HYPERHARMONIC SERIES INVOLVING HURWITZ ZETA FUNCTION

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Abstract. We show that the sum of the series formed by the so-called hyperharmonic numbers can be expressed in terms of the Riemann zeta function. These results enable us to reformulate Euler's formula involving the Hurwitz zeta function. In additon, we improve Conway and Guy's formula for hyperharmonic numbers.

1. Hyperharmonic numbers

Introduction. In 1996, J. H. Conway and R. K. Guy in [CG] have defined the notion of hyperharmonic numbers.

The n-th harmonic number of order 0 is

$$H_n^{(0)} = \frac{1}{n} \quad (n > 0),$$

and for all $r > 0$ let

$$H_n^{(r)} = \sum_{k=1}^{n} H_k^{(r-1)}$$

be the n-th hyperharmonic number of order r. In special, the hyperharmonic numbers of order 1 are simply called harmonic numbers:

$$H_n^{(1)} = H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

It turned out that these numbers have many combinatorial connections. Getting deeper insight, see [BGG] and the references given there.

$H_n^{(r)}$ can be expressed by binomial coefficients and ordinary harmonic numbers [CG]:

(1)
$$H_n^{(r)} = \binom{n+r-1}{r-1}(H_{n+r-1} - H_{r-1}).$$

In this paper we give a new proof of (1) and, in addition, we provide a more general formula.

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Moreover, we investigate sums of series involving hyperharmonic numbers. Among others, we prove that for the Hurwitz zeta function

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m} = \sum_{n=1}^{\infty} H_n^{(r-1)} \zeta(m,n).$$

As a special case, we get the next curious identities for Riemann zeta and Hurwitz zeta function

$$\sum_{k=1}^{\infty} \frac{\zeta(2,k)}{k} = 2\zeta(3), \tag{2}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(3,k)}{k} = \frac{5}{4}\zeta(4). \tag{3}$$

We close our introduction with the relation between hyperharmonics and the so-called r-Stirling numbers.

r-Stirling numbers. The r-Stirling number of the first kind with parameters n and k, denoted by $\left[{n \atop k}\right]_r$, is the number of permutations of the set $\{1, \dots, n\}$ having k disjoint, non-empty cycles, in which the elements 1 through r are restricted to appear in different cycles $(n \ge k \ge r)$.

The following identity integrates the hyperharmonic- and the r-Stirling numbers of the first kind.

$$\frac{\left[{n+r \atop r+1}\right]_r}{n!} = H_n^{(r)}.$$

This equality will be used in the special case $r = 1$ [GKP]:

$$\frac{\left[{n+1 \atop 2}\right]}{n!} := \frac{\left[{n+1 \atop 2}\right]_1}{n!} = H_n. \tag{4}$$

2. A relation between hyperharmonic numbers

Generating functions. Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence. Then the function

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

is called the generating function of $(a_n)_{n\in\mathbb{N}}$. If $a_n = H_n$ we get that (see [GKP, BGG])

$$\sum_{n=0}^{\infty} H_n z^n = -\frac{\ln(1-z)}{1-z}, \tag{5}$$

and in general (cf. [D])

$$\sum_{n=0}^{\infty} H_n^{(r)} z^n = -\frac{\ln(1-z)}{(1-z)^r}. \tag{6}$$

Beside these we also need Newton's binomial formula

$$\frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} z^n.$$

We provide a more general form of (1).

