

$$\begin{aligned}&\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F(\widehat{X},\widehat{Y},\widehat{Z}) \overset{(57)}{=} \nabla_{X^{\mathsf{v}}}\left(\frac{1}{F}\eta\right)(\widehat{Y},\widehat{Z})\\ &\quad= X^{\mathsf{v}}\left(\frac{1}{F}\right)\eta(\widehat{Y},\widehat{Z}) + \frac{1}{F}\nabla_{X^{\mathsf{v}}}(g-\ell_\flat\otimes\ell_\flat)(\widehat{Y},\widehat{Z})\\ &\quad= -\frac{1}{F^2}(X^{\mathsf{v}}F)\eta(\widehat{Y},\widehat{Z}) + \frac{1}{F}\left(\nabla_{X^{\mathsf{v}}}g\right)(\widehat{Y},\widehat{Z}) - \frac{1}{F}\left(\nabla_{X^{\mathsf{v}}}\ell_\flat\right)(\widehat{Y})\ell_\flat(\widehat{Z})\\ &\qquad - \frac{1}{F}\ell_\flat(\widehat{Y})\left(\nabla_{X^{\mathsf{v}}}\ell_\flat\right)(\widehat{Z}) = \frac{1}{F}\nabla^{\mathsf{v}}g(\widehat{X},\widehat{Y},\widehat{Z}) - \frac{1}{F^2}\nabla^{\mathsf{v}}F(\widehat{X})\eta(\widehat{Y},\widehat{Z})\\ &\qquad - \frac{1}{F}\left(\nabla^{\mathsf{v}}\ell_\flat\right)(\widehat{X},\widehat{Y})\ell_\flat(\widehat{Z}) - \frac{1}{F}\ell_\flat(\widehat{Y})\nabla^{\mathsf{v}}\ell_\flat(\widehat{X},\widehat{Z}) \overset{(51),(55),(58)}{=} \frac{2}{F}\mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z})\\ &\qquad - \frac{1}{F^2}\left(\ell_\flat\otimes\eta(\widehat{X},\widehat{Y},\widehat{Z}) + \ell_\flat\otimes\eta(\widehat{Z},\widehat{X},\widehat{Y})\right) + \ell_\flat\otimes\eta(\widehat{Y},\widehat{X},\widehat{Z})\\ &\quad= \frac{2}{F}\mathcal{C}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) - \frac{1}{F^2}\underset{(\widehat{X},\widehat{Y},\widehat{Z})}{\mathfrak{S}}\ell_\flat\otimes\eta(\widehat{X},\widehat{Y},\widehat{Z}),\end{aligned}$$

taking into account in the last step that

$$\begin{aligned}\nabla^{\mathsf{v}}\ell_\flat(\widehat{X},\widehat{Z}) &= \frac{1}{F}\eta(\widehat{X},\widehat{Z}) = \frac{1}{F}(g(\widehat{X},\widehat{Z}) - (X^{\mathsf{v}}F)(Z^{\mathsf{v}}F))\\ &= \frac{1}{F}(g(\widehat{Z},\widehat{X}) - (Z^{\mathsf{v}}F)(X^{\mathsf{v}}F)) = \nabla^{\mathsf{v}}\ell_\flat(\widehat{Z},\widehat{X}).\end{aligned}$$

□

Remark. Let Sym denote the symmetrizer defined by

$$(\mathrm{Sym}\widetilde{\mathbf{A}})(\widetilde{X},\widetilde{Y},\widetilde{Z}) := \underset{(\widetilde{X},\widetilde{Y},\widetilde{Z})}{\mathfrak{S}} \mathbf{A}(\widetilde{X},\widetilde{Y},\widetilde{Z}),$$

if $\mathbf{A} \in \mathcal{T}_3^0(\overset{\circ}{\pi})$. Let $\lambda := \frac{\nabla^{\mathsf{v}} F}{F}$, $\mu := \nabla^{\mathsf{v}}\nabla^{\mathsf{v}} F$. Then

$$\frac{1}{F^2}\ell_\flat \otimes \eta = \frac{\nabla^{\mathsf{v}} F}{F} \otimes \mu,$$

and (59) may be written in the more concise form

$$\nabla^{\mathsf{v}}\mu = \frac{2}{F}\mathcal{C}_\flat - \mathrm{Sym}(\lambda \otimes \mu). \tag{60}$$

If (M, F) is a Finsler manifold, then the 2-form

$$\frac{1}{2}d(\nabla F^2 \circ \mathbf{j}) = dd_{\mathbf{J}}E$$

is (fibrewise) non-degenerate on $\overset{\circ}{T}M$ by (F_3). So there exists a unique map

$$S\colon TM \to TTM$$

defined to be zero on $o(M)$, and defined on $\overset{\circ}{T}M$ to be the unique vector field such that

$$i_S dd_{\mathbf{J}}E = -dE. \tag{61}$$

Then, actually, *S is a spray over M*, i.e., has the properties (C_1)-(C_6). A very instructive, but quite forgotten proof of this fundamental fact may be found in F. Warner's quoted paper [25]. The spray S will be called the *canonical spray* of the Finsler manifold (M, F). If $\mathcal{H}$ is the Ehresmann connection associated to S according to (9), then

(i) *$\mathcal{H}$ is homogeneous and torsion-free*;
(ii) *$\mathcal{H}$ is conservative* in the sense that

$$dF \circ \mathcal{H} = 0 \;\Leftrightarrow X^{\mathsf{h}}F = X^{\mathsf{h}}E = 0,\; X \in \mathfrak{X}(M). \tag{62}$$

Property (i), as we have already learnt, holds for any Ehresmann connection associated to a spray. Here the new and surprising phenomenon is property (ii), which expresses that *the Finsler function (and hence the energy function) is a first integral of the horizontally lifted vector fields.* We call this Ehresmann connection the *canonical connection* of the Finsler manifold. We note that other terms - *Barthel connection*, *Cartan's nonlinear connection*, *Berwald connection* - are also frequently used in the literature. A recent index-free proof of (ii) can be found in [22]. In the next section we shall show that *the canonical connection of a Finsler manifold (M, F) is unique* in the sense that there is only one Ehresmann connection over M which satisfies conditions (i), (ii).

Warning. The *Berwald connection* $\mathcal{H}\colon TM \times_M TM \to TTM$ associated to the canonical spray of a Finsler manifold and the *Berwald derivative* $\nabla =$

$(\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ induced by $\mathcal{H}$ are essentially different objects: the latter is a covariant derivative operator ('linear connection') in a (special) vector bundle. It will sometimes be mentioned as the *Finslerian Berwald derivative*.

