A REGULARITY THEOREM FOR COMPOSITE FUNCTIONAL EQUATIONS

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ABSTRACT. In this paper a regularity theorem for the functional equation

$$\begin{aligned} f(x+y)-f(x) \quad &+ \quad \sum_{i=1}^{n} \varphi_i[g_i(y+z_i)-g_i(y)] \\ &= \quad \sum_{i=1}^{n} \psi_i[g_i(y+x+z_i)-g_i(y+x)-g_i(y+z_i)+g_i(y)] \end{aligned}$$

is proved.

In the regularity theory of functional equations, problems are considered, where the unknown functions have "weak" regularity properties (monotonicity, measurability, etc.) and a "stronger" regularity (e.g. differentiability) is proved. The most general and significant results in this field are due to A. Járai. (cf. e.g. [8], [9], [10]; [6], [11], [12]). In the cited papers functional equations are studied in which the unknown functions are not substituted into each other. Regularity properties of composite functional equations, motivated by a problem of J. Aczél [1], were first studied by Zs. Páles [16], [17]. Developing these ideas and methods, in [18] and [19], the functional equation

$$f(x+y)-f(x)+\varphi(g(y+z)-g(y)) = \psi(g(x+y+z)-g(y+z)-g(x+y)+g(y)) \qquad (x,y,z>0,\ x+y+z<\alpha)$$

was considered, where $0<\alpha\le\infty$, $f,g: I\to\mathbb{R}$, $\varphi: J\to\mathbb{R}$, $\psi: H\to\mathbb{R}$ and $I=]0,\alpha[$, $J=\{g(y+z)-g(y) \mid y,z>0,\ y+z<\alpha\}$, $H=\{g(y+x+z)-g(y+x)-g(y+z)+g(y) \mid x,y,z>0,\ y+x+z<\alpha\}$, and it was shown that the strict monotonicity of the solutions f,g,φ,ψ of the equation above implies that f and g are strictly convex or strictly concave, differentiable functions, the sets J and H are open intervals and the functions φ and ψ are also differentiable. In [18] and [19] the functional equation above was also completely solved. A crucial step of the argument therein was to obtain the differentiability properties of the unknown functions. Then differentiating the equation with respect to the variables, one can eliminate the subexpressions containing composite functions. The functional equation so obtained can be handled with common and general methods of the theory of noncomposite functional equations.

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In the present paper, extending the methods used in [18] and [19], we generalize the regularity result above for a broad class of functional equations. Our investigation has been motivated by some recent applications of functional equations of this type in social and behavioral sciences (c.f. [2], [3], [4], [14] and [15]). Some of the problems dealt with in these papers were reduced to functional equations that are special cases of the equation treated in our main theorem. Therefore, this theorem can be useful to determine the solutions of similar concrete equations. The result given here was also announced during the 7$^{\text{th}}$ International Conference on Functional Equations and Inequalities, Złockie, Poland, 1999 (cf. [7]).

Our result reads as follows.

Theorem. *Let us consider the functional equation*

$$\begin{aligned} f(x+y)-f(x) \quad &+ \quad \sum_{i=1}^{n}\varphi_i[g_i(y+z_i)-g_i(y)] \\ &= \quad \sum_{i=1}^{n}\psi_i[g_i(y+x+z_i)-g_i(y+x)-g_i(y+z_i)+g_i(y)] \qquad (1) \\ &\qquad\qquad (x,y,z_i>0,\ x+y+z_i<\alpha;\ i=1,\dots,n), \end{aligned}$$

where n is a positive integer, $0<\alpha\le\infty$, $f,g_i:I\to\mathbb{R}$, $\varphi_i:J_i\to\mathbb{R}$, $\psi_i:H_i\to\mathbb{R}$ and

$$\begin{aligned} I &=]0,\alpha[, \\ J_i &= \{g_i(y+z_i)-g_i(y) \mid y,z_i>0,\ y+z_i<\alpha\}, \\ H_i &= \{g_i(y+x+z_i)-g_i(y+x)-g_i(y+z_i)+g_i(y) \mid x,y,z_i>0,\ y+x+z_i<\alpha\}, \end{aligned}$$

for $i=1,\dots,n$. Suppose, that the functions $\varphi_1,\dots,\varphi_n$, $\psi_1,\dots,\psi_n$, $g_1,\dots,g_n$ and f satisfy (1), furthermore, $\varphi_1,\dots,\varphi_n$; $\psi_1,\dots,\psi_n$ and $g_1,\dots,g_n$ are strictly monotonic, and the elements of each n-tuple in the same way.
Then