Theorem 1. *We have*

$$\binom{k+r-1}{k} H_n^{(k+r)} = \binom{n+k}{n} H_{n+k}^{(r)} - \binom{n+k+r-1}{n} H_k^{(r)}$$

Proof. Let us consider the generating function of the hyperharmonic numbers in (6). If we differentiate the left hand side k-times and make some rearragement, we obtain

$$\frac{d^k}{dz^k}\left\{-\frac{\ln(1-z)}{(1-z)^r}\right\} = \frac{(k+r-1)!}{(r-1)!}\frac{(H_{k+r-1} - H_{r-1} - \ln(1-z))}{(1-z)^{k+r}}.$$

We can write this in terms of Newton's binomial series and generating function of hyperharmonic numbers as follows:

$$\frac{d^k}{d^k z}\left\{-\frac{\ln(1-z)}{(1-z)^r}\right\} =$$

$$k!\sum_{n=0}^{\infty}\left[\binom{n+k+r-1}{n} H_k^{(r)} + \binom{k+r-1}{k} H_n^{(k+r)}\right] z^n.$$

On the other hand, if we view the generating function as a power series, we obtain

$$\frac{d^k}{dz^k}\left\{\sum_{n=1}^{\infty} H_n^{(r)} z^n\right\} = k!\sum_{n=0}^{\infty}\binom{n+k}{k} H_{n+k}^{(r)} z^n.$$

Comparing the coefficients of both sides gives the statement. □

If we substitute $r = 1$ and $k = r - 1$ in the formula above, we get back Conway and Guy's result (1).

3. Asymptotic approximation

To have the exact asymptotic behaviour of hyperharmonic numbers we need the following inequality from [CG].

$$(7) \qquad \frac{1}{2(n+1)} + \ln(n) + \gamma < H_n < \frac{1}{2n} + \ln(n) + \gamma \quad (n \in \mathbb{N}),$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant.

Lemma 2. *For all $n \in \mathbb{N}$ and for a fixed order $r \geq 2$ we have*

$$H_n^{(r)} \sim \frac{1}{(r-1)!}\left(n^{r-1}\ln(n)\right),$$

that is, the quotient of the left and right hand side tends to 1 as $n \to \infty$.

Proof. For the binomial coefficient in (1),

$$\binom{n+r-1}{r-1} \sim \frac{1}{(r-1)!} n^{r-1}.$$

For the convenience let us introduce the variable $t := r-1$. We would like to estimate the factor $H_{n+t} - H_t$ in (1). According to (7), we get that

$$\ln(n+t) - \ln(t\sqrt{e}) < H_{n+t} - H_t < \ln(n+t),$$

whence

$$1 - \frac{\ln(t\sqrt{e})}{\ln(n+t)} < \frac{H_{n+t} - H_t}{\ln(n+t)} < 1.$$

The limit of the left-hand side is 1 as n tends to infinity. Therefore (remember that $t = r-1$)

$$H_{n+r-1} - H_{r-1} \sim \ln(n+r-1) \sim \ln(n).$$

Collecting the results above we get the statement of the Lemma. □

Corollary 3. *We have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m} < +\infty,$$

whenever $m > r$.

4. A connecion with the Hurwitz zeta function

The Hurwitz zeta function is defined as

$$\zeta(m,n) = \sum_{p=0}^{\infty} \frac{1}{(n+p)^m}.$$

We point out that the sums involving hyperharmonic numbers can be transformed into the form as in the next theorem.

Theorem 4. *If $r \geq 1$ and $m \geq r+1$, then*

$$\sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^m} = \sum_{n=1}^{\infty} H_n^{(r-1)} \zeta(m,n).$$

Proof. We transform the left hand side as

$$\frac{H_1^{(r)}}{1^m} + \frac{H_2^{(r)}}{2^m} + \frac{H_3^{(r)}}{3^m} + \cdots$$

$$= \frac{H_1^{(r-1)}}{1^m} + \frac{H_1^{(r-1)} + H_2^{(r-1)}}{2^m} + \frac{H_1^{(r-1)} + H_2^{(r-1)} + H_3^{(r-1)}}{3^m} + \cdots$$

$$= H_1^{(r-1)} \sum_{p=1}^{\infty} \frac{1}{p^m} + H_2^{(r-1)} \sum_{p=1}^{\infty} \frac{1}{(p+1)^m} + H_3^{(r-1)} \sum_{p=1}^{\infty} \frac{1}{(p+2)^m} + \cdots$$

$$= \sum_{n=1}^{\infty} H_n^{(r-1)} \sum_{p=1}^{\infty} \frac{1}{(p+n-1)^m},$$

and the result comes. □

In the case $r = 1$, this theorem and Euler's summation formula give the identities (2) and (3). And, in general,

$$\sum_{n=1}^{\infty} \frac{H_n}{n^m} = \sum_{k=1}^{\infty} \frac{\zeta(m,k)}{k}.$$

5. Generating functions, Euler sums and Hypergeometric Series

In this section we introduce the notions needed in what follows.

