Preprint typeset in JHEP style - HYPER VERSION

Phase Structure and Compactness

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Abstract: In order to study the influence of compactness on low-energy properties, we compare the phase structures of the compact and non-compact two-dimensional multi-frequency sine-Gordon models. It is shown that the high-energy scaling of the compact and non-compact models coincides, but their low-energy behaviors differ. The critical frequency $\beta^2 = 8\pi$ at which the sine-Gordon model undergoes a topological phase transition is found to be unaffected by the compactness of the field since it is determined by high-energy scaling laws. However, the compact two-frequency sine-Gordon model has first and second order phase transitions determined by the low-energy scaling: we show that these are absent in the non-compact model.

Keywords: Field Theories in Lower Dimensions, Renormalization Group.

Contents

1. Introduction

The computation in a completely controlled way of renormalization group (RG) flows in gauge theories is at date a challenging issue. A major reason for such difficulties is the fact that, one must adopt a regularization scheme which incorporates a gauge invariant cutoff even for approximated treatments of exact RG equations. A related problem is the determination of critical properties and phases of compact field theories, since, compactness can be considered as one of the simplest realization of the gauge symmetry [1] and a general treatment for the controlled computation of renormalization flows in gauge theories would apply to the RG study of compact field theories.

Sic stantibus rebus, in order to study the influence of compactness on low-energy properties, it would be then relevant to compare the phase structure of a field theory with the fields being respectively compact and non-compact. In this paper we perform such comparison for the multi-frequency sine-Gordon (SG) model in $1+1$ dimensions. Several reasons lead us to the choice and use of such a model for the purpose of studying the effects of compactness: first, the single-frequency SG is a paradigmatical example of integrable field theory [2], very well studied in the last four decades. Second, the rich phase structure of the compact double-frequency SG has been the subject of intense study [3, 4, 5, 6, 7, 8, 9], and, last but not least, the non-compact multi-frequency SG model (MFSG) can be studied using non-perturbative RG methods [10].

The importance of the SG model stems from the fact that it is directly related to interacting fermionic field theories through bosonization [11]. In low dimensions exact bosonization rules enable one to reformulate fermionic and gauge models in terms of elementary scalar fields. The equivalence between the massive Thirring model and the sine-Gordon (SG) scalar theory [12] is a well-known example. Two-dimensional gauge models like the multi-flavor quantum electrodynamics (QED_2) [13, 14, 15] and the single-flavor quantum chromodynamics (QCD_2) [16] can also be rewritten as a multi-component SG theory where the SG fields are coupled by a mass-matrix. It has been also shown that various aspects of the low-energy QCD_2 with multi-flavors (and with unequal quark masses) can be described by the so-called generalized SG model [17] of which reduced sub-model is the MFSG model with non-compact field variable.

Moreover, the SG model, the simplest non-trivial quantum field theory which can be used to study confinement phenomena, has already received a considerable attention in several areas of physics. For example, in string theory the SG model is assumed to be related to the classical string on specific manifolds [18] and the possible contribution of new type of SG models to brane profile has also been studied [19] in the framework of Randall-Sudrum [20] theory. SG type model has recently been investigated in 3+1 dimensions in the context of axion physics [21]. Furthermore, the SG model is used as a textbook example for integrable systems and it has many applications in condensed matter and statistical physics as well, e.g. coupled SG models were successfully used to describe the vortex dynamics of layered high transition temperature superconductors [22]. Another attractive property of low-dimensional SG models is that they provide us an excellent playground to test and compare various types of non-perturbative methods [23]. For example, SG type models have already been investigated in the framework of Integrable and Conformal Field Theory (CFT) [24] and the exact functional RG treatment for these periodic models has also been developed [25, 22, 15, 14, 26].

Our goal in this paper is to consider the non-compact MFSG model by means of the functional RG approach and to compare our findings to those obtained by other methods for the non-compact and compact models, as well. In particular, we investigate the influence of the compact or non-compact nature of the field variable on the low-energy behavior of the MFSG model whose action reads as [3, 4, 5, 6, 7, 8, 9]

$$\mathcal{S}_{\rm MFSG} = \int \mathrm{d}^2 x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \sum_i^n \mu_i \cos(\beta_i \phi + \delta_i) \right] \tag{1.1}$$

which contains n cosine terms where ϕ is a real scalar field, $\beta_i \in \mathbb{R}$ are the frequencies, $\beta_i \neq \beta_j$ if $i \neq j$, μ_i are the coupling constants (of dimension mass^2 at the classical level) and $\delta_i \in \mathbb{R}$ are the phases in the terms of the potential. Let us note that the MFSG model is usually defined on the two-dimensional Minkowski space, however, in this paper we use the Euclidean action which is more convenient for an RG study and it is generally assumed to be suitable for mapping out the phase structure of the model.

Two cases can be distinguished according to the periodicity properties of the model. The first one is the rational case, when the potential is a trigonometric function: the ratios of the frequencies β_i are rational and consequently, the potential is periodic. Let the period of the potential be $2\pi\beta$ in this case. Then the target space of the field ϕ can be compactified: $\phi \equiv \phi + 2k\beta\pi$, where $k \in \mathbb{N}$ can be chosen arbitrarily. The model obtained in this way is called the k-folded multi-frequency SG model [5]. The other case is the irrational one, when the potential is not periodic. We restrict our attention to the rational case in the present paper.

At the quantum level the theory can be considered as a perturbation of its high-energy/ultraviolet (UV) limiting conformal field theory [3, 4, 5, 6, 7, 8, 9]

$$\mathcal{S}_{\rm MFSG} = \mathcal{S}_{\rm CFT} + \mathcal{S}_{\rm pert}, \tag{1.2}$$

where

$$\mathcal{S}_{\rm CFT} = \int \mathrm{d}^2 x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi,$$
$$\mathcal{S}_{\rm pert} = -\frac{1}{2} \int \mathrm{d}^2 x \sum_{i=1}^{n} (\mu_i e^{i\delta_i} V_{\beta_i} + \mu_i e^{-i\delta_i} V_{-\beta_i}),$$

with the vertex operator $V_\omega =: e^{i\omega\phi}$: which corresponds to a primary field with conformal dimensions $\Delta^{\pm}_{\omega} = \Delta_\omega = \frac{\omega^2}{8\pi}$ in the UV limit and the upper index $\pm$ corresponds to the left/right conformal algebra and : : denotes the conformal normal ordering. Correspondingly, the dimensions of the couplings in the UV limit at the quantum level are $[\mu_i] = ({\rm mass})^{2-2\Delta_i}$ with $\Delta_i \equiv \Delta_{\beta_i}$.

