On a problem of Recaman and its generalization

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Abstract

We solve some cases of a conjecture of Pomerance concerning reduced residue systems modulo k consisting of the first $\varphi(k)$ primes not dividing k. We cover the case when k is a prime, thus giving a complete solution to a problem of Recaman.

Key words: the problem of Recaman, the problem of Pomerance, Jacobsthal function, primes in residue classes
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Dedicated to Professor K. Győry on the occasion of his 70th birthday

1 Introduction

Let $k > 1$ be an integer and denote by $\varphi(k)$ Euler's totient function. We say that k is a P-integer if the first $\varphi(k)$ primes coprime to k form a reduced residue system modulo k. Note that a prime p is a P-integer if and only if the first p primes form a complete residue system modulo p. In 1980, Pomerance [3] showed that there are only finitely many P-integers. Thereby he qualitatively solved the problem of finitely many prime P-integers which was raised earlier in 1978 by Recaman [4]. In his paper Pomerance conjectured that the largest

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P-integer is $k = 30$. It is easy to check that the only P-integers less than or equal to 30 are $k = 2, 4, 6, 12, 18, 30$.

In this paper we prove the conjecture of Pomerance in two "opposite" extremal cases: when k is composed of "large" prime factors (i.e. when all the prime divisors of k are above $\log(k)$), and when k is composed of "small" prime factors (i.e. k is the product of all primes $\leq x$ for some x). As a trivial consequence of the first result we get a complete quantitative solution for the problem of Recaman. Further, we verify the conjecture of Pomerance for all $k < 5.5 \cdot 10^5$. We note that Pomerance's finiteness result for P-integers [3] in principle can be made effective: one can possibly get an explicit upper bound for P-integers k. However, according to our calculations, this bound is rather huge, and it seems that to cover the remaining gap some additional (theoretical and/or computational) arguments are needed. So the complete resolution of the problem of Pomerance still remains an open quest; we plan to attack it in a future paper.

The proofs of our results depend on some properties of the Jacobsthal function $g(m)$ as in [3]. Among others we use the exact values of $g(m)$ when m is the product of first $h \leq 46$ primes, which were recently obtained by Hagedorn [1]. Further, we apply several formulas of Rosser and Schoenfeld [5], concerning various functions involving primes.

2 Main results

Our first result solves Recaman's problem completely.

Theorem 1 *The only prime P-integer is* 2.

In fact Theorem 1 is a trivial consequence of the following much more general result. For $k > 1$ let $\ell(k)$ be the least prime divisor of k.

Theorem 2 *Let $k > 1$ be an integer with $\ell(k) > \log(k)$. Then k is a P-integer if and only if $k \in \{2, 4, 6\}$.*

For fixed positive integer r and positive real X write

$$N_r := \{n \mid \omega(n) = r\} \quad \text{and} \quad N_r(X) := \{n \in N_r \mid n \leq X\},$$

where $\omega(n)$ denotes the number of distinct prime divisors of n. Further, for any positive real x, we let $\log_1(x) = \log(x)$ and for $t \geq 2$, $\log_t(x) = \log(\log_{t-1}(x))$.

By a result of Landau it is known that

$$|N_r(X)| \sim \frac{X(\log_2(X))^{r-1}}{\log(X)(r-1)!}$$

(see Theorem 437, p. 368 of [2]). Let $N_r'(X)$ denote the set of integers n in $N_r(X)$ with $\ell(n) \leq \log(n)$. Then for any $n \in N_r'(X)$ we have $\ell(n) \leq \log(X)$ and $n/\ell(n) \in N_{r-1}(X/\ell(n))$. Applying Landau's result to $N_{r-1}(X/p)$ for every $p \leq \log(X)$, and noting that $\frac{(\log_2(x))^{r-2}}{\log(x)}$ is a decreasing function of x for sufficiently large x, we find that

$$|N_r'(X)| \leq c_1 \sum_{p \leq \log(X)} \frac{\frac{X}{p}\left(\log_2\left(\frac{X}{p}\right)\right)^{r-2}}{\log\left(\frac{X}{p}\right)(r-2)!} \leq c_2 \frac{X(\log_2(X))^{r-2}}{\log(X)(r-2)!} \sum_{p \leq \log(X)} \frac{1}{p} \leq$$

$$\leq c_3 \frac{X(\log_2(X))^{r-2}}{\log(X)(r-2)!} \log_3(X)$$

where c_1, c_2 and c_3 are absolute constants. Thus we see that almost all integers in N_r has $\ell(n) > \log(n)$. In particular, k is not a P-integer whenever k is the product of twin primes.

Our third theorem verifies the conjecture of Pomerance for integers k being the products of the first few primes.