Using the Finslerian h-Berwald covariant derivative, we define the *Landsberg tensor* $\mathbf{P}$ of a Finsler manifold (M, F) by the following formula:

$$\mathbf{P} := -\frac{1}{2}\nabla^{\mathsf{h}} g. \tag{63}$$

Lemma 6.2. *For all vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have*

$$\nabla^{\mathsf{h}} g(\widehat{X}, \widehat{Y}, \widehat{Z}) = (\mathbf{iB}(\widehat{X}, \widehat{Y})\widehat{Z})E, \tag{64}$$

therefore the Berwald curvature and the Landsberg tensor of a Finsler manifold are related by

$$\nabla^{\mathsf{v}} E \circ \mathbf{B} = -2\mathbf{P}. \tag{65}$$

Proof.

$$\begin{aligned}
\nabla^{\mathsf{h}} g(\widehat{X}, \widehat{Y}, \widehat{Z}) &= (\nabla_{X^{\mathsf{h}}} g)(\widehat{Y}, \widehat{Z}) \\
&= X^{\mathsf{h}} g(\widehat{Y}, \widehat{Z}) - g(\nabla_{X^{\mathsf{h}}}\widehat{Y}, \widehat{Z}) - g(\widehat{Y}, \nabla_{X^{\mathsf{h}}}\widehat{Z}) \\
&\overset{(47),(48)}{=} X^{\mathsf{h}}(Y^{\mathsf{v}}(Z^{\mathsf{v}}E))) - (\mathbf{i}\nabla_{X^{\mathsf{h}}}\widehat{Y})Z^{\mathsf{v}}E + (\mathbf{i}\nabla_{\mathbf{i}\nabla_{X^{\mathsf{h}}}\widehat{Y}}\widehat{Z})E \\
&\quad - Y^{\mathsf{v}}(\mathbf{i}\nabla_{X^{\mathsf{h}}}\widehat{Z})E + \mathbf{J}[Y^{\mathsf{v}}, \mathcal{H}\nabla_{X^{\mathsf{h}}}\widehat{Z}]E \\
&= X^{\mathsf{h}}(Y^{\mathsf{v}}(Z^{\mathsf{v}}E)) - [X^{\mathsf{h}}, Y^{\mathsf{v}}]Z^{\mathsf{v}}E - Y^{\mathsf{v}}([X^{\mathsf{h}}, Z^{\mathsf{v}}]E) \\
&\quad + \mathbf{J}[Y^{\mathsf{v}}, \mathcal{H}\mathcal{V}[X^{\mathsf{h}}, Z^{\mathsf{v}}]].
\end{aligned}$$

Here, as we have already seen in the proof of 3.1,

$$\mathbf{J}[Y^{\mathsf{v}}, \mathcal{H}\mathcal{V}[X^{\mathsf{h}}, Z^{\mathsf{v}}]] = [Y^{\mathsf{v}}, [X^{\mathsf{h}}, Z^{\mathsf{v}}]] = \mathbf{iB}(\widehat{Y}, \widehat{X})\widehat{Z} = \mathbf{iB}(\widehat{X}, \widehat{Y})\widehat{Z}$$

therefore

$$\begin{aligned}
\nabla^{\mathsf{h}} g(\widehat{X}, \widehat{Y}, \widehat{Z}) = (\mathbf{iB}(\widehat{X}, \widehat{Y})\widehat{Z})E + Y^{\mathsf{v}}(X^{\mathsf{h}}(Z^{\mathsf{v}}E)) \\
- Y^{\mathsf{v}}(X^{\mathsf{h}}(Z^{\mathsf{v}}E) - Z^{\mathsf{v}}(X^{\mathsf{h}}E)) = \mathbf{iB}(\widehat{X}, \widehat{Y})\widehat{Z},
\end{aligned}$$

taking into account that $\mathcal{H}$ is conservative. Thus we have proved relation (64). Relation (65) is merely a reformulation of (64). □

Corollary 6.3. *The Landsberg tensor of a Finsler manifold has the following properties:*

(i) it is totally symmetric*;*
(ii) $\delta \in \left\{\widetilde{X}, \widetilde{Y}, \widetilde{Z}\right\} \Rightarrow \mathbf{P}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = 0$*;*
(iii) $\nabla_C \mathbf{P} = 0$*, i.e.,* $\mathbf{P}$ is homogeneous of degree zero.

Indeed, (i) and (ii) are immediate consequences of (64) and the corresponding property of the Berwald tensor. Taking into account our calculations in the proof of 3.5, we get for any vector fields X, Y, Z on M

$$\begin{aligned}(\nabla_C \mathbf{P})(\widehat{X},\widehat{Y},\widehat{Z}) &= C(\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z})) = -\frac{1}{2}C(\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z})E \\ &= -\frac{1}{2}([C,\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z}]E + (\mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z})CE) = 0,\end{aligned}$$

which proves (iii).

Corollary 6.4. *If g is the metric tensor, S is the canonical spray of a Finsler manifold, then $\nabla_S g = 0$.*

Proof. For any sections $\widetilde{X},\widetilde{Y}$ in $\mathrm{Sec}(\overset{\circ}{\pi})$,

$$(\nabla_S g)(\widetilde{X},\widetilde{Y}) = \left(\nabla^{\mathsf{h}}_\delta g\right)(\widetilde{X},\widetilde{Y}) = \nabla^{\mathsf{h}} g(\delta,\widetilde{X},\widetilde{Y}) = -2\mathbf{P}(\delta,\widetilde{X},\widetilde{Y}) \overset{6.3\,\text{(ii)}}{=} 0. \quad \square$$

Now we can easily deduce an important relation between the Cartan tensor and the Landsberg tensor of a Finsler manifold.

Proposition 6.5. $\nabla_S \mathcal{C}_\flat = \mathbf{P}$.

Proof. Let X, Y, Z be vector fields on M. Applying the hv-Ricci formula (24), property (53) and Corollary 6.3 (ii), we obtain:

$$\begin{aligned}2\,(\nabla_S \mathcal{C}_\flat)(\widehat{X},\widehat{Y},\widehat{Z}) &= \nabla^{\mathsf{h}}_\delta \nabla^{\mathsf{v}} g(\widehat{X},\widehat{Y},\widehat{Z}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}} g(\delta,\widehat{X},\widehat{Y},\widehat{Z}) \\ &= \nabla^{\mathsf{v}}\nabla^{\mathsf{h}} g(\widehat{X},\delta,\widehat{Y},\widehat{Z}) = \nabla_{X^{\mathsf{v}}}\nabla^{\mathsf{h}} g(\delta,\widehat{Y},\widehat{Z}) = -2\,(\nabla_{X^{\mathsf{v}}}\mathbf{P})(\delta,\widehat{Y},\widehat{Z}) \\ &= -2X^{\mathsf{v}}\mathbf{P}(\delta,\widehat{Y},\widehat{Z}) + 2\mathbf{P}(\nabla_{X^{\mathsf{v}}}\delta,\widehat{Y},\widehat{Z}) = 2\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z}). \quad \square\end{aligned}$$

Now we are able to prove in an index-free manner Proposition 3.1 in [2].

Proposition 6.6. *If the Landsberg tensor of a Finsler manifold depends only on the position, then it vanishes identically, i.e., $\nabla^{\mathsf{v}}\mathbf{P} = 0$ implies that $\mathbf{P} = 0$.*

Proof. Applying the preceding Proposition, the Ricci formula (24), and taking into account Corollary 3.4, for any vector fields X, Y, Z on M we have

$$\begin{aligned}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z}) &= (\nabla_S \mathcal{C}_\flat)(\widehat{X},\widehat{Y},\widehat{Z}) = (\nabla^{\mathsf{h}}\mathcal{C}_\flat)(\delta,\widehat{X},\widehat{Y},\widehat{Z}) = \frac{1}{2}(\nabla^{\mathsf{h}}\nabla^{\mathsf{v}} g)(\delta,\widehat{X},\widehat{Y},\widehat{Z}) \\ &= \frac{1}{2}(\nabla^{\mathsf{v}}\nabla^{\mathsf{h}} g)(\widehat{X},\delta,\widehat{Y},\widehat{Z}) = -(\nabla^{\mathsf{v}}\mathbf{P})(\widehat{X},\delta,\widehat{Y},\widehat{Z}) = 0. \quad \square\end{aligned}$$

By the *stretch tensor* of a Finsler manifold we mean the type $\binom{0}{4}$ tensor Σ along $\overset{\circ}{\tau}$ given by

$$\frac{1}{2}\mathbf{\Sigma}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) := \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) - \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{Y},\widetilde{X},\widetilde{Z},\widetilde{U})$$

([2],[3]). Next we verify by an index-free argument the following result of [2]:

Proposition 6.7. *If the stretch tensor of a Finsler manifold depends only on the position, then it vanishes identically, i.e.,* $\nabla^{\mathsf{v}}\Sigma = 0$ *implies that* $\Sigma = 0$.