It was shown by semiclassical (mean-field/Landau-Ginzburg) analysis [3] and by means of form factor perturbation and truncated conformal space approaches [3, 6, 8, 9] that (first and second order) phase transitions occur in the compact MFSG model as the coupling constants are tuned appropriately (assuming that $n > 1$). For example, according to the semi-classical results [3], the double-frequency SG model (for $\delta_1 = 0$, $\delta_2 = \pi/2$ and $\beta_2 = \beta_1/2$)

$$V_{\rm DFSG}(\phi) = -\mu_1 \cos(\beta_1 \phi) + \mu_2 \sin\left(\frac{\beta_1}{2}\phi\right) \tag{1.3}$$

undergoes a second order (Ising-type) phase transition at $\mu_2 = 4\mu_1$ [3]. This second order phase transition was found to appear for all frequencies $0 < \beta^2 < 8\pi$ beyond the semi-classical level, see e.g. the phase diagram in Fig. 7.5 of [6] which was determined by form factor perturbation theory and truncated conformal space approach. The Ising-type phase transition was also confirmed by renormalization group techniques based on operator product expansion in real space [27]. Let us note, that in this case the field variable is defined as a compact variable. It was also argued that the MFSG model reduces to the classical (single-frequency) SG model in the limit of $\delta_i \to 0$ for $i = 1, 2, ..., \infty$ and $\mu_i \to 0$ for $i = 2, 3, ..., \infty$. It was also shown that the SG model defined by the action which contains non-compact field variable [25, 26] belongs to the universality class of the two-dimensional Coulomb gas and the two-dimensional XY model, consequently, its phase transition at $\beta_c^2 = 8\pi$ is a topological or Kosterlitz-Thouless-Berezinskii (KTB) type one [28]. It is also known that the classical SG model with a compact field variable also possesses a topological phase transition at $\beta_c^2 = 8\pi$. Therefore, on the one hand, the MFSG model with compact field variable has Ising-type phase transitions (for $n > 1$), on the other hand in case of a single cosine (for $n = 1$) with compact and non-compact fields the model has a topological phase transition. Consequently, it represents an excellent toy model to study the influence of the compactness on the phase structure and the low-energy behavior of the model.

Our goal is to compare the UV/IR scaling behavior and the phase structure of the MFSG models with compact and non-compact field variables: we study the MFSG theory with non-compact fields by means of the functional RG method in the local potential approximation, discussing the comparison with the available results for the compact model [3, 4, 5, 6, 7, 8, 9, 27]. The structure of our paper is the following: a brief introduction of RG equations used for the renormalization of the non-compact MFSG model is given in Section 2. In Section 3, the connection between RG equations and symmetries of the MFSG model is discussed. The UV and IR scaling laws of the non-compact MFSG model are determined and compared to those of the compact model in Sections 4 and 5, respectively. Finally, Section 6 presents the summary and our concluding remarks.

2. Renormalization Group Approach

In this section we briefly discuss the functional RG equations used for the renormalization of the MFSG model. The differential RG transformations are realized via a blocking construction [29], the successive elimination of the degrees of freedom which lie above the running UV momentum cutoff k. Consequently, the effective theory described by the blocked action contains quantum fluctuations whose frequencies are smaller than the momentum cutoff. This procedure generates the functional RG flow equation [30, 31, 32]

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2)}[\phi] + R_k\right)^{-1} k\partial_k R_k$$

for the effective action $\Gamma_k[\phi]$ when various types of regulator functions R_k are used, where $\Gamma_k^{(2)}[\phi]$ denotes the second functional derivative of the effective action (see e.g. [10]). Here R_k is a properly chosen IR regulator function which fulfils a few basic constraints to ensure that Γ_k approaches the bare action in the UV limit ($k \to \Lambda$) and the full quantum effective action in the IR limit ($k \to 0$). Indeed, various renormalization schemes are constructed in such a manner that the RG flow starts at the bare action and provides the effective action in the IR limit, so that the physical predictions (e.g. fixed points and critical exponents) are independent of the renormalization scheme particularly used [35, 33, 34].

Since RG equations are functional partial differential equations it is not possible to solve them in general, hence, approximations are required. One of the commonly used systematic approximation is the truncated derivative expansion where the effective action is expanded in powers of the derivative of the field [35, 33, 34],

$$\Gamma_k[\phi] = \int_x \left[V_k(\phi) + Z_k(\phi)\frac{1}{2}(\partial_\mu\phi)^2 + ...\right].$$

In the local potential approximation (LPA) higher derivative terms are neglected and the wave-function renormalization is set equal to constant, i.e. $Z_k \equiv 1$. In this paper we use two types of RG equations (i.e. two different IR regulators R_k), namely the Wegner-Houghton [36] and the Polchinski [37] RG approaches. However, let us note that in the LPA, the two-dimensional Wegner-Houghton RG equation is mathematically equivalent (see e.g. [33]) to the effective average action RG equation [30, 31] with the power-law regulator $R_k(p^2) \equiv p^2(p^2/k^2)^{-b}$ [32] with $b = 1$ and the functional Callan-Symanzik RG equation [38].

2.1 Wegner–Houghton, effective average action and functional Callan-Symanzik RG equations

In this section, we consider three types of RG equations, namely the Wegner–Houghton, the effective average action with power-law regulator ($b = 1$) and the functional Callan-Symanzik RG equations which have the same form in LPA for $d = 2$ dimensions [33].

The blocking in momentum space, i.e. the integration over the field fluctuations with momenta of the magnitude between the UV scale Λ and zero is performed in successive blocking steps over infinitesimal momentum intervals $k \to k - \Delta k$ each of which consists of the splitting the field variable, $\phi = \varphi + \phi'$ in such a manner that φ and ϕ' contain Fourier modes with $|p| < k - \Delta k$ and $k - \Delta k < |p| < k$, respectively and the integration over ϕ' leads to the Wegner–Houghton (WH) RG equation [36]

$$(2 + k\partial_k)\,\tilde{V}_k(\phi) = -\frac{1}{4\pi}\ln\left(1 + \tilde{V}_k''(\phi)\right) \tag{2.1}$$

with $\tilde{V}_k''(\phi) = \partial_\phi^2 \tilde{V}_k(\phi)$ for the dimensionless local potential $\tilde{V}_k = k^{-2}V_k$ for $d = 2$ dimensions in the leading order of the derivative expansion, in the LPA when ϕ reduces to a constant. (Below we suppress the notation of the field-dependence of the local potential and use notations with tilde for dimensionless quantities where the dimension is taken away by the appropriate power of the gliding cutoff k.) The differentiation with respect to the field variable and the multiplication with $1+\tilde{V}_k''$ leads to the derivative form of the WH–RG equation [33]

$$(2 + k\partial_k)\tilde{V}_k' = -\tilde{V}_k''(2 + k\partial_k)\tilde{V}_k' - \frac{1}{4\pi}\tilde{V}_k'''. \tag{2.2}$$

This equation is obtained by assuming the absence of instabilities for the modes around the gliding cutoff k. The WH-RG scheme which uses the sharp gliding cutoff k can also account for the spinodal instability, which appears when the restoring force acting on the field fluctuations to be eliminated vanishes, $1 + \tilde{V}_k''(\phi) = 0$ at some finite scale $k_{\rm SI}$ and the resulting condensate generates tree-level contributions to the evolution equation. The saddle point ϕ_0' for the single blocking step $k \to k-\Delta k$ is obtained by minimizing the action, $S_{k-\Delta k}[\phi] = \min_{\phi_0'}\left(S_k[\phi + \phi_0']\right)$. The restriction of the space of saddle-point configurations to that of the plane waves $\phi_0' = \rho\cos(k_1 x)$ gives [39]

$$\tilde{V}_{k-\Delta k}(\phi) = \min_{\rho}\left[\rho^2 + \frac{1}{2}\int_{-1}^{1} du\tilde{V}_k(\phi + 2\rho\cos(\pi u))\right] \tag{2.3}$$

in LPA, where the minimum is sought for the amplitude ρ only. It was shown that the tree-level RG equation (2.3) leads to the local potential [33]

$$\tilde{V}_{k\to 0} = -\frac{1}{2}\phi^2 + c\phi + \text{const}, \tag{2.4}$$

which can also be obtained as the solution of $1 + \tilde{V}''_{k\to 0}(\phi) = 0$ (in case of a $\phi \to -\phi$ symmetry the linear term vanishes, $c = 0$). Therefore, if SI occurs during the RG flow at some scale $k_{\rm SI} > 0$, then Eqs. (2.1) or (2.2) should be applied only for scales $k > k_{\rm SI}$, and the tree-level renormalization Eq.(2.3) or Eq.(2.4) should be performed at scales $k < k_{\rm SI}$.