Theorem 3 *Let k be the product of the primes $\leq x$ for some $x \geq 2$. Then k is a P-integer if and only if $k \in \{2, 6, 30\}$.*

Finally, we formulate a statement concerning the solution of the problem of Pomerance for "small" values of k. Our main motivation of doing so is that this result will be very useful in the proof of Theorem 2.

Proposition 4 *Suppose that $1 < k < 5.5 \cdot 10^5$. Then k is a P-integer if and only if $k \in \{2, 4, 6, 12, 18, 30\}$.*

3 Lemmas

We need many lemmas of different types to prove our theorems. We shall make use of several estimates of Rosser and Schoenfeld [5] concerning various functions related to prime numbers. Further, we need certain results due to Stevens [6] and Hagedorn [1] about the Jacobsthal function. Finally, we need a theorem of Pomerance about primes in residue classes modulo m.

3.1 Lemmas concerning functions involving primes

The following four lemmas are estimates from Rosser and Schoenfeld [5] which we need later on.

Lemma 5 *Let p_n denote the n-th prime. Then*
(i) $p_n > n(\log(n) + \log_2(n) - \frac{3}{2})$ *for* $n > 1$*;*
(ii) $p_n < n(\log(n) + \log_2(n))$ *for* $n \geq 6$*.*

Lemma 6 *For any $x \geq 59$ we have*

$$\frac{x}{\log(x)}\left(1 + \frac{1}{2\log(x)}\right) < \pi(x) < \frac{x}{\log(x)}\left(1 + \frac{3}{2\log(x)}\right).$$

Lemma 7 *For $x \geq 2$ write $\vartheta(x) = \sum_{p \leq x} \log(p)$. For any $x \geq 563$ we have*

$$x\left(1 - \frac{1}{2\log(x)}\right) < \vartheta(x) < x\left(1 + \frac{1}{2\log(x)}\right).$$

Lemma 8 *For any $x > 1$ we have*

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) < \frac{0.56146}{\log(x)}\left(1 + \frac{1}{2\log^2(x)}\right).$$

Note that here 0.56146 could be replaced by any number exceeding $e^{-\gamma}$, where γ is Euler's constant.

3.2 Lemmas about the Jacobsthal function

For $n \geq 1$ the Jacobsthal function $g(n)$ is defined as the smallest integer such that any sequence of $g(n)$ consecutive integers contains an element which is coprime to n. This function has been studied by many authors, and good lower as well as upper bounds are known (see e.g. [6], [3] and [1] for history). Further, the exact values of $g(n)$ when n is the product of the first $h < 50$ primes is given in Table 1 of [1].

It was observed by Jacobsthal that for integers k with $\ell(k) > \log(k)$ we have $g(k) = \omega(k) + 1$. Further, $g(k) \geq \omega(k) + 1$ is obviously valid for any k. We shall use these assertions throughout the paper without any further reference.

Our first lemma concerning the Jacobsthal function is a reformulation of the Theorem of Stevens [6].

Lemma 9 *We have* $g(k) \le 2\omega(k)^{2+2e\log(\omega(k))}$ *for all* $k > 1$.

The next lemma is Proposition 1.1 of Hagedorn [1].

Lemma 10 *We have*

$$g\left(\prod_{i=1}^{h} p_i\right) \ge 2p_{h-1} \quad \textit{for} \ \ h > 2.$$

3.3 A result of Pomerance

Let k and l be positive integers with $\gcd(k,l) = 1$. Denote by $p(k,l)$ the least prime $p \equiv l \pmod{k}$. We write $P(k)$ for the maximal value of $p(k,l)$ for all l. Observe that k is a P-integer if and only if $P(k)$ equals the $\varphi(k)$-th prime not dividing k. Since the number of primes dividing k is $\omega(k)$, we get that if k is a P-integer then

$$p_{\varphi(k)} \le P(k) \le p_{\varphi(k)+\omega(k)}$$

holds. Note also that since $\varphi(k) + \omega(k) \le k$, we have $P(k) \le p_k$ whenever k is a P-integer.

To prove the finiteness of k's which are P-integers, Pomerance [3] derived a lower bound for $P(k)$ which (for large k) turns to be larger than standard upper bounds for $p_{\varphi(k)+\omega(k)}$, obtained by using estimates from [5]. This lower bound of Pomerance is based upon the following result from [3].

Lemma 11 *Let* k *and* m *be integers with* $0 < m \le \frac{k}{1+g(k)}$ *and* $\gcd(m,k) = 1$. *Then* $P(k) > (g(m) - 1)k$.

4 Proofs

Since in the proof of Theorem 2 we use Proposition 4, we start with the proof of the latter result.

