A Simple Way to Compute the Number of Vehicles That Are Required to Operate a Periodic Timetable*

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Abstract

We consider the following planning problem in public transportation: Given a periodic timetable, how many vehicles are required to operate it?

In [9], for this sequential approach, it is proposed to first expand the periodic timetable over time, and then answer the above question by solving a flow-based aperiodic optimization problem.

In this contribution we propose to keep the compact periodic representation of the timetable and simply solve a particular perfect matching problem. For practical networks, it is very much likely that the matching problem decomposes into several connected components. Our key observation is that there is no need to change any turnaround decision for the vehicles of a line during the day, as long as the timetable stays exactly the same.

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1 Introduction

During the last decades, public transport has become one of the classic fields for applied mathematical optimization [1]. Typically, the planning process is subdivided into line planning, timetabling, vehicle scheduling etc. Timetabling, in particular computing a periodic timetable for instance for bus networks, is still attracting several teams of researchers [2, 4, 5, 10].

The design of public transportation services is pursuing several objectives, of course. One is operating efficiency, where a key performance indicator is the number of vehicles that are required for operation.

In this paper, we restrain ourselves to the classical sequential approach of planning. In particular, having fixed the line plan as well as the timetable, the next task is to compute a vehicle schedule, in particular defining the number of vehicles required to operate the given timetable. This is essentially what in [9] is denoted "the traditional approach".

In more detail, we are considering the following setting, right as in [9]:

- We restrict ourselves to periodic timetables, where we denote the common period time of all lines as T.
- For a given line plan and periodic timetable, we want to compute the number of required vehicles, i.e., evaluate a so-called LTS-plan, according to [9].

We agree that in general a vehicle schedule is aperiodic. Hence, it makes most sense for software providers such as IVU or GIRO to develop and promote highly specialized algorithms on a commercial basis.

Yet, in our paper we show that the aperiodicity of optimum vehicle schedules is just a result of aperiodic timetables. In practice, this may be due to extra peak-hour trips and/or shorter trip durations during night hours. In contrast, as long as the underlying timetable is fully periodic, we prove that one can always find a vehicle schedule with a minimal number of vehicles, even when restricting the vehicle schedule to perform the very same turnaround activities of the vehicles over the entire day. In a sense, this turns out to be a consequence of the structure of bipartite matching polytopes. So, to compute the number of vehicles that are required to operate a given periodic timetable, in contrast to the procedure that is reported in [9], actually there is no need to expand (or, roll out) the periodic timetable for the number N of periods that are needed to cover a whole planning horizon (e.g., a day), and then perform a full vehicle schedule optimization from scratch, e.g., using a flow-based model. Rather, staying with the much more compact periodic representation turns out to be absolutely sufficient. Although we are aware that in several earlier contributions, minimization of operating cost had been done pretty much in this way (e.g. [6, 8]), we were not able to detect any justification in those papers that was equivalent to the one we are proposing here.

Notice that with respect to practice, this result is not only relevant, if a timetable stays the same over the entire day. Rather, if the peak in the number of vehicles was not induced by some single trips without any periodically recurring "copies", and if the trip lengths of the lines are relatively small (e.g., at most two hours) compared to the duration of the peak traffic time for which the periodic timetable is valid (say from 2 p.m. until 7 p.m.), already then our result applies.

The paper is organized as follows: At first, we shortly recall the setting of periodic timetabling. Second, we consider the task of periodic vehicle scheduling for a given fixed periodic timetable. Our goal is to prove in Theorem 12 that there is no advantage to compute the minimum number of vehicles on an expanded aperiodic network (as it is necessary for general vehicle scheduling), given that the underlying timetable is 100% periodic. To build the bridge from the periodic model to the expanded aperiodic model, we consider an expanded (or rolled-out) periodic version as an intermediate step, serving as a theoretical benchmark. Let us emphasize that the attribute "simple" in the title of this paper refers to the result itself rather than to the contents of its proof.

2 Periodic timetabling

The basis for our timetabling model is the *periodic event scheduling problem* (PESP) from [12]. Since we are focusing on computing the number of vehicles that are required to operate a periodic timetable, we are only considering activities that are associated with vehicles. The main player is an *event-activity network* $\mathcal{N} = (G, T, \ell)$, where G is a directed graph with node set V and arc set A satisfying the following properties:

- Each node $v \in V$ is either a *departure node* or an *arrival node*, so that the set of nodes of G decomposes as $V = V_{\text{dep}} \,\dot{\cup}\, V_{\text{arr}}$.
- The set A of arcs is the disjoint union of a set $A_d \subseteq V_{\text{dep}} \times V_{\text{arr}}$ of *driving arcs* and a set $A_t \subseteq V_{\text{arr}} \times V_{\text{dep}}$ of *turnaround arcs*. In particular, G is a bipartite graph.
- Each departure node has exactly one outgoing arc, and arrival nodes have exactly one ingoing arc, i.e., their respective driving arcs.

The event-activity network comes with a *period time* $T \in \mathbb{N}$. Moreover, we consider for each arc $a = (v, w) \in A$ its time duration $\ell_a \in [0, \infty)$. For a driving arc $vw \in A_d$, the quantity ℓ_{vw} denotes the time required to travel along vw. Similarly, if $vw \in A_t$ is a turnaround arc, then ℓ_{vw} measures the waiting time from the arrival at v until the departure at w. We assume that $\ell_{vw} > 0$ holds for driving arcs, later we will even motivate $\ell_{vw} \in (0, T]$.

A *periodic timetable* for an event-activity network $\mathcal{N} = (G, T, \ell)$ is a vector $\pi \in [0, T)^V$ such that

$$\pi_w - \pi_v \equiv \ell_{vw} \mod T \quad \text{for all } vw \in A.$$

In the case of technical minimum turnaround times (e.g., 3 min for subways), for a network with $T = 10$ an arrival at $\pi_v = 5$ and departure at $\pi_w = 6$ could yield a value $\ell_{vw} = 11$, because the train that arrives at minute five is not ready for departure at minute six, and thus has to wait until the next departure ten minutes later. This value is larger than the period time and does not equal the positive immediate difference

$$\pi_w - \pi_v = 1 \neq 11 = \ell_{vw} > T = 10.$$

We therefore define the *periodic offset* of an arc $vw \in A$ as

$$p_{vw} := \frac{\ell_{vw} - (\pi_w - \pi_v)}{T} \quad \in \mathbb{Z}_{\geq 0}. \tag{1}$$

An example of an event-activity network with a periodic timetable is given in Figure 1. Notice that compared to periodic timetabling, where an optimal timetable is sought, here we are using a kind of simplified notation. Since in the setting that we are investigating the timetable is the input, and thus fixed, there is no need to elaborate on any minimum time durations serving as timetabling constraints. In fact, our values ℓ_{vw} are just the well-known periodic tensions [7].

