On a functional equation arising from comparison of utility representations

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Abstract

We solve the functional equation $F_1(t) - F_1(t+s) = F_2[F_3(t) + F_4(s)]$ for real functions defined on intervals, assuming that F_2 is positive valued and strictly monotonic and that F_3 is continuous. The equation arose from the equivalence problem of utility representations under assumptions of separability, homogeneity and segregation (e-distributivity).

Key words: Functional equation, utility representation, binary gamble, convexity.

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1 Background, introduction

Utility representations furnish background to the functional equation that we solve in this paper. With $f, g \in X$ (X a set of valued consequences) and Γ an event, $(f, \Gamma; g)$ is an uncertain alternative (a binary gamble), in which the holder (gambler) receives f if Γ occurs and receives g if it does not. There exists in X a "no change" consequence e. A transitive and connected preference order (weak order) $\succsim$ is assumed to exist between gambles. It is assumed that $(f, \Gamma; f) \sim f$ (idempotence), in the sense that the gamble $(f, \Gamma; f)$ is identified with the consequence f. So the weak order extends to the set X of consequences (it can also be extended to gambles with other gambles as consequences) and it makes sense to talk both about gambles and about consequences $\succsim e$. A "utility function" U maps the set of all such gambles and consequences onto the half-open real interval $[0, k[$ ($k \in]0, \infty]$), and a "weighting function" W maps the set of all events onto the closed interval $[0, 1]$. They are strictly increasing in the sense that, for $f, g \succsim e$, we have $U(f) \geq U(g)$ if, and only if, $f \succsim g$ (in particular, $U(e) = 0$) and, for $f \succ g \succsim e$, we have $W(\Gamma_1) \geq W(\Gamma_2)$ if, and only if, $(f, \Gamma_1; g) \succsim (f, \Gamma_2; g)$. We have a "utility representation" if $U[(f, \Gamma; g)]$ is, for $f \succsim g \succsim e$, a function M of $U(f), U(g)$ and $W(\Gamma)$ alone: $U[(f, \Gamma; g)] = M[U(f), U(g), W(\Gamma)]$.

For consequences there is also a "joint receipt" operation $\oplus$, strictly increasing in the first term, meaning that, for $f, f', g \succsim e$, we have $f \oplus g \succsim f' \oplus g$ if, and only if, $f \succsim f'$. The "no change" consequence e is a left unit: $e \oplus g = g$ for all $g \succsim e$.

Under the further restrictions that, for $f, g \succsim e$, "e-distributivity" $(f, \Gamma; e) \oplus g \sim (f \oplus g, \Gamma; g)$ (called also "segregation") and "separability" $U[(f, \Gamma; e)] = U(f)W(\Gamma)$ hold, and M is homogeneous in its first two variables, it was proved in [8, Theorem 4] that there exists a strictly increasing continuous function $\lambda : [1, \infty[\to [0, \infty[$, with $\lambda(1) = 0$, such that the following utility representation holds for $f \succsim g \succ e$:

$$U(f, \Gamma; g) = U(g)\lambda^{-1}\left[W(\Gamma)\lambda\left(\frac{U(f)}{U(g)}\right)\right]. \tag{1}$$

Here we examine when two such representations, with λ and $\tilde{\lambda}$, are equivalent in the sense that there exist two order preserving homeomorphisms $G :]0, k[\to]0, \tilde{k}[$ ($\tilde{k} \in]0, \infty]$) and $H : [0, 1] \to [0, 1]$ such that $\widetilde{U} = G \circ U$ and $\widetilde{W} = H \circ W$. Thus

$$G[U(g)]\tilde{\lambda}^{-1}\left[H(W(\Gamma))\tilde{\lambda}\left(\frac{G[U(f)]}{G[U(g)]}\right)\right] = G\left(U(g)\lambda^{-1}\left[W(\Gamma)\lambda\left(\frac{U(f)}{U(g)}\right)\right]\right) \tag{2}$$

holds for $f \succsim g \succ e$. (The formulas for λ, $\tilde{\lambda}$, G and H will be stated in the concluding

Section 4). Writing $p = U(f)$, $q = U(g)$, $w = W(\Gamma)$, we get

$$G(q)\tilde{\lambda}^{-1}\left[H(w)\tilde{\lambda}\left(\frac{G(p)}{G(q)}\right)\right] = G\left(q\lambda^{-1}\left[w\lambda\left(\frac{p}{q}\right)\right]\right)$$

for $k > p \geq q > 0$ and $w \in [0, 1]$. With $z = p/q \geq 1$, this equation goes over into

$$G(q)\tilde{\lambda}^{-1}\left[H(w)\tilde{\lambda}\left(\frac{G(qz)}{G(q)}\right)\right] = G(q\lambda^{-1}\left[w\lambda\left(z\right)\right]).$$

