On the maximum of r-Stirling numbers

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Abstract

Determining the location of the maximum of Stirling numbers is a well developed area. In this paper we give same results for the so-called r-Stirling numbers which are natural generalizations of Stirling numbers.

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1 Introduction

The Stirling number of the first kind $\left[{n \atop m}\right]$ gives the number of permutations of n elements formed by exactly m disjoint cycles. They satisfy the recurrence relation

$$\left[{n \atop 0}\right] = \delta_{0n}, \quad \left[{n \atop m}\right] = (n-1)\left[{n-1 \atop m}\right] + \left[{n-1 \atop m-1}\right], \tag{1}$$

As an equivalent definition, the numbers $\left(\left[{n \atop k}\right]\right)_{k=0}^{n}$ are the coefficients of the next polynomial:

$$\sum_{k=0}^{n} \left[{n \atop k}\right] x^k = x(x+1)(x+2)\cdots(x+n-1). \tag{2}$$

The Stirling number of the second kind, denoted by $\left\{{n \atop m}\right\}$, enumerates the number of partitions of a set with n elements consisting of m disjoint, nonempty

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sets. The following recurrence relation holds

$$\left\{ {n \atop 0} \right\} = \delta_{0n}, \quad \left\{ {n \atop m} \right\} = m \left\{ {n-1 \atop m} \right\} + \left\{ {n-1 \atop m-1} \right\}. \tag{3}$$

An alternative definition can be given by the formula

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)(x-2)\cdots(x-k+1). \tag{4}$$

An excellent introduction to these numbers can be found in [9].

A sequence $a_1, a_2, \ldots, a_n$ is said to be unimodal [25] if its members rise to a maximum and then decrease, that is, there exist an index k such that

$$a_1 \le a_2 \le \cdots \le a_k,$$

and

$$a_k \ge a_{k+1} \ge \cdots \ge a_n.$$

A stronger property, called log-concavity, implies the unimodality. The sequence $a_1, a_2, \ldots, a_n$ is called log-concave when

$$a_k^2 \ge a_{k+1} a_{k-1} \quad (k = 2, \ldots, n-1), \tag{5}$$

and it is called strongly log-concave when there is strict inequality in the above expression.

Newton's inequality [18] gives a simple test to verify the strong log-concavity.

Theorem 1 (Newton's inequality) *If the polynomial $a_1 x + a_2 x^2 + \cdots + a_n x^n$ has only real roots then*

$$a_k^2 \ge a_{k+1} a_{k-1} \frac{k}{k-1} \frac{n-k+1}{n-k} \quad (k = 2, \ldots, n-1).$$

This immediately implies the strict version of (5).

Considering (2), an immediate consequence is that the sequence $\left(\left[{n \atop k} \right] \right)_{k=1}^{n}$ is strictly log-concave for all n. According to the work of Hammersley [11] and Erdős [8], much more is true. Namely, the index K_n of the maximal Stirling number of the first kind is unique for all fixed $n > 2$:

$$\left[{n \atop 1} \right] < \left[{n \atop 2} \right] < \cdots < \left[{n \atop K_n - 1} \right] < \left[{n \atop K_n} \right] > \left[{n \atop K_n + 1} \right] > \cdots > \left[{n \atop n} \right].$$

Moreover, the maximizing index is determined by

$$K_n = \left[\log(n+1) + \gamma - 1 + \frac{\zeta(2) - \zeta(3)}{\log(n+1) + \gamma - \frac{3}{2}} + \frac{h}{\left(\log(n+1) + \gamma - \frac{3}{2}\right)^2}\right],$$

where $[x]$ denotes the integer part of x, ζ is the Riemann zeta function, $\gamma = 0.5772\dots$ is the Euler-Mascheroni constant and $-1.1 < h < 1.5$. As Erdős remarked, this can be simplified when $n > 188$:

$$\left[\log n - \frac{1}{2}\right] < K_n < [\log n]. \tag{6}$$

The situation changes for Stirling numbers of the second kind. There is no exact closed form for the maximizing index K_n. What is more, we do not know whether it is unique or not. Although K_2 is not unique, since $\left\{ {2 \atop 1} \right\} = \left\{ {2 \atop 2} \right\} = 1$, in 1973 Wegner [23] conjectured that for all $n \geq 3$ the index K_n is unique. According to the paper [4], there is no counterexample for $3 < n < 10^6$. One thing is certain, the Stirling numbers of the second kind form a strongly log-concave sequence [4,7,18,20].