For $d = 2$ dimensions the effective average action (EAA) RG equation with power-law regulator can be written in the LPA as

$$(2 + k\partial_k)\tilde{V}_k = -\frac{1}{4\pi}\int_0^{\Lambda^2/k^2} {\rm d}y \frac{(-b)y^{-b}\, y}{y(1+y^{-b}) + \tilde{V}''_k} \tag{2.5}$$

with $y = p^2/k^2$. For arbitrary parameter value b, the propagator on the right hand side of Eq. (2.5) may develop a pole at some scale $k_{\rm SI}$ and at some value of the field ϕ for which $\tilde{V}''_k(\phi) = -C(b) = -b/(b-1)^{(b-1)/b}$ holds, which signals the occurring of SI. The infrared singularity of the functional RG equation is supposed to be related to the convexity of the effective action for theories within a phase of spontaneous symmetry breaking [31]. It was shown that in such a case one has to seek the local potential for $k < k_{\rm SI}$ by minimizing Γ_k in the subspace of inhomogeneous (soliton like) field configurations and ends up with the result [31, 10]

$$\tilde{V}_{k\to 0} = -\frac{1}{2}C(b)\phi^2 + c\phi + \text{const.} \tag{2.6}$$

It is worthwhile noticing that Eq. (2.5) with the power-law regulator leads to the WH-RG equation (2.1), and Eq. (2.6) leads to Eq. (2.4) for $b = 1$ as well as for $b \to \infty$ in the limit $\Lambda \to \infty$. This feature holds only for $d = 2$.

In the functional Callan-Symanzik (CS) type internal space RG method [38], the successive elimination of the field fluctuations is performed in the space of the field variable (internal space) as opposed to the usual RG methods where the blocking transformations are realized in either the momentum or the real (external) space. The functional CS–RG equation for the one-component scalar field theory for dimensions $d = 2$ in the LPA reads

$$(2 + \lambda\, \partial_\lambda)\, \tilde{V}_\lambda = -\tfrac{1}{4\pi} \ln\left(1 + \tilde{V}''_\lambda\right) \tag{2.7}$$

with the control parameter λ. This equation is mathematically equivalent to the two-dimensional WH–RG equation in the LPA assuming the equivalence of the scales $\lambda \equiv k$. However, for dimensions $d \neq 2$ the functional CS–RG and the WH–RG differ from each other. Assuming the above mentioned equivalence of the scales λ and k, there occurs the same singularity in the right hand side of (2.7) as the one in the WH-RG approach. Therefore, the functional CS–RG signals the SI with the vanishing of the argument of the logarithm in the right hand side of (2.7). The solution of (2.7) provides the scaling laws down to the scale $k_{\rm SI}$ and one has to turn to the tree-level renormalization with the help of the WH-RG approach in order to determine the IR scaling laws.

Since the WH–RG, EAA–RG with power-law regulator ($b = 1$) and the functional CS–RG equations have the same form in LPA for $d = 2$ dimensions, in this paper we refer to them as the WH–RG equation.

2.2 Polchinski's RG equation

In Polchinski's RG (P–RG) method [37] the realization of the differential RG transformations is based on a non-linear generalization of the blocking procedure using a smooth momentum cutoff. In the infinitesimal blocking step the field variable ϕ is split again into the sum of a slowly oscillating IR and a fast oscillating UV components, but both fields contain now low- and high-frequency modes, as well, due to the smoothness of the cutoff. Above the moving momentum scale k the propagator for the IR component is suppressed by a properly chosen smooth regulator function $K(y)$ with $y = p^2/k^2$, $K(y) \to 0$ if $y >> 1$, and $K(y) \to 1$ if $y << 1$. The P–RG equation in LPA for $d = 2$ dimensions reads as

$$(2 + k\partial_k)\tilde{V}_k = -[\tilde{V}_k']^2 K_0' + \tilde{V}_k'' I_2, \tag{2.8}$$

where $K' = \partial_y K(y)$, $K_0' = \partial_y K(y)|_{y=0}$ and $I_2 = (1/4\pi)\int_0^\infty dy K'(y) = -1/4\pi$. The parameter K_0' can be eliminated by the rescaling of the potential and the field variable, consequently, it does not influence the physics. In order to make the comparison of the RG flows obtained by various RG methods straightforward, we choose $K_0' = -1$ for which the linearized forms of Eq. (2.1) and Eq. (2.8) and the UV scaling laws obtained by WH–RG and P–RG are identical. Then the differentiation of both sides of Eq. (2.8) with respect to the field variable ϕ yields [33]

$$(2 + k\partial_k)\tilde{V}_k' = 2\tilde{V}_k''\tilde{V}_k' - \frac{1}{4\pi}\tilde{V}_k''' \tag{2.9}$$

being independent of the regulator function $K(y)$ and differing of the WH–RG equation (2.2) by the term $-\tilde{V}_k'' k\partial_k \tilde{V}_k'$ and by the opposite sign for the non-linear term. Let us note, that the P–RG method treats all quantum fluctuations below and above the scale k on the same footing due to the usage of the smooth cutoff. Therefore, even if there occurs a scale $k_{\rm SI}$ at which $1 + \tilde{V}_k''$ exhibits zeros, no singular behavior is expected in case of the P–RG equation, consequently Eq.(2.9) can be applied above ($k > k_{\rm SI}$) and below ($k < k_{\rm SI}$) the scale $k_{\rm SI}$ with the price of the SI being unnoticed.

3. Symmetries and Renormalization

As a rule, the solution of the RG equations is sought for in a restricted functional subspace [10]. Since the RG equations retain the symmetries of the bare action, the functional subspace should be chosen keeping the symmetries of the bare action unbroken. Furthermore, even this – generally infinite dimensional – subspace is reduced to a finite dimensional one by the truncation of the appropriate series expansion of the blocked potential. For example, the potential can be expanded in powers of the field variable $V_k(\phi) = \sum_{n=1}^{N} c_n(k)\,\phi^n$ with a truncation at the power N and the scale-dependence is encoded in the coupling constants $c_n(k)$. In this case one has to check whether the results obtained are independent of N. It is known that $O(M)$ scalar models can be considered in Taylor expanded form only if $M > 1$ (for $M = 1$, strong oscillatory behavior of the critical exponents in terms of N is observed) [10]. Similarly, the truncated Fourier expanded form can be a straightforward approximation for scalar models with periodicity in internal space [25, 26].

Let us now turn to the symmetries of MFSG models if the ratios of the frequencies are rational. Then the bare potential is periodic in the internal space, let be its period $2\pi\beta$, and one has to look for the solution of the RG equations among the periodic functions with such a period. The bare potential may have however further symmetries as well. For example, the MFSG models can exhibit a reflection symmetry besides periodicity. Three cases can be distinguished.

- Let us suppose that the bare potential of the MFSG model contains a single cosine mode with $\delta_1 = 0$

$$\tilde{V}_\Lambda(\phi) = \tilde{\mu}_1 \cos(\beta\,\phi)\,. \tag{3.1}$$

  In this case the model has a discrete reflection symmetry ($\phi \to -\phi$), which is preserved by the WH–RG and P–RG equations. Since the RG transformations generate higher harmonics, one is inclined to look for the solution in its Fourier decomposed form

$$\tilde{V}_k(\phi) = \sum_{n=0}^{N} \tilde{u}_n(k)\, \cos(n\,\beta\,\phi)\,, \tag{3.2}$$

  exhibiting periodicity in the internal space. The dimensionless couplings are represented by the Fourier amplitudes $\tilde{u}_n(k)$ (with $\tilde{u}_1(k=\Lambda) = \tilde{\mu}_1$) and the 'frequency' β is a scale-independent, dimensionless parameter in the LPA.