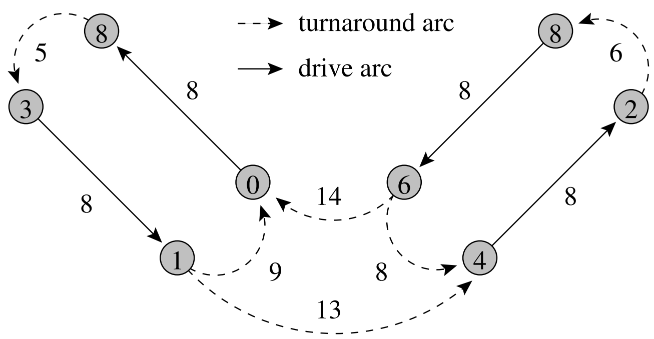

Figure 1 Example event-activity network ($T = 10$) with a periodic timetable.

3 Periodic vehicle scheduling

Let $\mathcal{N} = (G, T, \ell)$ be an event-activity network with a periodic timetable π. What is the minimal number of vehicles required to operate the timetable?

To answer this question, we define a *periodic vehicle schedule* as a collection S of directed cycles in G such that each driving arc $a \in A_d$ is contained in exactly one cycle in S. Moreover, we define the *length* resp. *periodic offset* of a directed cycle γ in G as

$$\ell(\gamma) := \sum_{a \in \gamma} \ell_a \quad \text{resp.} \quad p(\gamma) := \sum_{a \in \gamma} p_a.$$

▶ **Lemma 1.** *Let $\mathcal{N}$ be an event-activity network and let $\gamma = (v_1, \dots, v_k, v_1)$ be a directed cycle in G. If $\mathcal{N}$ admits a periodic timetable π, then $\ell(\gamma) = p(\gamma) \cdot T$ is a positive integer multiple of T.*

Proof. By definition of π and p,

$$\begin{aligned}
\ell(\gamma) = \sum_{a \in \gamma} \ell_a &= \ell_{v_1 v_2} + \dots + \ell_{v_{k-1} v_k} + \ell_{v_k v_1} \\
&= \pi_{v_2} - \pi_{v_1} + \dots + \pi_{v_k} - \pi_{v_{n-1}} + \pi_{v_1} - \pi_{v_k} + T p_{v_1 v_2} + \dots + T p_{v_k v_1} \\
&= T \cdot \sum_{a \in \gamma} p_a \\
&= T \cdot p(\gamma).
\end{aligned}$$
◀

In fact, this is a special case of the well-known cycle periodicity constraints in periodic timetabling [7]. This means that a vehicle driving on a cycle γ of a periodic vehicle schedule S can periodically continue after a time of $\ell(\gamma)$. Since each driving arc has to be covered in every period, the cycle γ requires in total $\ell(\gamma)/T = p(\gamma)$ vehicles. The *number of vehicles* $n(S)$ associated to a periodic vehicle schedule S is thus

$$n(S) := \frac{1}{T} \sum_{\gamma \in S} \ell(\gamma) = \frac{1}{T} \sum_{\gamma \in S} \sum_{a \in \gamma} \ell_a = \sum_{\gamma \in S} \sum_{a \in A} p_a = \sum_{\gamma \in S} p(\gamma).$$

In other words, in any periodic schedule S we can obtain the number of required vehicles either by summing up all cycle lengths and dividing by the period time, or by counting for each cycle the "jumps" to the next period. Notice already, that later we will translate this optimal compact periodic solution to optimal solutions for both, the expanded aperiodic vehicle scheduling problem as well as the expanded periodic model as an intermediate step.

The goal is now to compute a periodic vehicle schedule S such that $n(S)$ is minimal. We call this the *minimal periodic vehicle schedule problem.* This problem has an easy reformulation as a minimum cost circulation problem, where the variables x_a indicate whether the arc a is used in the optimal vehicle schedule:

$$\begin{aligned} \text{Minimize} \quad & \sum_{a \in A} p_a x_a \\ \text{s.t.} \quad & \sum_{u\colon uv \in A} x_{uv} = \sum_{w\colon vw \in A} x_{vw}, && v \in V, \\ & x_a = 1, && a \in A_d, \\ & x_a \in \{0,1\}, && a \in A_t. \end{aligned} \tag{2}$$

▶ **Lemma 2.** *The integer program* (2) *solves the minimal periodic vehicle schedule problem.*

Proof. This follows directly from plugging in the definitions of periodic vehicle schedules and their minimal number of vehicles into the standard integer programming formulation for minimum cost circulations. ◀

A closer inspection of the IP (2) yields the following: Since all driving arcs are covered exactly once, their cost is fixed in the objective. As every driving arc $a \in A_d$ requires at least $\lfloor \frac{\ell_a}{T} \rfloor$ vehicles, we may assume w.l.o.g. that for any driving arc $a \in A_d$ holds $\ell_a \in [0, T)$, which by (1) implies $p_a \in \{0, 1\}$. In turn, we remember to add $\lfloor \frac{\ell_a}{T} \rfloor$ vehicles for each shortened driving arc a. Furthermore, as any departure (arrival) node has only one outgoing (ingoing) arc, which is a driving arc, the flow conservation conditions may be replaced by

$$\begin{aligned} \sum_{u\colon uv \in A_t} x_{uv} &= 1, && v \in V_{\text{dep}}, \\ \sum_{w\colon vw \in A_t} x_{vw} &= 1, && v \in V_{\text{arr}}. \end{aligned}$$

In the end, we arrive at the following minimum weight perfect matching problem:

$$\begin{aligned} \text{Minimize} \quad & \sum_{a \in A_t} p_a x_a + \sum_{a \in A_d} p_a \\ \text{s.t.} \quad & \sum_{a \in A_t\colon v \in a} x_a = 1, && v \in V, \\ & x_a \in \{0,1\}, && a \in A_t. \end{aligned} \tag{3}$$

In other words, we have established the following:

▶ **Lemma 3.** *Let $\mathcal{N} = (G, T, \ell)$ be an event-activity network with periodic timetable π. Let $G_t = (V, A_t)$ be the subgraph of G where all driving arcs are removed. There is a one-to-one correspondence*

$$\{\textit{perfect matchings in } G_t\} \leftrightarrow \left\{ \begin{array}{c} \textit{circulations in } G \textit{ covering} \\ \textit{all driving arcs exactly once} \end{array} \right\}.$$

Moreover, a minimum weight perfect matching w.r.t. ℓ (or p) in G_t corresponds to a minimum cost circulation w.r.t. ℓ (or p) in G.