The papers [10,12,14–16,23] contain a number of estimations for K_n. The most exact (without any approximative term) was given by Wegner [23]:

$$\begin{aligned} K_n &< \frac{n}{\log n - \log\log n} \qquad (n \geq 3), \\ \frac{n}{\log n} &< K_n \qquad\qquad\qquad (n \geq 18). \end{aligned} \tag{7}$$

Asymptotic properties of the maximizing index were proved in [19,21] and even by statistical tools in [13]:

$$K_n \sim \frac{n}{\log n},$$

in the sense that their quotient tends to 1 as n tends to infinity.

We remark that this approximation can be given using the result in [4]. It is shown that

$$K_n \in \{\lfloor e^{r(n)} - 1 \rfloor, \lceil e^{r(n)} - 1 \rceil\},$$

where $r(n)$ was defined implicitly by the equation

$$r(n)e^{r(n)} = n.$$

Since $r(n)$ is known as the Lambert W function [5] we can use its approximation [5, p. 349]:

$$W(z) = \log z - \log\log z + \frac{\log\log z}{\log z} + O\left(\frac{\log\log z}{\log z}\right)^2 \quad (z > 3).$$

This means that

$$e^{r(n)} = e^{W(n)} = \frac{n}{\log n} e^{\frac{\log\log n}{\log n}} \cdots \sim \frac{n}{\log n}.$$

2 Notion of r-Stirling numbers

In the previous section we have introduced the problems on the maximum of Stirling numbers and presented the solutions. Now we extend the problem to the natural generalization of Stirling numbers as follows.

For any positive integer r the symbol $\left[{n \atop m}\right]_r$ denotes the number of those permutations of the set $\{1, 2, \ldots, n\}$ that have m cycles such that the first r element are in distinct cycles. The recurrence relation is the same that of ordinary Stirlings

$$\begin{aligned}
\left[{n \atop m}\right]_r &= 0, && n < r, \\
\left[{n \atop m}\right]_r &= \delta_{mr}, && n = r, \\
\left[{n \atop m}\right]_r &= (n-1)\left[{n-1 \atop m}\right]_r + \left[{n-1 \atop m-1}\right]_r, && n > r.
\end{aligned} \tag{8}$$

A double generating function is given in [3]:

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left[{n+r \atop k+r}\right] x^k\right)\frac{z^n}{n!} = \frac{1}{(1-z)^{r+x}}. \tag{9}$$

Let us introduce the r-Stirling numbers of the second kind. $\left\{{n \atop m}\right\}_r$ denotes the number of those partitions of the set $\{1, 2, \ldots, n\}$ that have m nonempty, disjoint subsets, such that the first r elements are in distinct subsets. The usual recurrence is again the same.

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_r = 0, \qquad n < r,$$
$$\begin{Bmatrix} n \\ m \end{Bmatrix}_r = \delta_{mr}, \qquad n = r, \tag{10}$$
$$\begin{Bmatrix} n \\ m \end{Bmatrix}_r = m\begin{Bmatrix} n-1 \\ m \end{Bmatrix}_r + \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}_r, \qquad n > r.$$

The identity (4) turns to be

$$(x+r)^n = \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r x(x-1)\cdots(x-k+1). \tag{11}$$

One can identify the ordinary Stirlings to r-Stirlings via

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_0 = \begin{bmatrix} n \\ m \end{bmatrix}_1,$$
$$\begin{Bmatrix} n \\ m \end{Bmatrix} = \begin{Bmatrix} n \\ m \end{Bmatrix}_0 = \begin{Bmatrix} n \\ m \end{Bmatrix}_1.$$

A nice, introductory paper was written by Broder [3].