- If the bare potential of the MFSG model contains a single sine mode (i.e. $\delta_1 = 3\pi/2$)

$$\tilde{V}_\Lambda(\phi) = \tilde{\mu}_1 \sin(\beta\,\phi)\,, \tag{3.3}$$

  the model has another discrete Z_2 symmetry ($\phi \to -\pi/\beta - \phi$) which is preserved by the RG equations. The potential is antisymmetric but the RG equations are not, consequently, one has to look for the solution of the RG equations as

$$\tilde{V}_k(\phi) = \sum_{n=0}^{N} \left[\tilde{u}_{2n}(k)\cos(2n\beta\phi) + \tilde{v}_{2n+1}(k)\sin((2n+1)\beta\phi)\right] \tag{3.4}$$

  with the dimensionless Fourier amplitudes $\tilde{u}_{2n}(k)$ and $\tilde{v}_{2n+1}(k)$ (and $\tilde{v}_1(k=\Lambda) = \tilde{\mu}_1$). Let us note that the double-frequency SG model (1.3) belongs to this case, too.

- Finally, if the bare potential of the MFSG model contains both cosine and sine modes (i.e. $\delta_1 = 0$ and $\delta_2 = 3\pi/2$)

$$\tilde{V}_\Lambda(\phi) = \tilde{\mu}_1 \cos(\beta\,\phi) + \tilde{\mu}_2 \sin(\beta\,\phi)\,, \tag{3.5}$$

  the model has no Z_2 symmetry, consequently, all the Fourier modes are generated during the RG flow and the solution has the general form

$$\tilde{V}_k(\phi) = \sum_{n=0}^{N} \left[\tilde{u}_n(k)\cos(n\beta\phi) + \tilde{v}_n(k)\sin(n\beta\phi)\right], \tag{3.6}$$

  with the dimensionless Fourier amplitudes $\tilde{u}_n(k)$ and $\tilde{v}_n(k)$ (and $\tilde{u}_1(k=\Lambda) = \tilde{\mu}_1$, $\tilde{v}_1(k=\Lambda) = \tilde{\mu}_2$).

Since Eq.(3.6) represents the blocked potential for the most general MFSG model with rational frequency ratios, let us further discuss that case. Inserting the ansatz (3.6) into the derivative form of the WH–RG equation (2.2) one can read off RG flow equations for the Fourier amplitudes, i.e. for the scale-dependent dimensionless couplings $\tilde{u}_n(k)$, $\tilde{v}_n(k)$ which read as

$$(2+k\partial_k)n\tilde{u}_n = \tfrac{\beta^2}{4\pi}n^3\tilde{u}_n + \tfrac{\beta^2}{2}\textstyle\sum_{s=1}^N \left(sA^{(1)}_{n,s}(2+k\partial_k)\tilde{u}_s + sA^{(4)}_{n,s}(2+k\partial_k)\tilde{v}_s\right), \quad (3.7)$$

$$(2+k\partial_k)n\tilde{v}_n = \tfrac{\beta^2}{4\pi}n^3\tilde{v}_n - \tfrac{\beta^2}{2}\textstyle\sum_{s=1}^N \left(sA^{(2)}_{n,s}(2+k\partial_k)\tilde{u}_s + sA^{(3)}_{n,s}(2+k\partial_k)\tilde{v}_s\right), \quad (3.8)$$

where

$$\begin{aligned}
A^{(1)}_{n,s}(k) &= (n-s)^2\tilde{u}_{|n-s|} - (n+s)^2\tilde{u}_{n+s}\Theta(n+s\le N),\\
A^{(2)}_{n,s}(k) &= \mathrm{sgn}(s-n)\ \ (n-s)^2\tilde{v}_{|n-s|} + (n+s)^2\tilde{v}_{n+s}\Theta(n+s\le N),\\
A^{(3)}_{n,s}(k) &= -(n-s)^2\tilde{v}_{|n-s|} - (n+s)^2\tilde{v}_{n+s}\Theta(n+s\le N),\\
A^{(4)}_{n,s}(k) &= \mathrm{sgn}(s-n)\ \ (n-s)^2\tilde{v}_{|n-s|} - (n+s)^2\tilde{v}_{n+s}\Theta(n+s\le N),
\end{aligned}$$

with $\mathrm{sgn}(x) = 1$ if $x > 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$, and $\Theta(n \le N) = 1$ if $n \le N$ and $\Theta(n \le N) = 0$ if $n > N$. Let us note that Eq. (2.2) and, consequently, Eq. (3.7), Eq. (3.8) are valid unless SI arises.

Using the same machinery in the framework of the P–RG, one obtains from Eq. (2.9) the flow equations for $\tilde{u}_n(k)$ and $\tilde{v}_n(k)$

$$(2+k\partial_k)n\tilde{u}_n = \frac{\beta^2}{4\pi}n^3\tilde{u}_n - \beta^2\sum_{s=1}^N \left(sA^{(1)}_{n,s}\tilde{u}_s + sA^{(4)}_{n,s}\tilde{v}_s\right), \quad (3.9)$$

$$(2+k\partial_k)n\tilde{v}_n = \frac{\beta^2}{4\pi}n^3\tilde{v}_n + \beta^2\sum_{s=1}^N \left(sA^{(2)}_{n,s}\tilde{u}_s + sA^{(3)}_{n,s}\tilde{v}_s\right), \quad (3.10)$$

where $A^{(i)}_{n,s}$, $(i = 1, 2, 3, 4)$ are the same as those obtained for the WH–RG equation. Let us note, that the P–RG method does not take into account SI, consequently, Eq. (3.9) and Eq. (3.10) are valid at all scales k.

Let us end this section with the remark that the strong reduction of the functional subspace, in particular the truncation of the expansion of the blocked potential in a series of base functions may become unreliable when the blocked action becomes almost degenerate, i.e. $1+\tilde{V}''_k$ approaches zero. This motivates a direct numerical solution of the RG equation for the blocked potential which avoids any assumption on the functional subspace where the solution is sought for and any truncated series expansion in some base functions [10, 40, 41]. Therefore, we solved the RG equations (2.2) and (2.9) directly by using a computer algebraic program with periodic boundary conditions and the bare initial potential was chosen as a harmonic function.

4. UV scaling

Before the study of the low-energy/IR behavior of the MFSG model, let us first discuss the high-energy/UV scaling. This can be achieved by the linearization of the RG equations

(3.7), (3.8) and (3.9), (3.10) around the UV Gaussian fixed point ($\tilde{V}^*(\phi) \equiv 0$) which results in the following uncoupled set of differential equations

$$(2 + k\partial_k)\tilde{u}_n = \frac{\beta^2}{4\pi} n^2 \tilde{u}_n, \tag{4.1}$$

$$(2 + k\partial_k)\tilde{v}_n = \frac{\beta^2}{4\pi} n^2 \tilde{v}_n, \tag{4.2}$$

which is independent of the RG method used. The solution

$$\tilde{u}_n(k) = \tilde{u}_n(\Lambda) \left(\frac{k}{\Lambda}\right)^{\frac{\beta^2}{4\pi} n^2 - 2}, \tag{4.3}$$

$$\tilde{v}_n(k) = \tilde{v}_n(\Lambda) \left(\frac{k}{\Lambda}\right)^{\frac{\beta^2}{4\pi} n^2 - 2}, \tag{4.4}$$

gives the same UV scaling as that obtained for the compact MFSG model in [27]. These UV scaling laws can be understood if one considers the MFSG model as a perturbation of the corresponding CFT, c.f. the discussion below Eq. (1.2). One should conclude that the kinetic term of the action suppresses the large amplitude ($\phi^2 \gg 1/p^2$) quantum fluctuations with large momentum ($\Lambda^2 > p^2 > k^2$) close to the UV cutoff and, therefore the UV scaling laws are not influenced by the compactness of the field variable.