Proof. Let X, Y, Z, U, V be vector fields on M.

Step 1 By our assumption

$$\begin{aligned}
0 &= (\nabla^{\mathsf{v}}\Sigma)(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\widehat{V}) = X^{\mathsf{v}}(\Sigma(\widehat{Y},\widehat{Z},\widehat{U},\widehat{V}))\\
&= 2(X^{\mathsf{v}}(\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{U},\widehat{V}) - \nabla^{\mathsf{h}}\mathbf{P}(\widehat{Z},\widehat{Y},\widehat{U},\widehat{V})))\\
&= 2(\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\widehat{V}) - \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Z},\widehat{Y},\widehat{U},\widehat{V}))),
\end{aligned}$$

hence

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\widehat{V}) = \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Z},\widehat{Y},\widehat{U},\widehat{V}).$$

Since this is a tensorian relation, we also have

$$\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\delta) = \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Z},\widehat{Y},\widehat{U},\delta). \tag{66}$$

Now, by the Ricci formula (24) and Corollary 6.3 (ii),

$$\begin{aligned}
\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\delta) &= \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{X},\widehat{Z},\widehat{U},\delta)\\
&\overset{(66)}{=} \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U},\delta) \overset{(24)}{=} \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U},\delta),
\end{aligned}$$

so $\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(.,.,.,.,\delta)$ is symmetric in its first and third variables:

$$\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\delta) = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U},\delta). \tag{67}$$

Step 2 We show that

$$\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U}) + \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\delta) = 0. \tag{68}$$

We start out the identity $\mathbf{P}(\widehat{Z},\widehat{U},\delta) = 0$. Operating on both sides first by Y^{v}, and next by X^{h}, we obtain

$$\begin{aligned}
0 = Y^{\mathsf{v}}(\mathbf{P}(\widehat{Z},\widehat{U},\delta)) &= \nabla^{\mathsf{v}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{U},\delta) + \mathbf{P}(\widehat{Z},\widehat{U},\widehat{Y})\\
&= \mathbf{P}(\widehat{Y},\widehat{Z},\widehat{U}) + \nabla^{\mathsf{v}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{U},\delta);
\end{aligned}$$

$$\begin{aligned}
0 &= X^{\mathsf{h}}(\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{U})) + X^{\mathsf{h}}(\nabla^{\mathsf{v}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{U},\delta))\\
&= (\nabla^{\mathsf{h}}\mathbf{P})(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U}) + (\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P})(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\delta)\\
&\quad + \mathbf{P}(\nabla_{X^{\mathsf{h}}}\widehat{Y},\widehat{Z},\widehat{U}) + \mathbf{P}(\widehat{Y},\nabla_{X^{\mathsf{h}}}\widehat{Z},\widehat{U}) + \mathbf{P}(\widehat{Y},\widehat{Z},\nabla_{X^{\mathsf{h}}}\widehat{U})\\
&\quad + \nabla^{\mathsf{v}}\mathbf{P}(\nabla_{X^{\mathsf{h}}}\widehat{Y},\widehat{Z},\widehat{U},\delta) + \nabla^{\mathsf{v}}\mathbf{P}(\widehat{Y},\nabla_{X^{\mathsf{h}}}\widehat{Z},\widehat{U},\delta) + \nabla^{\mathsf{v}}\mathbf{P}(\widehat{Y},\widehat{Z},\nabla_{X^{\mathsf{h}}}\widehat{U},\delta).
\end{aligned}$$

Since, for example,

$$\begin{aligned}
\nabla^{\mathsf{v}}\mathbf{P}(\nabla_{X^{\mathsf{h}}}\widehat{Y},\widehat{Z},\widehat{U},\delta) &= [X^{\mathsf{h}},Y^{\mathsf{v}}]\mathbf{P}(\widehat{Z},\widehat{U},\delta) - \mathbf{P}(\widehat{Z},\widehat{U},\nabla_{[X^{\mathsf{h}},Y^{\mathsf{v}}]}\delta)\\
&= -\mathbf{P}(\widehat{Z},\widehat{U},\mathcal{V}[X^{\mathsf{h}},Y^{\mathsf{v}}]) = -\mathbf{P}(\widehat{Z},\widehat{U},\nabla_{X^{\mathsf{h}}}\widehat{Y}) = -\mathbf{P}(\nabla_{X^{\mathsf{h}}}\widehat{Y},\widehat{Z},\widehat{U}),
\end{aligned}$$

the last six terms cancel in pairs on the right-hand side of the above relation. So we get (68).

Step 3 Interchanging $\widehat{X}$ and $\widehat{Y}$ in (68), we find

$$\begin{aligned}0 &= \nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{X},\widehat{Z},\widehat{U}) + \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(\widehat{Y},\widehat{X},\widehat{Z},\widehat{U},\delta)\\ &= \nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{X},\widehat{Z},\widehat{U}) + \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U},\delta),\end{aligned}$$

since, by Step 1, the second term does not change under the permutations

$$(\widehat{Y},\widehat{X},\widehat{Z}) \to (\widehat{Y},\widehat{Z},\widehat{X}) \to (\widehat{X},\widehat{Z},\widehat{Y}) \to (\widehat{X},\widehat{Y},\widehat{Z}).$$

The last relation and (68) imply that

$$\nabla^{\mathsf{h}}\mathbf{P}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U}) = \nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{X},\widehat{Z},\widehat{U})$$

whence $\Sigma = 0$. □

Proposition 6.8. *Let (M,F) and $(M,\overline{F})$ be Finsler manifolds, and let the geometric data arising from $\overline{F}$ be distinguished by bar. Suppose that the canonical sprays S and $\overline{S}$ of (M,F) and $(M,\overline{F})$ are projectively related, namely $\overline{S} = S - 2PC$. Then*

$$2P = \frac{S\overline{F}}{\overline{F}}, \tag{69}$$

and

$$\nabla_S\overline{\mathcal{C}}_\flat = P\overline{\mathcal{C}}_\flat + \overline{\mathbf{P}}. \tag{70}$$