The generating function

$$\frac{1}{m!}\left(-\ln(1-z)\right)^m = \sum_{n=1}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{z^n}{n!} \tag{8}$$

can be found in [GKP, B].

The well known polylogarithm functions can also be considered as generating functions belong to $a_n = \frac{1}{n^k}$ (for a fixed k).

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \quad (k = 1, 2, \dots).$$

Euler sums. The general Euler sum is an infinite sum whose general term is a product of harmonic numbers divided by some power of n, see the comprehensive paper [FS]. The sum

$$\sum_{n=1}^{\infty} \frac{H_n}{n^m} = \frac{1}{2}(m+2)\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(m-k)\zeta(k+1)$$

was derived by Euler (see [BB] and the references given there). Related series were studied by De Doelder in [dD] and Shen [S], for instance.

Hypergeometric series. The Pochhammer symbol is defined by

$$(x)_n = x(x+1)\cdots(x+n-1), \tag{9}$$

with special cases $(1)_n = n!$ and $(x)_1 = x$. The definition of the hypergeometric function (or hypergeometric series) is the following:

$${}_nF_m\left(\begin{array}{cccc} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_m \end{array} \middle| \, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_n)_k}{(b_1)_k (b_2)_k \cdots (b_m)_k} \frac{z^k}{k!}.$$

This function will appear in the sum of the hyperharmonic numbers.

We shall need one more statement.

Lemma 5. *We have*

$$\int \frac{\ln(z)}{(1-z)z} dz = \mathrm{Li}_2(1-z) + \frac{1}{2}\ln^2(z),$$

and for all $2 \le r \in \mathbb{N}$

$$\int \frac{\ln(z)}{(1-z)z^r}dz = \int \frac{\ln(z)}{(1-z)z^{r-1}}dz - \frac{\ln(z)}{(r-1)z^{r-1}} - \frac{1}{(r-1)^2 z^{r-1}}, \tag{10}$$

or, equivalently,

$$\int \frac{\ln(z)}{(1-z)z^r}dz = \mathrm{Li}_2(1-z) + \frac{1}{2}\ln^2(z) - \sum_{k=1}^{r-1}\left(\frac{\ln(z)}{kz^k} + \frac{1}{k^2 z^k}\right).$$

up to additive constants

Proof. The definition of $\mathrm{Li}_2(z)$ readily gives that

$$\mathrm{Li}_2'(1-z) = \frac{\ln(z)}{1-z}.$$

Moreover,

$$\left[\frac{1}{2}\ln^2(z)\right]' = \frac{\ln(z)}{z},$$

whence

$$\mathrm{Li}_2'(1-z) + \left[\frac{1}{2}\ln^2(z)\right]' = \frac{\ln(z)}{(1-z)z}.$$

The first statement is proved. The second one also can be deduced by differentiation. The derivative of the right-hand side of (10) has the form

$$\frac{\ln(z)}{(1-z)z^{r-1}} - \frac{(r-1)z^{r-2} - (r-1)^2\ln(z)z^{r-2}}{(r-1)^2(z^{r-1})^2} - \frac{-(r-1)}{(r-1)^2 z^r} = \frac{\ln(z)}{z^r(1-z)},$$

as we want. □

6. The summation formula

For the sake of simplicity, we introduce the notations

$$S(r,m) := \sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n^m},$$

and

$$B(k,m) := {}_{m+1}F_m\left(\begin{array}{ccccc} 1, & 1, & \ldots, & 1, & k+1 \\ 2, & 2, & \ldots, & 2 & \end{array}\middle| 1\right).$$

After these introductory steps we are ready to deduce a recursion formula for $S(r,m)$.

