A FUNCTIONAL EQUATION AND ITS APPLICATION TO THE CHARACTERIZATION OF GAMMA DISTRIBUTIONS

FRUZSINA MÉSZÁROS

Dedicated to the 70th birthday of Professor Zoltán Daróczy

ABSTRACT. The functional equation

$$f\,(x)\,g\,(y) = p\,(x+y)\,q\left(\frac{x}{y}\right)$$

is investigated for almost all $(x,\,y) \in \mathbb{R}_+^2$ and for the measurable functions f, g, p, $q : \mathbb{R}_+ \to \mathbb{R}_+$. This equation is related to the Lukács characterization of the gamma distribution.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+$ be the set of positive real numbers. The functional equation

$$f\,(x)\,g\,(y) = p\,(x+y)\,q\left(\frac{x}{y}\right), \qquad (x,y) \in \mathbb{R}_+^2 \tag{1}$$

with f, g, p, $q : \mathbb{R}_+ \to \mathbb{R}$, was investigated by J. A. Baker in [1] and by K. Lajkó in [10], where the results of [2] and [3] were applied.

The purpose of this paper is to determine all measurable solutions f, g, p, $q : \mathbb{R}_+ \to \mathbb{R}_+$ of equation (1) satisfied almost everywhere. This result can be used for a characterization of gamma distributions. Our proof is based on the following theorem of A. Járai (see [5], [6]):

Theorem 1 (Járai)**.** *Let Z be a regular topological space, Z_i $(i = 1, 2, \ldots, n)$ be topological spaces and T be a first countable topological space. Let Y be an open subset of $\mathbb{R}^k$, X_i an open subset of $\mathbb{R}^{r_i}$, $(i = 1, 2, \ldots, n)$ and D an open subset of $T \times Y$. Let furthermore $T' \subset T$ be a dense subset, $F\colon T' \to Z$, $g_i\colon D \to X_i$ and $h\colon D \times Z_1 \times \ldots \times Z_n \to Z$. Suppose that the function f_i is almost everywhere defined on X_i (with respect to the r_i-dimensional Lebesgue measure) with values in Z_i $(i = 1, 2, \ldots n)$ and the following conditions are satisfied:*

(1) for all $t \in T'$ and for almost all $y \in D_t = \{y \in Y : (t,y) \in D\}$

$$F(t) = h(t, y, f_1(g_1(t,y)), \ldots, f_n(g_n(t,y))); \tag{2}$$

(2) for each fixed y in Y, the function h is continuous in the other variables;

(3) f_i is Lebesgue measurable on X_i $(i = 1, 2, \ldots, n)$;

2000 *Mathematics Subject Classification.* 39B22, 60E05, 62E10.

Key words and phrases. Gamma distributions, characterizations of probability distributions, measurable solution a. e.

This research has been supported by the Hungarian Scientific Research Fund (OTKA), Grant NK 68040.

(4) g_i *and the partial derivative* $\frac{\partial g_i}{\partial y}$ *are continuous on* D $(i = 1, 2, \dots, n)$*;*
(5) for each $t \in T$ *there exist a* y *such that* $(t, y) \in D$ *and the partial derivative* $\frac{\partial g_i}{\partial y}$ *has the rank* r_i *at* $(t, y) \in D \quad (i = 1, 2, \dots, n)$.

Then there exists a unique continuous function $\widetilde{F}$ *such that* $F = \widetilde{F}$ *almost everywhere on* T*, and if* F *is replaced by* $\widetilde{F}$ *then equation* (2) *is satisfied almost everywhere on* D.

2. The measurable solution of (1) for a.e. $(x, y) \in \mathbb{R}_+^2$

First we prove the following

Lemma 1. *If the measurable functions* $f, g, p, q : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy equation* (1) *for almost all* $(x, y) \in \mathbb{R}_+^2$*, then there exist unique continuous functions* $\tilde{f}, \tilde{g}, \tilde{p}, \tilde{q} : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\tilde{f} = f$*,* $\tilde{g} = g$*,* $\tilde{p} = p$ *and* $\tilde{q} = q$ *almost everywhere, and if* f, g, p, q *are replaced by* $\tilde{f}, \tilde{g}, \tilde{p}, \tilde{q}$ *respectively, then* (1) *is satisfied everywhere on* $\mathbb{R}_+^2$.

Proof. First, by the help of Járai's Theorem, we prove that there exist unique continuous function $\tilde{p}$ which is almost everywhere equal to p on $\mathbb{R}_+$ and replacing p by $\tilde{p}$, equation (1) is satisfied almost everywhere.