The question arises immediately: what is true from the results of the first section with respect to r-Stirling numbers? In the following sections we give the answer.

3 Results for r-Stirling numbers of the first kind

Theorem 2 *The sequence* $\left(\begin{bmatrix} n+r \\ k+r \end{bmatrix}\right)_{k=0}^{n}$ *is strongly log-concave (and thus unimodal).*

Proof. Let us define the following polynomial:

$$P_{n,r}(x) := \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r x^k. \tag{12}$$

It is worth to shift the indices by r to avoid the redundant zeros, since $\begin{bmatrix} n \\ k \end{bmatrix}_r = 0$ if $n < r$. The exponential generating function of $P_{n,r}(x)$ is given in (9), whence

$$\frac{1}{(1-z)^{r+x}} = \sum_{n=0}^{\infty} \binom{r+x-1+n}{n} z^n = \sum_{n=0}^{\infty} \frac{P_{n,r}(x)}{n!} z^n.$$

Comparing the coefficients,

$$P_{n,r}(x) = n!\binom{r+x-1+n}{n} = (x+r)(x+r+1)\cdots(x+r+n-1). \quad (13)$$

Therefore the roots of $P_{n,r}(x)$ are real. Applying Newton's inequality, the proof is complete. $\square$

In what follows let $K^1_{n,r}$ denote the maximizing index of the sequence $\left(\left[{n \atop k}\right]_r\right)_{k=0}^n$ (the upper index 1 refers to the kind). To find the estimation of $K^1_{n,r}$ we have to remark that the numbers $\left[{n \atop k}\right]$ for a fixed n, are the elementary symmetric functions of the numbers $1, \ldots, n$, while the numbers $\left[{n \atop k}\right]_r$ are the elementary symmetric functions of the numbers $r, \ldots, n$ (see [3,8]). That is, for a fixed n, the (0-)Stirling numbers are the sums of the products of the first n natural numbers taken k at a time and r-Stirling numbers are the sums of the products of the $r, \ldots, n$ natural numbers taken k at a time. This was detailed in [3]:

$$\left[{n \atop n-k}\right]_r = \sum_{r \le i_1 < i_2 < \cdots < i_k < n} i_1 i_2 \cdots i_k \quad (n, k \ge 0). \quad (14)$$

Now we cite a theorem of Erdős and Stone [8]:

Theorem 3 (P. Erdős and A. H. Stone) *Let $u_1 < u_2 < \cdots$ be an infinite sequence of positive real numbers such that*

$$\sum_{i=1}^{\infty} \frac{1}{u_i} = \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} \frac{1}{u_i^2} < \infty.$$

Denote by $\Sigma_{n,k}$ the sum of the product of the first n of them taken k at a time and denote by K_n the largest value of k for which $\Sigma_{n,k}$ assumes its maximum value. Then

$$K_n = n - \left[\sum_{i=1}^{n} \frac{1}{u_i} - \sum_{i=1}^{n} \frac{1}{u_i^2}\left(1 + \frac{1}{u_i}\right)^{-1} + o(1)\right].$$

It is obvious from (14) that

$$\left[{n \atop k}\right]_r = \Sigma_{n-r,n-k} \quad (15)$$

with the sequence $u_1 = r, u_2 = r+1, \ldots$. As a consequence, we get the parallel result of (6):

Theorem 4 *The largest index for which the sequence $\left(\left[{n \atop k}\right]\right)_{k=0}^n$ assumes its* *maximum is given by the approximation*

$$K^1_{n,r} = r + \left[\log\left(\frac{n-1}{r-1}\right) - \frac{1}{r} + o(1)\right].$$

Proof. If we choose $u_1 = r, u_2 = r+1, \dots$ then, by (15), the maximizing index $K^1_{n,r}$ equals to

$$\begin{aligned} & r + \left[\frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n-1} - \sum_{i=1}^{\infty} \frac{1}{(r+i-1)(r+i)} + o(1)\right] \\ = & r + \left[\log\left(\frac{n-1}{r-1}\right) - \frac{1}{r} + o(1)\right], \end{aligned}$$

since it is well known that

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \gamma + o(1).$$

The additive term r comes from the fact that the first nonzero symmetric function belongs to the index $k = r$ in the sequence $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{k=0}^{n}$. $\square$