Treating q as a parameter, this is a Pexider equation in the variables w and z. Its general continuous, strictly increasing solution is of the form

$$\tilde{\lambda}\left(\frac{G(qz)}{G(q)}\right) = \mu(q)\lambda(z)^{\rho}, \qquad H(w) = w^{\rho} \tag{3}$$

for some “constants” $\mu(q) > 0$ and $\rho > 0$ (cf. [1, Section 3.1.1] and [2, Section 5]). The first equation is trivially satisfied if $z = 1$, so we will assume $z > 1$. We define

$$F_1(t) = -\ln G(e^t),\ F_2(u) = \ln\ \tilde{\lambda}^{-1}(e^u),\ F_3(t) = \ln \mu(e^t),\ F_4(s) = \rho \ln \lambda(e^s), \tag{4}$$

with $s = \ln z$, $t = \ln q$, and arrive at the functional equation

$$F_1(t) - F_1(t+s) = F_2[F_3(t) + F_4(s)]. \tag{5}$$

This functional equation has been encountered before by A. Lundberg [7] and by J. Aczél, Gy. Maksa, C. T. Ng and Zs.Páles [3] under different conditions. It has been solved in [3] on a domain suitable for the current motivation. However, strict monotonicity was assumed for F_3. Here the continuity of F_3 is assumed instead. In what follows we shall consider the equation on a more general domain.

For general background to the concepts underlying the formulation of decision making under uncertainty, that gives rise to our functional equation problem, see R. D. Luce [6].

2 The main functional equation and associated equations

Given $a, b \in [-\infty, \infty]$ $(a < b)$, the functional equation

$$F_1(t) - F_1(t+s) = F_2[F_3(t) + F_4(s)] \qquad (t \in]a, b[\,,\ s \in]0, b-t[\,) \tag{6}$$

is considered under the following assumptions:

(A1) $F_1 :]a, b[\to \mathbb{R}$,
(A2) $F_3 :]a, b[\to \mathbb{R}$ is continuous,
(A3) $F_4 :]0, b-a[\to \mathbb{R}$,
(A4) $F_2 : I \to]0, \infty[$ is strictly monotonic, where

$$I = \{F_3(t) + F_4(s) \mid t \in]a, b[\,,\ s \in]0, b-t[\,\}.$$

Here and later the following customary conventions are used to interprete the intervals $]0, b-t[$ and $]0, b-a[$: $b - (-\infty) := \infty$ for finite b, and $\infty - \omega := \infty$ both for finite ω and for $\omega = -\infty$.

Lemma 1. *Suppose* (6) *holds and the conditions* (A1)–(A4) *are satisfied. Then there exists a* $c \in [a, b]$ *such that* F_3 *is strictly monotonic on* $]a, c[$ *and constant on* $]c, b[\,$.

PROOF. If F_3 is strictly monotonic on $]a, b[\,$, then the assertion holds with $c = b$. Now suppose that F_3 is not strictly monotonic on $]a, b[\,$.

Let $t_1 < t_2$ in $]a, b[$ be such that $F_3(t_1) = F_3(t_2)$. Let t_3 be a point in $]t_1, t_2[$ where F_3 has a maximum or minimum, say maximum, within $[t_1, t_2]$; if F_3 is constantly maximum on an interval $[t_3', t_3'']$ then choose $t_3 = (t_3' + t_3'')/2$. Thus $F_3(t_3) \geq F_3(t_1) = F_3(t_2)$. Then there exist arbitrarily close t_4, t_5 such that $t_1 < t_4 < t_3 < t_5 < t_2$ and $F_3(t_4) = F_3(t_5)$. Thus, by (6) we have

$$F_1(t_4) - F_1(t_4 + s) = F_1(t_5) - F_1(t_5 + s) \qquad (s \in]0, b - t_5[).$$

Fixing $s = s_0 \in]0, b - t_5[$ we get

$$F_1(t_4) - F_1(t_4 + s_0) = F_1(t_5) - F_1(t_5 + s_0),$$

and replacing s by $s_0 + s$ gives

$$F_1(t_4) - F_1(t_4 + s_0 + s) = F_1(t_5) - F_1(t_5 + s_0 + s).$$

Subtracting the former from the latter we obtain

$$F_1(t_4 + s_0) - F_1(t_4 + s_0 + s) = F_1(t_5 + s_0) - F_1(t_5 + s_0 + s) \quad (s \in]0, b - t_5 - s_0[).$$

Putting this back into (6) results in

$$F_2[F_3(t_4 + s_0) + F_4(s)] = F_2[F_3(t_5 + s_0) + F_4(s)] \qquad (s \in]0, b - t_2 - s_0[\,).$$

Because F_2 is injective, that gives

$$F_3(t_4+s_0)=F_3(t_5+s_0) \qquad \text{for all } s_0\in\,]0,b-t_2[.$$

This fact can be rephrased as follows: If $F_3(t_1)=F_3(t_2)$ for fixed $t_1<t_2$ then F_3, continuous by (A2), is periodic with arbitrarily small period t_5-t_4 on the interval $[t_3,b[$ (while t_4 depends on how small a period we want, t_3 depends only upon $[t_1,t_2]$). So F_3 is constant at least on $[t_3,b[$. Let $c\in[a,b[$ be the smallest number for which F_3 is constant on $]c,b[$; then F_3 is strictly monotonic on $]a,c[$. □

In the following lemma we investigate (6) in the case when the function F_3 is nonconstant.