Note that the matching formulation is of rather local nature: It suffices to compute a perfect matching for every weakly connected component of G_t. Since the turnaround arcs usually stem from turnarounds at certain stations, this means that we can compute a minimal periodic vehicle schedule by optimizing the transitions at every station. Of course, several stations might be connected by longer unloaded trips.

The following theorem summarizes the different ways to solve the minimal periodic vehicle schedule problem:

▶ **Theorem 4.** *For an event-activity network $\mathcal{N} = (G, T, \ell)$ with periodic timetable π, the number $n(S_{\min})$ of vehicles of a minimal periodic vehicle schedule is given by:*

(a) *The cost of a minimum cost circulation in G w.r.t. ℓ covering all driving arcs exactly once, divided by T.*

(b) *The sum of periodic offsets of the arcs occurring in a minimum cost circulation in G w.r.t. ℓ covering all driving arcs exactly once.*

(c) *The sum of the weights ℓ_a of a minimum weight perfect matching of the turnaround arcs in G w.r.t. ℓ plus the travel times of all driving arcs, divided by T.*

(d) *The sum of periodic offsets p_a occurring in a minimum weight perfect matching of the turnaround arcs in G w.r.t. ℓ plus the periodic offsets of all driving arcs.*

4 Periodic expansion

In this section, we describe a procedure to expand an event-activity network in a periodic way. This construction will be of use for the proof of our main result Theorem 12, the optimality proof for a periodic vehicle scheduling solution in an expanded aperiodic context.

At first, we define for any $x \in \mathbb{R}$ and $N \in \mathbb{N}$ the expression $[x]_N$ as the unique real number $y \in [0, N)$ with $x \equiv y \bmod N$. For example, $[-8]_{10} = 2$.

Let $\mathcal{N} = (G, T, \ell)$ be an event-activity network with periodic timetable π. For any positive integer N, we define another event-activity network, namely the *N-th periodic expansion* $\mathcal{N}^{(N)} = (G^{(N)}, T^{(N)}, \ell^{(N)})$ as follows:

- The node set of $G^{(N)}$ is $V^{(N)} := V \times \{0, 1, \ldots, N-1\}$. A node (v, i) is called a departure (arrival) node iff v is a departure (arrival) node.
- For each driving arc $vw \in A_d$, add to the arc set $A^{(N)}$ of $G^{(N)}$ the driving arcs
$$((v, i), (w, [i + p_{vw}]_N)), \quad i = 0, \ldots, N-1.$$
- For each turnaround arc $vw \in A_t$, add to $A^{(N)}$ turnaround arcs
$$((v, i), (w, j)), \quad i, j = 0, \ldots, N-1.$$
- The duration of an arc $((v, i), (w, j)) \in A^{(N)}$ is set to
$$\ell^{(N)}_{(v,i),(w,j)} := \ell_{vw} + [j - i - p_{vw}]_N \cdot T.$$
- $T^{(N)} := N \cdot T$.

▶ Remark. Some observations:

(a) Up to notation, $\mathcal{N}^{(1)}$ is the same as $\mathcal{N}$.

(b) Each driving arc in $\mathcal{N}$ has N copies in $\mathcal{N}^{(N)}$, whereas each turnaround arc has N^2 copies. In fact, take a periodic turnaround arc $vw \in A_t$. For each of the N expanded occurrences of its periodic arrival event v, we keep the possibility to continue on *any* of the N copies of the respective expanded departure event w. At first sight, it could appear that some of these expanded arcs point backward in time. Yet, since $\mathcal{N}^{(N)}$ is still a periodic model, these arcs have positive durations, too, when considering their actual endpoints one period $N \cdot T$ later.

(c) Let vw be an arc in $\mathcal{N}$. Then the value of $\ell^{(N)}$ of any arc $((v,i),(w,j))$ is at least ℓ_{vw}, and the arcs $((v,i),(v,[i+p_{vw}]_N))$ for $i = 0,\dots,N-1$ are precisely the arcs whose duration is exactly ℓ_{vw}.

Periodic timetables extend in a natural way to the N-th periodic expansion:

▶ **Lemma 5.** *Let π be a periodic timetable for $\mathcal{N}$. Define $\pi^{(N)} \in [0, N\cdot T)^{V^{(N)}}$ via*

$$\pi^{(N)}_{(v,i)} := \pi_v + i \cdot T, \quad (v,i) \in V^{(N)}.$$

Then $\pi^{(N)}$ is a periodic timetable for $\mathcal{N}^{(N)}$ for the periodic tension values $\ell^{(N)}_{(v,i),(w,j)}$.

Proof. Let $((v,i),(w,j)) \in A^{(N)}$. We need to show that $\pi^{(N)}_{(w,j)} - \pi^{(N)}_{(v,i)} - \ell^{(N)}_{(v,i),(w,j)}$ is an integer multiple of $N \cdot T$. Plugging in the definitions,

$$\begin{aligned}\pi^{(N)}_{(w,j)} - \pi^{(N)}_{(v,i)} - \ell^{(N)}_{(v,i),(w,j)} &= \pi_w + j\cdot T - \pi_v - i \cdot T - \ell_{vw} - [j-i-p_{vw}]_N \cdot T \\ &= (j - i - p_{vw}) \cdot T - [j-i-p_{vw}]_N \cdot T \\ &\equiv 0 \mod N \cdot T,\end{aligned}$$

as i, j, p_{vw} are all integers and π is a periodic timetable for $\mathcal{N}$. ◀

In the remainder of this section, we establish that $n(S_{\min}) = n(S^{(N)}_{\min})$, where $S_{\min}$ denotes a minimal vehicle schedule for $\mathcal{N}$, $S^{(N)}_{\min}$ a minimal vehicle schedule for the N-th periodic expansion $\mathcal{N}^{(N)}$ of $\mathcal{N}$, and $n(\cdot)$ the number of vehicles of the respective schedules. We first prove that $n(S^{(N)}_{\min}) \leq n(S_{\min})$.