Example 5 *We give an elementary application: the maximal element of the sequence* $\left(\begin{bmatrix} 30 \\ k \end{bmatrix}_3\right)_{k=0}^{30}$ *belongs to the index*

$$K^1_{30,3} = 3 + \left[\log\left(\frac{30-1}{3-1}\right) - \frac{1}{3} + o(1)\right] = 5.$$

Indeed,

$$\begin{bmatrix} 30 \\ 5 \end{bmatrix}_3 = 1.259 \cdot 10^{31}$$

is maximal, as one can see with any computer algebra system using the recurrence relations (8)*.*

4 Results for r-Stirling numbers of the second kind

To formulate our results, we prove the following theorem.

Theorem 6 *The sequence* $\left(\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r\right)_{k=0}^{n}$ *is strongly log-concave.*

Proof. As before, we define the polynomial

$$B_{n,r}(x) := \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r x^k. \tag{16}$$

Using the recurrence relation (10),

$$\begin{aligned} B_{n,r}(x) &= \sum_{k=0}^{n}(k+r)\left\{ n-1+r \atop k+r \right\}_r x^k + \sum_{k=0}^{n}(k+r)\left\{ n-1+r \atop k-1+r \right\}_r x^k \\ &= \frac{1}{x^{r-1}} \sum_{k=0}^{n-1}(k+r)\left\{ n-1+r \atop k+r \right\}_r x^{k+r-1} + xB_{n-1,r}(x) \\ &= \frac{1}{x^{r-1}} \frac{\partial}{\partial x}\left(x^r B_{n-1,r}(x)\right) + xB_{n-1,r}(x). \end{aligned}$$

From this we get a recurrence relation to the polynomials $B_{n,r}(x)$:

$$B_{n,r}(x) = x\left(\frac{\partial}{\partial x}B_{n-1,r}(x) + B_{n-1,r}(x)\right) + rB_{n-1,r}(x). \tag{17}$$

This equation implies the identity

$$e^x x^r B_{n,r}(x) = x\frac{\partial}{\partial x}\left(e^x x^r B_{n-1,r}(x)\right). \tag{18}$$

Moreover, by the definition (16), $B_{n-1,r}(x) > 0$ if $x \geq 0$. We prove the remaining part by induction. Since $B_{1,r}(x) = x + r$, its root is real (and negative). Now assume that all of the roots of $B_{n-1,r}(x)$ are real and negative.

So Rolle's theorem gives that on the right hand side of (18) there are $n-1$ negative roots beside the root $x = 0$ with multiplicity r. Because the function on the left hand side must have exactly $n+r$ finite roots, the missing one can not be complex. Since $B_{n,r}(x) > 0$ if $x \geq 0$, it must be negative, too. Newton's theorem completes the proof. □

Remark 7 *The Bell polynomials are defined as*

$$B_n(x) = \sum_{k=0}^{n}\left\{ n \atop k \right\}x^k.$$

The Bell numbers are $B_n = B_n(0)$. *Therefore the definition* (16) *can be considered as a generalization of these numbers and polynomials in the special case* $B_n(x) = B_{n,0}(x)$.

We continue with the following lemma which is a partial generalization of the so-called Bonferroni-inequality (see [23,24]).

Lemma 8 *We have*

$$\frac{(m+r)^n}{m!} - \frac{(m-1+r)^n}{(m-1)!} < \left\{ n+r \atop m+r \right\}_r < \frac{(m+r)^n}{m!},$$

for all $n \geq m > 0$.