Lemma 2. *Suppose that* (6) *holds with (A1)–(A4) and that F_3 is nonconstant. Then the following properties follow:*

(S0) There exists a $c\in\,]a,b]$ such that F_3 is strictly monotonic on $]a,c[$ and constant on $]c,b[$,
(S1) F_1 is strictly decreasing,
(S2) F_1 is convex or concave on $]a,b[$, strictly convex or strictly concave on $]a,c[$, and affine on $]c,b[$,
(S3) the left derivative F'_{1-} exists, is negative and monotonic on $]a,b[$, and satisfies $F'_{1-}(t+s)\neq F'_{1-}(t)$ for all $t\in\,]a,c[$, $s\in\,]0,b-t[$,
(S4) $J=\{F_1(t)-F_1(t+s)\mid t\in\,]a,b[\,,s\in\,]0,b-t[\,\}$ is an open interval,
(S5) F_4 is differentiable,
(S6) F_2^{-1} is differentiable on J,
(S7) the left derivative F'_{3-} exists on $]a,b[$,
(S8) the following differential-functional equation holds:

$$F_4'(s)[F'_{1-}(t+s)-F'_{1-}(t)]=F'_{3-}(t)F'_{1-}(t+s) \qquad (t\in\,]a,b[\,,\ s\in\,]0,b-t[\,),$$

(S9) F'_{3-} is everywhere positive or everywhere negative on $]a,c[$ (for short, we say: F'_{3-} is sign preserving on $]a,c[$) , and it vanishes on $]c,b[$,
(S10) F_4' is sign preserving on $]0,b-a[$.

PROOF. The first property (S0) is due to Lemma 1. By assumption (A4), F_2 is positive valued. This implies (S1).

By (A4), and by the monotonicity of F_3 seen from (S0), the right hand side of equation (6), as function of t, is either increasing for all fixed s or decreasing for all fixed s. Thus, for $s\in\,]0,b-a[$, the functions

$$t\mapsto F_1(t)-F_1(t+s) \qquad (t\in\,]a,b-s[\,) \tag{7}$$

are also monotonic. Suppose that they are decreasing. Then, for $s \in]0,(b-a)/2[$, we have $F_1(t) - F_1(t+s) \geq F_1(t+s) - F_1((t+s)+s)$, that is,

$$2F_1(t+s) \leq F_1(t) + F_1(t+2s) \qquad (t \in]a, b-2s[\,). \tag{8}$$

Because s can be chosen arbitrarily in $]\,0,(b-a)/2\,[$, this inequality means that F_1 is Jensen-convex on $]a,b[$. Furthermore, by (S0), we see that inequality (8) holds in the strict form $2F_1(t+s) < F_1(t)+F_1(t+2s)$ for $t, t+2s \in]a,c[$; and $2F_1(t+s) = F_1(t)+F_1(t+2s)$ for $t, t+2s \in]c,b[$. Thus, F_1 is strictly Jensen-convex on $]a,c[$ and Jensen-affine on $]c,b[$. The monotonicity of F_1 yields its local boundedness on $]a,b[$, so, by the Bernstein-Doetsch theorem ([4], [5, Chapter VI]), it is convex on $]a,b[$, strictly convex on $]a,c[$, and affine on $]c,b[$. Had we assumed that the function in (7) is increasing, we would have come to the same conclusion with convexity replaced by concavity. This proves (S2).

We shall restrict the arguments about (S3) to convex F_1, as the concave case is similar. Using well-known properties of convex functions (cf. e.g. [5, Chapter VII]), we get that (i) F_1 is continuous on $]a,b[$, (ii) its left derivative F'_{1-} exists at every point of $]a,b[$, and is monotonic increasing (not yet strictly) on $]a,b[$, (using e.g. Theorem B in [9, p. 5]), (iii) F'_{1-} is nonpositive on $]a,b[$ in view of (S1), and (iv) F_1 is differentiable everywhere except for at most countably many places in $]a,b[$. We now argue for the second assertion in (S3), that F'_{1-} is indeed negative. Suppose, to the contrary, that $F'_{1-}(d) = 0$ at some $d \in]a,b[$. Then, by (ii) and (iii), F'_{1-} vanishes on $]d,b[$. By [5, Chapter VII, Theorem 4.2], F_1 is differentiable and constant on $]d,b[$. This contradicts (S1). To show the third assertion in (S3), suppose, to the contrary, that $F'_{1-}(t_0+s_0) = F'_{1-}(t_0)$ for some $t_0 \in]a,c[$, $s_0 \in]0,b-t_0[$. Then, by (ii), F'_{1-} is constant on $]t_0,t_0+s_0[$. This implies that F_1 is affine on $]t_0,t_0+s_0[$: a contradiction to the strict convexity of F_1 on $]a,c[$. This proves (S3).

The continuity of F_1 yields that the set J defined in (S4) is an interval. The strict monotonicity of F_1 implies that J is open.