▶ **Lemma 6.** *Let $\mathcal{N}$ be an event-activity network with a periodic timetable π and a periodic vehicle schedule S using $n(S)$ vehicles. For any positive integer N, the timetable $\pi^{(N)}$ on $\mathcal{N}^{(N)}$ can be operated with $n(S)$ vehicles.*

Proof. Let M be a perfect matching of the turnaround arcs in G, resulting in a periodic vehicle schedule S using $n(S)$ vehicles. Then

$$M^{(N)} := \{((v,i),(w,[i+p_{vw}]_N) \mid vw \in M,\ i = 0,\dots,N-1\}$$

is a perfect matching of the turnaround arcs in $G^{(N)}$. By the previous remark, the arcs of $M^{(N)}$ have the same turnaround time as their counterpart in M. Moreover, every driving arc in G has N copies with the same travel time in $G^{(N)}$. By Theorem 4, $M^{(N)}$ leads hence to a periodic vehicle schedule whose number of vehicles is

$$\frac{1}{N\cdot T}\left(\sum_{a\in M^{(N)}} \ell^{(N)}_a + \sum_{a \in A_d^{(N)}} \ell^{(N)}_a\right) = \frac{1}{N \cdot T}\left(N \cdot \sum_{a\in M} \ell_a + N \cdot \sum_{a \in A_d} \ell_a\right) = n(S). \quad ◀$$

▶ **Theorem 7.** *Let $\mathcal{N}$ be an event-activity network with a periodic timetable π. For any positive integer N, the number of vehicles of a minimal periodic vehicle schedule w.r.t. $\pi^{(N)}$ on $\mathcal{N}^{(N)}$ equals the number of vehicles of a minimal periodic vehicle schedule w.r.t. π on $\mathcal{N}$.*

Proof. By Lemma 6, here it remains to show that $n(S_{\min}) \leq n(S^{(N)}_{\min})$, where $S^{(N)_{\min}}$ denotes a minimal periodic vehicle schedule w.r.t. π on $\mathcal{N}^{(N)}$, and $S_{\min}$ for the initial unexpanded periodic network $\mathcal{N}$. By Theorem 4, $S^{(N)}_{\min}$ induces a perfect matching $M^{(N)}$ of the turnaround arcs, with corresponding binary variables $x^{(N)}_a$ for $a \in A^{(N)}_t$ set to 1 in the integer programming formulation (3).

We define a – possibly fractional – periodic vehicle schedule S_{frac} w.r.t. π on $\mathcal{N}$ as follows: For each turnaround arc $vw \in A_t$, set the value of its matching variable x_{vw} as

$$x_{vw} := \frac{1}{N} \cdot \sum_{i,j=0}^{N-1} x^{(N)}_{(v,i),(w,j)} \tag{4}$$

By the definition of $A^{(N)}$ and by the matching property of $M^{(N)}$, S_{frac} indeed constitutes a – possibly fractional – periodic vehicle schedule w.r.t. π on $\mathcal{N}$, i.e., a fractional perfect matching in the bipartite graph of the turnaround arcs A_t of G. By Remark 4, the travel time along any arc used by S_{frac} is at most the travel time of any of its counterparts in $S^{(N)}_{\min}$. This implies that the total cost of S^{frac} is at most $n(S^{(N)}_{\min})$:

$$\begin{aligned} n(S_{\text{frac}}) &= \frac{1}{T}\left(\sum_{vw\in A_d} \ell_{vw} + \sum_{vw\in A_t} x_a \ell_{vw}\right) \\ &\overset{(4)}{=} \frac{1}{T}\left(\sum_{vw\in A_d} \frac{1}{N}\sum_{i=0}^{N-1} \ell_{vw} + \sum_{vw\in A_t} \frac{1}{N}\sum_{i,j=0}^{N-1} x^{(N)}_{(v,i),(w,j)}\ell_{vw}\right) \\ &\leq \frac{1}{T}\left(\frac{1}{N}\sum_{vw\in A_d}\sum_{i=0}^{N-1} \ell^{(N)}_{(v,i),(v,[i+p_{vw}]_N)} + \frac{1}{N}\sum_{vw\in A_t}\sum_{i,j=0}^{N-1} x^{(N)}_{(v,i),(w,j)}\ell^{(N)}_{(v,i)(w,j)}\right) \\ &= n(S^{(N)}_{\min}). \end{aligned}$$

Recall several elementary results as they are collected, e.g., in the book of Schrijver [11]:

- As the subgraph (V, A_t) of G is bipartite, the constraints $x_a \geq 0$ and $\sum_{a\in\delta(v)} x_a = 1$ (i.e., the incidence matrix) already determine the perfect matching polytope [11, Theorem 18.1].
- The incidence matrix of any directed graph is totally unimodular [11, Theorem 13.9].
- For a totally unimodular matrix together with an integer right-hand-side vector, their associated polyhedron is integer [11, Theorem 5.20].

Now, due to the integrality of the perfect matching polytope (i.e., the assignment problem polytope), we find an optimal integral perfect matching M in the bipartite graph of the turnaround arcs A_t. This induces a minimal periodic vehicle schedule $S_{\min}$ w.r.t. π on $\mathcal{N}$. Since S_{frac} is a fractional solution of this perfect matching polytope, we finally find

$$n(S_{\min}) \leq n(S_{\text{frac}}) \leq n(S^{(N)}_{\min}).$$

Since Lemma 6 asserts $n(S^{(N)}_{\min}) \leq n(S_{\min})$, this finishes the proof. ◀

5 Aperiodic vehicle scheduling

The standard way to compute the minimal number of vehicles required to operate a – not necessarily periodic – timetable is to use a network flow model [3, Â§2.4]. For a periodic timetable, the first step is to *expand* (or *roll out*) the timetable for a sufficient amount of time, e.g., a day.

