Multi-Source Multi-Sink Nash Flows over Time*

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—— **Abstract** ——————————————————————————

Nash flows over time describe the behavior of selfish users eager to reach their destination as early as possible while traveling along the arcs of a network with capacities and transit times. Throughout the past decade, they have been thoroughly studied in single-source single-sink networks for the deterministic queuing model, which is of particular relevance and frequently used in the context of traffic and transport networks. In this setting there exist Nash flows over time that can be described by a sequence of static flows featuring special properties, so-called 'thin flows with resetting'. This insight can also be used algorithmically to compute Nash flows over time. We present an extension of these results to networks with multiple sources and sinks which are much more relevant in practical applications. In particular, we come up with a subtle generalization of thin flows with resetting, which yields a compact description as well as an algorithmic approach for computing multi-terminal Nash flows over time.

1 Introduction

With the emergence of novel navigation and vehicle technologies (including, e.g., self-driving/smart vehicles) along with the availability of massive amounts of data in todays and future traffic and transportation networks, increasing attention is given to the mathematical modeling and algorithmic solution of the interplay of individual agents in such networks. We study the behavior of selfish users who wish to travel through a traffic or transportation network. While there is already a vast amount of literature and results on steady states of such systems (see, e.g., Roughgarden [13] and the references therein), much less is known about the often more realistic but also much more complex situation of such systems evolving and changing over time.

Flows over time.   Flows over time provide an excellent mathematical model for agents (flow particles) traveling through a network over time, with capacities and transit times (delays) on the arcs. Flows over time have been introduced in a seminal paper by Ford and Fulkerson [4] and can also be found in their classic textbook [5]. For a given single-source single-sink network with capacities and transit times on the arcs and a given time horizon, they show how to efficiently construct a maximum flow over time, that is, a way of sending as much flow as possible from the source to the sink within the given time horizon. The underlying algorithm is based on a static min-cost flow computation in the given network where arc transit times are interpreted as costs. A decomposition of the static flow into flows along source-sink-paths then provides an optimal strategy for sending flow over time from the source to the sink by using each path as long as possible. Surprisingly, and in contrast to the situation known for classic (i.e., static) network flows, the problem of balancing given supplies and demands in a network with several sources and/or sinks by sending flow within a given time horizon turns out to be considerably more difficult and complicated. Following the work of Ford and Fulkerson, it took almost four decades before Hoppe and Tardos [8] came up with an efficient algorithm for solving this transshipment over time problem; see also Hoppe's PhD thesis [7]. Their algorithm, however, while being theoretically efficient, relies on parametric submodular function minimization, leading to unpleasant and usually unrealistic running times for networks of practical sizes. Only recently, Schlöter and Skutella [14] presented a slight improvement of this result. Another somewhat surprising evidence for the increased difficulty of flow over time problems compared to static flow problems is the fact that the computation of (fractional) multicommodity flows over time constitutes an NP-hard problem [6]. We refer to [15] for a recent survey on and thorough introduction to flows over time.

Nash equilibria for the deterministic queuing model.   The flow over time problems discussed in the previous paragraph are all based on the assumption that flow particles are controlled by a central authority who decides the route choices and schedules of the particles. In most realistic traffic situations, however, the lack of coordination among flow particles necessitates an additional game theoretic perspective. We assume that each flow particle is an individual agent that seeks to arrive at a destination in the least possible time. Such models have mostly been studied in the transportation literature; see, e.g., the book by Ran and Boyce [12] for an overview.

In this paper we study Nash equilibria for flows over time in the deterministic queuing model that is also at the core of many large-scale agent-based traffic simulations such as, e.g., MATSim; see [9]. Here the actual transit time of a flow particle along an arc is the sum of the arc's free-flow transit time plus the waiting time spent in a queue that builds up whenever more flow tries to use an arc than the arc's capacity can handle. In particular, the first-in-first-out (FIFO) principle holds. We refer to Section 3 for a detailed definition.

For a single-source single-sink network, Koch and Skutella [10] characterize Nash flows over time featuring a special and very useful structure: Their derivatives are piece-wise constant, therefore constituting a sequence of particular static source-sink flows, so-called *thin flows with resetting*. Exploiting this key concept of thin flows with resetting, Cominetti, Correa, and Larré [2] provide a constructive proof for the existence and uniqueness of equilibria in this setting, using a fixed-point formulation. Furthermore, for the more general case of multiple origin-destination pairs, they provide a non-constructive existence proof. For the single-source single-sink setting, Cominetti, Correa, and Olver [3] show that, for networks with sufficient capacity, a dynamic equilibrium reaches a steady state in finite time.

Our contribution. Our structural and algorithmic understanding of Nash flows over time is limited to the very restrictive special case of single-source single-sink networks. Moreover, in contrast to the classical case of static flows, single-commodity flows over time in multi-source multi-sink networks with given supplies and demands cannot easily be reduced by introducing a super-source and a super-sink; see, e.g., the work of Hoppe and Tardos [8] discussed above. Nevertheless, we show that such a reduction is possible, albeit non-trivial, when considering a particularly meaningful model of Nash flows over time in such networks. This leads to an interesting generalization of the structural and algorithmic results known for the single-source single-sink case; see [10, 2, 3]. In particular, we present an appropriate generalization of 'thin flows with resetting' and prove that a Nash flow over time can be described and algorithmically obtained via a sequence of these static flows. As another interesting aspect of this work, we show how to get rid of the identification of flow particles with the time they enter the network which has been used in previous work on the single-source single-sink case. In our more general model, all flow is waiting in front of the sources of the network right from the beginning, a subtle point that turns out to be crucial for being able to handle multiple source nodes.

Outline. In Section 2 we informally describe several different settings for dynamic routing games with multiple sources and sinks and identify a suitable model for our purposes. Section 3 introduces the necessary concepts and notations for describing Nash flows over time. Then, Section 4 explains how to deal with multiple source nodes. Finally, in Section 5 multiple sinks are considered as well.

2 Settings for routing games with multiple sources and sinks

There are several different settings for dynamic routing games when considering multiple sources and multiple sinks. We discuss the most meaningful interpretations in the following.

Nash flows over time are mainly motivated by dynamic traffic assignments which naturally lead to the consideration of multiple commodities with independent origin-destination-pairs (s_i, t_i) and inflow rates $r_i \geq 0$, for $i = 1, \ldots, n$. At each origin s_i, a flow enters the network with rate r_i and every infinitesimal small particle of this flow has the goal to reach destination t_i as early as possible while considering all other particles from the past and the future. For every commodity, there are time dependent in- and outflow rates for every arc that must satisfy flow conservation at every node. A dynamic equilibrium then consists of a flow over time with n commodities, where each particle chooses a combination of fastest routes from s_i to t_i as strategy. Note that queues build up on arcs whenever the inflow rate exceeds the arc's capacity. This causes a delay of all subsequent particles, therefore influencing the traversing time of all routes using this arc. Cominetti et al. [2] prove that these dynamic equilibria exist by using variational inequalities for the path-based formulation. Unfortunately, the known techniques for single commodity flows are not sufficient for analyzing or algorithmically constructing such dynamic multi-commodity Nash flows over time. The fact that each commodity has different earliest arrival times at the nodes is the main difficulty as this causes cyclic interdependencies between the commodities. Each particle entering the network has to take into account not only all flow that previously entered the network, but also flow entering the network subsequently; an illustrative example is given in the left part of Figure 1.

When we relax the pairing of origins and destinations, however, the route choice of each particle only depends on flow that previously entered the network. We stick to individual inflow rates for the sources, but instead of matching the sources to destinations, we consider
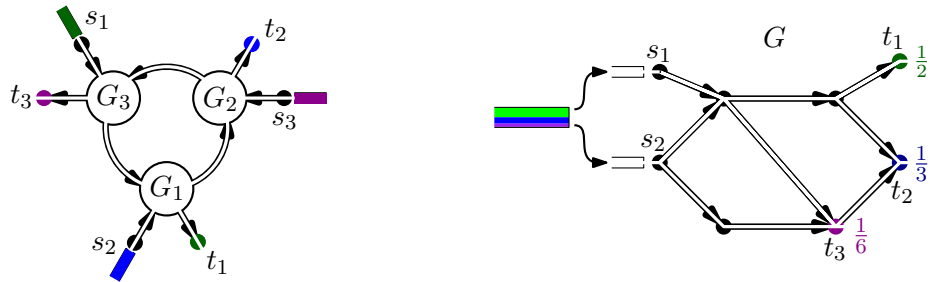
Figure 1 *Left:* Illustration of cyclic interdependencies of commodities with different origin-destination-pairs. The waiting times for flow from s_1 to t_1 within subnetwork G_1 depend on flow starting later from s_2 to t_2. The cyclic symmetry implies that a particle has to take into account not only previous but also future flow from all sources. *Right:* An example of the setting considered in this article. Each flow particle may choose whether to enter the network at s_1 or s_2, but the flow is partitioned according to the demands at the sinks. Here one half of the flow has t_1 as its destination, one third wants to reach t_2, and the rest of one sixth aims at t_3.