Since F_2 is strictly monotonic, (6) can be written in the form

$$F_2^{-1}[F_1(t) - F_1(t+s)] = F_3(t) + F_4(s) \qquad (t \in]a,b[\,,\ s \in]0,b-t[\,). \tag{9}$$

The function F_2^{-1} is also strictly monotonic, therefore, by Lebesgue's theorem, it is differentiable almost everywhere on J. According to (S2), F_1 is differentiable on $]a,b[$ except at at most countably many points. Furthermore, by the strict monotonicity of F_3 on $]a,c[$, the function defined in (7) is strictly monotonic on the nonempty, open interval $]a,c[\cap]a,b-s_0[$ for each fixed $s_0 \in]0,b-a[$. Thus there exists a $t_0 \in]a,c[\cap]a,b-s_0[$ such that F_1 is differentiable at t_0+s_0 and F_2^{-1} is differentiable at $F_1(t_0) - F_1(t_0+s_0)$. That is, for $t = t_0$ the left hand side of (9) is differentiable with respect to s at s_0. Therefore, F_4 is also differentiable at s_0. As s_0 can be taken arbitrarily in $]a,b[$, this proves (S5).

Let $z_0 = F_1(t_0) - F_1(t_0 + s_0) \in J$ $(t_0 \in]a, b[,\ \ s_0 \in]0, b - t_0[)$ be given. Then $s_0 = F_1^{-1}[F_1(t_0) - z_0] - t_0$. By continuity, there exists a neighborhood $T_0 \times Z_0$ of (t_0, z_0) such that $F_1^{-1}[F_1(t) - z] - t > 0$ and $F_1^{-1}[F_1(t) - z] < b$ for all $(t, z) \in T_0 \times Z_0$. Taking an element $t_1 \in T_0$ such that the strictly monotonic F_1^{-1} is differentiable at $F_1(t_1) - z_0$, and writing $t = t_1$ and $s = F_1^{-1}[F_1(t_1) - z] - t_1$ in (9), we get

$$F_2^{-1}(z) = F_3(t_1) + F_4(F_1^{-1}[F_1(t_1) - z] - t_1) \qquad (z \in Z_0).$$

By the differentiability of F_4 and by the choice of t_1, the right hand side of this equation is differentiable with respect to z at z_0 and that implies the differentiability of F_2^{-1} at z_0. Because z_0 is arbitrary in J, this proves (S6).

The left derivative of F_1 exists on $]a, b[$ and F_2 is differentiable on J, therefore the left derivative of F_3 exists on $]a, b[$ by (9), and (S7) is proved.

Differentiating equation (9) with respect to t and s from the left, we get

$$(F_2^{-1})'[F_1(t) - F_1(t+s)]\ [F'_{1-}(t) - F'_{1-}(t+s)] = F'_{3-}(t)$$

and

$$-(F_2^{-1})'[F_1(t) - F_1(t+s)]\ F'_{1-}(t+s) = F'_4(s),$$

repectively, for all $t \in]a, b[\,,\ s \in]0, b-t[\,$. Multiplying the first equation by $F'_{1-}(t+s)$ and the second by $F'_{1-}(t+s) - F'_{1-}(t)$, and adding, we obtain the equation in (S8).

The function F_3 is constant on $]c, b[$, thus its derivative is 0 there; and as F_3 is monotonic on $]a, b[$, F'_{3-} is either nonnegative on $]a, b[$ or nonpositive on $]a, b[$. In order to prove the other part of (S9), suppose that there exists a $t_0 \in]a, c[$ such that $F'_{3-}(t_0) = 0$. By (S3), we have $F'_{1-}(t_0 + s) \neq F'_{1-}(t_0)$ for all $s \in]0, b - t_0[$. So the equation in (S8) implies that $F'_4(s) = 0$ for all $s \in]0, b - t_0[$. Since F_3 is strictly monotonic on $]a, c[$, there exits a $t_1 \in]a, t_0[$ such that $F'_{3-}(t_1) \neq 0$. Thus, the equation in (S8) yields that $F'_{1-}(t_1 + s) = 0$ for all $s \in]0, b - t_0[$ which contradicts (S3). This proves (S9).

Finally, if F'_4 were not sign preserving then, by the intermediate value property of functions that are derivatives ([10, Theorem 5.12]), there would exist an $s_0 \in]0, b - a[$ such that $F'_4(s_0) = 0$. Then, using (S8) and (S9), we would obtain $F'_{1-}(t + s_0) = 0$ for $t \in]a, b - s_0[$. This contradiction to (S3) proves (S10). $\square$

Now we solve the functional equation (S8) in Lemma 2. Let

$$\psi := F'_{1-}, \qquad \varphi := F'_{3-}, \qquad \chi := F'_4. \tag{10}$$

Then the equation becomes

$$\chi(s)[\psi(t+s)-\psi(t)] = \varphi(t)\psi(t+s) \qquad (t\in\,]a,b[\,,\ s\in\,]0,b-t[\,), \tag{11}$$

where

(A5) $\psi :]a,b[\to]-\infty,0[\,$,
(A6) $\varphi :]a,b[\to\mathbb{R}$ is sign preserving on $]a,c[$ and 0 on $]c,b[$, for a $c\in]a,b]$,
(A7) $\chi :]0,b-a[\to\mathbb{R}$ is sign preserving.