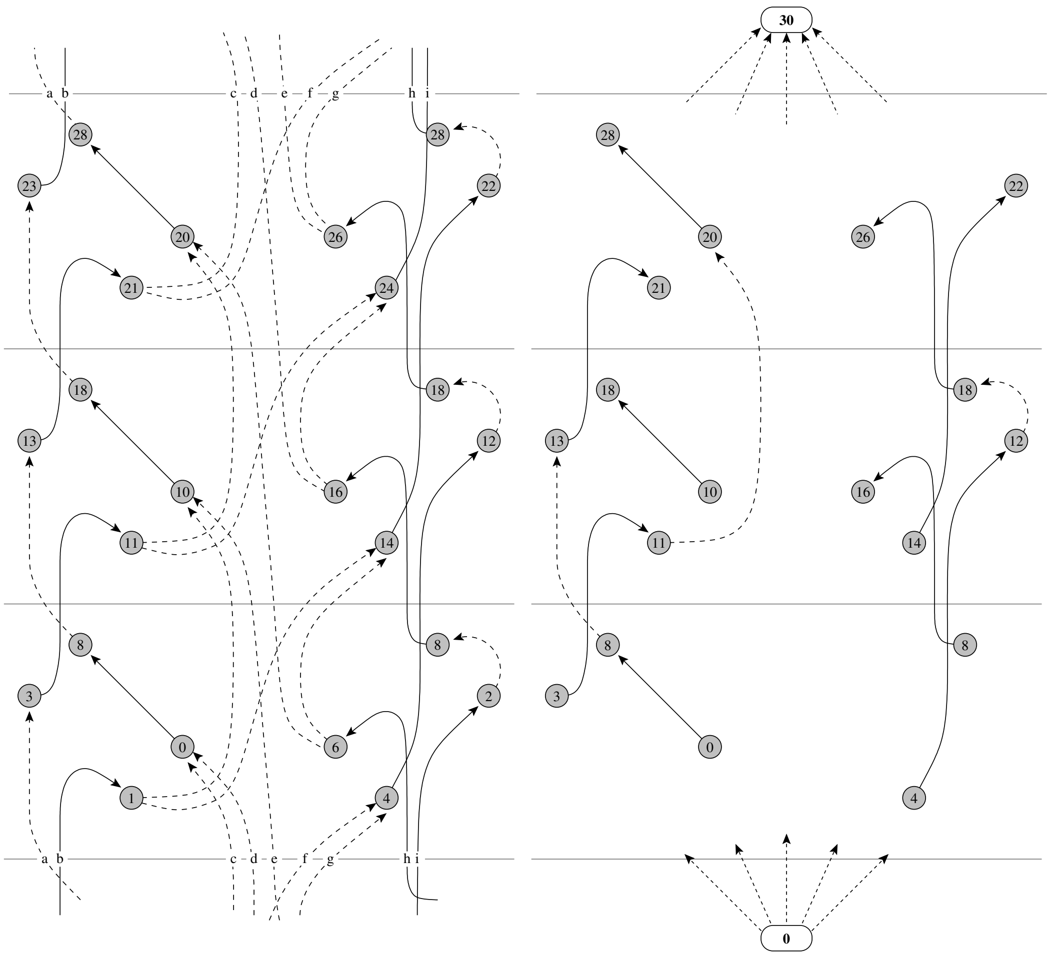

Figure 2 The first $N = 3$ layers of the periodic expansion with selected turnaround activities of the event-activity network in Figure 1 on the left, and its aperiodic counterpart on the right.

We formalize this process as follows: Starting from an event-activity network $\mathcal{N}$ with periodic timetable π, we construct the N*-th aperiodic expansion* $\mathcal{N}^{[N]} = (G^{[N]}, T^{[N]}, \ell^{[N]})$ with node set $V^{[N]}$ and arc set $A^{[N]}$ according to the following rules, see also Figure 2:

- Initialize $\mathcal{N}^{[N]}$ as the N-th periodic expansion $\mathcal{N}^{(N)}$.
- Delete all arcs $((v, i), (w, j))$ with $p^{(N)}_{(v,i),(w,j)} \geq 1$, i.e., those that leave the periodically expanded graph at time $N \cdot T$ and re-enter it at time zero.
- Remove departure nodes with out-degree zero and arrival nodes with in-degree zero, together with any incident turnaround arcs.
- Add a super-source s and arcs from s to all remaining departure nodes (v, i) with length $\ell^{[N]}_{s,(v,i)} = \pi^{(N)}_{(v,i)}$.
- Introduce a super-sink t. Add arcs from all remaining arrival nodes (w, j) to t with $\ell^{[N]}_{(w,j),t} = N \cdot T - \pi^{(N)}_{(w,j)}$.
- Finally make an extra arc (t, s) with $\ell^{[N]}_{ts} = 0$.

Deleting arcs with positive periodic offset $p^{(N)}$ means intuitively that all arcs $((v, i), (w, j))$ with $\pi_v + i \cdot T > \pi_w + j \cdot T$ ("backward in time") are omitted, as well as arcs whose duration $\ell^{(N)}_{(v,i),(w,j)}$ is at least $N \cdot T$ ("jump to the next period"). If we delete a driving arc, then we also remove the corresponding departure and arrival nodes. The arc (t, s) is the only arc in the aperiodic expansion that is allowed to go “backward in time”. Moreover, think of the deletion of a turnaround arc $((w, j), (v, i))$ as a kind of replacing it with the new pull-in arc $((w, j), t)$ together with the new pull-out arc $(s, (v, i))$.

▶ Remark.

(a) Every arc of the form $((v, i), (w, j)) \in A^{[N]}$ satisfies $p^{(N)}_{(v,i),(w,j)} = 0$ and hence $\ell^{(N)}_{(v,i),(w,j)} = \pi^{(N)}_{(w,j)} - \pi^{(N)}_{(v,i)} \in [0, N \cdot T)$.

(b) Suppose that γ is a directed cycle in $\mathcal{N}^{[N]}$ containing an arc of positive duration. Then γ contains also the arc from t to s, as $\pi^{(N)}$ increases along γ and the arc (t, s) is the only way to decrease $\pi^{(N)}$ again.