(a)   **(b)**   **(c)**
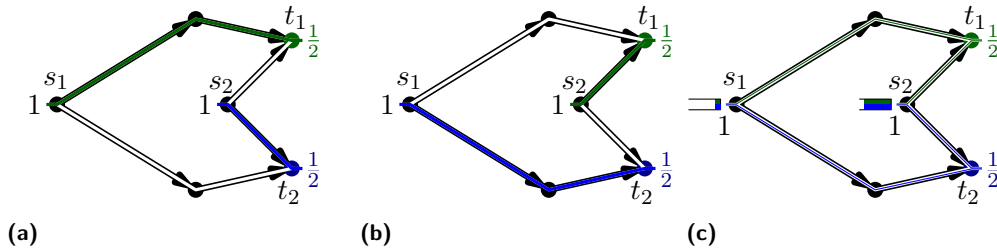
Figure 2 Assume that, at each point in time, the total inflow is equally divided according to the demands. Then, in this symmetric instance, a possible dynamic equilibrium sends all flow from s_1 to t_1 and from s_2 to t_2 (a). Alternatively, the destinations might be swapped (b). Equilibrium (a), however, heavily benefits sink t_2 by serving it earlier as the path from s_1 to t_1 is longer than the path from s_2 to t_2. Symmetrically, equilibrium (b) benefits t_1. This is the reason for considering queues in front of the sources such that a Nash flow over time is forced to behave symmetrically in this instance, that is, sinks with the same demand are treated equally in every equilibrium; see (c).

m sinks $t_1, \ldots, t_m$ with demands $d_1, \ldots, d_m \geq 0$, such that $d_1 + \cdots + d_m = 1$. The value d_j denotes the share of the total flow entering the network that has t_j as destination. In terms of traffic networks this means that each road user has a predetermined destination, but may choose between multiple origins to enter the network; see right side of Figure 1. In order to obtain well defined Nash flows over time with unique arrival times we exclude situations as described in Figure 2 by considering queues in front of the sources. In other words, there is essentially one flow $\mathbb{R}_{\geq 0}$ consisting of a continuum of infinitesimally small particles $\phi \in \mathbb{R}_{\geq 0}$, where each splittable particle chooses, in the order given by $<$, a convex combination of fastest routes from the sources to the sinks as strategy. The sum of the coefficients of all paths to sink t_j has to be equal to demand d_j. Each particle is then split according to these coefficients and each part is sent along its route. How these choices of strategies can be constructed, and what structure these Nash flows over time have, is discussed in this paper.

In the case of one source and multiple sinks with given demands, the two settings presented above are equivalent: given a multi-origin-destination instance with one source but n commodities, we can construct an equivalent multi-source multi-sink instance by setting the inflow at the source to $r := r_1 + \cdots + r_n$ and the demand at sink t_j to $d_j := r_j/r$. It is easy to see that these settings are also equivalent in the case of multiple sources and one sink.
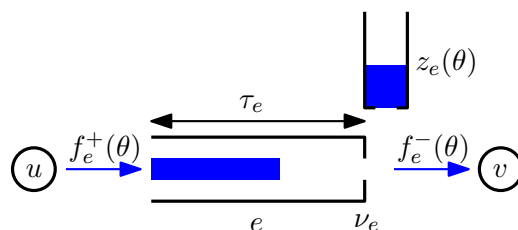
Figure 3 A snapshot of arc $e = uv$ at time θ, with transit time τ_e, capacity ν_e, inflowrate $f_e^+(\theta)$, outflowrate $f_e^-(\theta)$, and queue $z_e(\theta)$.

3 Flow dynamics

In this section we present all necessary definitions of a *fluid queuing network*. The model is a modified version used by Koch and Skutella [10] and Cominetti et al. [1, 2, 3] that matches the multi-source multi-sink setting.

Throughout this paper we consider a directed graph $G = (V, E)$ with transit times $\tau_e \geq 0$ and capacities $\nu_e > 0$ on every arc $e \in E$, a set of $n \geq 1$ sources $S^+ = \{ s_1, \ldots, s_n \} \subseteq V$ with inflow rates $r_1, \ldots, r_n > 0$, and a set of $m \geq 1$ sinks $S^- = \{ t_1, \ldots, t_m \} \subseteq V$. The corresponding demands will be introduced in Section 5. We assume that every node is reachable by a source and can itself reach at least one sink. Furthermore, we assume that the sum of transit times along every directed cycle is positive.

Flows over time. A flow over time is specified by locally integrable and bounded functions $f_e^+ : [0, \infty) \to [0, \infty)$ for every arc e. These *inflow functions* describe the rate of flow entering the arcs for every point in time $\theta \in [0, \infty)$. We set $f_e^+(\theta) := 0$ for $\theta < 0$.

For every arc e there is a bottleneck given by its capacity ν_e at the head of the arc.[1] When flow enters e it immediately starts to traverse this arc, which takes τ_e time. If the rate of flow trying to leave e exceeds the capacity ν_e, the flow builds up a queue in front of the bottleneck which is described by a function $z_e : [0, \infty) \to [0, \infty)$. Note that the queue does not have any physical dimension in the network, and is therefore called *point queue*. Whenever there is a positive queue the outflow rate operates at capacity rate ν_e. This leads to the following evolution of the queue starting with $z(0) = 0$,

$$z_e'(\theta) := \begin{cases} f_e^+(\theta - \tau_e) - \nu_e & \text{if } z_e(\theta) > 0 \\ \max \{ f_e^+(\theta - \tau_e) - \nu_e, 0 \} & \text{if } z_e(\theta) = 0. \end{cases} \tag{1}$$

This determines a unique queue function z_e [2], which is characterized later on. The outflow rate function $f_e^- : [0, \infty) \to [0, \infty)$ is defined by

$$f_e^-(\theta) := \begin{cases} \nu_e & \text{if } z_e(\theta) > 0, \\ \min \{ f_e^+(\theta - \tau_e), \nu_e \} & \text{if } z_e(\theta) = 0. \end{cases} \tag{2}$$

A *flow over time* is given by a family of inflow functions $(f_e^+)_{e \in E}$ that *conserve flow* at every $v \in V \setminus S^-$, which means that the following equation holds for almost all $\theta \in [0, \infty)$:

$$\sum_{e \in \delta^+(v)} f_e^+(\theta) - \sum_{e \in \delta^-(v)} f_e^-(\theta) = \begin{cases} 0 & \text{if } v \in V \setminus S^+, \\ r_i & \text{if } v = s_i \in S^+. \end{cases} \tag{3}$$

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[1] The dynamics are exactly the same if the bottleneck is located at the tail of the arc or anywhere between tail and head.

This ensures that the network does not leak at intermediate vertices and that the amount of flow entering through source s_i matches the inflow rate r_i.

The *cumulative in- and outflow* of an arc e is the total amount of flow that has entered or left e up to some point in time θ and is defined by $F_e^+(\theta) := \int_0^\theta f_e^+(\xi)\,d\xi$ and $F_e^-(\theta) := \int_0^\theta f_e^-(\xi)\,d\xi$. The amount of flow in the queue of an arc e at time θ equals the difference between the amount of flow that has entered the queue before time θ and the flow that has left the queue up to this point in time. The former can be described by the amount of flow that has entered arc e at time $\theta - \tau_e$. In short, $z_e(\theta) = F_e^+(\theta - \tau_e) - F_e^-(\theta)$; see Lemma 14 in A.1. Since f_e^+ and f_e^- are bounded, the functions F_e^+, F_e^-, and z_e are Lipschitz continuous, and therefore almost everywhere differentiable due to Rademacher's theorem [11]. Considering that $f_e^+(\theta)$ and $f_e^-(\theta)$ are non-negative and $z_e'(\theta) \geq -\nu_e$ for all θ it follows that F_e^+ and F_e^- are non-decreasing and z_e cannot decrease faster than with slope $-\nu_e$.

We identify the flow with the non-negative reals $\mathbb{R}_{\geq 0}$, that is, each $\phi \in \mathbb{R}_{\geq 0}$ corresponds to an infinitesimally small flow particle. The natural ordering $\leq$ corresponds to the priority among the flow particles when entering the network, i.e., particle ϕ has priority over all $\phi' > \phi$. Consequently, all flow that wants to enter the network through the same source does this in order of priority. Note that the flow represented by the non-negative reals has a width of 1. That is, there is exactly one unit of flow associated with every unit interval $[a, a+1] \subseteq \mathbb{R}_{\geq 0}$. To distinguish between flow and time we write $\mathbb{R}_{\geq 0}$ for the ordered set of flow particles, mostly denoted by ϕ or φ, and $[0, \infty)$ for the time whose elements are *points in time*, often denoted by θ or ϑ.

A family of locally integrable functions $f_i \colon \mathbb{R}_{\geq 0} \to [0, 1]$, for $i = 1, \dots, n$, is called *inflow distribution* if $\sum_{i=1}^n f_i(\phi) = 1$ for almost all $\phi \in \mathbb{R}_{\geq 0}$ and if each *cumulative source inflow* $F_i(\phi) := \int_0^\phi f_i(\varphi)\,d\varphi$ is unbounded for $\phi \to \infty$. The function $f_i(\phi)$ describes the fraction of particle ϕ that enters the network trough s_i. The cumulative source inflow functions have to be unbounded in order to guarantee that the inflow rates at the sources never run dry.