Proof of Proposition 4. Let k be arbitrary with $1 < k < 5.5 \cdot 10^5$. Let $q_1 < q_2 < q_3 < \ldots$ be the primes $> tk$ with $t = 1$ if k is even and $t = 2$ if k is odd, respectively. We find the first index i such that $q_i - tk$ is a prime. For all k in the considered interval we found $i \le 34$. If $k+2$ is a prime then let $q = k+2$, otherwise set $q = q_i$ with the above defined index i. A calculation with Maple based upon Lemma 5 ensures that for $k > 210$ we have $q \le p_{\varphi(k)}$. Thus there exist two primes $\le p_{\varphi(k)}$ being coprime to k in the same residue class modulo k, which proves that k is not a P-integer in this case. Finally, for $k \le 210$ we check by Maple the first $\varphi(k)$ primes not dividing k to get the assertion of the proposition. $\square$

Proof of Theorem 2. Let k be a P-integer with $\ell(k) > \log k$. Assume first that $k \geq 10^{90}$. We split the proof of this case into two parts. Suppose first that $k < (\omega(k)+2)^{20}$. Then, since we know that $\omega(k)\log(\ell(k)) \leq \log(k)$, we obtain

$$\omega(k) \leq \frac{\log(k)}{\log_2(k)}.$$

Hence using our assumption for k we get

$$k < \left(\frac{\log(k)}{\log_2(k)} + 2\right)^{20}.$$

This implies that $k < 10^{90}$, which is a contradiction, and the statement follows in this case. Suppose next that we have $k \geq (\omega(k)+2)^{20}$. Let

$$h = \left\lfloor \frac{0.92\log(k)}{\log_2(k)} \right\rfloor + 1.$$

Then

$$h < \frac{0.946\log(k)}{\log_2(k)} < \log(k).$$

Hence by Lemma 5 (ii)

$$p_h < 0.946\log(k) < \log(k).$$

Let m be the product of the first h primes coprime to k. Since $p_h < \log(k) < \ell(k)$, by assumption, we see that m is indeed the product of all the first h primes. Hence

$$m < p_h^h < e^{0.946\log(k)} < \frac{k}{\omega(k)+2}$$

since we assumed $\omega(k)+2 \leq k^{\frac{1}{20}}$. Thus by Lemmas 10 and 11, we have

$$P(k) > (g(m)-1)k \geq (2p_{h-1}-1)k.$$

Now

$$h-1 \geq 0.92\frac{\log(k)}{\log_2(k)} - 1 > 0.894\frac{\log(k)}{\log_2(k)}.$$

Hence by Lemma 5 (i)

$$p_{h-1} \geq X\left(\log(X) + \log_2(X) - \frac{3}{2}\right)$$

where $X = 0.894\frac{\log(k)}{\log_2(k)}$. Let

$$F(k) = 2X\left(\log(X) + \log_2(X) - \frac{3}{2}\right)k - k\log(k) - k\log_2(k) - k.$$

Then $F(k) = k\log(k)f(k)$ with

$$f(k) := \frac{1.788}{\log_2(k)}\left(\log(X) + \log_2(X) - \frac{3}{2}\right) - 1 - \frac{\log_2(k)}{\log(k)} - \frac{1}{\log(k)}.$$

Observe that $f(k)$ is an increasing function of k and hence $f(k) \geq f(10^{90})$, since $k \geq 10^{90}$. As $f(10^{90}) \geq 0.0803$, we find that $F(k) > 0$ which implies that $P(k) > k\log(k) + k\log_2(k) > p_k \geq p_{\varphi(k)+\omega(k)}$. Hence k is not a P-integer. This contradiction proves the theorem for $k \geq 10^{90}$ with $\ell(k) > \log(k)$.

Assume now that $k < 10^{90}$. By Proposition 4 we may suppose that $5.5 \cdot 10^5 \leq k < 10^{90}$. We divide the interval $[5.5 \cdot 10^5, 10^{90})$ into sub-intervals and assign a value h to each interval as follows. Let $v_0 = 10^{90}$. The largest integer h such that $p_h < \log(10^{90})$ is 46. We set our initial sub-interval as $[u_0, v_0) = [10^{87}, 10^{90})$, $\alpha_0 = 87$ and $h_0 = h = 46$. For any k with $\ell(k) > \log(k)$ in this interval we have $g(k) = \omega(k) + 1 < \log(k) + 1 < 209$. We check that

$$m_0 := \prod_{j=1}^{46} p_j < \frac{10^{87}}{210} \leq \frac{k}{g(k)+1}.$$

Now we proceed inductively. Let $i \geq 1$ and take $h_i = h_0 - i$. We define the sub-interval $[u_i, v_i)$ as $[10^{\alpha_i}, 10^{\alpha_{i-1}})$ satisfying the following properties:

$$p_{h_i} < \alpha_i \log(10) \tag{1}$$

and

$$m_i := \prod_{j=1}^{h_0-i} p_j < \frac{10^{\alpha_i}}{(\alpha_{i-1}\log(10) + 2)}. \tag{2}$$