Theorem 6. *If* $r \ge 2$ *and* $m \ge r+1$*, then*

$$S(r,m) = S(1,m) + \sum_{k=1}^{r-1}\frac{1}{k}\left[S(k,m-1) - B(k,m)\right].$$

Proof. We begin with the generating function (6):

$$\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n}z^n=-\int\frac{\ln(1-z)}{z(1-z)^r}dz.$$

From the previous lemma

$$-\int\frac{\ln(1-z)}{z(1-z)^r}dz=\int\frac{\ln(z)}{(1-z)z^r}dz=$$

$$=\mathrm{Li}_2(z)+\frac{1}{2}\ln^2(1-z)-\sum_{k=1}^{r-1}\left(\frac{\ln(1-z)}{k(1-z)^k}+\frac{1}{k^2(1-z)^k}\right).$$

According to (6) and (8) one can write

$$\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n}z^n=\mathrm{Li}_2(z)+\sum_{n=1}^{\infty}\begin{bmatrix}n\\2\end{bmatrix}\frac{z^n}{n!}-$$

$$\sum_{k=1}^{r-1}\left(\frac{1}{k}(-1)\sum_{n=0}^{\infty}H_n^{(k)}z^n+\frac{1}{k^2}\sum_{n=0}^{\infty}\binom{n+k-1}{n}z^n\right).$$

Let us deal with the second term. The Stirling numbers of the first kind satisfy the recurrence relation [GKP]

$$\begin{bmatrix}n\\k\end{bmatrix}=(n-1)\begin{bmatrix}n-1\\k\end{bmatrix}+\begin{bmatrix}n-1\\k-1\end{bmatrix}\quad(n>0).$$

From this

$$\begin{bmatrix}n+1\\2\end{bmatrix}=n\begin{bmatrix}n\\2\end{bmatrix}+\begin{bmatrix}n\\1\end{bmatrix}=n\begin{bmatrix}n\\2\end{bmatrix}+(n-1)!\quad(n>0).$$

Now, (4) can be rewritten as follows

$$H_n=\frac{1}{n!}\begin{bmatrix}n+1\\2\end{bmatrix}=\frac{1}{(n-1)!}\begin{bmatrix}n\\2\end{bmatrix}+\frac{1}{n},$$

whence

$$\frac{1}{n!}\begin{bmatrix}n\\2\end{bmatrix}=\frac{H_n}{n}-\frac{1}{n^2}.$$

Therefore the second sum is

$$\sum_{n=1}^{\infty}\begin{bmatrix}n\\2\end{bmatrix}\frac{z^n}{n!}=\sum_{n=1}^{\infty}\frac{H_n}{n}z^n-\sum_{n=1}^{\infty}\frac{z^n}{n^2}.$$

Since the last member equals to $\mathrm{Li}_2(z)$, it cancels the first member of the sum above. Hence

$$\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n}z^n=\sum_{n=1}^{\infty}\frac{H_n}{n}z^n+\sum_{k=1}^{r-1}\left(\frac{1}{k}\sum_{n=0}^{\infty}H_n^{(r)}z^n-\frac{1}{k^2}\sum_{n=0}^{\infty}\binom{n+k-1}{n}z^n\right).$$

An easy induction shows that (after dividing with z, integrating, and repeating these steps $(m-1)$-times and finally substituting $z=1$)

(11)
$$\sum_{n=1}^{\infty}\frac{H_n^{(r)}}{n^m}=S(1,m)+\sum_{k=1}^{r-1}\left(\frac{1}{k}S(r,m-1)-\frac{1}{k^2}\sum_{n=1}^{\infty}\binom{n+k-1}{n}\frac{1}{n^{m-1}}\right).$$