Current shortest paths network.   Given a flow over time $(f_e^+)_{e \in E}$ together with an inflow distribution $(f_i)_{i=1}^n$, the *arc travel time* for arc e is the function $T_e \colon [0, \infty) \to [0, \infty)$ that maps the entrance time θ to the exit time $T_e(\theta)$. More precisely, if a particle enters e at time θ, it traverses the arc first, which takes τ_e time, and then queues up and has to wait in line for $z_e(\theta + \tau_e)/\nu_e$ time units. Hence, $T_e(\theta) := \theta + \tau_e + z_e(\theta + \tau_e)/\nu_e$. We require the flow to satisfy the *first in first out (FIFO) condition* on every arc, that is, no particle can overtake other flow on an arc or in a queue. Suppose flow particle ϕ enters e at time θ, then the amount of flow which has entered e before ϕ is exactly the amount of flow that leaves e before time $T_e(\theta)$ when ϕ leaves the arc. In short $F_e^+(\theta) = F_e^-(T_e(\theta))$; see Lemma 14 in A.1.

For every $i = 1, \dots, n$, the *source arrival time function* maps each particle $\phi \in \mathbb{R}_{\geq 0}$ to the time it arrives at s_i and is given by $T_i(\phi) := F_i(\phi)/r_i$. Given an s_i-v path $P = (e_1, e_2, \dots, e_k)$ the *arrival time function* $T_P \colon \mathbb{R}_{\geq 0} \to [0, \infty)$ maps the particle ϕ to the time at which ϕ arrives at v if it traverses the path P, i.e., $T_P(\phi) := T_{e_k} \circ T_{e_{k-1}} \circ \cdots \circ T_{e_1} \circ T_i(\phi)$. Since the functions z_e and F_i are Lipschitz continuous, the same holds for T_e, T_i, and T_P. Note that the queue length z_e cannot decrease faster than with slope $-\nu_e$ and, therefore all these T-functions are nondecreasing. Furthermore, all T-functions go to infinity for $\phi \to \infty$ since the queue lengths z_e are non-negative and the F_i are unbounded.

The *earliest arrival time function* $\ell_v \colon \mathbb{R}_{\geq 0} \to [0, \infty)$ of node $v \in V$ maps each particle ϕ to the earliest time $\ell_v(\phi)$ it can possibly reach node v. We have $\ell_v(\phi) = \min_{P \in \mathcal{P}_v} T_P(\phi)$,

where $\mathcal{P}_v$ is the set of all paths from some source $s \in S^+$ to v. Note that these node-labels are also Lipschitz continuous, nondecreasing, and unbounded and that they are the unique solutions to the following Bellman equations:

$$\begin{aligned}
\ell_{s_i}(\phi) &= \min\left(\{\,T_i(\phi)\,\} \cup \{\,T_e(\ell_u(\phi)) \mid e = us_i \in E\,\}\right) & \text{for } i = 1, \dots, n, \\
\ell_v(\phi) &= \min_{e = uv \in E} T_e(\ell_u(\phi)) & \text{for } v \in V \setminus S^+.
\end{aligned} \tag{4}$$

This is well defined since all cycles in G have by assumption positive travel times.

For a fixed particle ϕ we call an arc $e = uv$ *active for ϕ* if $\ell_v(\phi) = T_e(\ell_u(\phi))$ holds. With E'_ϕ we denote the set of all active arcs for particle ϕ and the subgraph $G'_\phi = (V, E'_\phi)$ is called the *current shortest paths network*. Note that the current shortest paths network is always acyclic since the sum of transit times of each directed cycle is positive.

4   Multi-source single-sink Nash flows over time

For this section we only consider fluid queuing networks with exactly one sink t. A flow over time together with an inflow distribution corresponds to a strategy profile, where the strategy of each particle consists of a convex combination of S^+-t-paths. The following definition characterizes a Nash equilibrium.

▶ **Definition 1** (Nash flow over time). A tuple $f = ((f_e^+)_{e \in E}, (f_i)_{i=1}^n)$ consisting of a flow over time and an inflow distribution is a *Nash flow over time*, also called *dynamic equilibrium*, if the following two *Nash flow conditions* hold:

$$\begin{aligned}
\ell_{s_i}(\phi) &= T_i(\phi) & \text{for all } i = 1, \dots, n \text{ and almost all } \phi \in \mathbb{R}_{\geq 0}, & \quad\text{(N1)} \\
f_e^+(\theta) > 0 &\implies \theta \in \ell_u(\Phi_e) & \text{for all arcs } e = uv \in E \text{ and almost all } \theta \in [0, \infty), & \quad\text{(N2)}
\end{aligned}$$

where $\Phi_e := \{\,\phi \in \mathbb{R}_{\geq 0} \mid e \in E'_\phi\,\}$ is the set of flow particles for which arc e is active.

Figuratively speaking, these two conditions mean, that entering the network through a source s_i is always a fastest way to reach s_i (N1) and that a Nash flow over time uses only active arcs (N2), and therefore only shortest paths to t. More precisely, particle ϕ reaches t at time $\ell_t(\phi)$ by using active arcs only, and $\ell_t(\phi)$ is the earliest time ϕ can possibly reach t under the assumption that the routes of all previous particles $\varphi < \phi$ are fixed. Since this is true for all particles, a Nash flow over time is indeed a Nash equilibrium.

▶ **Lemma 2.** *A tuple $f = ((f_e^+)_{e \in E}, (f_i)_{i=1}^n)$ of a flow over time and an inflow distribution is a Nash flow over time if, and only if, we have $F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$ and $F_i(\phi) = \ell_{s_i}(\phi) \cdot r_i$ for all arcs $e = uv \in E$, every $i = 1, \dots, n$, and all particles $\phi \in \mathbb{R}_{\geq 0}$.*

Proof. "$\Rightarrow$": Let $\xi \in [0, \phi]$ be the particle of largest value with $F_e^+(\ell_u(\xi)) = F_e^-(\ell_v(\phi))$. This ξ exists because of the intermediate value theorem, together with the fact that $F_e^+ \circ \ell_u$ is continuous and the following inequality, which follows by the monotonicity of F_e^- and Lemma 14:

$$F_e^+(\ell_u(0)) = 0 \quad \leq \quad F_e^-(\ell_v(\phi)) \quad \leq \quad F_e^-(T_e(\ell_u(\phi))) = F_e^+(\ell_u(\phi)).$$

Note that the second inequality is true because of $\ell_v(\phi) \leq T_e(\ell_u(\phi))$. In the case of $\xi = \phi$ we are done since $F_e^-(\ell_v(\phi)) = F_e^+(\ell_u(\xi)) = F_e^+(\ell_u(\phi))$. Suppose $\xi < \phi$. For all particles $\varphi \in (\xi, \phi]$ we know that $T_e(\ell_u(\varphi)) \neq \ell_v(\phi)$ because, otherwise, we had with Lemma 14 (ii) that $F_e^+(\ell_u(\varphi)) = F_e^-(T_e(\ell_u(\varphi))) = F_e^-(\ell_v(\phi))$ which would contradict the maximality

of ξ. Hence, e is not active for particles in $(\xi, \phi]$ which implies $f_e^+(\theta) = 0$ for almost all $\theta \in \ell_u((\xi, \phi]) = (\ell_u(\xi), \ell_u(\phi)]$ since f is a Nash flow over time. This leads to

$$F_e^+(\ell_u(\phi)) - F_e^-(\ell_v(\phi)) = F_e^+(\ell_u(\phi)) - F_e^+(\ell_u(\xi)) = \int_{\ell_u(\xi)}^{\ell_u(\phi)} f_e^+(\vartheta)\, \mathrm{d}\vartheta = 0,$$

which finishes the first part. The second part follows directly from $\ell_{s_i}(\phi) = T_i(\phi) = F_i(\phi)/r_i$ for all $i = 1, \dots, n$.