Proof. Since $\overline{S}$ is horizontal with respect to the canonical connection of $(M,\overline{F})$, we obtain

$$0 = \overline{S}\,\overline{F} = (S - 2PC)\overline{F} = S\overline{F} - 2P\overline{F},$$

so (69) is valid. To prove the second relation, let X, Y, Z be arbitrary vector fields on M. Then, applying (41) and (53),

$$\begin{aligned}&\left(\nabla_S\overline{\mathcal{C}}_\flat\right)(\widehat{X},\widehat{Y},\widehat{Z})\\ &\quad= S(\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z})) - \overline{\mathcal{C}}_\flat(\nabla_S\widehat{X},\widehat{Y},\widehat{Z}) - \overline{\mathcal{C}}_\flat(\widehat{X},\nabla_S\widehat{Y},\widehat{Z}) - \overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\nabla_S\widehat{Z})\\ &\quad= \overline{S}(\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z})) + 2PC(\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z})) - \overline{\mathcal{C}}_\flat(\overline{\nabla}_{\overline{S}}\widehat{X},\widehat{Y},\widehat{Z})\\ &\qquad- \overline{\mathcal{C}}_\flat(\widehat{X},\overline{\nabla}_{\overline{S}}\widehat{Y},\widehat{Z}) - \overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\overline{\nabla}_{\overline{S}}\widehat{Z}) + 3P\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z}).\end{aligned}$$

Since $\overline{\mathcal{C}}_\flat$ is homogeneous of degree -1,

$$C\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) = \left(\nabla_C\overline{\mathcal{C}}_\flat\right)(\widehat{X},\widehat{Y},\widehat{Z}) = -\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z}),$$

so we get

$$\left(\nabla_S\overline{\mathcal{C}}_\flat\right)(\widehat{X},\widehat{Y},\widehat{Z}) = \overline{\nabla}_{\overline{S}}\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) + P\overline{\mathcal{C}}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) \overset{6.5}{=} (\overline{\mathbf{P}} + P\overline{\mathcal{C}}_\flat)(\widehat{X},\widehat{Y},\widehat{Z}),$$

thus proving Proposition 6.8. □

7. Rapcsák's equations: some consequences and applications

Following the terminology of [24] we say that a spray is *Finsler-metrizable in a broad sense* or, after Z. Shen [16], *projectively Finslerian*, if there exists a Finsler function whose canonical spray is projectively related to the given spray. If, in particular, the projective relation is trivial in the sense that the projective factor vanishes, then the spray will be called *Finsler metrizable in a natural sense* or *Finsler variational.* In this section we deal with different conditions concerning both types of metrizability of a spray.

Lemma 7.1. *Let a spray S over M be given. Let $\mathcal{H}$ be the Ehresmann connection associated to S, and let $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ be the horizontal projector associated to $\mathcal{H}$. Then for any smooth function F on $\overset{\circ}{T}M$ we have*

$$2d_{\mathbf{h}}F = d(F - CF) - i_S dd_{\mathbf{J}}F + d_{\mathbf{J}} i_S dF. \tag{71}$$

This important relation was found by J. Klein, see [8], section 3.2. Its validity may be checked by brute force, evaluating both sides of (71) on vertical lifts X^{v} and complete lifts X^{c}, $X \in \mathfrak{X}(M)$. It is possible, however, to verify (71) also by a more elegant, completely 'argumentum-free' reasoning, see again [8], and [19].

Proposition 7.2. *Let $(M, \overline{F})$ be a Finsler-manifold with energy function $\overline{E} := \frac{1}{2}\overline{F}^2$ and canonical spray $\overline{S}$, given by $i_{\overline{S}} dd_{\mathbf{J}}\overline{E} = -d\overline{E}$ on $\overset{\circ}{T}M$. Suppose S is a further spray over M, and let $\mathcal{H}$ be the Ehresmann connection associated to S.*

S is projectively related to $\overline{S}$, if and only if, for each vector field $X \in \mathfrak{X}(M)$ we have

$$(\mathrm{R}_1)\quad 2X^{\mathsf{h}}\overline{F} = X^{\mathsf{v}}(S\overline{F}) \quad (X^{\mathsf{h}} := \mathcal{H}\widehat{X}).$$

Proof. Let $\overline{\mathcal{H}}$ be the canonical connection of $(M, \overline{F})$. Suppose first that S and $\overline{S}$ are projectively related, namely $\overline{S} = S - 2PC$, $P \in C^\infty(\overset{\circ}{T}M) \cap C^1(TM)$. If $X \in \mathfrak{X}(M)$, $X^{\mathsf{h}} = \mathcal{H}(\widehat{X})$, $X^{\overline{\mathsf{h}}} = \overline{\mathcal{H}}(\widehat{X})$, then

$$X^{\overline{\mathsf{h}}} = X^{\mathsf{h}} - PX^{\mathsf{v}} - (X^{\mathsf{v}}P)C$$

by (37). By Proposition 6.8 we have $2P = \frac{S\overline{F}}{\overline{F}}$. Since $\overline{\mathcal{H}}$ is conservative, then we obtain:

$$\begin{aligned}
0 = 2X^{\overline{\mathsf{h}}}\overline{F} &= (2X^{\mathsf{h}} - \frac{S\overline{F}}{\overline{F}}X^{\mathsf{v}} - X^{\mathsf{v}}(\frac{S\overline{F}}{\overline{F}})C)\overline{F} \\
&= 2X^{\mathsf{h}}\overline{F} - \frac{S\overline{F}}{\overline{F}}(X^{\mathsf{v}}\overline{F}) - \frac{1}{\overline{F}}X^{\mathsf{v}}(S\overline{F})\overline{F} + \frac{1}{\overline{F}^2}(S\overline{F})(X^{\mathsf{v}}\overline{F})\overline{F} = 2X^{\mathsf{h}}\overline{F} - X^{\mathsf{v}}(S\overline{F}).
\end{aligned}$$

This proves the validity of (R_1) if S and $\overline{S}$ are projectively related.

Conversely, suppose that (R_1) is satisfied. Then, for all $X \in \mathfrak{X}(M)$,

$$2X^{\mathsf{h}}\overline{E} = 2\overline{F}(X^{\mathsf{h}}\overline{F}) \overset{(\mathrm{R}_1)}{=} \overline{F}(X^{\mathsf{v}}(S\overline{F})) = X^{\mathsf{v}}(S\overline{E}) - (X^{\mathsf{v}}\overline{F})(S\overline{F}). \tag{72}$$

Since $\overline{E} - C\overline{E} = -\overline{E}$, we obtain by Lemma 7.1

$$d_{\mathbf{J}} i_S d\overline{E} - 2d_{\mathbf{h}}\overline{E} = i_S dd_{\mathbf{J}}\overline{E} + d\overline{E}. \tag{73}$$

Next we prove that the left-hand side of (73) equals to $\frac{S\overline{F}}{\overline{F}} i_C dd_{\mathbf{J}}\overline{E}$.