Proof. Equation (11) yields that

$$(m+r)^n = \sum_{k=0}^{m} \left\{ {n+r \atop k+r} \right\}_r \frac{m!}{(m-k)!}.$$

Hence

$$\frac{(m+r)^n}{m!} = \sum_{k=0}^{m-1} \left\{ {n+r \atop k+r} \right\}_r \frac{1}{(m-k)!} + \left\{ {n+r \atop m+r} \right\}_r,$$

therefore the inequality on the right hand side is valid. Applying (11) again, we get

$$\begin{aligned}\left\{ {n+r \atop m+r} \right\}_r &> \frac{(m+r)^n}{m!} - \sum_{k=0}^{m-1} \left\{ {n+r \atop k+r} \right\}_r \frac{1}{(m-1-k)!} \\ &= \frac{(m+r)^n}{m!} - \frac{(m-1+r)^n}{(m-1)!}.\end{aligned}$$

□

Since the sequence $\left(\left\{ {n \atop k} \right\}_r \right)_{k=r}^{n}$ is strongly log-concave, there exist an index $K_{n,r}^2$ for which

$$\cdots < \left\{ {n \atop K_{n,r}^2 - 1} \right\}_r \leq \left\{ {n \atop K_{n,r}^2} \right\}_r > \left\{ {n \atop K_{n,r}^2 + 1} \right\}_r > \cdots$$

Now we give estimations of the maximizing index $K_{n,r}^2$ for r-Stirling numbers of the second kind.

Theorem 9 *Let* $K_{n,r}^2$ *be the greatest maximizing index shown above. Then*

$$\begin{aligned}K_{n,r}^2 &< \frac{n-r}{\log(n-r) - \log\log(n-r)} \qquad (n \geq r+3), \\ \frac{n-r}{\log(n-r)} &< K_{n,r}^2 \qquad (n \geq r + \max\{18, \log 2/\log(1+1/r)\}).\end{aligned}$$

Proof. It is convenient again to shift the indices. To prove the upper estimation, we apply equation (32) of [3]:

$$\left\{ {n+r \atop k+r} \right\}_r = \sum_{j=0}^{n} \binom{n}{j} \left\{ {j \atop k} \right\}_1 r^{n-j},$$

therefore

$$\left\{ {n+r \atop k+r} \right\}_r - \left\{ {n+r \atop k-1+r} \right\}_r = \sum_{j=0}^{n} \binom{n}{j} \left[\left\{ {j \atop k} \right\}_1 - \left\{ {j \atop k-1} \right\}_1 \right] r^{n-j}.$$

The terms $\left\{ {j \atop k} \right\}_1 - \left\{ {j \atop k-1} \right\}_1$ are surely negative if $k > K^2_{n,1}$ because of the strong log-concavity of Stirling numbers of the second kind and the fact that $K^2_{n,1} \geq K^2_{n-1,1}$ for all n (see [7,20]). Thus $\left\{ {n+r \atop k+r} \right\}_r < \left\{ {n+r \atop k-1+r} \right\}_r$ for all $k > K^2_{n,1}$, whence $K^2_{n+r,r} < K^2_{n,1}$ follows. Wegner's estimation in (7) validates the upper estimation.

To prove the lower estimation, we use the generalized Bonferroni's inequality stated in Lemma 8. above. For the sake of simplicity, let us define M by $K^2_{n+r,r}$. Then

$$\begin{aligned} 0 &> \left\{ {n+r \atop M+1} \right\}_r - \left\{ {n+r \atop M} \right\}_r \\ &\geq \frac{(M+1)^n}{(M+1-r)!} - \frac{M^n}{(M-r)!} - \frac{M^n}{(M-r)!} \\ &= \frac{1}{(M-r)!} \left(\frac{(M+1)^n}{M+1-r} - 2M^n \right). \end{aligned} \tag{19}$$

Let us introduce the function

$$f_{n,r}(x) := \frac{x^{n-1}}{1-\frac{r}{x}} - 2(x-1)^n,$$

and its logarithm

$$g_{n,r}(x) := \log f_{n,r}(x) = (n-1)\log x - n\log(x-1) - \log 2 - \log\left(1 - \frac{r}{x}\right).$$

Then it is obvious that the last part of (19) can be written in the form

$$\frac{1}{(M-r)!} \left(\frac{(M+1)^n}{M+1-r} - 2M^n \right) = \frac{1}{(M-r)!} f_{n,r}(M+1) \tag{20}$$