"$\Leftarrow$": Given a particle ϕ and an arc $e = uv$ such that e is not active for ϕ, i.e., $\ell_v(\phi) < T_e(\ell_u(\phi))$. The continuity of ℓ_v and $T_e \circ \ell_u$ implies that there is an $\varepsilon > 0$ with $\ell_v(\phi + \varepsilon) < T_e(\ell_u(\phi - \varepsilon))$ and e is not active for all particles in $[\phi - \varepsilon, \phi + \varepsilon]$. This, the fact that f_e^+ and f_e^- are non-negative, and Lemma 14 gives us

$$
\begin{aligned}
0 \;\leq\; & \int_{\ell_u(\phi-\varepsilon)}^{\ell_u(\phi+\varepsilon)} f_e^+(\vartheta)\, \mathrm{d}\vartheta = \int_{T_e(\ell_u(\phi-\varepsilon))}^{T_e(\ell_u(\phi+\varepsilon))} f_e^-(\vartheta)\, \mathrm{d}\vartheta \leq \int_{\ell_v(\phi+\varepsilon)}^{T_e(\ell_u(\phi+\varepsilon))} f_e^-(\vartheta)\, \mathrm{d}\vartheta \\
\;=\; & F_e^-(T_e(\ell_u(\phi + \varepsilon))) - F_e^-(\ell_v(\phi + \varepsilon)) = F_e^+(\ell_u(\phi + \varepsilon)) - F_e^-(\ell_v(\phi + \varepsilon)) \overset{(ii)}{=} 0.
\end{aligned}
$$

Hence, $f_e^+(\theta) = 0$ for almost all $\theta \in [\ell_u(\phi - \varepsilon), \ell_u(\phi + \varepsilon)]$. In other words, for almost all $\theta \in [0, \infty)$ it holds that $\theta \notin \ell_u(\Phi_e) \Rightarrow f_e^+(\theta) = 0$. This is true because for $\theta \geq \ell_u(0)$ we find a particle ϕ with $\ell_u(\phi) = \theta$, due to the fact, that ℓ_u is continuous and unbounded, and for all $\theta < \ell_u(0)$ we have $f_e^+(\theta) = 0$, since no flow can reach u faster than $\ell_u(0)$. Finally, we get $\ell_{s_i}(\phi) = T_i(\phi)$ since $\ell_{s_i}(\phi) \cdot r_i = F_i(\phi) = T_i(\phi) \cdot r_i$ for all $i = 1, \dots, n$. This shows that f is a Nash flow over time, which finishes the proof.   ◀

Lemma 2 motivates to consider the *underlying static flow* for every particle ϕ, which is defined by $x_e(\phi) := F_e^+(\ell_u(\phi)) = F_e^-(\ell_v(\phi))$ and $x_i(\phi) := F_i(\phi) = \ell_{s_i}(\phi) \cdot r_i$. For a fixed ϕ this is indeed a static S^+-t-flow since the integral of (3) over $[0, \ell_v(\phi)]$ yields

$$
\sum_{e \in \delta^+(v)} x_e(\phi) - \sum_{e \in \delta^-(v)} x_e(\phi) = \begin{cases} 0 & \text{if } v \in V \setminus (S^+ \cup \{t\}), \\ \ell_{s_i}(\phi) \cdot r_i = x_i(\phi) & \text{if } v = s_i \in S^+. \end{cases} \tag{5}
$$

Let x_e', x_i', and ℓ_v' denote the derivative functions, which exist almost everywhere, since the x- and ℓ-functions are Lipschitz continuous. It is possible to determine the inflow function of every arc $e = uv$ from these derivatives, since $x_e'(\phi) = f_e^+(\ell_u(\phi)) \cdot \ell_u'(\phi)$. Moreover, the inflow distribution is given by $f_i(\phi) = \ell_{s_i}'(\phi) \cdot r_i$. Consequently, a Nash flow over time is completely characterized by these derivatives. Note that differentiating (5) yields that $x'(\phi)$ also forms a static S^+-t-flow, which has very specific properties that are characterized in the following.

Thin flows with resetting for multiple sources and a single sink.   A thin flow with resetting is a static flow defined on a subgraph of G characterizing the strategies of particles in a flow interval of a Nash flow over time. The definition of thin flows with resetting given in this article generalizes the thin flows with resetting introduced in [10] and the normalized thin flows with resetting from [2], in order to suit the multi-source setting.

Let $E' \subseteq E$ be a subset of arcs such that the subgraph $G' = (V, E')$ is acyclic and every node is reachable by a source within G'. Note that not every node needs to be able to reach sink t. Additionally, we consider a subset of arcs $E^* \subseteq E'$, called *resetting arcs*. Moreover, let $K(E', x_1', \dots, x_n')$ be the set of all static S^+-t-flows in G' with inflow x_i' at source s_i for $x_i' \geq 0$ and $x_1' + \cdots + x_n' = 1$.

▶ **Definition 3** (Thin flow with resetting). A vector $(x_i')_{i=1}^n$, with $x_i' \geq 0$ and $x_1' + \cdots + x_n' = 1$, and a static flow $(x_e')_{e \in E} \in K(E', x_1', \ldots, x_n')$ together with a node labeling $(\ell_v')_{v \in V}$ is called *thin flow with resetting* on $E^* \subseteq E'$ if:

$$\ell_{s_i}' = x_i'/r_i \qquad\qquad \text{for all } i = 1, \ldots, n, \tag{TF1}$$

$$\ell_{s_i}' \leq \min_{e = us_i \in E'} \rho_e(\ell_u', x_e') \qquad \text{for all } i = 1, \ldots, n, \tag{TF2}$$

$$\ell_v' = \min_{e = uv \in E'} \rho_e(\ell_u', x_e') \qquad \text{for all } v \in V \backslash S^+, \tag{TF3}$$

$$\ell_v' = \rho_e(\ell_u', x_e') \qquad\qquad \text{for all } e = uv \in E' \text{ with } x_e' > 0, \tag{TF4}$$

$$\text{where} \qquad \rho_e(\ell_u', x_e') := \begin{cases} x_e'/\nu_e & \text{if } e = uv \in E^*, \\ \max\{\ell_u', x_e'/\nu_e\} & \text{if } e = uv \in E' \backslash E^*. \end{cases}$$

The next theorem states that the derivatives of a Nash flow over time f form almost everywhere a thin flow with resetting on the arcs with positive queues. Recall that E_ϕ' is the subset of arcs that are active for ϕ, and let $E_\phi^* := \{\, e = uv \in E \mid z_e(\ell_u(\phi) + \tau_e) > 0 \,\}$ be the set of arcs where the particle ϕ would experience a queue.

▶ **Theorem 4.** *For a Nash flow over time $((f_e^+)_{e \in E}, (f_i)_{i=1}^n)$, the derivative labels $(x_i'(\phi))_{i=1}^n$ and $(x_e'(\phi))_{e \in E_\phi'}$ together with $(\ell_v'(\phi))_{v \in V}$ form a thin flow with resetting on E_ϕ^* in the current shortest paths network $G_\phi' = (V, E_\phi')$, for almost all $\phi \in \mathbb{R}_{\geq 0}$.*

The intuitive idea is that x_e'/ν_e describes the *congestion* of arc e and $\rho_e(\ell_u', x_e')$ is the *congestion* of all paths to v using e. The higher this congestion is, the longer it will take for following particles to reach v, which is captured by a high derivative of the earliest arrival time ℓ_v'. If we have $\ell_v' < \rho_e(\ell_u', x_e')$ this means that e leaves the current shortest paths network, and therefore it cannot be used by following particles, i.e., $x_e' = 0$. A detailed proof is given in A.2.

The reverse of Theorem 4 is also true in the sense that we can use thin flows with resetting to construct a Nash flow over time. For this we first show that there always exists a thin flow with resetting for any acyclic graph and any subset of resetting arcs.

▶ **Theorem 5.** *Consider an acyclic graph $G' = (V, E')$ with sources S^+, sink t, capacities ν_e, and a subset of arcs $E^* \subseteq E'$ and suppose every node is reachable by a source. Then there exists a thin flow $((x_i')_{i=1}^n, (x_e')_{e \in E}, (\ell_v')_{v \in V})$ with resetting on E^*.*

The proof is essentially given in [2] and is only slightly modified to fit the new definition of a thin flow with resetting. The key idea is to use a set-valued function in order to apply the Kakutan'i fixed-point theorem.

Constructing Nash flows. Note that in a dynamic equilibrium no particle can overtake any other particle, and therefore the choice of strategy for ϕ only depends on the strategies of the particles in $[0, \phi)$. So we may assume that the particles decide in order of priority. More precisely, given a Nash flow over time up to some $\phi \in \mathbb{R}_{\geq 0}$, it is possible to extend it by using a thin flow on the G_ϕ' with resetting on E_ϕ^*.

A *restricted Nash flow over time* on $[0, \phi]$ is a Nash flow over time where only the particles in $[0, \phi]$ are considered, i.e., for $i = 1, \ldots, n$ we have $f_i(\varphi) = 0$ for all $\varphi > \phi$ and for each arc $e = uv \in E$ we have $f_e(\ell_u(\theta)) = 0$ for all $\theta > \ell_u(\phi)$. But the Nash flow conditions (N1) and (N2) are satisfied for almost all particles in $[0, \phi]$ and almost all times in $[0, \ell_u(\phi)]$.