5. IR scaling

In the IR domain neither the kinetic term nor the periodic potential terms are able to suppress the contributions of the large-amplitude quantum fluctuations with small momenta. Therefore, the compact and non-compact MFSG models are expected to behave differently in the IR domain. There are two ways to determine the IR scaling of the non-compact MFSG model in the LPA, (i) either the partial differential equations (2.2) and (2.9) have to be solved numerically by a computer algebraic program using the initial condition (1.1), (ii) or one can find the solution of the ordinary differential equations (3.7), (3.8) and (3.9), (3.10) which are obtained by inserting the ansatz (3.6) into Eqs.(2.2), (2.9). In the latter case, besides the LPA, we use a further approximation, namely the truncation of the Fourier expansion of the potential.

According to our experiences concerning the renormalization of SG type models based on previous publications [25, 14, 15, 22, 33], it is expected that the RG equations obtained by using the truncated Fourier decomposition of the periodic potential, is always applicable, except the situation if one would like to decide unambiguously whether SI appears or not in the RG flow. SI is related to the singularity of the RG flow, consequently, in some cases it could be important to solve the partial differential (RG) equations without using any further approximations in order to be able to decide whether SI can be avoided or not. Since the P–RG method does not take account for SI it is more convenient to use this method first to consider the IR behavior of the non-compact MFSG model.

5.1 Polchinski RG approach

Let us first discuss the IR effective theory of the MFSG model in the framework of the P–RG method by solving Eqs. (3.9), (3.10) numerically with the most general ansatz (3.6).

Qualitatively different IR scaling behaviors of the MFSG model are observed below and above $\beta_c^2 = 8\pi$. For $\beta^2 > 8\pi$, every Fourier amplitudes are found to be irrelevant in the limit $k \to 0$, i.e. they are decreasing coupling constants independently of the initial conditions, see Fig. 1. Consequently, for $\beta^2 > 8\pi$, the non-compact MFSG model is a free massless theory in the IR limit independently of whether the bare initial potential possesses a Z_2 symmetry (Eqs. (3.1), (3.3)) or not (Eq.(3.5)).

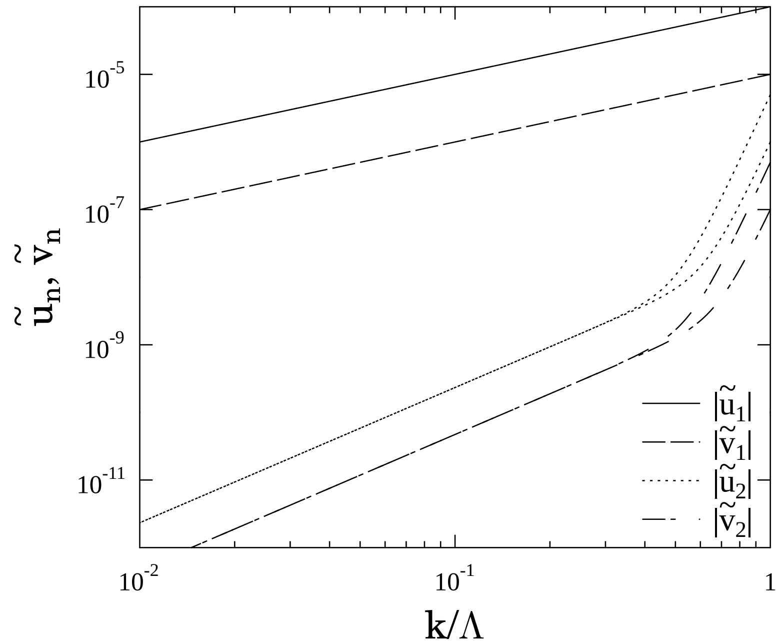

Figure 1: The scaling of the first few Fourier amplitudes of the non-compact MFSG model is obtained by the P–RG method solving Eqs. (3.9), (3.10) numerically for $\beta^2 = 12\pi$ with various initial conditions for the higher harmonics.

A further important result of the RG analysis is that the Fourier amplitudes of the non-compact MFSG model show up the IR scaling behavior

$$\tilde{u}_n(k) = f_n \left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)}, \tag{5.1}$$

$$\tilde{v}_n(k) = g_n \left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)}, \tag{5.2}$$

which differs from that obtained in the UV regime, Eqs. (4.3) and (4.4). Here the welljustified approximations

$$\sum_{s=1}^{N} sA^{(1)}_{n,s}\tilde{u}_s \approx +\sum_{s=1}^{n-1} s(n-s)^2 f_{n-s}f_s \left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$
$$\sum_{s=1}^{N} sA^{(2)}_{n,s}\tilde{u}_s \approx -\sum_{s=1}^{n-1} s(n-s)^2 g_{n-s}f_s \left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$
$$\sum_{s=1}^{N} sA^{(3)}_{n,s}\tilde{v}_s \approx -\sum_{s=1}^{n-1} s(n-s)^2 f_{n-s}g_s \left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$
$$\sum_{s=1}^{N} sA^{(4)}_{n,s}\tilde{v}_s \approx -\sum_{s=1}^{n-1} s(n-s)^2 g_{n-s}g_s \left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$

result in the following recursion relations for the constants f_n and g_n,

$$f_n = -\frac{\beta^2 \sum_{s=1}^{n-1} s(n-s)^2 (f_{n-s}f_s - g_{n-s}g_s)}{n\left[2 + n\left(\frac{\beta^2}{4\pi} - 2\right) - \frac{\beta^2}{4\pi}n^2\right]}, \tag{5.3}$$

$$g_n = +\frac{\beta^2 \sum_{s=1}^{n-1} s(n-s)^2 (g_{n-s}f_s + f_{n-s}g_s)}{n\left[2 + n\left(\frac{\beta^2}{4\pi} - 2\right) - \frac{\beta^2}{4\pi}n^2\right]}. \tag{5.4}$$

Let us note that in case of the fundamental modes (i.e. for $n = 1$) the UV and IR scalings coincide. The IR scaling of the model is determined by two independent parameters, $f_1 = \tilde{u}_1(\Lambda)$ and $g_1 = \tilde{v}_1(\Lambda)$ since for $n > 1$ the constants f_n and g_n are fixed by Eqs.(5.3), (5.4). Therefore, the IR behavior of the model is independent of the initial conditions for the higher harmonics, see Fig. 1, and depends on either $\tilde{u}_1(\Lambda)$ or $\tilde{v}_1(\Lambda)$ if the model has a Z_2 symmetry and in the absence of the reflection symmetry the IR physics is determined by both $\tilde{u}_1(\Lambda)$ and $\tilde{v}_1(\Lambda)$.

For $\beta^2 < 8\pi$, one has to distinguish three scaling regimes in case of the non-compact MFSG model (i) the UV (ii) the IR (iii) and the deep IR scaling behavior, see Fig. 2. The UV (4.3), (4.4) and the IR (5.1), (5.2) scaling laws are given by the same expressions as those obtained in the strong coupling phase ($\beta^2 > 8\pi$). However, if $\beta^2 < 8\pi$, according to the IR scaling law, every Fourier amplitude becomes relevant (increasing) coupling in the IR domain. Even more important difference is that a qualitatively new behavior is found in the deep IR limit ($k \to 0$), namely, at a certain momentum scale k_c the Fourier amplitudes of the non-compact MFSG model become constants, see Fig. 2. Therefore, if $\beta^2 < 8\pi$, the dimensionless IR effective potential of the non-compact model is non-trivial.