The sign preserving solutions of (11) were determined in [3] for the case $b=c=\infty$. Here we solve it under the somewhat weaker conditions (A5)–(A7) and for arbitrary $a<c\le b$. Our method is similar to that in [3]. In what follows we write $\Pi_+(\,]a,b[)$, and $\Pi_-(\,]a,b[\,)$ for the set of all pairs $(C,D)\in\mathbb{R}\times\mathbb{R}$, $C\neq 0$, for which the function $t\mapsto D+e^{Ct}$ is everywhere positive on $]a,b[$, or everywhere negative on $]a,b[$, respectively. We define $\Pi(\,]a,b[\,)=\Pi_-(\,]a,b[\,)\cup\Pi_+(\,]a,b[\,)$ and

$$\sigma(C,D)=\begin{cases} 1, & \text{if } (C,D)\in\Pi_+(\,]a,b[\,)\\ -1, & \text{if } (C,D)\in\Pi_-(\,]a,b[\,).\end{cases}$$

Theorem 1. *Let $a<b$ in $[-\infty,\infty]$ be given. For $c=b$, the functions ψ, φ, χ with the properties* (A5)–(A7) *solve the functional equation* (11) *if, and only if, they are, for all $t\in]a,b[$, $s\in]0,b-a[$, either of the form*

$$\psi(t)=\frac{A}{D+e^{Ct}}, \qquad \varphi(t)=\frac{Be^{Ct}}{D+e^{Ct}}, \qquad \chi(s)=\frac{B}{1-e^{Cs}}, \tag{12}$$

where A, B, C and D are constants with $BC\neq 0$, $(C,D)\in\Pi(\,]a,b[\,)$ and $A\,\sigma(C,D)<0$; or, if $]a,b[\neq\mathbb{R}$, of the form

$$\psi(t)=\frac{P}{t+R}, \qquad \varphi(t)=\frac{Q}{t+R}, \qquad \chi(s)=-\frac{Q}{s}, \tag{13}$$

where P, Q and R are constants with $Q\neq 0$ and either $P>0, R\le -b$, or $P<0, R\ge -a$. No function satisfies (11) *and* (A5)–(A7) *if $c<b$ in* (A6).

PROOF. It can be easily shown, that the functions in (12) and (13) satisfy (11) and fulfill the conditions (A5)–(A7) with $c=b$.

In order to prove that (11) has no other solutions with these properties, we define

$$\ell=\frac{1}{\psi}, \qquad m=\frac{\varphi}{\psi}, \qquad n=-\frac{1}{\chi} \tag{14}$$

and write (11) in the form

$$\ell(t+s) = \ell(t) + m(t)n(s) \qquad (t \in]a,b[\,,\ s \in]0, b-t[\,), \tag{15}$$

where ℓ and n are sign preserving on $]a,b[$ or $]0,b-a[$, respectively, while m is sign preserving on $]a,c[$, and 0 on $]c,b[$. (Note that "sign preserving" includes that the function has no zero on that interval).

By the monotonicity of $\psi := F'_{1-}$ in (S3), the function ℓ is monotonic on $]a,b[$ (though not necessarily strictly monotonic at this stage; when the theorem is proved we will have ℓ strictly monotonic and m sign preserving on all of $]a,b[$, that is, $c = b$). Thus ℓ is integrable on all finite closed subintervals of $]a,b[$. Furthermore, $n(s) \neq 0$, so m is also locally integrable.

Fix $t_1 < t_2$ in $]a,c[$ and integrate (15) with respect to t from t_1 to t_2 to get

$$\int_{t_1+s}^{t_2+s} \ell = \int_{t_1}^{t_2} \ell + n(s) \int_{t_1}^{t_2} m \qquad (s \in]0, b-t_2[). \tag{16}$$

Here $\int_{t_1}^{t_2} m \neq 0$ because m is sign preserving on $]a,c[$ and $a < t_1 < t_2 < c$. The left hand side of (16) is continuous in s, so n is continuous on $]0,b-t_2[$. As t_2 can be arbitrarily close to a, we get the continuity of n on its domain $]0,b-a[$. With equation (15) this gives the continuity of ℓ on $]a,b[$. Since n is nowhere 0, the continuity of m on $]a,b[$ also follows.

Hence we get the continuity of ℓ, m, n from local integrability. Now the left hand side of (16) is differentiable, so n is differentiable too and, by (15) so is ℓ. Repeated application of the same standard steps gives that all three functions are C^∞.