For any $X \in \mathfrak{X}(M)$ an easy calculation shows that, on the one hand,

$$(d_{\mathbf{J}} i_S d\overline{E} - 2d_{\mathbf{h}}\overline{E})(X^{\mathsf{v}}) = 0 = \frac{S\overline{F}}{\overline{F}} i_C dd_{\mathbf{J}}\overline{E}(X^{\mathsf{v}}).$$

On the other hand,

$$\begin{aligned}(d_{\mathbf{J}} i_S d\overline{E} - 2d_{\mathbf{h}}\overline{E})(X^{\mathsf{c}}) &= \mathbf{J}X^{\mathsf{c}}(i_S d\overline{E}) - 2(\mathbf{h}X^{\mathsf{c}})\overline{E} \\ &= X^{\mathsf{v}}(S\overline{E}) - 2X^{\mathsf{h}}E \overset{(72)}{=} (X^{\mathsf{v}}\overline{F})(S\overline{F}),\end{aligned}$$

while

$$\begin{aligned}\frac{S\overline{F}}{\overline{F}} i_C dd_{\mathbf{J}}\overline{E}(X^{\mathsf{c}}) &= \frac{S\overline{F}}{\overline{F}} dd_{\mathbf{J}}\overline{E}(C, X^{\mathsf{c}}) \\ &= \frac{S\overline{F}}{\overline{F}} \left(C d_{\mathbf{J}}\overline{E}(X^{\mathsf{c}}) - X^{\mathsf{c}} d_{\mathbf{J}}\overline{E}(C) - d_{\mathbf{J}}\overline{E}[C, X^{\mathsf{c}}]\right) \\ &= \frac{S\overline{F}}{\overline{F}} C(X^{\mathsf{v}}\overline{E}) = \frac{S\overline{F}}{\overline{F}}([C, X^{\mathsf{v}}]\overline{E} + X^{\mathsf{v}}(C\overline{E})) \\ &= \frac{S\overline{F}}{\overline{F}}(X^{\mathsf{v}}\overline{E}) = (S\overline{F})(X^{\mathsf{v}}\overline{F}),\end{aligned}$$

hence

$$d_{\mathbf{J}} i_S d\overline{E} - 2d_{\mathbf{h}}\overline{E} = \frac{S\overline{F}}{\overline{F}} i_C dd_{\mathbf{J}}\overline{E}, \tag{74}$$

as we claimed. (73) and (74) imply that

$$i_S dd_{\mathbf{J}}\overline{E} + d\overline{E} = i_{\frac{S\overline{F}}{\overline{F}}C} dd_{\mathbf{J}}\overline{E}$$

whence

$$i_{S - \frac{S\overline{F}}{\overline{F}}C} dd_{\mathbf{J}}\overline{E} = -d\overline{E}.$$

Since $\overline{S}$ is uniquely determined on $\overset{\circ}{T}M$ by the 'Euler-Lagrange equation' $i_{\overline{S}} dd_{\mathbf{J}}\overline{E} = -d\overline{E}$, we conclude that $\overline{S} = S - \frac{S\overline{F}}{\overline{F}}C$, which proves the Proposition. □

(R_1) provides a necessary and sufficient condition for the Finsler-metrizability of a spray in a broad sense. In terms of classical tensor calculus, it was first formulated by A. Rapcsák [15], so it will be quoted as *Rapcsák's equation* for $\overline{F}$ with respect to S.

Now we derive a 'more intrinsic' expression of (R_1), showing that it can also be written in the form

(R_2) $\nabla_S \nabla^{\mathsf{v}}\overline{F} = \nabla^{\mathsf{h}}\overline{F}$.

Indeed, for any vector field $X \in \mathfrak{X}(M)$ we have

$$\begin{aligned}\nabla_S\nabla^{\mathsf{v}}\overline{F}(\widehat{X}) &= S(X^{\mathsf{v}}\overline{F}) - \nabla^{\mathsf{v}}\overline{F}(\nabla_S\widehat{X}) \\ &= S(X^{\mathsf{v}}\overline{F}) - (\mathbf{i}\nabla_S\widehat{X})\overline{F} = S(X^{\mathsf{v}}\overline{F}) - \mathbf{v}[S,X^{\mathsf{v}}]\overline{F} \\ &\overset{(9)}{=} S(X^{\mathsf{v}}\overline{F}) - (\mathbf{v}X^{\mathsf{c}})\overline{F} = S(X^{\mathsf{v}}\overline{F}) - X^{\mathsf{c}}\overline{F} + X^{\mathsf{h}}\overline{F} \\ &= [S,X^{\mathsf{v}}]\overline{F} + X^{\mathsf{v}}(S\overline{F}) - X^{\mathsf{c}}\overline{F} + X^{\mathsf{h}}\overline{F} \\ &\overset{(9)}{=} (X^{\mathsf{c}} - 2X^{\mathsf{h}})\overline{F} + X^{\mathsf{v}}(S\overline{F}) - X^{\mathsf{c}}\overline{F} + X^{\mathsf{h}}\overline{F} = X^{\mathsf{v}}(S\overline{F}) - X^{\mathsf{h}}\overline{F},\end{aligned}$$

hence

$$\nabla_S\nabla^{\mathsf{v}}\overline{F} = \nabla^{\mathsf{h}}\overline{F} \Leftrightarrow X^{\mathsf{v}}(S\overline{F}) - X^{\mathsf{h}}\overline{F} = X^{\mathsf{h}}\overline{F} \text{ for all } X \in \mathfrak{X}(M).$$

This proves the equivalence of (R_1) and (R_2). Rapcsák equations (R_1), (R_2) have several further equivalents, we collect here some of them:

(R_3) $i_S dd_{\mathbf{J}}\overline{F} = 0$;
(R_4) $i_\delta\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\overline{F} = \nabla^{\mathsf{h}}\overline{F}$;
(R_5) $d_{\mathbf{h}}d_{\mathbf{J}}\overline{F} = 0$;
(R_6) $\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\overline{F}(\widetilde{X},\widetilde{Y}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\overline{F}(\widetilde{Y},\widetilde{X})$;
(R_7) $\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\overline{F}(\widetilde{X},\widetilde{Y}) = \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\overline{F}(\widetilde{Y},\widetilde{X})$

(in (R_6) and (R_7) $\widetilde{X}$ and $\widetilde{Y}$ are arbitrary sections along $\mathring{\tau}$).

Details on a proof of the equivalence of conditions (R_1)-(R_7) can be found in [24], [19], [20], [18]. We note only that the equivalence of (R_6) and (R_7) is an immediate consequence of the Ricci identity (22), while the equivalence of (R_2) and (R_4) follows from the identity $i_\delta\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\overline{F} = \nabla_S\nabla^{\mathsf{v}}\overline{F}$.

Proposition 7.3 (Criterion for Finsler variationality). *Let S be a spray over M, and let $\mathcal{H}$ be the Ehresmann connection associated to S. S is the canonical spray of a Finsler manifold $(M,\overline{F})$, if and only if, $d\overline{F}\circ\mathcal{H} = 0$.*

Proof. The necessity is obvious since the canonical connection of $(M,\overline{F})$ is conservative. To prove the sufficiency, suppose that $d\overline{F}\circ\mathcal{H} = 0$. Then for all $X \in \mathfrak{X}(M)$ we have

$$d\overline{F}\circ\mathcal{H}(\widehat{X}) = d\overline{F}(X^{\mathsf{h}}) = X^{\mathsf{h}}\overline{F} = 0. \tag{75}$$

Since the horizontal lifts X^{h}, $X \in \mathfrak{X}(M)$ generate the $C^\infty(\mathring{T}M)$-module of $\mathcal{H}$-horizontal vector fields, this implies that for any $\mathcal{H}$-horizontal vector field ξ on $\mathring{T}M$ we have $\xi\overline{F} = 0$. In particular, S is also $\mathcal{H}$-horizontal, so $S\overline{F} = 0$ holds too. Then Rapcsák's equation (R_1) is valid trivially: both sides of the relation vanish identically. By Proposition 7.2, from this it follows that $\overline{S}$ and S are projectively related:

$$\overline{S} = S - 2PC,\ P \in C^1(TM)\cap C^\infty(\mathring{T}M).$$

However, $P \overset{(69)}{=} \frac{1}{2}\frac{S\overline{F}}{\overline{F}} = 0$, so we obtain the required equality $\overline{S} = S$. □