The last step is the transformation of the last member.

$$\sum_{n=1}^{\infty}\binom{n+k-1}{n}\frac{1}{n^{m-1}}=\frac{1}{(k-1)!}\sum_{n=1}^{\infty}\frac{(n)_k}{n^m},$$

because of the definition of the Pochhammer symbol (9). On the other hand, the definition of $B(k,m)$ yields that

$$B(k,m)=\sum_{n=0}^{\infty}\frac{(n!)^m}{(n+1)!^m}\frac{(k+1)_n}{n!}=\sum_{n=0}^{\infty}\frac{1}{(n+1)^m}\frac{(k+1)_n}{n!}.$$

The next conversion should be applied:

$$\frac{k!(k+1)_n}{n!}=\frac{(k+n)!}{n!}=(n+1)(n+2)\cdots(n+k)=(n+1)_k.$$

It means that the equality

$$B(k,m)=\frac{1}{k!}\sum_{n=0}^{\infty}\frac{(n+1)_k}{(n+1)^m}=\frac{1}{k!}\sum_{n=1}^{\infty}\frac{(n)_k}{n^m} \quad (12)$$

holds. That is,

$$\sum_{n=1}^{\infty}\binom{n+k-1}{n}\frac{1}{n^{m-1}}=kB(k,m).$$

Considering this and (11) the result follows. □

7. Tables of the low-order sums

In the following tables we collect the low-order results of the Summation Theorem. We used the following identities which can be easily derived from (9) and (12).

$$B(1,m)=\zeta(m-1),$$
$$B(2,m)=\frac{1}{2}\left(\zeta(m-1)+\zeta(m-2)\right),$$
$$B(3,m)=\frac{1}{6}\zeta(m-3)+\frac{1}{2}\zeta(m-2)+\frac{1}{3}\zeta(m-1).$$

$S(2,m)$

Power of n	Closed form	Approx. value
$m=3$	$\frac{\pi^4}{72}-\frac{\pi^2}{6}+2\zeta(3)$	2.112083781
$m=4$	$\frac{\pi^4}{72}+3\zeta(5)-\zeta(3)\left(1+\frac{\pi^2}{6}\right)$	1.284326055

$m = 5$	$\frac{\pi^6}{540} - \frac{\pi^4}{90} - \frac{1}{2}\zeta(3)^2 + 3\zeta(5) - \frac{\pi^2}{6}\zeta(3)$	1.109035642
$m = 6$	$\frac{\pi^6}{540} + 4\zeta(7) - \frac{\pi^4}{90}\zeta(3) - \frac{1}{2}\zeta(3)^2 - \zeta(5)\left(1 + \frac{\pi^2}{6}\right)$	1.047657410
$m = 7$	$\frac{\pi^8}{4200} - \frac{\pi^6}{945} - \zeta(5)\zeta(3) + 4\zeta(7) - \frac{\pi^2}{6}\zeta(5) - \frac{\pi^4}{90}\zeta(3)$	1.022090029
$m = 8$	$\frac{\pi^8}{4200} + 5\zeta(9) - \frac{\pi^6}{945}\zeta(3) - \frac{\pi^4}{90}\zeta(5) - \zeta(5)\zeta(3) - \zeta(7)\left(1 + \frac{\pi^2}{6}\right)$	1.010557246
$m = 9$	$\frac{\pi^{10}}{34020} - \frac{\pi^8}{9450} - \zeta(7)\zeta(3) - \frac{1}{2}\zeta(5)^2 + 5\zeta(9) - \frac{\pi^2}{6}\zeta(7) - \frac{\pi^6}{945}\zeta(3) - \frac{\pi^4}{90}\zeta(5)$	1.005133570
$m = 10$	$\frac{\pi^{10}}{34020} + 6\zeta(11) - \frac{\pi^8}{9450}\zeta(3) - \frac{\pi^6}{945}\zeta(5) - \frac{1}{2}\zeta(5)^2 - \frac{\pi^4}{90}\zeta(7) - \zeta(7)\zeta(3) - \zeta(9)\left(1 + \frac{\pi^2}{6}\right)$	1.002522063