Since all previous results carry over to restricted Nash flows over time, the earliest arrival times $(\ell_v)_{v \in V}$ are well-defined for particles in $[0, \phi]$, and therefore it is possible to

determine $G'_\phi = (V, E'_\phi)$ and E^*_ϕ; see Lemma 15 in A.3. To extend a restricted Nash flow over time, we first compute a thin flow on G'_ϕ with resetting on E^*_ϕ, and then extend the labels linearly as follows. For some $\alpha > 0$ we get for all $v \in V$, $e \in E$, $i = 1, \dots, n$, and $\varphi \in (\phi, \phi+\alpha]$ that

$$\ell_v(\varphi) := \ell_v(\phi) + (\varphi - \phi) \cdot \ell'_v \quad \text{and} \quad \begin{aligned} x_e(\varphi) &:= x_e(\phi) + (\varphi - \phi) \cdot x'_e, \\ x_i(\varphi) &:= x_i(\phi) + (\varphi - \phi) \cdot x'_i. \end{aligned}$$

Based on this we can extend the inflow function and the inflow distribution, which gives us

$$f^+_e(\theta) := \frac{x'_e}{\ell'_u} \quad \text{for } \theta \in (\ell_u(\phi), \ell_u(\phi+\alpha)] \qquad \text{and} \qquad f_i(\varphi) := x'_i = \ell'_{s_i} \cdot r_i \quad \text{for } \varphi \in (\phi, \phi+\alpha]$$

for all $e = uv \in E$ and all $i = 1, \dots, n$. Note that in the case of $\ell'_u = 0$ the time interval is empty. Furthermore, it turns out that $f^-_e(\theta) = x'_e/\ell'_v$ for all $\theta \in (\ell_v(\phi), \ell_v(\phi + \alpha)]$, which is formally shown in Lemma 16 in A.4. This extended flow over time together with the extended inflow distribution is called α-extension and it extends the Nash flow over time as long as the α stays within the the following bounds:

$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \geq \tau_e \qquad \text{for all } e = uv \in E^* \qquad (6)$$
$$\ell_v(\phi) - \ell_u(\phi) + \alpha(\ell'_v - \ell'_u) \leq \tau_e \qquad \text{for all } e = uv \in E \backslash E'. \qquad (7)$$

The first inequality ensures that no flow can traverse an arc faster than its transit time. It holds with equality when the queue of e vanishes at time $\ell_u(\phi + \alpha)$. The second inequality makes sure that all non-active arcs are unattractive for all particles in $[\phi, \phi + \alpha)$. When it holds with equality the arc e becomes active for $\phi + \alpha$. When such an event occurs we must compute a new thin flow with resetting because either a resetting arc has become non-resetting or a non-active arc has become active. It is easy to see that there exists an $\alpha > 0$ that satisfies these inequalities since $\ell_v(\phi) > \ell_u(\phi) + \tau_e$ for arcs $e \in E^*_\phi$ and $\ell_v(\phi) < \ell_u(\phi) + \tau_e$ for arcs $e \notin E'_\phi$, as it is stated in Lemma 15 in A.3.

▶ **Lemma 6.** *The α-extension forms a flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., satisfy the Bellman equations (4) for all $\varphi \in (\phi, \phi + \alpha]$.*

The flow conservation follows immediately from the flow conservation of x' and the Bellman equations are shown by distinguishing three cases. If the arc is non-active it stays non-active during the extended interval. For active, but non-resetting arcs that do not build up a queue, we obtain $\ell_v(\phi + \xi) \leq T_e(\ell_u(\phi + \xi))$ from (TF3) with equality if $\ell'_v = \rho_e(\ell'_u, x'_e)$. The same is true for resetting arcs or arcs that build up a queue, even though, the proof is a bit more technical. For a detailed proof we refer to A.5.

▶ **Theorem 7.** *Given a restricted Nash flow over time $((f^+_e)_{e \in E}, (f_i)^n_{i=1})$ on $[0, \phi]$ and $\alpha > 0$ satisfying (6) and (7), the α-extension is a restricted Nash flow over time on $[0, \phi + \alpha]$.*

Proof. We have $\sum^n_{i=1} f_i(\theta) = \sum^n_{i=1} x'_i = 1$ for all $\theta \in (\phi, \phi+\alpha]$, which shows that $(f_i)^n_{i=1}$ is a restricted inflow distribution. Lemma 2 yields $F^+_e(\ell_u(\varphi)) = F^-_e(\ell_v(\varphi))$ and $F_i(\varphi) = \ell_{s_i}(\varphi) \cdot r_i$ for all $\varphi \in [0, \phi]$, so for $\xi \in (0, \alpha]$ it holds that

$$F^+_e(\ell_u(\phi + \xi)) = F^+_e(\ell_u(\phi)) + x'_e/\ell'_u \cdot \xi \cdot \ell'_u = F^-_e(\ell_v(\phi)) + x'_e/\ell'_v \cdot \xi \cdot \ell'_v = F^-_e(\ell_v(\phi + \xi)),$$
$$F_i(\phi + \xi) = F_i(\phi) + \xi \cdot x'_i = \ell_{s_i}(\phi) \cdot r_i + \xi \cdot \ell'_{s_i} \cdot r_i = \ell_{s_i}(\phi + \xi) \cdot r_i.$$

Again with Lemma 2 we have that the α-extension is a restricted Nash flow on $[0, \phi + \alpha]$.   ◀

Finally, we show that this construction leads to a Nash flow over time.

▶ **Theorem 8.** *There exists a Nash flow over time with multiple sources and a single sink.*

Proof. In the first part we show that these α-extensions lead to a restricted Nash flow on $[0, \infty)$. In the second part we prove, that all cumulative source inflow functions are unbounded, which shows that we have, indeed, a Nash flow over time.

The process starts with the empty flow over time and the zero flow distribution, i.e., a restricted Nash flow over time for $[0, 0]$. By applying Theorem 7 iteratively and choosing α maximal according to (6) and (7), we obtain a sequence of restricted Nash flows over time f^i for $[0, \phi_i]$ for $i = 1, 2, \ldots$, with strictly increasing $(\phi_i)_{i=1}^\infty$. In the case that this sequence has a finite limit, say ϕ_∞, we define a restricted Nash flow over time f^∞ for $[0, \phi_\infty]$ by using the point-wise limit of the x- and ℓ-labels, which exists due to monotonicity and Lipschitz continuity of these functions. Then the process can be restarted from this limit point.

Let $\mathcal{P}_G$ be the set of all particles $\phi \in \mathbb{R}_{\geq 0}$ for which there exists a restricted Nash flow over time on $[0, \phi]$ constructed as described above. The set $\mathcal{P}_G$ cannot have a maximal element because this could be extended by using Theorem 7. But it also cannot have an upper bound since the limit of any convergent sequence would be contained in this set. Therefore, there exists an unbounded increasing sequence $(\phi_i)_{i=1}^\infty \in \mathcal{P}_G$. From the corresponding restricted Nash flows over time we can construct the restricted Nash flow over time f on $[0, \infty)$ by taking the point-wise limit of the x- and ℓ-labels.

It remains to show that the inflow distribution of this restricted Nash flow over time is unbounded. For this we first show that the earliest arrival time ℓ_t is unbounded. There cannot be an upper bound B on ℓ_t since the flow rate into t is bounded by $N := \sum_{e \in \delta^-(t)} \nu_e$ and with the FIFO principle we obtain that no particle $\phi > N \cdot B$ reaches t before time $\phi/N > B$. Next, we show that all ℓ-labels are unbounded. Suppose this is not true. Since every node can reach t there would be an arc $e = uv$, where ℓ_u is bounded and ℓ_v is not. Since T_e is Lipschitz continuous $T_e \circ \ell_u$ would be bounded as well. But this contradicts that $\ell_v(\phi) \leq T_e(\ell_u(\phi))$ goes to infinity for $\phi \to \infty$. Hence, $F_i(\phi) = \ell_{s_i}(\phi) \cdot r_i$ is unbounded for every $i = 1, \ldots, n$, which completes the proof. ◀

5 Multiple sinks with demands

In this section we consider a graph $G = (V, E)$ as before except that it can have multiple sinks $S^- := \{ t_1, \ldots, t_m \}$ and demands $d_1, \ldots, d_m > 0$ with $d_1 + \cdots + d_m = 1$. We show how to construct a Nash flow over time in G where a share of d_j of the flow has t_j as destination.

Sub-flow over time decomposition. In the following we define a sub-flow over time, which is, intuitively, a colored proportion of a flow over time satisfying flow conservation. Given a flow over time $f = (f_e^+)_{e \in E}$ with queue functions $(z_e)_{e \in E}$, we consider a family of locally integrable and bounded inflow functions $g = (g_e^+)_{e \in E}$ with $g_e^+(\theta) \leq f_e^+(\theta)$ for almost all $\theta \in [0, \infty)$. The corresponding outflow functions are obtained by the following consideration. For a point in time $\vartheta \in [0, \infty)$ let $T_e^{-1}(\vartheta)$ be all times at which a particle could enter e in order to leave it at time ϑ. Whenever $T_e^{-1}(\vartheta)$ is not a singleton it is a proper interval and by (11) we have that $f_e^+(\theta) = 0$ for almost all $\theta \in T_e^{-1}(\vartheta)$. The *sub-outflow function* for arc $e \in E$ is defined as

$$g_e^-(\vartheta) := \begin{cases} f_e^-(\vartheta) \cdot \frac{g_e^+(\theta)}{f_e^+(\theta)} & \text{if } f_e^+(\theta) > 0 \text{ and } T_e^{-1}(\vartheta) = \{ \theta \}, \\ 0 & \text{else.} \end{cases} \tag{8}$$

In other words, if $g_e^+(\theta)/f_e^+(\theta) \in [0, 1]$ is the inflow share of g at time θ, then the outflow share of g has the same value at time $T_e(\theta)$. We call $g = (g_e^+)_{e \in E}$ a *sub-flow over time* of f if for every $v \in V \setminus S^+$ and almost all $\theta \in [0, \infty)$ we have

$$\sum_{e \in \delta^-(v)} g_e^-(\theta) - \sum_{e \in \delta^+(v)} g_e^+(\theta) \leq \sum_{e \in \delta^-(v)} f_e^-(\theta) - \sum_{e \in \delta^+(v)} f_e^+(\theta). \tag{9}$$

Intuitively, this means that at every non-source node v the sub-flow over time g can at most "lose" as much flow as f does. Furthermore, we say g *conserves flow at node* $v \in V \setminus S^+$ if $\sum_{e \in \delta^-(v)} g_e^-(\theta) - \sum_{e \in \delta^+(v)} g_e^+(\theta) = 0$ holds for almost all $\theta \in [0, \infty)$. Note that if f conserves flow at some node v, then g does so as well. We say g is an S^+-t_j-*sub-flow over time* if it conserves flow at all nodes in $V \setminus \{ t_j \}$.