Differentiating equation (15) with respect to s we get

$$\ell'(t+s) = m(t)n'(s) \qquad (t \in]a,b[\,,\ s \in]0, b-t[\,).$$

The nonzero differentiable solutions of this Pexider equation are

$$\ell'(t) = a_1 a_2 e^{Ct}, \qquad m(t) = a_1 e^{Ct}, \qquad n'(s) = a_2 e^{Cs} \qquad (t \in]a,b[\,,\ s \in]0, b-a[\,),$$

where C, $a_1 \neq 0$ and $a_2 \neq 0$ are constants (cf. e.g. [1, Sections 3.1.1 and 4.2.1]). Integrating ℓ' and n', and using (15), we get in the case $C \neq 0$

$$\ell(t) = \frac{a_1 a_2}{C} e^{Ct} + a_3, \qquad m(t) = a_1 e^{Ct}, \qquad n(s) = \frac{a_2}{C} e^{Cs} - \frac{a_2}{C}$$

with a constant a_3, and in the case $C = 0$ we get

$$\ell(t) = a_1 a_2 t + a_4, \qquad m(t) = a_1, \qquad n(s) = a_2 s$$

with a constant a_4. Taking (14) into consideration and defining

$$A = \frac{C}{a_1 a_2}, \quad B = \frac{C}{a_2}, \quad D = \frac{C a_3}{a_1 a_2}, \quad P = \frac{1}{a_1 a_2}, \quad Q = \frac{1}{a_2}, \quad R = \frac{a_4}{a_1 a_2},$$

we get that the solutions of (11) are of the forms (12) and (13). The assumptions (A5)–(A7) yield the restrictions on the constants in the theorem. In particular, in order that ψ be negative, $A\,\sigma(C, D) < 0$ and either $P > 0$, $R \le -b$ or $P < 0$, $R \ge -a$ have to hold. □

3 Solutions of equation (6)

Finally, we determine the solutions of our main equation (6).

(Throughout, $A_1, A_2, A_3, A, B, C, D, P, Q, R, C_1, C_3, C_4$ are constants).

Theorem 2. *Let $a < b$ be in $[-\infty, \infty]$. Assume that the functions F_1, F_2, F_3, F_4 solve equation* (6) *and satisfy the properties* (A1)–(A4)*. Then F_3 is either constant or strictly monotonic. The general solution of* (6) *under the above assumptions are:*

I. *If F_3 is constant then*

$$F_1(t) = A_1 t + A_2 \qquad (t \in \,]a, b[\,), \tag{17}$$

$$F_3(t) = A_3 \qquad (t \in \,]a, b[\,), \tag{18}$$

F_4 *is strictly monotonic* (19)

$$F_2(u) = -A_1 F_4^{-1}(u - A_3) \qquad (u \in I) \tag{20}$$

with $A_1 < 0$.

II. *If F_3 is strictly monotonic then either*

$$F_1(t) = -\frac{A}{CD} \ln |De^{-Ct} + 1| + C_1 \qquad (t \in \,]a, b[\,), \tag{21}$$

$$F_3(t) = \frac{B}{C} \ln |D + e^{Ct}| + C_3 \qquad (t \in \,]a, b[\,), \tag{22}$$

$$F_4(s) = -\frac{B}{C} \ln |1 - e^{-Cs}| + C_4 \qquad (s \in \,]0, b-a[\,), \tag{23}$$

$$F_2(u) = \frac{A}{CD} \ln \left(1 - \sigma(C,D)\, D \operatorname{sign} C\, e^{-\frac{C}{B}(u - C_3 - C_4)}\right) \qquad (u \in I), \tag{24}$$

with $BCD \neq 0$, $(C, D) \in \Pi(\,]a,b[\,)$ *and* $A\,\sigma(C,D) < 0$; *or*

$$F_1(t) = -\frac{A}{C} e^{-Ct} + C_1 \qquad (t \in]a,b[), \tag{25}$$

$$F_3(t) = Bt + C_3 \qquad (t \in]a,b[), \tag{26}$$

$$F_4(s) = -\frac{B}{C} \ln |1 - e^{-Cs}| + C_4 \qquad (s \in \,]0, b-a[\,), \tag{27}$$

$$F_2(u) = -\frac{A}{|C|} e^{-\frac{C}{B}(u - C_3 - C_4)} \qquad (u \in I), \tag{28}$$

with $BC \neq 0$, $A < 0$; *or, if* $]a,b[\neq \mathbb{R}$,

$$F_1(t) = P \ln |t + R| + C_1 \qquad (t \in \,]a,b[\,), \tag{29}$$

$$F_3(t) = Q \ln |t + R| + C_3 \qquad (t \in \,]a,b[\,), \tag{30}$$

$$F_4(s) = -Q \ln s + C_4 \qquad (s \in \,]0, b-a[\,), \tag{31}$$

$$F_2(u) = -P \ln \left(1 - \operatorname{sign} P\, e^{-\frac{u - C_3 - C_4}{Q}}\right) \qquad (u \in I), \tag{32}$$

with $Q \neq 0$ *and either* $P > 0$, $R \leq -b$ *or* $P < 0$, $R \geq -a$.

PROOF.