Let us analyze the sensitivity of the IR theory on the initial conditions in order to map out the phase structure. If the bare action has no Z_2 symmetry (see Eq.(3.5)) then the deep IR effective potential depends on a single parameter, namely, the ratio of the initial values of the fundamental modes, $r = \tilde{u}_1(\Lambda)/\tilde{v}_1(\Lambda)$ which remains unchanged during the RG flow, see Fig. 3. Let us remind that in the strong coupling regime ($\beta^2 > 8\pi$) the IR behavior of the model (without a Z_2 symmetry) is determined by two independent parameters ($\tilde{u}_1(\Lambda)$

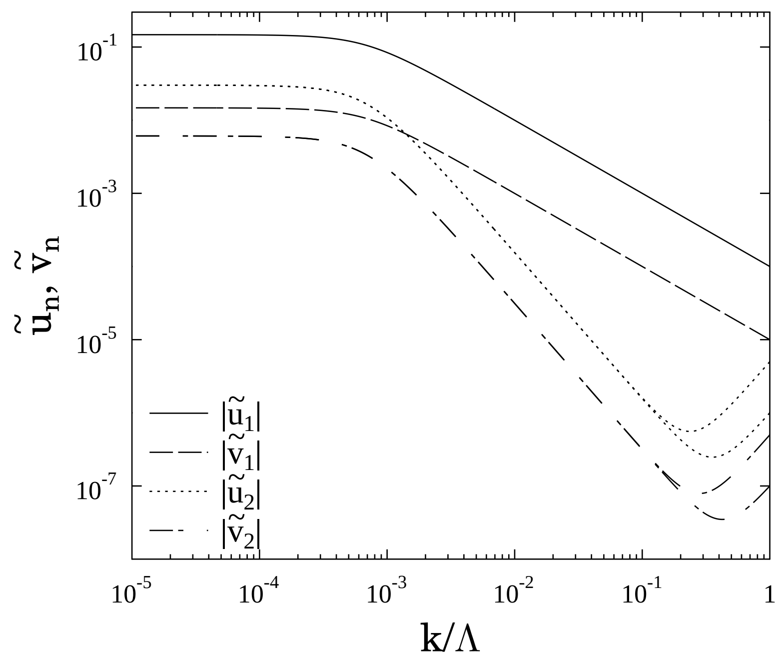

Figure 2: The scaling of the first few Fourier amplitudes of the non-compact MFSG model is obtained by the P–RG method solving Eqs. (3.9), (3.10) numerically for $\beta^2 = 4\pi$ with various initial conditions for the higher harmonics. At the momentum scale $k_c \sim 3 \times 10^{-4}$, the Fourier amplitudes become constants.

and $\tilde{v}_1(\Lambda)$). In the presence of Z_2 symmetry (see (3.1) for $r = \infty$ or (3.3) for $r = 0$), the deep IR potential is superuniversal, i.e. it is independent of any initial conditions [25, 33].

Again, for $\beta^2 > 8\pi$, if the action has a reflection symmetry, the IR scaling is determined by a single parameter, i.e. the initial value of the fundamental mode, (either $\tilde{u}_1(\Lambda)$ or $\tilde{v}_1(\Lambda)$). Therefore, the non-compact MFSG model has two phases separated by the critical value $\beta_c^2 = 8\pi$. As a consequence of the superuniversal, and universal behavior, no other phase transition can be identified in the non-compact model.

Finally let us consider the IR behavior of the non-compact MFSG model by solving directly the P–RG equation (2.9). When the solution of the partial differential equation (2.9) had been obtained it was expanded in Fourier series. For $\beta^2 = 4\pi$ the UV, IR and the deep IR scaling of the first few Fourier amplitudes coincide to that of obtained by the numerical solution of Eqs. (3.9), (3.10) which are plotted in Fig. 2. Therefore, there is an excellent quantitative agreement between the results obtained by solving Eqs. (3.9), (3.10) and by solving Eq.(2.9) directly. This shows that in case of the P–RG method the RG flow seems to avoid the SI and one can look for the solution of the RG equations in its Fourier decomposed form.

If one tries to determine the IR behavior of the MFSG model by an RG method, like the WH–RG approach, which has a singular structure, and consequently, SI could appear in the RG flow it could be important to solve the partial differential RG equation obtained in the LPA without using any further approximations in order to be able to decide whether SI can be avoided or not.

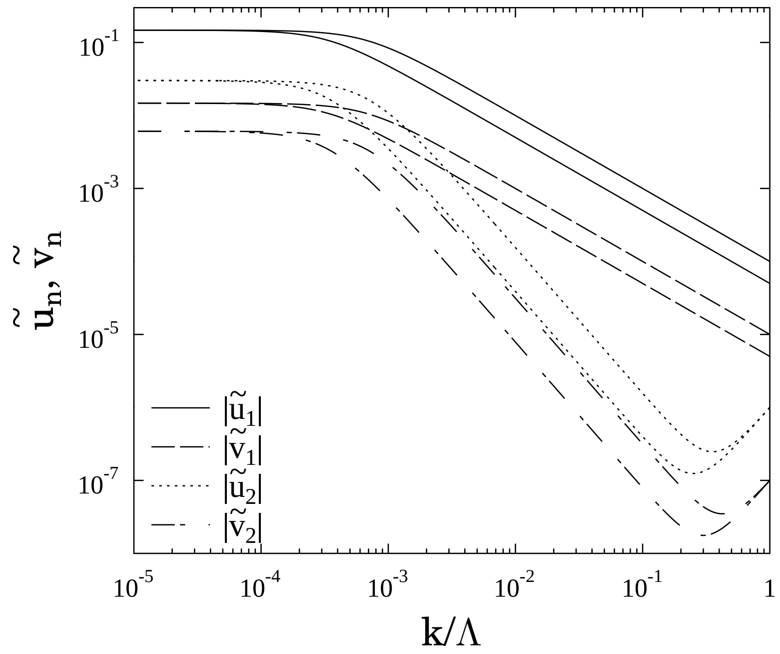

Figure 3: In this figure we show that the IR effective potential of the non-compact MFSG model for $\beta^2 = 4\pi$ depends on only the ratio of the Fourier amplitudes of the fundamental cosine and sine modes. If the P–RG equation has been solved with various initial conditions for $\tilde{u}_1(\Lambda)$ and $\tilde{v}_1(\Lambda)$ but keeping their ratio fixed, then one obtains the same deep IR behavior.

5.2 Wegner-Houghton RG approach

Let us consider the IR effective theory of the MFSG model in the framework of the WH–RG method by solving Eqs. (3.7), (3.8) numerically. For $\beta^2 > 8\pi$, similarly to the results obtained by the P–RG method, the Fourier amplitudes are irrelevant in the limit $k \to 0$, independently of the initial conditions, see Fig. 4. The numerical solution of the WH–RG equation provides once again the IR scaling laws given by (5.1) and (5.2). Similarly to the P–RG flow, this IR behavior can also be obtained by using the IR approximations

$$\sum_{s=1}^{N} sA^{(1)}_{n,s}(2+k\partial_k)\tilde{u}_s \approx +\sum_{s=1}^{n-1} s(n-s)^2 f_{n-s} f_s \left[2+s\left(\frac{\beta^2}{4\pi}-2\right)\right]\left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$

$$\sum_{s=1}^{N} sA^{(2)}_{n,s}(2+k\partial_k)\tilde{u}_s \approx -\sum_{s=1}^{n-1} s(n-s)^2 g_{n-s} f_s \left[2+s\left(\frac{\beta^2}{4\pi}-2\right)\right]\left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$

$$\sum_{s=1}^{N} sA^{(3)}_{n,s}(2+k\partial_k)\tilde{v}_s \approx -\sum_{s=1}^{n-1} s(n-s)^2 f_{n-s} g_s \left[2+s\left(\frac{\beta^2}{4\pi}-2\right)\right]\left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$

$$\sum_{s=1}^{N} sA^{(4)}_{n,s}(2+k\partial_k)\tilde{v}_s \approx -\sum_{s=1}^{n-1} s(n-s)^2 g_{n-s} g_s \left[2+s\left(\frac{\beta^2}{4\pi}-2\right)\right]\left(\frac{k}{\Lambda}\right)^{n(\frac{\beta^2}{4\pi}-2)},$$