Given an inflow distribution $(f_i)_{i=1}^n$ and a number $\gamma \in [0, 1]$, a family of locally integrable functions $(g_i)_{i=1}^n$ with $g_i(\phi) \leq f_i(\phi)$ is called *sub-inflow distribution of value* γ if we have $\sum_{i=1}^n g_i(\phi) = \gamma$ for almost all $\phi \in \mathbb{R}_{\geq 0}$. To ensure that sub-flow is conserved at the sources we require the net flow leaving a source s_i at time $T_i(\phi)$ to be equal to the amount of flow distributed to s_i at time $T_i(\phi)$, which is $r_i \cdot g_i(\phi)/f_i(\phi) = g_i(\phi)/T_i'(\phi)$, whenever $f_i(\phi) > 0$ and 0 otherwise. More precisely, we say a sub-inflow distribution *matches* a sub-flow over time if for almost all $\phi \in \mathbb{R}_{\geq 0}$ and all $i = 1, \ldots, n$ we have

$$T_i'(\phi) \cdot \left(\sum_{e \in \delta^+(s_i)} g_e^+(T_i(\phi)) - \sum_{e \in \delta^-(s_i)} g_e^-(T_i(\phi)) \right) = g_i(\phi).$$

In this case we also say that the sub-flow over time *conserves flow* at s_i and that the sub-flow over time g *has value* γ.

▶ **Definition 9** (Sub-flow over time decomposition). A family of sub-flows over time $(g_e^{j+})_{e \in E}$ and matching sub-inflow distributions $(g_i^j)_{i=1}^n$ of value γ_j, for $j = 1, \ldots, m$, is called a *sub-flow over time decomposition* of f with values $\gamma_1, \ldots, \gamma_m$ if $\sum_{j=1}^m \gamma_j = 1$ and

$$g_e^{1+}(\theta) + \cdots + g_e^{m+}(\theta) = f_e^+(\theta) \quad \text{for all } e \in E \text{ and almost all } \theta \in [0, \infty).$$

Note that (8) implies $\sum_{j=1}^m g_e^{j-}(\xi) = f_e^-(\xi)$ for all $e \in E$ and almost all $\xi \in [0, \infty)$.

Nash flows over time with multiple sinks and demands. These sub-flow over time decompositions allow us to formalize Nash flows over time in the setting of multiple sinks with demands. Note that for the sake of clarity we omit the $+$ and simply write f_e and g_e^j for the inflow functions for the remaining of this paper.

▶ **Definition 10** (Nash flow over time with demands). A tuple $f = ((f_e)_{e \in E}, (f_i)_{i=1}^n)$ consisting of a flow over time and an inflow distribution is a *Nash flow over time with demands* $d_1, \ldots, d_m$ if it satisfies the Nash flow conditions (N1) and (N2) from Definition 1 and, furthermore, has a sub-flow over time decomposition $((g_e^j)_{e \in E}, (g_i^j)_{i=1}^n)_{j=1}^m$, such that $(g_e^j)_{e \in E}$ is an S^+-t_j-sub-flow over time of value d_j for all $j = 1, \ldots, m$.

To construct a Nash flow over time with demands we add a super sink t to the graph and use a single-sink Nash flow over time as constructed in Section 4. For this let $\nu_{\min} := \min_{e \in E} \nu_e$ and $r_{\min} := \min_{i=1,\ldots,n} r_i$ be the minimal capacity/inflow rate and $\sigma := \min \{ \nu_{\min}, r_{\min} \}$. For all $j = 1, \ldots, m$ we define $\delta_j := \min_{s \in S^+} d(s, t_j)$, where $d(s, t_j)$ is the length of a shortest s-t_j-path according to the transit times. Furthermore, let $\delta_{\max} := \max_{j=1,\ldots,m} \delta_j$ be the

maximal distance to some sink t_j from its nearest source. We extend G by a super sink t and m new arcs $e_j := (t_j, t)$ with

$$\tau_{e_j} := \delta_{\max} - \delta_j \quad \text{and} \quad \nu_{e_j} := 1/2 \cdot d_j \cdot \sigma. \tag{10}$$

We denote the *extended graph* by $\bar{G} := (\bar{V}, \bar{E})$ with $\bar{V} := V \cup \{\, t \,\}$ and $\bar{E} := E \cup \{\, e_1, \ldots, e_m \,\}$.

Note, that the new capacities are strictly smaller than all original capacities and all inflow rates and that they are proportional to the demands. Furthermore, we choose the transit times such that all new arcs are in the current shortest paths network for particle $\phi = 0$. The reason for the choice of σ is that for every thin flow with resetting (x', ℓ') in G we have $\ell'_v \leq 1/\sigma$ for all $v \in V$, which is shown in Lemma 17 in A.6.

We obtain a Nash flow over time with demands f by using a single-sink Nash flow over time $\bar{f}$ in $\bar{G}$, which exists due to Theorem 8. To prove this we first show that if all new arcs are active for some particle ϕ then there is a static flow decomposition of the thin flow with resetting x' with $x'_{e_j} = d_j$. This is formalized in the following lemma, where we write $x'\big|_E$ for the restriction of x' to the original graph G and $|\cdot|$ for the flow value of a static flow.

▶ **Lemma 11.** *Consider a thin flow with resetting (x', ℓ') in $\bar{G}$ where $\{\, e_1, \ldots, e_m \,\} \subseteq E'$, then there exists a static flow decomposition $x'\big|_E = x'^1 + \cdots + x'^m$ such that each static flow x'^j conserves flow on all $v \in V \setminus (S^+ \cup \{\, t_j \,\})$ and $\big|x'^j\big| = d_j$ for $j = 1, \ldots, m$.*

Proof. Let $\mathcal{P}$ be the set of all S^+-t-paths in the current shortest paths network $G' = (V, E')$. Note, that G' is always acyclic and x' can, therefore, be described by the path vector $(x'_P)_{P \in \mathcal{P}}$ due to the well-known flow decomposition theorem. For $j = 1, \ldots, m$ let $\mathcal{P}_j$ be the set of all S^+-t-paths that contain e_j. These sets form a partition of $\mathcal{P}$ since every path has to use exactly one of the new arcs. By setting $x'^j := \sum_{P \in \mathcal{P}_j} x'_P\big|_E$ we obtain the desired decomposition of x', because $x'_P\big|_E$ for $P \in \mathcal{P}_j$ conserves flow on all nodes except the ones in $S^+ \cup \{\, t_j \,\}$ and the same is true for their sums.

Since x'^j sends $\big|x'^j\big|$ flow units from S^+ over e_j to t_j we have $\big|x'^j\big| = x'_{e_j}$. It remains to show that $x'_{e_j} = d_j$ for all $j = 1, \ldots, m$. Suppose this is not true. Since x' sends exactly $d_1 + \cdots + d_m = 1$ flow units from S^+ to t, there has to be an index $a \in \{\, 1, \ldots, m \,\}$ with $x'_{e_a} > d_a$ and an index $b \in \{\, 1, \ldots, m \,\}$ with $x'_{e_b} < d_b$.