Let a, b be given and suppose that F_1, F_2, F_3 and F_4 satisfy equation (6) and conditions (A1)–(A4). According to Lemma 1, there exists a $c \in [a, b]$ such that F_3 is strictly monotonic on $[a, c[$ and constant on $]c, b[$. The last statement in Theorem 1 implies (since, by (10), c in (A6) and in Theorem 1 is identical with c in Lemma 1) that, under the assumption that F_3 is nonconstant, equation (6) has no solutions if $c < b$. Therefore, F_3 is either constant or strictly monotonic, thus, the first statement of our theorem is proved.

Substitution shows that the functions listed above fulfill (A1)–(A4) and (6).

Now we prove that (6) has no other solutions under these assumptions.

In the case I, when F_3 is constant, say $F_3 = A_3$, equation (6) reduces to the Pexider equation

$$F_1(t) - F_1(t+s) = F_2[A_3 + F_4(s)]. \tag{33}$$

By (A4), F_2 is positive valued and strictly monotonic. Therefore, F_1 is strictly decreasing, so equation (33) implies (17) $F_1(t) = A_1 t + A_2$ with $A_1 < 0$ and $F_2[A_3 + F_4(s)] = A_1 s$, that is, (20). Therefore (19) is also valid.

In the following we consider the case II, where F_3 is strictly monotonic. By Lemma 2, the functions F_1 and F_3 are differentiable from the left on $]a, b[$, F_4 is differentiable on $]0, b-a[$, the functions ψ, φ, χ introduced in (10) fulfill the properties (A5)–(A7) and they satisfy (11). By Theorem 1 with (10), we have, for all $t \in]a, b[$, $s \in]0, b-a[$, either

$$F'_{1-}(t) = \frac{A}{D + e^{Ct}}, \qquad F'_{3-}(t) = \frac{Be^{Ct}}{D + e^{Ct}}, \qquad F'_4(s) = \frac{B}{1 - e^{Cs}}, \tag{34}$$

where $BC \neq 0$, $(C, D) \in \Pi(]a, b[)$, and $A\,\sigma(C, D) < 0$; or, if $]a, b[\neq \mathbb{R}$,

$$F'_{1-}(t) = \frac{P}{t + R}, \qquad F'_{3-}(t) = \frac{Q}{t + R}, \qquad F'_4(s) = -\frac{Q}{s}, \tag{35}$$

where $Q \neq 0$ and either $P > 0,\ R \leq -b$ or $P < 0,\ R \geq -a$. According to (S2) in Lemma 2, F_1 is convex or concave and its one sided derivative F'_{1-} is continuous by (34) and (35), thus it is differentiable on $]a, b[$ (cf. [5, Chapter VII, Theorem 4.2]).So, by (S6) and (9), F_3 is also differentiable on $]a, b[$. Therefore F_1, F_3, F_4 can be obtained by integrating the corresponding functions in (34) and (35).

Integration in (34) gives (21), (22), (23) if $D \neq 0$, and (25), (26), (27) if $D = 0$.

Taking $D \neq 0$ and substituting F_1, F_3, F_4 into (6), we get

$$\frac{A}{CD} \ln \left| \frac{De^{-C(t+s)} + 1}{De^{-Ct} + 1} \right| = F_2 \left(-\frac{B}{C} \ln \left| \frac{1 - e^{-Cs}}{D + e^{Ct}} \right| + C_3 + C_4 \right) \tag{36}$$

for all $t \in]a, b[$, $s \in]0, b-t[$. Observing

$$\frac{De^{-C(t+s)} + 1}{De^{-Ct} + 1} = 1 - D\frac{1 - e^{-Cs}}{D + e^{Ct}}, \qquad \frac{De^{-C(t+s)} + 1}{De^{-Ct} + 1} > 0,$$

$$\left| \frac{1 - e^{-Cs}}{D + e^{Ct}} \right| = \sigma(C, D)\,\mathrm{sign}\, C \frac{1 - e^{-Cs}}{D + e^{Ct}} \qquad (t \in]a, b[\,,\ s \in]0, b-t[\,),$$

we see that equation (36) yields (24) for $D \neq 0$.

If $D=0$, equation (6) yields

$$-\frac{A}{C}\frac{e^{Cs}-1}{e^{C(t+s)}} = F_2\left(-\frac{B}{C}\ln\frac{|1-e^{Cs}|}{e^{Ct}} + C_3 + C_4\right) \qquad (t\in\,]a,b[\,,\; s\in\,]0,b-t[\,). \tag{37}$$

Since $\operatorname{sign} C = \operatorname{sign}(e^{Cs}-1)$, we can write (37) as

$$-\frac{A}{|C|}\frac{|e^{Cs}-1|}{e^{C(t+s)}} = F_2\left(-\frac{B}{C}\ln\frac{|1-e^{Cs}|}{e^{Ct}} + C_3 + C_4\right)$$

which yields that F_2 is of the form (28). The positivity of F_2 gives $A<0$.