Remark. In his excellent textbook [10], written in a 'semi-classical style', D. Laugwitz formulates and proves the following theorem: *The paths of a system* $(x^i)'' + 2H^i(x, x') = 0$ $(i \in \{1, \dots n\})$ *are the geodesics of a Finsler function* F*, if and only if,* F *is invariant under the parallel displacement*

$$\frac{d\xi^i}{dt} + H^i_r(x, \xi)\frac{dx^r}{dt} = 0,\ H^i_r := \frac{\partial H^i}{\partial y^r}$$

associated with the sytem of paths. ([10], Theorem 15.8.1.) Here we slightly modified Laugwitz's formulation and notation. The 'system', actually a SODE, is given in a chart $(\tau^{-1}(\mathcal{U}), (x^i, y^i))$ on TM, induced by a chart $(\mathcal{U}, (u^i))$ on M:

$$x^i := u^i \circ \tau = (u^i)^{\mathsf{v}},\ y^i := (u^i)^{\mathsf{c}}\ ;\ i \in \{1, \dots, n\}\,.$$

It may be easily seen that our Proposition 7.3 is just an intrinsic reformulation of Laugwitz's metrization theorem. Laugwitz's proof takes more than one page and applies a totally different argument.

Corollary 7.4 (The uniqueness of the canonical connection.)**.** *Let* $(M, \overline{F})$ *be a Finsler manifold. If* $\mathcal{H}$ *is a torsion-free, homogeneous Ehresmann connection over* M *such that* $d\overline{F} \circ \mathcal{H} = 0$*, then* $\mathcal{H}$ *is the canonical connection of* $(M, \overline{F})$*.*

Proof. Since $\mathcal{H}$ is torsion-free and homogeneous, Corollary 6 in section 3 of [19] assures that $\mathcal{H}$ is associated to a spray. Then the condition $d\overline{F} \circ \mathcal{H} = 0$ implies by the preceding Proposition that this spray is the canonical spray, and hence $\mathcal{H}$ is the canonical connection of $(M, \overline{F})$. □

Remark. The uniqueness proof presented here is based, actually, on the Rapcsák equations. The idea that they may be applied also in this context is due to Z. I. Szabó [17].

Our next results may be considered as *necessary conditions for the Finsler metrizability of a spray in a broad sense.*

Proposition 7.5. *Let* S *be a spray over* M*, and let* $\nabla = (\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ *be the Berwald derivative induced by the Ehresmann connection associated to* S*. If a Finsler function* $\overline{F}\colon TM \to \mathbb{R}$ *satisfies one (and hence all) of the Rapcsák equations with respect to* S*, then*

(76) $$\nabla_S \nabla^{\mathsf{v}} \nabla^{\mathsf{v}} \overline{F} = 0.$$

Proof. For any vector fields X, Y on M we have

$$\begin{aligned}
&\left(\nabla_S\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}\right)(\widehat{X},\widehat{Y}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}(\nabla^{\mathsf{v}}\overline{F})(\delta,\widehat{X},\widehat{Y})\\
&\overset{(24)}{=} \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}(\nabla^{\mathsf{v}}\overline{F})(\widehat{X},\delta,\widehat{Y}) + \nabla^{\mathsf{v}}\overline{F}(\mathbf{B}(\widehat{X},\delta)\widehat{Y})\\
&\overset{(21)}{=} \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}(\nabla^{\mathsf{v}}\overline{F})(\widehat{X},\delta,\widehat{Y}) = X^{\mathsf{v}}\left(\nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\overline{F}(\delta,\widehat{Y})\right) - \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\overline{F}(\widehat{X},\widehat{Y})\\
&\overset{(22)}{=} X^{\mathsf{v}}\left(\nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\overline{F}(\widehat{Y},\delta)\right) - \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\overline{F}(\widehat{Y},\widehat{X})\\
&= X^{\mathsf{v}}(Y^{\mathsf{v}}(S\overline{F}) - \nabla^{\mathsf{h}}\overline{F}(\widehat{Y})) - Y^{\mathsf{v}}(X^{\mathsf{h}}\overline{F})\\
&\overset{(\mathrm{R}_1)}{=} X^{\mathsf{v}}(2Y^{\mathsf{h}}\overline{F} - Y^{\mathsf{h}}\overline{F}) - Y^{\mathsf{v}}(X^{\mathsf{h}}\overline{F})\\
&= X^{\mathsf{v}}(Y^{\mathsf{h}}\overline{F}) - Y^{\mathsf{v}}(X^{\mathsf{h}}\overline{F}) = \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\overline{F}(\widehat{X},\widehat{Y}) - \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\overline{F}(\widehat{Y},\widehat{X}) \overset{(\mathrm{R}_7)}{=} 0. \qquad \square
\end{aligned}$$

Corollary 7.6. *Under the assumptions of the Proposition above, let*

$$\overline{\mu} := \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F} \overset{(57)}{=} \frac{1}{\overline{F}}\overline{\eta},$$

where $\overline{\eta}$ is the angular metric tensor of the Finsler manifold $(M,\overline{F})$. Then

$$\nabla_S\nabla^{\mathsf{v}}\overline{\mu} + \nabla^{\mathsf{h}}\overline{\mu} = 0. \tag{77}$$

Proof. Let $Y, Z \in \mathfrak{X}(M)$. Then by Proposition 7.5,

$$\nabla^{\mathsf{h}}\overline{\mu}(\delta,\widehat{Y},\widehat{Z}) = \nabla_S\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}(\widehat{Y},\widehat{Z}) = 0.$$

Operating on both sides by X^{v}, where $X \in \mathfrak{X}(M)$, we obtain

$$\begin{aligned}
0 &= X^{\mathsf{v}}(\nabla^{\mathsf{h}}\overline{\mu}(\delta,\widehat{Y},\widehat{Z})) = \left(\nabla_{X^{\mathsf{v}}}\nabla^{\mathsf{h}}\overline{\mu}\right)(\delta,\widehat{Y},\widehat{Z}) + \nabla^{\mathsf{h}}\overline{\mu}(\widehat{X},\widehat{Y},\widehat{Z})\\
&= \nabla^{\mathsf{v}}\nabla^{\mathsf{h}}\overline{\mu}(\widehat{X},\delta,\widehat{Y},\widehat{Z}) + \nabla^{\mathsf{h}}\overline{\mu}(\widehat{X},\widehat{Y},\widehat{Z})\\
&\overset{(24),(21)}{=} \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\overline{\mu}(\delta,\widehat{X},\widehat{Y},\widehat{Z}) + \nabla^{\mathsf{h}}\overline{\mu}(\widehat{X},\widehat{Y},\widehat{Z})\\
&= (\nabla_S\nabla^{\mathsf{v}}\overline{\mu} + \nabla^{\mathsf{h}}\overline{\mu})(\widehat{X},\widehat{Y},\widehat{Z}). \qquad \square
\end{aligned}$$