With Lemma 17 it follows that

$$\ell'_{t_b} \leq \frac{1}{\sigma} \overset{(10)}{<} \frac{d_a}{\nu_{e_a}} < \frac{x'_{e_a}}{\nu_{e_a}} \overset{(\text{TF4})}{\leq} \ell'_t \quad \text{as well as} \quad \frac{x'_{e_b}}{\nu_{e_b}} \overset{(10)}{=} \underbrace{\frac{x'_{e_b}}{d_b}}_{<1} \cdot \frac{2}{\sigma} < \underbrace{\frac{x'_{e_a}}{d_a}}_{>1} \cdot \frac{2}{\sigma} \overset{(10)}{=} \frac{x'_{e_a}}{\nu_{e_a}} \overset{(\text{TF4})}{\leq} \ell'_t.$$

But this is a contradiction, because (TF3) yields that $\ell'_t = \min_{j=1,\ldots,m} \rho_{e_j}(\ell'_{t_j}, x'_{e_j})$ and the last two equations show $\rho_{e_b}(\ell'_{t_b}, x'_{e_b}) < \ell'_t$. Hence, we have $\big|x'^j\big| = d_j$ for all $j = 1, \ldots, m$, which finishes the proof. ◀

▶ **Lemma 12.** *In a Nash flow over time $\bar{f}$ in $\bar{G}$ the new arcs $e_1, \ldots, e_m$ are active for all particles $\phi \in \mathbb{R}_{\geq 0}$.*

Proof. For particle $\phi = 0$ there are no queues yet, and therefore the exit time for each arc e is $T_e(\theta) = \theta + \tau_e$. Hence, $\ell_{t_j}(0) = \delta_j$ for all $j = 1, \ldots, m$ and by construction we have $\ell_t(0) = \ell_{t_j}(0) + \tau_{e_j} = T_{e_j}(\ell_{t_j}(0))$ for $j = 1, \ldots, m$. Therefore, all arcs e_j are active in the beginning and also during the first thin flow phase because by Lemma 11 we have $x'_{e_j} > 0$ for the first thin flow with resetting which implies that e_j stays active.

Suppose now for contradiction that there are particles for which not all new arcs are active. Let ϕ_0 be the infimum of these particles. By the consideration above we have $\phi_0 > 0$ and Lemmas 17 and 11 imply

$$f_{e_j}^+(\ell_{t_j}(\phi)) = \frac{x'_{e_j}}{\ell'_{t_j}} \geq x'_{e_j} \cdot \sigma = d_j \cdot \sigma \overset{(10)}{>} \nu_{e_j}$$

for almost all $\phi \in [0, \phi_0)$ and all $j = 1, \ldots, m$. Hence, (1) yields $z'_{e_j}(\ell_{t_j}(\phi) + \tau_{e_j}) = f_{e_j}^+(\ell_{t_j}(\phi)) - \nu_{e_j} > 0$ and, together with the fact that $\ell'_{t_j} > 0$ (due to the positive throughput of x' at t_j), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\phi} z_{e_j}(\ell_{t_j}(\phi) + \tau_{e_j}) = z'_{e_j}(\ell_{t_j}(\phi) + \tau_{e_j}) \cdot \ell'_{t_j} > 0.$$

In other words, a queue is building up within $[0, \phi_0)$, and therefore $z_{e_j}(\ell_{t_j}(\phi_0) + \tau_{e_j}) > 0$ for all $j = 1, \ldots, m$. But the continuity of z_{e_j} implies that there will be positive queues for all $\phi \in [\phi_0, \phi_0 + \varepsilon]$ for sufficiently small $\varepsilon > 0$. By Lemma 15 this implies that all new arcs are active during this interval contradicting that ϕ_0 is an infimum. ◄

By means of the previous lemmas we can finally prove that the Nash flow over time in $\bar{G}$ induces a Nash flow over time with demands in G.

▶ **Theorem 13.** *Let $\bar{f}$ be a S^+-t-Nash flow over time in $\bar{G}$. The flow over time $f := \bar{f}\big|_E$ on the original network together with the inflow distribution of $\bar{f}$ is a Nash flow over time with demands $d_1, \ldots, d_m$.*

Proof. We have to show that the thin flow decompositions of the particles in $\mathbb{R}_{\geq 0}$ correspond to a sub-flow over time decomposition of the Nash flow over time. Throughout this proof we denote $\delta^-(v)$ and $\delta^+(v)$ for the in- and out-going arcs of v within the original network G. Let $I := [a, b)$ be an interval such that the thin flow with resetting is constant (x', ℓ') for all particles in I. For every node v we denote by $I_v := [\ell_v(a), \ell_v(b))$ the interval of local times of particles in I. By Lemma 12 all new arcs $e_1, \ldots, e_m$ are active. Let $x'^1, \ldots, x'^m$ be the thin flow decomposition given by Lemma 11. The corresponding decomposition for the Nash flow over time with demands is constructed by setting

$$g_e^j(\theta) := \frac{x_e'^j}{\ell'_u} \qquad\qquad \text{for } \theta \in I_u$$

$$g_i^j(\phi) := \sum_{e \in \delta^+(s_i)} x_e'^j - \sum_{e \in \delta^-(s_i)} x_e'^j \qquad \text{for } \varphi \in I$$

for all $j = 1, \ldots, m$, every $e = uv \in E$, and all $i = 1, \ldots, n$. Note that if $\ell'_u = 0$ we have $\ell_u(a) = \ell_u(b)$, and therefore I_u is empty. By setting $g_e^j(\theta) := 0$ for all $\theta < \ell_u(0)$ we obtain well-defined functions g_e^j.

First, we show that g^j satisfies the sub-flow over time properties and conserves flow at all nodes except $S^+ \cup \{ t_j \}$ for all $\theta \in I_u$.

Given an arc $e = uv$ we obviously have for all $\theta \in I_u$ that

$$g_e^j(\theta) = x_e'^j/\ell'_u \leq x'_e/\ell'_u = f_e(\theta).$$

If $x'_e > 0$ we have $f_e(\theta) = x'_e/\ell'_u > 0$ for almost all $\theta \in I_u$ and by the definition of g^{j-} we get for almost all $\xi \in I_v = T_e(I_u)$ and $\theta \in I_u$, the unique value with $\xi = T_e(\ell_u(\phi))$, that

$$g_e^{j-}(\xi) = f_e^-(\xi) \cdot \frac{g_e^j(\theta)}{f_e(\theta)} = \frac{x'_e}{\ell'_v} \cdot \frac{x_e'^j}{\ell'_u} \cdot \frac{\ell'_u}{x'_e} = \frac{x_e'^j}{\ell'_v}.$$

But this equality also holds if $x'_e = 0$ because in this case it holds that $f_e(\theta) = 0$ for almost all $\theta \in I_u$, and therefore we have by definition that $g_e^{j-}(\xi) = 0$. The following equation shows that g^j conserves flow at all nodes $v \in V \backslash S^+ \cup \{ t_j \}$ for almost all $\theta \in I_v$

$$\sum_{e \in \delta^-(v)} g_e^{j-}(\theta) - \sum_{e \in \delta^+(v)} g_e^j(\theta) = \sum_{e \in \delta^-(v)} \frac{x'^j_e}{\ell'_v} - \sum_{e \in \delta^+(v)} \frac{x'^j_e}{\ell'_v} = \frac{1}{\ell'_v} \cdot \left(\sum_{e \in \delta^-(v)} x'^j_e - \sum_{e \in \delta^+(v)} x'^j_e \right) = 0,$$

where the last equality holds because of the flow conservation of x'^j at v. To show (9) it remains to prove it for t_j, which is true because for all $\theta \in I_{t_j}$ we have

$$\sum_{e \in \delta^-(t_j)} g_e^{j-}(\theta) - \sum_{e \in \delta^+(t_j)} g_e^j(\theta) = \frac{x_{e_j}}{\ell'_{t_j}} = \bar{f}_{e_j}(\theta) = \sum_{e \in \delta^-(t_j)} f_e^-(\theta) - \sum_{e \in \delta^+(t_j)} f_e(\theta).$$

Next, we show that $(g_i^j)_{i=1}^n$ is a matching sub-inflow distribution for all $j = 1, \ldots, m$ with values d_j for all $\phi \in I$. In the case of $T'(\phi) \overset{(N1)}{=} \ell'_{s_i} > 0$ it holds that

$$\left(\sum_{e \in \delta^+(s_i)} g_e^j(\ell_{s_i}(\phi)) - \sum_{e \in \delta^-(s_i)} g_e^{j-}(\ell_{s_i}(\phi)) \right) T_i'(\phi) = \left(\sum_{e \in \delta^+(s_i)} \frac{x'^j_e}{\ell'_{s_i}} - \sum_{e \in \delta^-(s_i)} \frac{x'^j_e}{\ell'_{s_i}} \right) \ell'_{s_i} = g_i^j(\phi).$$

In the case of $\ell'_{s_i} = 0$ this is also true since both sides are equal to 0. By Lemma 11 we obtain for all $\phi \in I$ that

$$\sum_{i=1}^n g_i^j(\phi) = \sum_{i=1}^n \left(\sum_{e \in \delta^+(s_i)} x'^j_e - \sum_{e \in \delta^-(s_i)} x'^j_e \right) = |x'^j| = d_j.$$

Finally, we show that the family $(g^j)_{j=1}^m$ together with the matching sub-inflow distributions fulfills the sub-flow over time decomposition conditions for all $\theta \in I_u$. Clearly, $\sum_{j=1}^m d_j = 1$ and for all $e = uv \in E$ we have

$$\sum_{j=1}^m g_e^j(\theta) = \sum_{j=1}^m \frac{x'^j_e}{\ell'_u} = \frac{x'_e}{\ell'_u} = f_e(\theta).$$

Note that all these previous conditions hold for all $\phi \in \mathbb{R}_{\geq 0}$ and all $\theta \in [0, \infty)$ because either $\theta < \ell_v(0)$, where all in and out flow at v is 0, or θ is element of the local times I_u of some particle interval I. Hence, $(g^j)_{j=1^m}$ is a sub-flow over time decomposition of f with values $d_1, \ldots, d_m$, where g^j is an S^+-t_j-sub-flow over time. Since $\bar{f}$ is a Nash flow over time f satisfies the Nash flow conditions (N1) and (N2) as well, and therefore f is a Nash flow over time with demands $d_1, \ldots, d_m$. ◀

6 Conclusion and outlook

We showed that the Nash flow over time introduced in [10] can be extended to our multi-terminal setting, for which we uncoupled the flow particles from their entering times and introduced inflow distributions instead. Furthermore, the proper definition of a sub-flow-structure and a super-sink-construction allowed us to have Nash flows over time with multiple sinks and demands. Nonetheless the much more challenging question about the structure of a dynamic equilibrium in a setting with multiple origin-destination-pairs remains open. There are also further interesting aspects that are unsolved in the original setting, such as the computational complexity of thin flows with resetting or the question if the number of thin flow phases is finite within a Nash flow over time. Last but not least the very interesting question if the price of anarchy is bounded or not remains open, despite some promising progress in recent time.