Let us consider the functions in (35). By integration we get (29), (30) and (31). Substituting F_1, F_3 and F_4 into (6), we obtain

$$P\ln\frac{|t+R|}{|t+R+s|} = P\ln\frac{|(t+R)/s|}{|(t+R)/s)+1|} = F_2\left(Q\ln\left|\frac{t+R}{s}\right| + C_3 + C_4\right)$$

for $t\in]a,b[$, $s\in]0,b-t[$. Since F'_{1-} is everywhere negative, we have $\operatorname{sign}(t+R) = -\operatorname{sign}P$ for all $t\in\,]a,b[$. Thus

$$\frac{t+R}{t+R+s} > 0 \qquad (t\in]a,b[\,, s\in]0,b-t[),$$

and the absolute value signs can be omitted on the left hand side of the equation above. Using these properties, a simple calculation gives (32). □

4 Conclusion

In section 1, equations (3), we found $H(w)=w^\rho$ ($\rho>0$) to be one of the homeomorphisms establishing the equivalence (2). We calculate now the other homeomorphism, G, first for $q\in]0,k[$, and then determine those which can be continuously extended to $q\in[0,k[$. By (4),

$$G(q) = e^{-F_1(\ln q)}, \qquad \mu(q) = e^{F_3(\ln q)}, \qquad \lambda(z) = e^{\frac{1}{\rho}F_4(\ln z)}, \qquad \tilde\lambda(v) = e^{F_2^{-1}(\ln v)}. \tag{38}$$

We assumed $\lambda(1)=0$ and continued with $z>1$. In order that λ, and also $\tilde\lambda$, be *continuous at 1* we need *the limit condition* $\lim_{z\to1+}\lambda(z) = \lim_{v\to1+}\tilde\lambda(v) = 0$.

Let first μ and thus F_3 be constant. If, as in (2), λ and $\tilde{\lambda}$ are strictly increasing and continuous on $[1,\infty[$ and $\lambda(1)=\tilde{\lambda}(1)=0$, then F_4 and F_2 are continuous and strictly increasing on $]0,\infty[$, so the assumptions in Theorem 2, yielding solution I, are satisfied. Thus we have (17) $F_1(t)=A_1t+A_2$ $(A_1<0)$. Therefore we get $G(q)=\gamma q^{1/\beta}$ $(\beta>0,\, \gamma>0)$. Furthermore, by (4) and by $\lim_{z\to 1+}\lambda(z)=0$, we have $\lim_{s\to 0+}F_4(s)=-\infty$, otherwise the continuous and strictly *increasing* F_4 and λ are arbitrary. Also, by (20) $F_2(u)=-A_1F_4^{-1}(u-A_3)$, that is,

$$\tilde{\lambda}(v)=\alpha\,\lambda(v^{\beta})^{\rho} \qquad (\alpha>0,\ \beta>0,\ \rho>0) \tag{39}$$

which implies $\lim_{v\to 1+}\tilde{\lambda}(v)=0$. This shows that *the pair of homeomorphisms*

$$G(q)=\gamma q^{1/\beta}\,, \qquad H(w)=w^{\rho} \qquad (\beta>0,\ \gamma>0,\ \rho>0) \tag{40}$$

gives an equivalent in the sense (2) *to any representation of the form (1).*

By Theorem 2 and (38) *there exist additional pairs of equivalent representations* (2) *whose connection differs from* (39) *or from* (40). We identify the representations here by the λ , $\tilde{\lambda}$ and the homeomorphisms G (always $H(w)=w^{\rho}$) that establish the equivalence $(\alpha,\beta,\gamma,\delta,\varepsilon,\varepsilon'$ and ρ are positive constants, A,B,C,D,P,Q,R are as in Theorem 2, II). They *are the following, and only these*:

$$\lambda(z)=\delta|1-z^{-C}|^{-\frac{B}{\rho C}}, \qquad \tilde{\lambda}(v)=\varepsilon|1-v^{\frac{CD}{A}}|^{-\frac{B}{C}}, \qquad G(q)=\gamma|Dq^{-C}+1|^{\frac{A}{CD}}, \tag{41}$$

$$\lambda(z)=\delta|1-z^{-C}|^{-\frac{B}{\rho C}}, \qquad \tilde{\lambda}(v)=\varepsilon'(\ln v)^{-\frac{B}{C}}, \qquad G(q)=\alpha e^{\frac{A}{C}q^{-C}}, \tag{42}$$

$$\lambda(z)=\delta(\ln z)^{-\frac{Q}{\rho}}, \qquad \tilde{\lambda}(v)=\varepsilon|1-v^{-\frac{1}{P}}|^{-Q}, \qquad G(q)=\beta|\ln q+R|^{-P}. \tag{43}$$

The restrictions in Theorem 2 (case II) and $\rho>0$ guarantee that G and H are strictly increasing. By (38), (42), (41), and (43), the limit condition *holds if, and only if,* in addition to $\rho>0$ and to the restrictions in Theorem 2 II *also* $Q<0$ *in (43) and* $BC<0$ *in (42) and in (41).*

Notice that (39) also holds for the pair λ, $\tilde{\lambda}$ in (41) but G differs there from (40). In (42) and (43), λ and $\tilde{\lambda}$ are not connected by (39). The pair in (43) is the mirror image of that in (42) while G and H are replaced by their inverses.

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