Theorem 7.7. *Let $(M,\overline{F})$ be a Finsler manifold with canonical spray $\overline{S}$; let $\overline{\lambda} := \frac{\nabla^{\mathsf{v}}\overline{F}}{\overline{F}}$, $\overline{\mu} := \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}$; and let $\overline{\mathcal{C}}_\flat$ and $\overline{\mathbf{P}}$ be the Cartan and the Landsberg tensor of $(M,\overline{F})$, respectively. If $\overline{F}$ satisfies one (and hence all) of the Rapcsák equations with respect to a spray S, and P is the projective factor between S and $\overline{S}$, then*

$$\nabla^{\mathsf{h}}\overline{\mu} = \mathrm{Sym}(\overline{\lambda}\otimes\overline{\mu}) + \frac{2}{\overline{\overline{F}}}(P\overline{\mathcal{C}}_\flat - \overline{\mathbf{P}}). \tag{78}$$

Proof.

$$
\begin{aligned}
\nabla^{\mathsf{h}}\overline{\mu} &\overset{(77)}{=} -\nabla_S\nabla^{\mathsf{v}}\overline{\mu} \overset{(60)}{=} -2\nabla_S\left(\frac{1}{\overline{F}}\overline{\mathcal{C}}_\flat\right) + \nabla_S\mathrm{Sym}(\overline{\lambda}\otimes\overline{\mu}) \\
&= \frac{2S\overline{F}}{\overline{F}^2}\overline{\mathcal{C}}_\flat - \frac{2}{\overline{F}}\nabla_S\overline{\mathcal{C}}_\flat + \mathrm{Sym}(\nabla_S\overline{\lambda}\otimes\overline{\mu} + \overline{\lambda}\otimes\nabla_S\overline{\mu}) \\
&\overset{(69),(70),(76)}{=} \frac{4P}{\overline{F}}\overline{\mathcal{C}}_\flat - \frac{2P}{\overline{F}}\mathcal{C}_\flat - \frac{2}{\overline{F}}\overline{\mathbf{P}} + \mathrm{Sym}(\nabla_S\overline{\lambda}\otimes\overline{\mu}) \\
&= \mathrm{Sym}(\nabla_S\overline{\lambda}\otimes\overline{\mu}) + \frac{2}{\overline{F}}(P\overline{\mathcal{C}}_\flat - \overline{\mathbf{P}}). \qquad \square
\end{aligned}
$$

Corollary 7.8. *If a Finsler function $\overline{F}$ satisfies a Rapcsák equation with respect to a spray S, and $(\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ is the Berwald derivative induced by S, then the tensor $\nabla^{\mathsf{h}}\overline{\mu} = \nabla^{\mathsf{h}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}$ is totally symmetric.*

Proof. The total symmetry of $\nabla^{\mathsf{h}}\overline{\mu}$ can be read from (78). □

Remark. Relation (78) is an intrinsic, index and argumentum free version of formula (2.4) in [1]. The total symmetry of $\nabla^{\mathsf{h}}\overline{\mu}$ (modulo a Rapcsák equation) may also be verified immediately, independently of (78).

Proposition 7.9. *Let S be a spray over M, and suppose that a Finsler function $\overline{F}\colon TM \to \mathbb{R}$ satisfies a Rapcsák equation with respect to S. If $\mathbf{R}$ is the curvature of the Ehresmann connection associated to $\mathcal{H}$, then*

$$
\underset{(\widehat{X},\widehat{Y},\widehat{Z})}{\mathfrak{S}} \overline{\mu}(\mathbf{R}(\widehat{X},\widehat{Y}),\widehat{Z}) = 0. \tag{79}
$$

Proof. Let $(\nabla^{\mathsf{h}}, \nabla^{\mathsf{v}})$ be the Berwald derivative determined by $\mathcal{H}$. First we show that

$$
\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) \tag{80}
$$

($\overline{\ell}_\flat := \nabla^{\mathsf{v}}\overline{F}$; $X, Y, Z \in \mathfrak{X}(M)$) is symmetric in its last two arguments. Indeed,

$$
\begin{aligned}
\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) &= \left(\nabla_{X^{\mathsf{h}}}(\nabla^{\mathsf{h}}\overline{\ell}_\flat)\right)(\widehat{Y},\widehat{Z}) \\
&= X^{\mathsf{h}}(\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Y},\widehat{Z})) - \nabla^{\mathsf{h}}\overline{\ell}_\flat(\nabla_{X^{\mathsf{h}}}\widehat{Y},\widehat{Z}) - \nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Y},\nabla_{X^{\mathsf{h}}}\widehat{Z}) \\
&\overset{(\mathrm{R}_6)}{=} X^{\mathsf{h}}(\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Z},\widehat{Y})) - \nabla^{\mathsf{h}}\overline{\ell}_\flat(\nabla_{X^{\mathsf{h}}}\widehat{Z},\widehat{Y}) - \nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Z},\nabla_{X^{\mathsf{h}}}\widehat{Y}) \\
&= \left(\nabla_{X^{\mathsf{h}}}(\nabla^{\mathsf{h}}\overline{\ell}_\flat)\right)(\widehat{Z},\widehat{Y}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{X},\widehat{Z},\widehat{Y}),
\end{aligned}
$$

which proves our claim.

Next we apply the Ricci identity (33) to (80):

$$
\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{X},\widehat{Y},\widehat{Z}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Y},\widehat{X},\widehat{Z}) - \overline{\ell}_\flat(\mathbf{H}(\widehat{X},\widehat{Y})\widehat{Z}) - \overline{\mu}(\mathbf{R}(\widehat{X},\widehat{Y}),\widehat{Z}).
$$

Interchanging $\widehat{X}$, $\widehat{Y}$ and $\widehat{Z}$ cyclically:

$$\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Y},\widehat{Z},\widehat{X}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Z},\widehat{Y},\widehat{X}) - \overline{\ell}_\flat(\mathbf{H}(\widehat{Y},\widehat{Z})\widehat{X}) - \overline{\mu}(\mathbf{R}(\widehat{Y},\widehat{Z}),\widehat{X}),$$
$$\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{Z},\widehat{X},\widehat{Y}) = \nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat(\widehat{X},\widehat{Z},\widehat{Y}) - \overline{\ell}_\flat(\mathbf{H}(\widehat{Z},\widehat{X})\widehat{Y}) - \overline{\mu}(\mathbf{R}(\widehat{Z},\widehat{X}),\widehat{Y}).$$

We add these three relations. Then, using the Bianchi identity (28) and the symmetry of $\nabla^{\mathsf{h}}\nabla^{\mathsf{h}}\overline{\ell}_\flat$ in its last two variables, relation (79) drops. □

Remark. In the language of classical tensor calculus, relation (79) was first formulated by A. Rapcsák [15]. For another index-free treatment, using Grifone's formalism, we refer to [24].