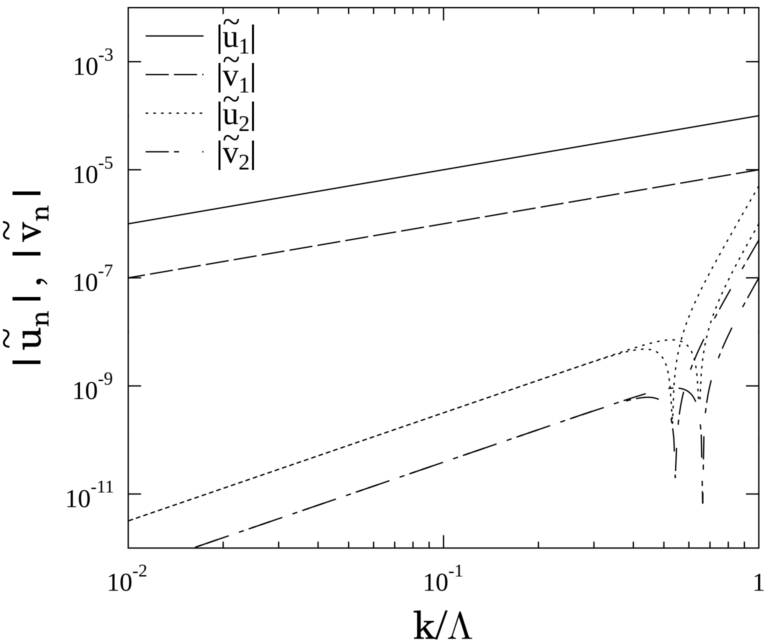

Figure 4: The scaling of the first few Fourier amplitudes of the non-compact MFSG model is obtained by the WH–RG method solving Eqs. (3.7), (3.8) numerically for $\beta^2 = 12\pi$ with various initial conditions for the higher harmonics. The peaks in the scaling of $\tilde{u}_2(k)$ and $\tilde{v}_2(k)$ indicate the change of their sign during the RG flow.

which result in the recursion relations for f_n and g_n,

$$f_n = +\frac{\beta^2 \sum_{s=1}^{n-1} s(n-s)^2 (f_{n-s} f_s - g_{n-s} g_s)[1 - s + \frac{s\beta^2}{8\pi}]}{n\left[2 + n\left(\frac{\beta^2}{4\pi} - 2\right) - \frac{\beta^2}{4\pi} n^2\right]}, \tag{5.5}$$

$$g_n = -\frac{\beta^2 \sum_{s=1}^{n-1} s(n-s)^2 (g_{n-s} f_s + f_{n-s} g_s)[1 - s + \frac{s\beta^2}{8\pi}]}{n\left[2 + n\left(\frac{\beta^2}{4\pi} - 2\right) - \frac{\beta^2}{4\pi} n^2\right]}. \tag{5.6}$$

This shows that the IR scalings of the non-compact MFSG model determined by the P–RG and the WH–RG methods are qualitatively the same for $\beta^2 > 8\pi$. The UV/IR scalings of the fundamental modes (i.e. for $n = 1$) coincide, independently of the RG method used, $f_1 = \tilde{u}_1(\Lambda)$ and $g_1 = \tilde{v}_1(\Lambda)$. The IR constants f_n and g_n of the higher harmonics (i.e. $n > 1$) are determined by the equations (5.5), (5.6) which predict the IR behavior similar to that obtained by the P–RG method. However, in case of the WH–RG method the f_n, g_n parameters have alternating signs for even and odd values of n. Therefore, if we use the same initial conditions, (e.g. all the bare Fourier amplitudes are positive) then in case of the WH–RG method, $\tilde{u}_2(k)$ and $\tilde{v}_2(k)$ change their signs during the RG flow, see Fig. 4. It is important to note that the IR scaling of the model (similarly to the P–RG method) is determined by two independent parameters ($\tilde{u}_1(\Lambda)$, $\tilde{v}_1(\Lambda)$), if the bare action has no Z_2 symmetry and depends on a single parameter (either $\tilde{u}_1(\Lambda)$ or $\tilde{v}_1(\Lambda)$) in case of a Z_2 symmetric bare action, and it is independent of the initial conditions of the higher harmonics, see Fig. 4. In conclusion, the WH–RG and the P–RG methods produce the same IR behavior for the MFSG model if $\beta^2 > 8\pi$.

For $\beta^2 < 8\pi$, the IR scaling behavior turns all the Fourier amplitudes into relevant coupling constants, consequently, the logarithm of the WH–RG equation (2.1) could become infinite, hence a SI could appear in the WH–RG flow. Indeed, in Fig. 5 the scaling of the coupling constants of the non-compact MFSG model is presented for $\beta^2 = 4\pi$ and the vertical line shows the appearance of SI. Beyond the momentum scale $k_{\rm SI}$, the WH–RG

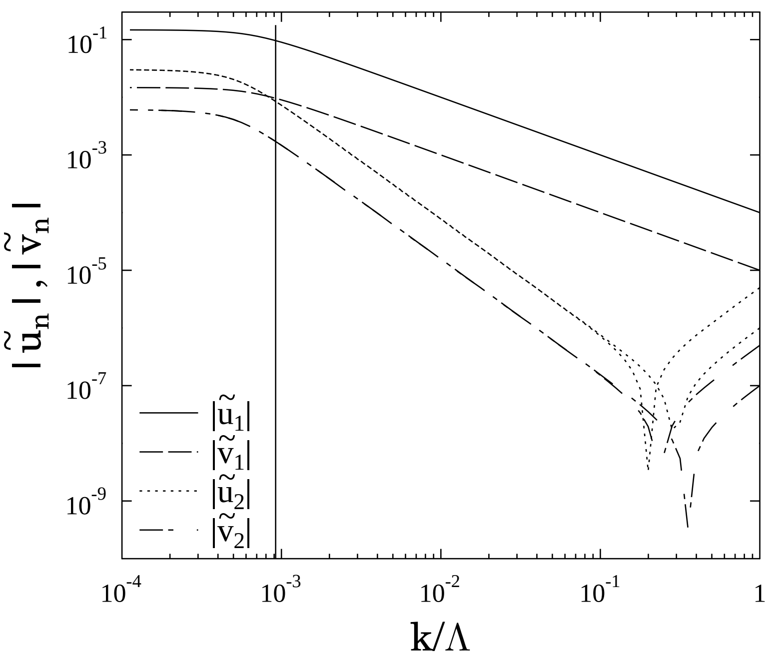

Figure 5: The scaling of the first few Fourier amplitudes of the non-compact MFSG model is obtained in the framework of the WH–RG method for $\beta^2 = 4\pi$ by solving numerically either Eqs. (3.7), (3.8) or Eq. (2.2). In the latter case, the partial differential equation (2.2) is solved by a computer algebraic code and then the solution is expanded in Fourier series. The vertical line indicates the momentum scale of SI ($k_{\rm SI}$) where Eqs. (3.7), (3.8) lose their validity. Above this scale, $k_{\rm SI} < k$, the results obtained by Eqs. (3.7), (3.8) and by Eq. (2.2) coincide. Below the scale of SI, $k < k_{\rm SI}$, the scaling of the Fourier amplitudes is determined by the direct integration of Eq. (2.2).