First, we determine the number of roots of $f_{n,r}(x)$. If $f_{n,r}(x) = 0$ then

$$\left(1 + \frac{1}{x-1}\right)^n = 2(x-r).$$

The left hand side is a strictly decreasing and the right hand side is a strictly increasing function of x, so there is at most one solution. But

$$f_{n,r}(r+1) > 0 \quad \text{if} \quad n > \frac{\log 2}{\log\left(1+\frac{1}{r}\right)}, \tag{21}$$

(according to (19), all of the interesting values of x are not less than $r+1$) and

$$\lim_{x\to\infty} f_{n,r}(x) = -\infty,$$

therefore $f_{n,r}(x)$ must have at least one root. Consequently, $f_{n,r}$ has exactly one root $Z_{n,r}$, say, and $f_{n,r}(x) > 0$ if $x < Z_{n,r}$ and $f_{n,r}(x) < 0$ if $x > Z_{n,r}$. Considering (20) we get that $M+1 > Z_{n,r}$.

On can easily see that the sign of $g_{n,r}(x)$ is the same as of $f_{n,r}(x)$ for all x and thus $g_{n,r}(Z_{n,r}) = 0$, too. We collect these results in the next formula:

$$f_{n,r}(x), g_{n,r}(x) \begin{cases} > 0 & r < x < Z_{n,r}, \\ = 0 & x = Z_{n,r}, \\ < 0 & x > Z_{n,r}, \end{cases}$$

if the condition under (21) holds for n.

The function $g_{n,r}(x)$ can be separated into the terms

$$g_{n,r}(x) = h_n(x) - \log\left(1 - \frac{r}{x}\right).$$

The function $h_n(x)$ was examined in the paper of Wegner [23] under the notation $g_{n,2}(x)$ and he proved that

$$h_n\left(\frac{n}{\log n} + 1\right) > 0 \quad (n \geq 18) \tag{22}$$

and thus the zero of h_n is greater than $\frac{n}{\log n} + 1$.

Since h_n has the same monotonicity as $g_{n,r}$ (see [23] again), the root of $g_{n,r}(x)$ is greater than the root of $h_n(x)$ because the second term $-\log(1 - r/x)$ is positive for $x > r$. Thus $\frac{n}{\log n} + 1 < Z_{n,r} < M + 1$. Collecting the necessary conditions on n (see (21),(22)) and considering that $M = K^2_{n+r,r}$, the proof is complete. □

Example 10 *We give an application of this case, too. The theorem states that*

$$12.78 = \frac{50}{\log 50} < K^2_{58,8} < \frac{50}{\log 50 - \log\log 50} = 19.62.$$

In fact,

$$\left\{ {58 \atop 19} \right\}_8 = 9.687 \cdot 10^{55},$$

and this is really the maximal.

Remark 11 *We mentioned in the proof of Theorem 9. that* $K^2_{n+1,1} = K^2_{n,1}$ *or* $K^2_{n,1} + 1$*. There are two proofs in [7] and [20]. The proof in [7] can be used without any modification to prove that*

$$K^2_{n+1,r} = K^2_{n,r} \text{ or } K^2_{n,r} + 1 \quad (r > 1).$$

5 An asymptotic formula for r-Stirling numbers of the second kind

Finally, Lemma 8. enables us to give another result. It is known [17] that

$$\left\{ {n \atop m} \right\} \sim \frac{m^n}{m!}.$$

Bonferroni's generalized result yields that this asymptotic formula has the form

$$\left\{ {n+r \atop m+r} \right\}_r \sim \frac{(m+r)^n}{m!}.$$

The proof is straightforward, since

$$1 - \left(\frac{m+r-1}{m+r} \right)^n m < \frac{m!}{(m+r)^n} \left\{ {n+r \atop m+r} \right\}_r < 1,$$

and the left hand side tends to 1 as n tends to infinity ($m = 0, 1, \dots$).