- *f is strictly convex or strictly concave and continuously differentiable on I;*
- *$g_1,\dots,g_n$ are all strictly convex or strictly concave on I;*
- *$J_1,\dots,J_n$ and $H_1,\dots,H_n$ are open intervals;*
- *$\varphi_1,\dots,\varphi_n$ and $\psi_1,\dots,\psi_n$ are differentiable on $J_1,\dots,J_n$ and $H_1,\dots,H_n$, respectively.*

Proof. First we prove that the functions g_i, $(i=1,\dots,n)$ are all strictly convex or strictly concave. Because of the symmetry of equation (1), it is enough to prove the statement for $i=1$. If we fix all the variables except for z_1 in (1), by the strict monotonicity of φ_1 and g_1 we get, that the left hand side of (1) is a strictly monotonic function of z_1. Therefore, the right hand side is strictly monotonic, too, so, for fixed $\bar{x},\bar{y}>0$, $\bar{x}+\bar{y}<\alpha$, the function

$$z\to g_1(\bar{x}+\bar{y}+z)-g_1(\bar{y}+z) \quad (z\in]0,\alpha-\bar{x}-\bar{y}[),$$

that is, $G_1:]0,\alpha-\bar{x}[\to\mathbb{R}$,

$$G_1(z)=g_1(\bar{x}+z)-g_1(z) \quad (z\in]0,\alpha-\bar{x}[),$$

is also strictly monotonic. (Moreover, the functions $G_i :]0, \alpha - \bar{x}[\to \mathbb{R}$,

$$G_i(z) = g_i(\bar{x} + z) - g_i(z) \quad (z \in]0, \alpha - \bar{x}[), \tag{2}$$

are strictly monotonic in the same way for each $i \in \{1, \dots, n\}$.) Let us suppose that G_1 is strictly monotone increasing. We have

$$\begin{aligned} g_1\left(\frac{s+t}{2}\right) - g_1(s) &< g_1\left(s + \frac{t-s}{2} + \frac{t-s}{2}\right) - g_1\left(s + \frac{t-s}{2}\right) \\ &= g_1(t) - g_1\left(\frac{s+t}{2}\right) \end{aligned}$$

for arbitrary $s, t \in]0, \alpha[$, $s < t$, which gives the strict Jensen convexity of g_1. By the theorem of Bernstein-Doetsch [5] (c.f. also [13], Chapter VI, §4), this fact and the monotonicity of g_1 implies the strict convexity of g_1. Obviously, if G_1 is strictly monotone decreasing then we get that g_1 is strictly concave. The functions G_i defined in (2) are strictly monotonic in the same way, thus, $g_1, \dots, g_n$ are all strictly convex or all strictly concave.

Now we show the strict convexity or concavity of f. Since $g_1, \dots, g_n$ are strictly convex or concave, for fixed $\bar{y}, \bar{z} > 0$, $\bar{y} + \bar{z} < \alpha$, the functions

$$x \to g_i(x + \bar{y} + \bar{z}) - g_i(x + \bar{y}) \quad (x \in]0, \alpha - \bar{y} - \bar{z}[)$$

are strictly monotonic in the same way for $i \in \{1, \dots, n\}$. The functions $\psi_1, \dots, \psi_n$ are also strictly monotonic in the same way, thus, the left hand side of (1) is a strictly monotonic function of x, that is,

$$x \to f(x + \bar{y}) - f(x) \quad (x \in]0, \alpha - \bar{y}[)$$

is strictly monotonic. Using a similar method as above, we get the strict Jensen convexity or concavity of f, which, together with the theorem of Bernstein-Doetsch, yields the strict convexity or concavity of f.

Thus, the functions $f, g_1, \dots, g_n : I \to \mathbb{R}$ are strictly convex or concave. Therefore, they are also continuous, there exists their left and right derivative at every point $x \in I$, moreover, they are differentiable everywhere but (at most) at countably many points in I. Since these functions are strictly monotonic, they are invertible, and their inverses also have the properties above (Cf. [13], Chapter VII.)

The continuity of $g_1, \dots, g_n$ gives that $J_1, \dots, J_n$ and $H_1, \dots, H_n$ are intervals, the strict monotonicity and the strict convexity or concavity of $g_1, \dots, g_n$ implies that these intervals are open.