With the substitution $t = x + y$ we get from (1) the equation

$$p\,(t) = \frac{f\,(t - y)\,g\,(y)}{q\left(\frac{t}{y} - 1\right)} \tag{3}$$

which is satisfied for almost all $(t, y) \in D$, where $D = \left\{(t, y) \in \mathbb{R}_+^2 \,|y < t\right\}$. By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}_+$ of full measure such that for all $t \in T'$ equation (3) is satisfied for almost every $y \in D_t = \{y \in \mathbb{R}_+ \,|(t, y) \in D\} = (0, t)$.

Let us define the functions g_1, g_2, g_3, h in the following way:

$$g_1\,(t, y) = t - y, \quad g_2\,(t, y) = y,$$
$$g_3\,(t, y) = \frac{t}{y} - 1, \quad h\,(t, y, z_1, z_2, z_3) = \frac{z_1 z_2}{z_3},$$

and let us now apply Theorem 1 of Járai to (3) with a suitable casting.

Hence the first assumption in Theorem 1 with respect to (3) holds. It is obvious that the conditions (2)–(5) of Theorem 1 are satisfied. So we get from Theorem 1 that there exists unique continuous function $\tilde{p}$ which is almost everywhere equal to p on $\mathbb{R}_+$ and $f, g, \tilde{p}, q$ satisfy equation (1) almost everywhere, which is equivalent to the equation

$$f\,(x)\,g\,(y) = \tilde{p}\,(x + y)\,q\left(\frac{x}{y}\right) \tag{4}$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

By a similar argument we can prove the same for the function q. From equation (4) with the substitution $t = \frac{x}{y}$ we get the equation

$$q\,(t) = \frac{f\,(ty)\,g\,(y)}{\tilde{p}\,(y\,(t + 1))},$$

which with a suitable casting

$$g_1\,(t, y) = ty, \quad g_2\,(t, y) = y,$$

$$g_3(t,y) = y(t+1), \quad h(t,y,z_1,z_2,z_3) = \frac{z_1 z_2}{z_3},$$

by Fubini's Theorem, and the fact that the assumptions of Theorem 1 are fulfilled again, gives us that there exists unique continuous function $\tilde{q}$ which is almost everywhere equal to q on $\mathbb{R}_+$ and $f, g, \tilde{p}$, $\tilde{q}$ satisfy equation (1) almost everywhere, i.e.

$$f(x)\,g(y) = \tilde{p}(x+y)\,\tilde{q}\left(\frac{x}{y}\right) \tag{5}$$

for almost all $(x,y) \in \mathbb{R}_+^2$.

There exist such x_0 and y_0 so that with the substitutions $x = x_0$ and $y = y_0$, respectively, we get from equation (5) that

$$f(x) = \frac{1}{g(y_0)}\tilde{p}(x+y_0)\,\tilde{q}\left(\frac{x}{y_0}\right)$$

holds for almost all $x \in \mathbb{R}_+$, and

$$g(y) = \frac{1}{f(x_0)}\tilde{p}(x_0+y)\,\tilde{q}\left(\frac{x_0}{y}\right)$$

holds for almost all $y \in \mathbb{R}_+$. Since $\tilde{p}, \tilde{q}$ are continuous, therefore there exist unique continuous functions $\tilde{f}$ and $\tilde{g}$, defined by the right-hand side of the last two equality, which are almost everywhere equal to f and g on $\mathbb{R}_+$, respectively, and if we replace f and g by $\tilde{f}$ and $\tilde{g}$, respectively, then the functional equation

$$\tilde{f}(x)\,\tilde{g}(y) = \tilde{p}(x+y)\,\tilde{q}\left(\frac{x}{y}\right) \tag{6}$$

is satisfied almost everywhere on $\mathbb{R}_+^2$.

Both side of (6) define continuous functions on $\mathbb{R}_+^2$, which are equal to each other on a dense subset of $\mathbb{R}_+^2$, therefore we obtain that (6) is satisfied everywhere on $\mathbb{R}_+^2$.

Further $f = \tilde{f}$, $g = \tilde{g}$, $p = \tilde{p}$, $q = \tilde{q}$ almost everywhere on their domains, respectively. $\square$

Hence, by the help of the measurable (continuous) solutions of equation (1) satisfied everywhere, we can give the solutions of the almost everywhere satisfied functional equation.

Theorem 2. *Suppose that the measurable (continuous) functions* $f, g, p, q : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy functional equation* (1) *almost everywhere, then*

$$\begin{aligned} f(x) &= A\exp[ax + b\ln x] && a.e. && x \in \mathbb{R}_+, \\ g(x) &= B\exp[ax + (c-b)\ln x] && a.e. && x \in \mathbb{R}_+, \\ p(x) &= C\exp[ax + c\ln x] && a.e. && x \in \mathbb{R}_+, \\ q(x) &= D\exp[b\ln x - c\ln(x+1)] && a.e. && x \in \mathbb{R}_+, \end{aligned}$$

where $a, b, c \in \mathbb{R}$ *and* $A, B, C, D \in \mathbb{R}_+$ *are arbitrary constants with* $AB = CD$.