equation loses its validity and one has to use the tree-level RG equation (2.3) which leads to the IR effective potential (2.4) in the deep IR limit ($k \to 0$). In order to preserve periodicity, the IR effective potential of the MFSG model has a parabola-shape for $\phi \in [-\pi/\beta, \pi/\beta]$ and such parabola sections are repeated along the ϕ axis. Let us analyze the sensitivity of the IR effective theory on the UV initial conditions. In case of a reflection symmetry $\phi \to -\phi$, the linear term vanishes in (2.4), i.e. $c = 0$, and the potential is superuniversal, i.e. independent of any initial conditions. If the bare action has another type of reflection symmetry $\phi \to -\phi - \pi/\beta$, then the constant in (2.4) is non-zero but fixed, i.e. $c = -\pi/2\beta$, consequently, the IR potential is again superuniversal. If the bare action of the MFSG model has no Z_2 symmetry then the deep IR behavior depends on a single parameter c. Therefore, in the framework of the WH–RG method if SI appears in the RG flow, the sensitivity of the IR behavior on the UV parameters is found to be the same as that obtained by the P–RG approach.

Finally, let us consider the IR scaling of the non-compact MFSG model by solving the WH–RG equation (2.2) by a computer algebraic code. The solution found is expanded in Fourier series in order to compare the results to those obtained by Eqs. (3.7), (3.8). For $\beta^2 = 4\pi$ the scalings of the first few Fourier amplitudes are plotted in Fig. 5. There is a quantitative agreement between the results obtained by Eq.(2.2) and Eqs. (3.7), (3.8) in the UV and IR scaling regimes. However, the important difference is that no SI is found in the RG flow when Eq.(2.2) is solved directly. This indicates that SI occurs in the WH–RG approach as an artifact due to the truncated Fourier-expansion applied to the almost degenerate blocked action of the MFSG model, at least for $\beta^2 = 4\pi$. On the other hand, it seems to support the Quantum Censorship conjecture to be at work in the MFSG model as well [41]. Let us emphasize that independently of whether the blocked action becomes degenerate or not, the sensitivity of the deep IR behavior of the MFSG model on the UV initial parameters is found to be the same. Consequently, the phase structure of the non-compact MFSG model is determined unambiguously and independently of the RG method used.

Let us note that if Quantum Censorship is really on work, then the WH–RG (2.2) and P–RG (2.9) partial differential equations and also their Fourier expanded forms, Eqs. (3.7), (3.8) and Eqs. (3.9), (3.10) retain their validity in the deep IR regime. When there the Fourier amplitudes take constant values at some momentum scale k_c, i.e. $\partial_k \tilde{V}_{(k<k_c)} = 0$ or $\partial_k \tilde{u}_n(k < k_c) = 0$, $\partial_k \tilde{v}_n(k < k_c) = 0$ hold, then Eqs. (3.7), (3.8) and Eqs. (3.9), (3.10) reduce to the same recursion equations except the sign of the non-linear term. Consequently, in the IR limit $k \to 0$ the WH–RG and the P–RG methods result in the same absolute values of the couplings $|\tilde{u}_n(0)|$ and $|\tilde{v}_n(0)|$ of the MFSG model.

6. Summary

In this paper we considered how the compactness of the field influences the renormalization and, consequently, the low-energy behavior of the theory. In particular, we compared the high-energy/UV and low-energy/IR behaviors of the two-dimensional multi-frequency sine–Gordon (MFSG) scalar field model defined by compact and non-compact field variables. We studied the renormalization of the MFSG model with a non-compact field in the framework of the functional renormalization group (RG) method using the local potential approximation (LPA), discussing the comparison with the results for the compact double- and multi- frequency sine Gordon.

We showed that the UV scaling of the compact and the non-compact MFSG models coincides but their IR behaviors are different. In the UV limit, the quantum fluctuations (with high frequency and small amplitude) do not feel the difference between the models defined by compact and non-compact fields but different behaviours are expected to appear in the IR limit due to the large-amplitude quantum fluctuations of the IR domain. On the one hand the critical frequency $\beta_c^2 = 8\pi$ at which the sine-Gordon model undergoes a topological phase transition is found to be unaffected by the compactness of the field since it is determined by the UV scaling laws. On the other hand, while it is known that the compact model has first and second order (Ising) type phase transitions which are determined by the IR scaling, we showed that these are absent in the non-compact model.

Indeed, the IR effective potential of the non-compact MFSG model was found to be different above and below $\beta_c^2 = 8\pi$. For $\beta^2 > 8\pi$, the deep IR behavior of the non-compact MFSG model with Z_2 symmetry (i.e. $\phi \to -\phi$ or $\phi \to -\pi/\beta - \phi$) depends on the UV initial condition for either the fundamental cosine or the fundamental sine mode, respectively, and for $\beta^2 < 8\pi$ it is superuniversal, i.e. independent of any initial conditions. If the non-compact MFSG model has no Z_2 symmetry, for $\beta^2 > 8\pi$ the IR effective potential depends on the UV initial conditions both for the fundamental cosine and sine modes (i.e. it depends on two independent parameters) and for $\beta^2 < 8\pi$ it is universal, i.e. depends on only a single parameter, namely the ratio $\tilde{u}_1(\Lambda)/\tilde{v}_1(\Lambda)$. Consequently, due to the superuniversal and universal IR behavior of the non-compact MFSG model, there is no room for first or second order phase transitions for $\beta^2 < 8\pi$.

These results were obtained by the functional renormalization group analysis of the non-compact MFSG model in the framework of the Polchinski and the Wegner–Houghton RG methods where the latter is mathematically equivalent to the effective average action RG with the power-law regulator ($b = 1$) and the functional Callan-Symanzik RG equation. The RG flow of the non-compact MFSG model was determined in two different ways (i) either the RG equations obtained in the LPA were solved numerically by a computer algebraic code and then the solution expanded in Fourier series, (ii) or first the RG equations were derived for the Fourier amplitudes and then those solved numerically. In the latter case, it was unavoidable to implement a further approximation besides the LPA, namely the truncation of the Fourier expansion of the potential. The sensitivity of the IR effective potential on the UV initial conditions, and consequently, the phase structure was found to be the same in both cases. Moreover, except the situation where the RG flow has a singularity, i.e. a spinodal instability (SI) appears in the IR limit, the scaling of the Fourier amplitudes obtained in the above mentioned two different ways, coincide. For $\beta^2 < 8\pi$, in case of the non-compact MFSG model a momentum scale was generated by the RG transformation in the deep IR regime (either the scale where the Fourier amplitudes become constants or the scale of SI). Below this momentum scale, the theory becomes superuniversal (if it has a Z_2 symmetry) or universal (if it has no Z_2 symmetry).

Finally, the classification of the IR scaling operators into relevant, marginal or irrelevant ones was also found to be different in case of the compact and the non-compact MFSG models. For the compact model, one can rely on the UV results even in the IR limit but for the non-compact case new types of scaling laws were observed in the IR domain which modify the classification of the scaling operators.

Acknowledgments

Fruitful discussions with G. Delfino, G. Mussardo and P. Sodano are warmly acknowledged. I.N., S.N., and K.S. acknowledge the support by the project TÁMOP 4.2.1-08/1-2008-003. The project is implemented through the New Hungary Development Plan co-financed by the European Social Fund, and the European Regional Development Fund. A.T. acknowledges support by the grants INSTANS (from ESF) and 2007JHLPEZ (from MIUR).

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