In the next step we prove that the function φ_i is differentiable on J_i for $i = 1, \dots, n$. Due to symmetry, we may assume that $i = 1$. For a $\bar{u} \in J_1$, there exist $\bar{x}, \bar{y}, \bar{z}_1 > 0$ such that $\bar{x} + \bar{y} + \bar{z}_1 < \alpha$ and $\bar{u} = g_1(\bar{y} + \bar{z}_1) - g_1(\bar{y})$. Thus, $\bar{z}_1 = g_1^{-1}(\bar{u} + g_1(\bar{y})) - \bar{y}$ and the function $(u, y) \to g_1^{-1}(u + g_1(y)) - y$ is defined in a neighbourhood of $(\bar{u}, \bar{y})$. Writing $g_1^{-1}(u + g_1(y)) - y$ for z_1 in (1) we get that, for fixed $\bar{z}_2, \dots, \bar{z}_n$, there exist some neighbourhoods U, V, W of $\bar{u}, \bar{x}, \bar{y}$, respectively, such that

$$\begin{aligned} \varphi_1(u) &= f(x) - f(x+y) \\ &\quad - \sum_{i=2}^{n} \varphi_i(g_i(y+\bar{z}_i) - g_i(y)) \\ &\quad + \psi_1[g_1(x + g_1^{-1}(u + g_1(y))) - u - g_1(y+x)] \\ &\quad + \sum_{i=2}^{n} \psi_i(g_i(y+x+\bar{z}_i) - g_i(y+x) - g_i(y+\bar{z}_i) + g_j(y)) \end{aligned} \tag{3}$$

holds for $(u, x, y) \in U \times V \times W$. We show that there exist $x \in V$ and $y \in W$ such that the right hand side of (3) is differentiable at $\bar{u}$ with respect to u. Let us denote the sets, where g_1, g_1^{-1} and ψ_1 are not differentiable, by A, B and C, respectively. Since g_1 and g_1^{-1} are strictly convex or concave, the sets A and B are countable, the strict monotonicity of ψ_1 and Lebesgue's theorem imply that the Lebesgue measure of C is 0. Because of the strict monotonicity of g_1, the set of the points $y \in W$, for which $\bar{u} + g_1(y) \in B$, is countable. Thus, there exists a $y^* \in W$, such that the function $u \to g_1^{-1}(u + g_1(y^*))$ is differentiable at $\bar{u}$ with respect to u. Obviously, the set $L \subset V$

$$L = \{x \in V \mid x + g_1^{-1}(\bar{u} + g_1(y^*)) \in A\}$$

is countable, so, for each (fixed) $x \in V \setminus L$ the function $u \to g_1(x + g_1^{-1}(\bar{u} + g_1(y^*)))$ is differentiable at $\bar{u}$ with respect to u. The strict convexity or concavity of g_1 and $\bar{u} \neq 0$ imply that the function $G_1 : V \to \mathbb{R}$,

$$G_1(x) = g_1(x + g_1^{-1}(\bar{u} + g_1(y^*))) - \bar{u} - g_1(x + y^*) \quad (x \in V)$$

is strictly monotonic in V. Therefore, $G_1(V \setminus L) = G_1(V) \setminus G_1(L)$ and $G_1(V)$ is open, so, $G_1(V \setminus L)$ has a positive measure. Thus, there exists an $x^* \in V \setminus L$ with the property $G_1(x^*) \notin C$. For this element, the function

$$u \to \psi_1(g_1(x^* + g_1^{-1}(u + g_1(y^*))) - u - g_1(x^* + y^*))$$

is differentiable at $\bar{u}$ with respect to u, which implies that the right hand side of (3) is differentiable at $\bar{u}$ with respect to u, therefore, φ_1 also has this property.

We show the differentiability of ψ_i on H_i $(i = 1, \ldots, n)$. Similarly as above, we consider only the case $i = 1$. For a fixed $\bar{u} \in H_i$, there exist $\bar{x},\ \bar{y},\ \bar{z}_1 > 0$ satisfying $\bar{x} + \bar{y} + \bar{z}_1 < \alpha$ and

$$\bar{u} = g_1(\bar{x} + \bar{y} + \bar{z}_1) - g_1(\bar{x} + \bar{y}) - g_1(\bar{y} + \bar{z}_1) - g_1(\bar{y}). \tag{4}$$