Define the sets of driving and turnaround arcs of $\mathcal{N}^{[N]}$ as $A^{[N]}_d := A^{(N)}_d \cap A^{[N]}$ and $A^{[N]}_t := A^{(N)}_t \cap A^{[N]}$, respectively. An *aperiodic vehicle schedule* is a collection $S^{[N]}$ of directed cycles in $\mathcal{N}^{[N]}$ such that each driving arc is contained in exactly one cycle of $S^{[N]}$.

By the previous remark, a vehicle starts at s, visits departure nodes and arrival nodes alternatingly until it reaches t, and finally goes back to s. The minimum number of vehicles $n(S^{[N]})$ of an aperiodic vehicle schedule $S^{[N]}$ is thus obtained by solving the following minimum cost circulation problem, see [3, Â§2.4]:

$$\begin{aligned} \text{Minimize} \quad & x_{ts} && \\ \text{s. t.} \quad & \sum_{u:\, uv \in A^{[N]}} x_{uv} = \sum_{w:\, vw \in A^{[N]}} x_{vw}, && v \in V, \\ & x_a = 1, && a \in A^{[N]}_d, \\ & x_a \in \mathbb{Z}_{\geq 0} && a \in A^{[N]} \setminus A^{[N]}_d. \end{aligned} \tag{5}$$

The *minimal aperiodic vehicle schedule* problem is to solve the above integer program, still for a given fixed timetable.

▶ **Lemma 8.** *Let $S^{[N]}$ be a minimal aperiodic vehicle schedule corresponding to an optimal solution x to the integer program* (5)*. Then the following numbers are equal:*

(a) $n(S^{[N]})$,

(b) $\dfrac{1}{N \cdot T} \sum_{a \in A^{[N]}} \ell^{[N]}_a x_a$,

(c) $\# A^{[N]}_d - \#\{a \in A^{[N]}_t \mid x_a = 1\}$,

(d) $\# A^{[N]}_d - \# M$, *where M is a maximum cardinality matching of* $(V^{[N]}, A^{[N]}_t)$,

(e) $\sum_{a=(s,v)} x_a = \sum_{a=(w,t)} x_a$.

Proof. If a feasible circulation x for (5) produces f units of flow on the t-s-arc, then it also contains f arc-disjoint paths from s to t. Let $q = (s, (v_1, i_1), \ldots, (v_k, i_k), t)$ be such an s-t-path. Then

$$\begin{aligned} \ell^{[N]}(q) &= \ell^{[N]}_{s,(v_1,i_1)} + \sum_{j=1}^{k-1} \ell^{[N]}_{(v_j,i_j),(v_{j+1},i_{j+1})} + \ell^{[N]}_{(v_k,i_k),t} \\ &= \pi^{(N)}_{(v_1,i_1)} + \sum_{j=1}^{k-1} \left(\pi^{(N)}_{(v_{j+1},i_{j+1})} - \pi^{(N)}_{(v_j,i_j)} \right) + N \cdot T - \pi^{(N)}_{(v_k,i_k)} \qquad = N \cdot T, \end{aligned}$$

by the definition of $\mathcal{N}^{[N]}$. In particular, $\sum_{a \in A^{[N]}} \ell^{[N]}_a x_a = f \cdot N \cdot T$. This shows (a) = (b).

Each simple cycle in a feasible circulation uses the arc from t to s, proceeds to a departure node, and then visits driving and turnaround activities alternatingly until it reaches its last driving activity, from which it goes back to t. In particular, for each such cycle γ holds

$$\#\{a \in A_d^{[N]} \mid a \in \gamma\} - \#\{a \in A_t^{[N]} \mid a \in \gamma\} = 1.$$

A minimum cost circulation decomposes into precisely $n(S^{[N]})$ such cycles, and covers each arc of $A_d^{[N]}$ precisely once. Summing over these cycles, we obtain (a) = (c).

Observe that $\{a \in A_t^{[N]} \mid x_a = 1\}$ is a matching of $(V^{[N]}, A_t^{[N]})$. Conversely, let M be any matching in $(V^{[N]}, A_t^{[N]})$. Consider the circulation consisting of the $\#A_d^{[N]}$ simple cycles $(s, (v, i), (w, j), t, s)$ for each driving arc $((v, i), (w, j)) \in A_d^{[N]}$. For each $a \in M$, connect the cycles of the driving arcs incident to a, thereby reducing the value of flow by one. This yields a circulation with value $\#A_d^{[N]} - \#M$.

Finally, (a) = (e) follows immediately from the structure of $\mathcal{N}^{[N]}$ and (5). ◀

▶ Remark. After $\mathcal{N}^{[N]}$ has been constructed, the number $n(S^{[N]})$ does neither depend on ℓ nor π. In other words, it is sufficient to look at feasible sequences of trips regardless of their actual duration.

Now, let's have a look at the cuts that are induced along the timelines $(i+1)T - \varepsilon$:

▶ **Lemma 9.** *Let $S^{[N]}$ be an aperiodic vehicle schedule with associated matching $M^{[N]}$ of $(V^{[N]}, A_t^{[N]})$. Then for any $i \in \{0, \dots, N-2\}$,*

$$n(S^{[N]}) \geq \sum_{a \in A_d} p_a + \#\{((v, i), (w, i+1)) \in M^{[N]}\}.$$

Proof. Let x be the corresponding solution to the IP (5). For small $\varepsilon > 0$, examine the flow x on all arcs at time $(i+1)T - \varepsilon$: At this point, there is one unit of flow on each driving arc departing before $(i+1)T$ and arriving at $(i+1)T$ or later. This means, there are p_a units of flow for each driving arc $a \in A_d$ in $\mathcal{N}$. Moreover, there is one unit of flow on each turnaround arc matched by $M^{[N]}$ with arrival before $(i+1)T$ and departure at $(i+1)T$ or later. In particular, this comprises turnaround arcs starting at some (v, i) and ending at some $(w, i+1)$. Finally, there is a non-negative flow on pull-in or pull-out arcs. ◀

We turn now to the comparison of periodic and aperiodic expansions:

▶ **Lemma 10.** *Let $\mathcal{N}$ be an event-activity network with periodic timetable π. Let $S^{[N]_{\min}}$ be a minimal aperiodic vehicle schedule on $\mathcal{N}^{[N]}$, and let $S^{(N)}$ be any periodic vehicle schedule on $\mathcal{N}^{(N)}$. Then $n(S_{\min}^{[N]}) \leq n(S^{(N)})$.*

Proof. Let $M^{(N)}$ be a perfect matching of the turnaround arcs in the N-th periodic expansion. By Theorem 4,

$$\begin{aligned} n(S^{(N)}) &= \sum_{a \in A_d^{(N)}} p_a^{(N)} + \sum_{a \in M^{(N)}} p_a^{(N)} \\ &\geq \#\{a \in A_d^{(N)} \mid p_a^{(N)} \geq 1\} + \#\{a \in M^{(N)} \mid p_a^{(N)} \geq 1\} \\ &= \#\{a \in A_d^{(N)} \mid p_a^{(N)} \geq 1\} + M^{(N)} - \#\{a \in M^{(N)} \mid p_a^{(N)} = 0\} \end{aligned}$$

Since $M^{(N)}$ is a perfect matching and in every directed cycle driving and (matched) turnaround arcs alternate, $\#M^{(N)} = \#A_d^{(N)}$, and we find

$$\begin{aligned} n(S^{(N)}) &\geq 2\#A_d^{(N)} - \#\{a \in A_d^{(N)} \mid p_a^{(N)} = 0\} - \#\{a \in M^{(N)} \mid p_a^{(N)} = 0\} \\ &= 2\#\{a \in A_d^{(N)} \mid p_a^{(N)} \geq 1\} + \#\{a \in A_d^{(N)} \mid p_a^{(N)} = 0\} \\ &\quad - \#\{a \in M^{(N)} \mid p_a^{(N)} = 0\}. \end{aligned} \tag{6}$$

The intersection $M^{(N)} \cap A_t^{[N]}$ is some matching in $\mathcal{N}^{[N]}$. We will compare this with a maximum cardinality matching $M^{[N]}$ of the turnaround arcs in the N-th aperiodic expansion $\mathcal{N}^{[N]}$. The matching $M^{(N)} \cap A_t^{[N]}$ contains all arcs a from $M^{(N)}$ with $p_a^{(N)} = 0$, except those being incident to a driving arc a with $p_a^{(N)} \geq 1$. Since any such driving arc can be incident to two turnaround arcs in $M^{(N)}$, this means

$$\#M^{[N]} \geq \#M^{(N)} \cap A_t^{[N]} \geq \#\{a \in M^{(N)} \mid p_a^{(N)} = 0\} - 2\#\{a \in A_d^{(N)} \mid p_a^{(N)} \geq 1\}. \tag{7}$$

Therefore, using (6) and (7), and then Lemma 8,

$$n(S^{(N)}) \geq \#\{a \in A_d^{(N)} \mid p_a^{(N)} = 0\} - \#M^{[N]} = \#A_d^{[N]} - \#M^{[N]} = n(S_{\min}^{[N]}).$$ ◀

The following lemma is an interesting fact about the interplay of minimum-weight perfect matchings and maximum-weight matchings in the N-th periodic expansion. The proof makes use of the structure of the 2-matching polytope of a bipartite graph.

▶ **Lemma 11.** *Let $M^{(N)}$ be a minimum-weight perfect matching w.r.t. $p^{(N)}$ of the turnaround arcs in the N-th periodic expansion $\mathcal{N}^{(N)}$. Let $q := \left\lceil \log_2 \left(\sum_{a \in A_t} p_a + 1 \right) \right\rceil$. If $N \geq 2^q$, then $M^{(N)}$ maximizes $\#\{a \in M \mid p_a^{(N)} = 0\}$ among all matchings of turnaround arcs in $\mathcal{N}^{(N)}$.*

Proof. Let $M^{(2)}$ be any matching of the turnaround arcs in the second periodic expansion of $\mathcal{N}$, giving rise to an incidence vector $x^{(2)} \in \{0,1\}^{A_t^{(2)}}$. Then the vector $x \in \{0,1,2\}^{A_t}$ with

$$x_{vw} := x^{(2)}_{(v,0),(w,0)} + x^{(2)}_{(v,0),(w,1)} + x^{(2)}_{(v,1),(w,0)} + x^{(2)}_{(v,1),(w,1)}, \quad vw \in A_t,$$

is a 2-matching of the turnaround arcs in $\mathcal{N}$. Since $\mathcal{N}$ is bipartite, the vertices of the 2-matching polytope correspond to matchings where each edge is taken twice [11, Theorem 31.10]. In particular, a matching maximizing the number of arcs with $p_a^{(2)} = 0$ in $\mathcal{N}^{(2)}$ can be found by considering instead a matching in $\mathcal{N}$. By construction of $\mathcal{N}^{(N)}$, any turnaround arc $a \in A_t$ produces $\max(2 - p_a, 0)$ copies in $\mathcal{N}^{(2)}$ with offset 0. We are hence interested in finding the maximum-weight matching in $\mathcal{N}$ w.r.t. the weight function $a \mapsto \max(2 - p_a, 0)$.

Repeating this process, we can analogously find for any $k \in \mathbb{N}$ the matching maximizing the number of turnaround arcs with $p_a^{(2^k)} = 0$ in $\mathcal{N}^{(2^k)}$ by computing a maximum-weight matching in $\mathcal{N}$ w.r.t. the weights $\max(2^k - p_a, 0)$, $a \in A_t$. If $2^k \geq \sum_{a \in A_t} p_a + 1$, then such a matching is automatically a perfect matching $M^{(1)}$ minimizing the periodic offsets p. Performing the construction of the proof of Lemma 6, we obtain from $M^{(1)}$ a perfect matching $M^{(2^k)}$ minimizing $p^{(2^k)}$. By Theorem 7, the weight of $M^{(1)}$ w.r.t. p equals the weight of $M^{(2^k)}$ w.r.t. $p^{(2^k)}$.