Let $k \in [u_i, v_i)$ with $\ell(k) > \log(k)$. Then $p_{h_i} < \log(k)$ and hence by the assumption on k, m_i is the product of the first h_i primes, and $\gcd(m_i, k) = 1$. Suppose that

$$g(m_i) - 1 - \alpha_{i-1}\log(10) - \log(\alpha_{i-1}\log(10)) > 0. \tag{3}$$

Then, since $k \leq 10^{\alpha_{i-1}}$, we find by Lemma 11 and Lemma 5 (ii) that

$$P(k) > k\log(k) + k\log_2(k) > p_k \geq p_{\varphi(k)+\omega(k)}$$

and hence $k \in [u_i, v_i)$ is not a P-integer.

In Table 1 we give the values $h_i = h$, $\alpha_i = \alpha$, and the exact value of $g(m)$ with $m = m_i$ from Table 1 of [4]. For these values, we check that (1), (2) and (3) are satisfied and hence we conclude that $k < 10^8$. Now consider k in Table 1

h	7	8	9	10	11	12	13	14	15	16
$g(m)$	26	34	40	46	58	66	74	90	100	106
α	8	9	10	13	14	17	18	19	21	24
h	17	18	19	20	21	22	23	24	25	26
$g(m)$	118	132	152	174	190	200	216	234	258	264
α	26	27	30	31	32	35	37	39	43	44
h	27	28	29	30	31	32	33	34	35	36
$g(m)$	282	300	312	330	354	378	388	414	432	450
α	45	47	48	55	56	57	60	61	65	66
h	37	38	39	40	41	42	43	44	45	46
$g(m)$	476	492	510	538	550	574	600	616	642	660
α	69	71	73	76	78	79	83	84	86	87

the intervals $[3 \cdot 10^7, 10^8)$ with $h = 7$ and $[5.5 \cdot 10^5, 3 \cdot 10^7)$ with $h = 6$ and $g(m) = 22$, respectively. Then conditions (1), (2) and (3) are satisfied again, showing that k is not a P-integer. Hence the statement follows. $\square$

Proof of Theorem 3. Assume first that $x \geq 1000$ and put $k = \prod_{p \leq x} p$. Set $m := \prod_{x<p\leq y} p$ with $y = 1.777x$. First we show that by these choices we have $m \leq k/(1 + g(k))$. This inequality can be rewritten as

$$1 + g(k) \leq \frac{\exp(2\vartheta(x))}{\exp(\vartheta(y))}.$$

Using Lemma 9, it is sufficient to show that

$$1 + 2\pi(x)^{2+2e\log(\pi(x))} \leq \exp(2\vartheta(x) - \vartheta(1.777x)).$$

With the help of Maple, by Lemmas 6 and 7 this can be seen to be true whenever $x \geq 12000$. For $1000 \leq x < 12000$ the assertion can be checked by calculating the exact values of the functions $\pi(x)$ and $\vartheta(x)$.

Now we show that (still with $x \geq 1000$) we have $(g(m) - 1)k \geq p_{\varphi(k)+\omega(k)}$. By Lemma 11 this implies the statement. To prove this, observe that $g(m) >$

$\omega(m) = \pi(y) - \pi(x)$. Hence using Lemma 5 (ii) it is sufficient to check that

$$\pi(1.777x) - \pi(x) \geq \left(\prod_{p \leq x} \left(1 - \frac{1}{p}\right) + \frac{\pi(x)}{\prod_{p \leq x} p} \right) (\vartheta(x) + \log(\vartheta(x)))$$

for $x \geq 1000$. Again, by the help of Lemmas 6, 7 and 8 this inequality can be verified for $x \geq 12000$ with Maple. Further, for $1000 \leq x < 12000$ the assertion can be proved by calculating the exact values of the expressions involved. Hence the statement is valid when $x \geq 1000$.

Assume now that $x < 1000$. Then we check the values of k one by one. For k given, let $q_1 = p_{\pi(k)+1}$ and $q_2 = p_{\pi(k)+2}$. A calculation by Maple shows that for $k > 30$ we have $q_2 \leq p_{\varphi(k)+\omega(k)}$, and also that one of $q_1 - k$, $q_2 - k$ is a prime. Finally, as $k = 2, 6, 30$ are P-integers indeed, the statement follows. □

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