$$S(3, m)$$

Power of n	Closed form	Approx. value
$m = 4$	$\frac{\pi^4}{48} - \frac{\pi^2}{8} - \frac{\pi^2}{6}\zeta(3) - \frac{1}{4}\zeta(3) + 3\zeta(5)$	1.628620203
$m = 5$	$\frac{\pi^6}{540} - \frac{\pi^4}{144} - \frac{\pi^2}{4}\zeta(3) - \frac{3}{4}\zeta(3) - \frac{1}{2}\zeta(3)^2 + \frac{9}{2}\zeta(5)$	1.180103635
$m = 6$	$\frac{\pi^6}{360} - \frac{\pi^4}{120} + 4\zeta(7) - \frac{\pi^2}{6}\zeta(5) - \frac{\pi^4}{90}\zeta(3) - \frac{3}{4}\zeta(3)^2 + \frac{1}{4}\zeta(5) - \frac{\pi^2}{12}\zeta(3)$	1.072362484
$m = 7$	$\frac{\pi^8}{4200} - \frac{\pi^6}{2520} - \frac{\pi^4}{60}\zeta(3) - \frac{1}{4}\zeta(3)^2 - \zeta(5)\zeta(3) - \zeta(5)\left(\frac{\pi^2}{4} + \frac{3}{4}\right) + 6\zeta(7)$	1.032351029
$m = 8$	$\frac{\pi^8}{2800} - \frac{\pi^6}{1260} - \zeta(3)\left(\frac{\pi^4}{180} + \frac{\pi^6}{945}\right) - \zeta(5)\left(\frac{\pi^2}{12} + \frac{\pi^4}{90}\right) - \frac{3}{2}\zeta(5)\zeta(3) + \zeta(7)\left(\frac{3}{4} - \frac{\pi^2}{6}\right) + 5\zeta(9)$	1.015179175

$$S(4, m)$$

Power of n	Closed form	Approx. value
$m = 5$	$\frac{\pi^6}{540} - \frac{\pi^4}{810} - \frac{11\pi^2}{216} - \zeta(3) - \frac{11\pi^2}{36}\zeta(3) - \frac{1}{2}\zeta(3)^2 + \frac{11}{2}\zeta(5)$	1.310990854
$m = 6$	$\frac{11\pi^6}{3240} - \frac{\pi^4}{80} - \zeta(3)\left(\frac{\pi^4}{90} + \frac{1\pi^2}{6} + \frac{11}{36}\right) + + \frac{11}{12}\zeta(3)^2 + \zeta(5)\left(\frac{59}{36} - \frac{\pi^2}{6}\right) + 4\zeta(7)$	1.103348021

$m = 7$	$\frac{\pi^8}{4200} - \frac{11\pi^4}{3240} + \frac{\pi^6}{2430} - \frac{1}{2}\zeta(3)^2 - \zeta(3)\left(\frac{11\pi^4}{540} - \frac{\pi^2}{36}\right) - \zeta(3)\zeta(5) - \zeta(5)\left(\frac{5}{6} - \frac{11\pi^2}{36}\right) + \frac{22}{3}\zeta(7)$	1.043816710
$m = 8$	$\frac{11\pi^8}{25200} - \frac{5\pi^6}{4536} - \zeta(3)\left(\frac{\pi^6}{945} + \frac{\pi^4}{90}\right) - \frac{1}{12}\zeta(3)^2 - \frac{11}{6}\zeta(3)\zeta(5) - \zeta(5)\left(\frac{\pi^4}{90} + \frac{\pi^2}{6} + \frac{11}{36}\right) + \zeta(7)\left(\frac{95}{36} - \frac{\pi^2}{6}\right) + 5\zeta(9)$	1.020093103

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