The strict convexity or concavity of g_1 implies that the function

$$G_1(s) = g_1(\bar{x} + s) - g_1(s) \quad (s \in]0, \alpha - \bar{x}[\tag{5}$$

is strictly monotonic and continuous. Writing $\bar{x} = x$ and $s = y + z_1$ in (1), we get

$$\begin{aligned} & f(\bar{x}+y) - f(\bar{x}) + \varphi_1[g_1(y+z_1) - g_1(y)] + \sum_{i=2}^{n} \varphi_i[g_i(y+z_i) - g_i(y)] \\ & = \psi_1[g_1(\bar{x}+s) - g_1(s) - g_1(\bar{x}+y) + g_1(y)] \\ & \quad + \sum_{i=2}^{n} \psi_i[g_i(y+\bar{x}+z_i) - g_i(y+\bar{x}) - g_i(y+z_i) + g_i(y)], \end{aligned}$$

that is,

$$\begin{aligned} & f(\bar{x}+y) - f(\bar{x}) + \varphi_1[g_1(s) - g_1(y)] + \sum_{i=2}^{n} \varphi_i[g_i(y+z_i) - g_i(y)] \\ & = \psi_1[G_1(s) - G_1(y)] \\ & \quad + \sum_{i=2}^{n} \psi_i[g_i(y+\bar{x}+z_i) - g_i(y+\bar{x}) - g_i(y+z_i) + g_i(y)], \end{aligned} \tag{6}$$

for $y, s \in]0, \alpha - \bar{x}[,\ s > y$ and $x, y, z_i > 0,\ x + y + z_i < \alpha$ $(i = 2, \dots, n)$. For $u = G_1(s) - G_1(y)$, we have $s = G_1^{-1}(G_1(y) + u)$ and, by (4) and (5), $\bar{u} = G_1(\bar{y} + \bar{z}) - G_1(\bar{y})$. Therefore, (3) implies that, for fixed $\bar{z}_1, \dots, \bar{z}_n$,

$$\begin{aligned} \psi_1(u) \ = \ & f(\bar{x}+y) - f(\bar{x} + \varphi[g_1(G_1^{-1}(G_1(y)+u)) - g_1(y)] + \sum_{i=2}^{n} \varphi_i[g_i(y+\bar{z}_i) - g_i(y)] \\ & - \sum_{i=2}^{n} \psi_i[g_i(y+\bar{x}+\bar{z}_i) - g_i(y+\bar{x}) - g_i(y+\bar{z}_i) + g_i(y)] \ ((u, y) \in U \times V) \end{aligned} \tag{7}$$

in some neighbourhoods U and V of $\bar{u}$ and $\bar{y}$. The set of the points, where g_1 is not differentiable, is countable. Therefore, by the strict monotonicity of G_1 and G_1^{-1}, there exists a $y^* \in V$, such that G_1^{-1} is differentiable at $G_1(y^*) + \bar{u}$ and g_1 is differentiable at $G_1^{-1}(G_1(y^*) + \bar{u})$ with respect to u. Thus, the right hand side of (7) is differentiable at $\bar{u}$ with respect to u, which implies the same property for ψ_1.

Finally, we prove the differentiability of f on I. Equation (1) implies that, for fixed $\bar{x}, \bar{y}, \bar{z}_1, \dots, \bar{z}_n > 0$ satisfying $\bar{x} + \bar{y} + \bar{z}_i < \alpha,$ $(i = 1, \dots, n)$, there exist some neighbourhoods V, W, Z_i of $\bar{x}, \bar{y}, \bar{z}_i,$ $(i = 1, \dots, n)$, respectively, such that

$$\begin{aligned} f(x) \ = \ & f(x+y) + \sum_{i=1}^{n} \varphi_i[g_i(y+z_i) - g_i(y)] \\ & - \sum_{i=1}^{n} \psi_i[g_i(y+x+z_i) - g_i(y+x) - g_i(y+z_i) + g_i(y)] \\ & \qquad ((x, y, z_i) \in V \times W \times Z_i;\ i = 1, \dots, n). \end{aligned} \tag{8}$$

The functions $g_1, \dots, g_n$ and f are differentiable everywhere but at countably many points, therefore, there exists a $y^* \in W$ such that $g_1, \dots, g_n$ and f are differentiable at $\bar{x} + y^*$ with respect to x. Similarly, there exists a $z_i^* \in Z_i$ such that g_i is differentiable at $y^* + \bar{x} + z_i^*$ for each $i \in \{1, \dots, n\}$. The functions $\psi_1, \dots, \psi_n$ are differentiable, thus, the right hand side of (8) is differentiable at $\bar{x}$ with respect to x, which implies the differentiability of f at $\bar{x}$. The convexity or concavity and the differentiability of f yield that f is continuously differentiable on I. □

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