Lemma 7.10. *Let a spray* $S\colon TM \to TTM$ *and a Finsler function* $\overline{F}\colon TM \to \mathbb{R}$ *be given. If* $\overline{\mu} := \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}$*;* $\mathbf{R}$ *is the curvature, and* $\mathbf{K}$ *is the Jacobi endomorphism of the Ehresmann connection associated to* S*, then relation (79) is equivalent to the condition*

$$\overline{\mu}(\mathbf{K}(\widehat{X}),\widehat{Y}) = \overline{\mu}(\widehat{X},\mathbf{K}(\widehat{Y}))\ ;\ X,Y \in \mathfrak{X}(M). \tag{81}$$

Proof. We recall that by (30), $\mathbf{K}$ and $\mathbf{R}$ are related by

$$\mathbf{K}(\widetilde{X}) = \mathbf{R}(\widetilde{X},\delta)\ ,\ \widetilde{X} \in \mathrm{Sec}(\mathring{\pi}).$$

(79)⇒(81) By assumption, for any vector fields X, Y on M we have

$$\overline{\mu}(\mathbf{R}(\widehat{X},\delta),\widehat{Y}) + \overline{\mu}(\mathbf{R}(\delta,\widehat{Y}),\widehat{X}) + \overline{\mu}(\mathbf{R}(\widehat{Y},\widehat{X}),\delta) = 0,$$

or, equivalently,

$$\overline{\mu}(\mathbf{K}(\widehat{X}),\widehat{Y}) - \overline{\mu}(\widehat{X},\mathbf{K}(\widehat{Y})) = \overline{\mu}(\mathbf{R}(\widehat{X},\widehat{Y}),\delta).$$

We show that the right-hand side vanishes.

$$\begin{aligned}
\overline{\mu}(\mathbf{R}(\widehat{X},\widehat{Y}),\delta) &= \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}(\delta,\mathbf{R}(\widehat{X},\widehat{Y}))\\
&= (\nabla_C\nabla^{\mathsf{v}}\overline{F})(\mathbf{R}(\widehat{X},\widehat{Y})) = C(\mathbf{i}\mathbf{R}(\widehat{X},\widehat{Y})\overline{F}) - \nabla^{\mathsf{v}}\overline{F}(\nabla_C(\mathbf{R}(\widehat{X},\widehat{Y})))\\
&= C(\mathbf{i}\mathbf{R}(\widehat{X},\widehat{Y})\overline{F}) - \nabla^{\mathsf{v}}\overline{F}(\nabla_C\mathbf{R}(\widehat{X},\widehat{Y}))\\
&\overset{(4.3)}{=} C(\mathbf{i}\mathbf{R}(\widehat{X},\widehat{Y})\overline{F}) - \mathbf{i}\mathbf{R}(\widehat{X},\widehat{Y})\overline{F}\\
&= [C,\mathbf{i}\mathbf{R}(\widehat{X},\widehat{Y})]\overline{F} = [C,[X,Y]^{\mathsf{h}} - [X^{\mathsf{h}},Y^{\mathsf{h}}]]\\
&= -[C,[X^{\mathsf{h}},Y^{\mathsf{h}}]] = [X^{\mathsf{h}},[Y^{\mathsf{h}},C]] + [Y^{\mathsf{h}},[C,X^{\mathsf{h}}]] = 0,
\end{aligned}$$

taking into account the homogeneity of the associated Ehresmann connection.

(81)⇒(79) We operate by X^{v} on both sides of the relation $\overline{\mu}(\mathbf{K}(\widehat{Y}),\widehat{Z}) = \overline{\mu}(\widehat{Y},\mathbf{K}(\widehat{Z}))$, and permute the variables cyclically. Then we obtain:

$$\begin{aligned}
X^{\mathsf{v}}(\overline{\mu}(\mathbf{K}(\widehat{Y}),\widehat{Z})) &= X^{\mathsf{v}}(\overline{\mu}(\widehat{Y},\mathbf{K}(\widehat{Z}))),\\
Y^{\mathsf{v}}(\overline{\mu}(\mathbf{K}(\widehat{Z}),\widehat{X})) &= Y^{\mathsf{v}}(\overline{\mu}(\widehat{Z},\mathbf{K}(\widehat{X}))),\\
Z^{\mathsf{v}}(\overline{\mu}(\mathbf{K}(\widehat{X}),\widehat{Y})) &= Z^{\mathsf{v}}(\overline{\mu}(\widehat{X},\mathbf{K}(\widehat{Y}))).
\end{aligned}$$

Applying the product rule,

$$\nabla^{\mathsf{v}}\overline{\mu}(\widehat{X},\mathbf{K}(\widehat{Y}),\widehat{Z}) - \nabla^{\mathsf{v}}\overline{\mu}(\widehat{X},\widehat{Y},\mathbf{K}(\widehat{Z})) = \overline{\mu}(\widehat{Y},\nabla^{\mathsf{v}}\mathbf{K}(\widehat{X},\widehat{Z})) - \overline{\mu}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{X},\widehat{Y}),\widehat{Z}),$$
$$\nabla^{\mathsf{v}}\overline{\mu}(\widehat{Y},\mathbf{K}(\widehat{Z}),\widehat{X}) - \nabla^{\mathsf{v}}\overline{\mu}(\widehat{Y},\widehat{Z},\mathbf{K}(\widehat{X})) = \overline{\mu}(\widehat{Z},\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{X})) - \overline{\mu}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{Z}),\widehat{X}),$$
$$\nabla^{\mathsf{v}}\overline{\mu}(\widehat{Z},\mathbf{K}(\widehat{X}),\widehat{Y}) - \nabla^{\mathsf{v}}\overline{\mu}(\widehat{Z},\widehat{X},\mathbf{K}(\widehat{Y})) = \overline{\mu}(\widehat{X},\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Z},\widehat{Y})) - \overline{\mu}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Z},\widehat{X}),\widehat{Y}).$$

Now we add these three relations. Since $\nabla^{\mathsf{v}}\overline{\mu} = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}$ is totally symmetric, we obtain

$$\begin{aligned} 0 &= \overline{\mu}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{X}) - \nabla^{\mathsf{v}}\mathbf{K}(\widehat{X},\widehat{Y}),\widehat{Z}) + \overline{\mu}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{Z},\widehat{Y}) \\ &\quad - \nabla^{\mathsf{v}}\mathbf{K}(\widehat{Y},\widehat{Z}),\widehat{X}) + \overline{\mu}(\nabla^{\mathsf{v}}\mathbf{K}(\widehat{X},\widehat{Z}) - \nabla^{\mathsf{v}}\mathbf{K}(\widehat{Z},\widehat{X}),\widehat{Y}) \\ &\overset{(4.6)}{=} 3(\overline{\mu}(\mathbf{R}(\widehat{X},\widehat{Y}),\widehat{Z}) + \overline{\mu}(\mathbf{R}(\widehat{Y},\widehat{Z}),\widehat{X}) + \overline{\mu}(\mathbf{R}(\widehat{Z},\widehat{X}),\widehat{Y})) \\ &= 3 \underset{(\widehat{X},\widehat{Y},\widehat{Z})}{\mathfrak{S}} \overline{\mu}(\mathbf{R}(\widehat{X},\widehat{Y}),\widehat{Z}). \end{aligned}$$ □

Corollary 7.11 (The self-adjointness condition)**.** *If a Finsler function $\overline{F}\colon TM \to \mathbb{R}$ satisfies a Rapcsák equation with respect to a spray over M, then the Jacobi endomorphism $\mathbf{K}$ determined by the spray is self-adjoint with respect to the symmetric type $\binom{0}{2}$ tensor $\overline{\mu} = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}\overline{F}$, i.e.,*

$$\overline{\mu}(\mathbf{K}(\widetilde{X}),\widetilde{Y}) = \overline{\mu}(\widetilde{X},\mathbf{K}(\widetilde{Y}))\ ;\ \widetilde{X},\widetilde{Y} \in \mathrm{Sec}(\overset{\circ}{\pi}).$$

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