6 Some notes on Darroch's theorem

The following useful theorem was proved by Darroch [2,6]:

Theorem 12 (J. N. Darroch) *Let* $A(x) = \sum_{k=0}^n a_k x^k$ *be a polynomial that has real roots only that satisfies* $A(1) > 0$*. In other words, A(x) has the form*

$$A(x) = a_n \prod_{j=1}^n (x + r_j),$$

where $r_j > 0$*. Let* K_n *be the leftmost maximizing index for the sequence* $a_0, a_1, \dots, a_n$ *and let*

$$\mu = \frac{A'(1)}{A(1)} = \sum_{j=1}^n \frac{1}{r_j + 1}.$$

Then we have

$$|K_n - \mu| < 1.$$

In the proof of Theorem 2. we can find that

$$P_{n,r}(x) := \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r x^k = (x+r)(x+r+1)\cdots(x+r+n-1),$$

therefore we immediately get the next

Corollary 13 *Darroch's theorem yields that*

$$\left| K^1_{n+r,r} - \left(\frac{1}{r+1} + \frac{1}{r+2} + \cdots + \frac{1}{r+n} \right) \right| < 1,$$

which is the same as the consequence of Erdős' theorem (Theorem 4.).

The case of Stirling numbers of the second kind is a bit more difficult. We proved earlier (see (17)) that

$$B'_{n,r}(x) = \frac{B_{n+1,r}(x)}{x} - \frac{rB_{n,r}(x)}{x} - B_{n,r}(x).$$

Thus

$$\mu = \frac{B'_{n,r}(1)}{B_{n,r}(1)} = \frac{B_{n+1,r}}{B_{n,r}} - (r+1).$$

Corollary 14 *We have*

$$\left| K^2_{n+r,r} - \left(\frac{B_{n+1,r}}{B_{n,r}} - (r+1) \right) \right| < 1,$$

which is a straight generalization of Harper's result [13].

7 Normality of r-Stirling numbers

As an other application of the real zero property of the polynomials (12) and (16) we prove that the coefficients of these polynomials – the r-Stirling numbers – are normally distributed.

Let $a_n(k)$ be a triangular array of nonnegative real numbers, $n = 1, 2, \ldots; k = 0, 1, \ldots, m$ (m depends on n). Let X_n be a random variable such that

$$P(X_n = k) = p_n(k) = \frac{a_n(k)}{\sum_{j=0}^{m} a_n(j)},$$

and let

$$g_n(x) = \sum_{k=0}^{n} p_n(k) x^k.$$

We use the notation $\tilde{X} = \frac{X-E(X)}{\sqrt{\mathrm{Var}(X)}}$. Finally, $X_n \to \mathcal{N}(0,1)$ means that X_n converges in distribution to the standard normal variable. One can read more on these notions in [22]. An application of the following theorem will be given.

Theorem 15 (E. A. Bender [1]) *Using the notations as above, if $g_n(x)$ has real roots only, and*

$$\sigma_n = \sqrt{Var(X_n)} = \sum_{i=1}^{m} \frac{r_i^{(n)}}{\left(r_i^{(n)}+1\right)^2} \to \infty,$$

then $\tilde{X}_n \to \mathcal{N}(0,1)$. Here $(-r_i)$'s are the roots of $g_n(x)$.

The Stirling numbers of the first and second kind are normal in this sense. These facts were proved by Goncharov and Harper, respectively [22]. We prove that these statements stand for r-Stirling numbers, too.

First, let $a_n(k) = \left[{n+r \atop k+r}\right]_r$. Then, because of (13),

$$\sigma_n = \sum_{k=0}^{n-1} \frac{k+r}{(k+r+1)^2} \to \infty.$$

So the conditions of Bender's theorem are fulfilled.

A result of Rucinski and Voigt [22, p. 223.] says that if

$$x^n = \sum_{k=0}^{n} a_n(k)(x-c_0)\cdots(x-c_{k-1})$$

holds for some nonnegative arithmetic progression $c_0, c_1, \ldots,$ then the array $a_n(k)$ is normal. Equation (11) with the substitution $x \rightsquigarrow x-r$ immediately yields that $a_n(k) = \left\{{n+r \atop k+r}\right\}_r$ is normal.

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