Finally let $N = 2^q + r$ for some $r \in \mathbb{N}$. Extending $M^{(1)}$ even further to a perfect matching $M^{(N)}$ in $\mathcal{N}^{(N)}$ yields in total $\sum_{a \in A_t} (2^q + r - p_a) = \sum_{a \in A_t} (2^q - p_a) + r\#A_d$ arcs with $p_a^{(N)} = 0$. If M is a matching in $\mathcal{N}^{(N)}$ maximizing $\mu := \#\{a \in M \mid p_a^{(N)} = 0\}$, then M matches at most $2r\#A_d$ vertices that do not appear in $\mathcal{N}^{(2^k)}$. As $M^{(2^k)}$ is maximum in $\mathcal{N}^{(2^k)}$, in particular $\mu - r\#A_d \leq \sum_{a \in A_t} (2^q - p_a)$, so that M has at most as many $p_a^{(N)} = 0$ arcs as $M^{(N)}$. ◀

We present now our main result, stating that rolling out and solving the minimal aperiodic vehicle schedule problem has no advantage over working on the periodic network itself:

▶ **Theorem 12.** *Let $\mathcal{N}$ be an event-activity network with periodic timetable π. Consider*

(a) *the number $n(S_{\min})$ of vehicles of a minimal periodic vehicle schedule $S_{\min}$ on $\mathcal{N}$ w.r.t. π,*

(b) *the number $n(S_{\min}^{(N)})$ of vehicles of a minimal periodic vehicle schedule $S_{\min}^{(N)}$ on the N-th periodic expansion $\mathcal{N}^{(N)}$ w.r.t. $\pi^{(N)}$, and*

(c) *the number $n(S_{\min}^{[N]})$ of vehicles of a minimal aperiodic vehicle schedule $S_{\min}^{[N]}$ on the N-th aperiodic expansion $\mathcal{N}^{[N]}$.*

Then $n(S_{\min}) = n(S_{\min}^{(N)}) \geq n(S_{\min}^{[N]})$. Moreover, $n(S_{\min}) = n(S_{\min}^{(N)}) = n(S_{\min}^{[N]})$ holds if $N \geq 2^q(2n(S_{\min})+1)$, where $q := \left\lceil \log_2 \left(\sum_{a\in A_t} p_a + 1\right)\right\rceil$.

Proof. The equality $n(S_{\min}) = n(S_{\min}^{(N)})$ has been established in Theorem 7. By Lemma 10, $n(S_{\min}^{(N)}) \geq n(S_{\min}^{[N]})$. Thus it remains to show that $n(S_{\min}^{[N]}) \geq n(S_{\min})$.

Fix a minimal aperiodic schedule $S_{\min}^{[N]}$. Let M be a minimum-weight perfect matching of the turnaround arcs in $\mathcal{N}$ w.r.t. the periodic offset p. Assume for the moment that

$$\begin{aligned} &p_a \in \{0,1\} \text{ for all } a \in A_t, \text{ and} \\ &M \text{ maximizes the number of arcs } a \text{ with } p_a = 0 \text{ among all matchings in } (V, A_t). \end{aligned} \tag{8}$$

By Lemma 8, the aperiodic schedule $S_{\min}^{[N]}$ uses at most $n(S_{\min}^{[N]}) \leq n(S_{\min})$ pull-out arcs and at most $n(S_{\min}^{[N]}) \leq n(S_{\min})$ pull-in arcs. Suppose now $N \geq 2n(S_{\min})+1$. Then, by the pigeonhole principle, we find an $i \in \{0,\dots,N-2\}$ such that no vertex (v,i) is preceded by a pull-out arc from s or followed by a pull-in arc to t.

Let $M^{[N]}$ be the matching in $(V^{[N]}, A_t^{[N]})$ corresponding to $S_{\min}^{[N]}$. By Lemma 9,

$$n(S_{\min}^{[N]}) \geq \sum_{a\in A_d} p_a + \#\{((v,i),(w,i+1)) \in M^{[N]}\}.$$

As there are neither pull-in nor pull-out arcs, all $\#A_d$ arrival vertices of the form (v,i) have to be matched by $M^{[N]}$. Moreover, each matching partner (w,j) of (v,i) has either $j=i$ or $j=i+1$ due to the assumption $p_a \in \{0,1\}$ in (8). Thus we can write

$$n(S_{\min}^{[N]}) \geq \sum_{a\in A_d} p_a + \#A_d - \{((v,i),(w,i)) \in M^{[N]}\}.$$

The set $\{((v,i),(w,i)) \in M^{[N]}\}$ yields naturally a matching in the unexpanded periodic network $\mathcal{N}$ using only turnaround arcs $a \in A_t$ with $p_a = 0$. With $\#M = \#A_d$, the assumptions (8) and Theorem 4,

$$n(S_{\min}^{[N]}) \geq \sum_{a\in A_d} p_a + \#A_d - \#\{a \in M \mid p_a = 0\} = \sum_{a\in A_d} p_a + \sum_{a\in M} p_a = n(S_{\min}).$$

Note that (8) might not be satisfied immediately. However, $\mathcal{N}$ can be replaced by its 2^q-th periodic expansion $\mathcal{N}^{(2^q)}$, where $q := \left\lceil \log_2 \left(\sum_{a\in A_t} p_a + 1\right)\right\rceil$: Then $2^q \geq p_a$ for each $a \in A_t$, so that a minimum-weight perfect matching $M^{(2^q)}$ constructed as in Lemma 6 uses only arcs a with $p_a^{(2^q)} \in \{0,1\}$. In particular, we can delete all arcs from $\mathcal{N}^{(2^q)}$ with $p_a^{(2^q)} \geq 2$, and still obtain the same perfect matching. Moreover, Lemma 11 now certifies the second assumption. In particular, for $N \geq 2^q(2n(S_{\min})+1)$, we finally obtain

$$n(S_{\min}^{[N]}) \geq n(S_{\min}^{(2^q)}) = n(S_{\min}).$$ ◀

6 Conclusion

To summarize, given that a public transportation network is to be operated with a purely periodic timetable, in order to compute the number of vehicles that are required to operate it, there is no need to expand the periodic network over time and solve a standard network flow model for vehicle scheduling. Rather, our results justify to keep the compact periodic structure and compute perfect matchings, where the graph is even likely to decompose and make the actual computation even easier. Moreover, this insight justifies that minimizing vehicle waiting time as early as in the step of optimizing the timetable itself, indeed points the timetable solution into the direction of a favorable efficient use of vehicles – right as it has already been common practice in several case studies.

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