Proof. The measurable (continuous) solutions of equation (6) satisfied everywhere (see [1] and [10]) and the previous lemma immediately gives our statement. $\square$

3. On characterization of gamma distributions

Let X be a gamma-distributed random variable with parameters λ and p, where λ and p are fixed positive numbers.

It is known that its density function is

$$f(x) = \begin{cases} \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0, \end{cases}$$

where

$$\Gamma(p) = \int_0^\infty u^{p-1} e^{-u}\, du$$

is the gamma function.

The well-known Lukács characterization of gamma distributions is the following: The independent random variables X and Y are gamma distributed (with the same scale parameter) if and only if $X + Y$ and $\frac{X}{Y}$ are independent (see [9] and [11]).

We shall use the transformation

$$\psi : \mathbb{R}_+^2 \to \mathbb{R}_+^2, \quad \psi(x, y) = \left(x + y, \frac{x}{y} \right). \tag{7}$$

Let X, Y be absolutely continuous and independent random variables with range in $\mathbb{R}_+$. Let us denote the densities by f_X, f_Y respectively. Then, by the Transformation Theorem (see [4]), the random variable $(U, V) = \psi(X, Y)$ is absolutely continuous with density function g defined by

$$g(u, v) := \frac{u}{(v+1)^2} f_X \left(\frac{uv}{v+1} \right) f_Y \left(\frac{u}{v+1} \right) \tag{8}$$

for almost all $(u, v) \in \mathbb{R}_+^2$.

If U and V are independent with density functions f_U, f_V respectively, then from (8) we get the functional equation

$$f_U(u) f_V(v) = \frac{u}{(v+1)^2} f_X \left(\frac{uv}{v+1} \right) f_Y \left(\frac{u}{v+1} \right) \qquad \text{a.e. } (u, v) \in \mathbb{R}_+^2 \tag{9}$$

for unknown density functions $f_X, f_Y, f_U, f_V : \mathbb{R}_+ \to \mathbb{R}_+$.

This equation can be transformed easily to equation

$$f(x)\, g(y) = p(x + y)\, q\left(\frac{x}{y} \right)$$

for almost all $(x, y) \in \mathbb{R}_+^2$, which was investigated before.

From Theorem 2, we get that

$$f(x) = A \exp[ax + b \ln x] \qquad \text{a.e. } x \in \mathbb{R}_+,$$

and

$$g(x) = B \exp[ax + (c - b) \ln x] \qquad \text{a.e. } x \in \mathbb{R}_+.$$

Hence

$$f_X(x) = A x^b e^{ax} \qquad \text{a.e. } x \in \mathbb{R}_+,$$

and

$$f_Y(x) = B x^{c-b-2} e^{ax} \qquad \text{a.e. } x \in \mathbb{R}_+,$$

where $-a$, $b + 1$, c, $A, B \in \mathbb{R}_+$ are arbitrary constants. That is, X is gamma distibuted with parameters $-a$ and $b+1$, Y is gamma distributed with parameters $-a$ and $c - b - 1$.

Remark 1. *Moreover, from Theorem 2, we get that*

$$f_U(x) = Cx^{c-1}e^{ax} \qquad a.e.\ x \in \mathbb{R}_+,$$

where $-a$*,* c*,* $C \in \mathbb{R}_+$ *are arbitrary constants. Hence* $U = X + Y$ *is gamma distributed with parameters* $-a$ *and* c*. This provides us the known property, that is, the sum of two gamma distributed independent random variables (with a common scale parameter) is gamma distributed as well.*

Remark 2. *The density function of* $V = X/Y$ *is*

$$f_V(x) = Dx^b(1+x)^{-c} \qquad a.e.\ x \in \mathbb{R}_+,$$

where $b+1$*,* c*,* $D \in \mathbb{R}_+$ *are arbitrary constants. Hence* V *has beta distribution of the second kind with parameters* $b+1$ *and* $c-b-1$*. (The density function of a beta distribution of the second kind with parameters* α *and* β *is* $f(x) = \frac{1}{B(\alpha,\beta)}x^{\alpha-1}(1+x)^{-\alpha-\beta}$*, where* α*,* β *are fixed positive numbers and* B *is the beta function, see* [7]*.)*

This yields again a property of the gamma distributed random variables: If X *and* Y *are independent gamma variates with shape parameters* α_1 *and* α_2*, respectively, then* $\frac{X}{Y}$ *has a beta distribution of the second kind with shape parameters* α_1 *and* α_2*.*

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Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P. O. Box 12, Hungary

E-mail address: `mefru@math.klte.hu`