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A   Appendix

A.1   Cumulative flows and queues

▶ **Lemma 14.** *For a given arc $e = uv \in E$ the following is true for all times $\theta \geq 0$:*
(i) $z_e(\theta) = F_e^+(\theta - \tau_e) - F_e^-(\theta)$
(ii) $F_e^+(\theta) = F_e^-(T_e(\theta))$

A proof can be found in [2, Section 2.2] and [10, Proposition 2].

A.2   Proof of Theorem 4

▶ **Theorem 4.** *For a Nash flow over time $((f_e^+)_{e \in E}, (f_i)_{i=1}^n)$, the derivative labels $(x_i'(\phi))_{i=1}^n$ and $(x_e'(\phi))_{e \in E_\phi'}$ together with $(\ell_v'(\phi))_{v \in V}$ form a thin flow with resetting on E_ϕ^* in the current shortest paths network $G_\phi' = (V, E_\phi')$, for almost all $\phi \in \mathbb{R}_{\geq 0}$.*

Proof. In Lemma 15 we showed that G'_ϕ and E^*_ϕ satisfy the preconditions. Furthermore, we have $x'_i(\phi) = f_i(\phi) \geq 0$ for all $i = 1, \ldots, n$ and $\sum_{i=1}^n x'_i(\phi) = \sum_{i=1}^n f_i(\phi) = 1$ for almost all $\phi \in \mathbb{R}_{\geq 0}$. It remains to show that the equations (TF1) to (TF4) are satisfied for almost all particles. For this let ϕ be a particle such that for all $e = uv$ the derivatives of x_e, ℓ_v, and $T_e \circ \ell_u$ exist and $x'_e(\phi) = f_e^+(\ell_u(\phi)) \cdot \ell'_u(\phi) = f_e^-(\ell_v(\phi)) \cdot \ell'_v(\phi)$, which is almost everywhere. From Lemma 2 follows (TF1) directly.

For (TF2) and (TF3) first note that since z_e is Lipschitz continuous, so is T_e. We thus obtain from (1) that the derivative of $T_e(\theta)$ is almost everywhere

$$T'_e(\theta) = \begin{cases} \frac{f_e^+(\theta)}{\nu_e} & \text{if } z_e(\theta + \tau_e) > 0, \\ \max\left\{ \frac{f_e^+(\theta)}{\nu_e}, 1 \right\} & \text{if } z_e(\theta + \tau_e) = 0. \end{cases} \tag{11}$$

In the case of $z_e(\ell_u(\phi) + \tau_e) > 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\phi} T_e(\ell_u(\phi)) = T'_e(\ell_u(\phi)) \cdot \ell'_u(\phi) \stackrel{(11)}{=} \left(\frac{f_e^+(\ell_u(\phi))}{\nu_e} \right) \cdot \ell'_u(\phi) = \frac{x'_e(\phi)}{\nu_e}$$

and if $z_e(\ell_u(\phi) + \tau_e) = 0$, it holds that

$$\frac{\mathrm{d}}{\mathrm{d}\phi} T_e(\ell_u(\phi)) \stackrel{(11)}{=} \max\left\{ \frac{f_e^+(\ell_u(\phi))}{\nu_e}, 1 \right\} \cdot \ell'_u(\phi) = \max\left\{ \frac{x'_e(\phi)}{\nu_e}, \ell'_u(\phi) \right\}.$$

Since the first case is equivalent to $e \in E^*_\phi$ and the second to $e \in E'_\phi \setminus E^*_\phi$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\phi} T_e(\ell_u(\phi)) = \rho_e(\ell'_u(\phi), x'_e(\phi)).$$

This equality together with the Bellman equations (4) and the differentiation rule for a minimum, i.e., $\ell'_v(\phi) = \min_{uv \in E'_\phi} T'_{uv}(\phi)$, provides

$$\ell'_{s_i}(\phi) \leq \min_{e=us_i \in E'_\phi} \rho_e(\ell'_u(\phi), x'_e(\phi)) \quad \text{and} \quad \ell'_v(\phi) = \min_{e=uv \in E'_\phi} \rho_e(\ell'_u(\phi), x'_e(\phi)).$$

For (TF4) suppose $x'_e(\phi) = f_e^-(\ell_v(\phi)) \cdot \ell'_v(\phi) > 0$. With (2) we obtain

$$\begin{aligned} \ell'_v(\phi) = \frac{x'_e(\phi)}{f_e^-(\ell_v(\phi))} &= \begin{cases} \dfrac{x'_e(\phi)}{\min\left\{ f_e^+(\ell_u(\phi)), \nu_e \right\}} & \text{if } z_e(\ell_u + \tau_e) = 0, \\ \dfrac{x'_e(\phi)}{\nu_e} & \text{else,} \end{cases} \\ &= \begin{cases} \max\left\{ \ell'_u, \dfrac{x'_e(\phi)}{\nu_e} \right\} & \text{if } e \in E'_\phi \setminus E^*_\phi, \\ \dfrac{x'_e(\phi)}{\nu_e} & \text{if } e \in E^*_\phi, \end{cases} \\ &= \rho_e(\ell'_u(\phi), x'_e(\phi)). \end{aligned}$$

This shows that the derivatives $(x'_i(\phi))_{i=1}^n$, $(x'_e(\phi))_{e \in E'_\phi}$, and $(\ell'_v(\phi))_{v \in V}$ form a thin flow with resetting. ◀

A.3 Characterization of active and resetting arcs

This lemma shows, among other facts, that every arc with a positive queue has to be active.

▶ **Lemma 15.** *Consider a Nash flow over time f with earliest arrival times $(\ell_v)_{v \in V}$. For every particle $\phi \in \mathbb{R}_{\geq 0}$, the following statements are true:*

(i) $E_\phi^* \subseteq E_\phi'$

(ii) $E_\phi' = \{ e = uv \in E \mid \ell_v(\phi) \geq \ell_u(\phi) + \tau_e \}$

(iii) $E_\phi^* = \{ e = uv \in E \mid \ell_v(\phi) > \ell_u(\phi) + \tau_e \}$

(iv) *The graph $G_\phi' = (V, E_\phi')$ is acyclic and every node is reachable by a source.*

For a proof we refer to [2, Proposition 2].

A.4 Extended outflow functions

▶ **Lemma 16.** *Let $((f_e^+)_{e \in E}, (f_i)_{i=1}^n)$ be a restricted Nash flow over time on $[0, \phi]$ and let $\alpha > 0$ satisfy (6) and (7). Then the outflow functions of the α-extension satisfy $f_e^-(\theta) = \frac{x_e'}{\ell_v'}$ for all $e = uv \in E$ and almost all $\theta \in (\ell_v(\phi), \ell_v(\phi + \alpha)]$ and $f_e^-(\theta) = 0$ for $\theta > \ell_v(\phi + \alpha)$.*

Proof. Note that throughout this proof $\ell_v(\varphi)$ for $\varphi > \phi$ is not the earliest arrival time, but the linear extension $\ell_v(\varphi) := \ell_v(\phi) + (\varphi - \phi) \cdot \ell_v'$. Let $I := (\phi, \phi + \alpha]$ be the flow of interest and $I_v := (\ell_v(\phi, \ell_v(\phi + \alpha)]$ for all nodes v. The particles in $[0, \phi]$ do not interfere with the outflow function f_v^+ within I_v, since otherwise the restricted Nash flow over time would not have chosen the fastest direction. We divide the proof into three cases.

Case 1: $x_e' = 0$.

Since $f_e^+(\theta) = x_e'/\ell_u' = 0$ for all $\theta \in I_u$ we have that $f_e^-(\theta) = 0 = x_e'/\ell_v'$ for all $\theta \in I_v$ and of course $f_e^-(\theta) = 0$ for $\theta > \ell_u(\phi + \alpha)$.

Case 2: $x_e' > 0$, $e \notin E^*$ and $x_e'/\nu_e \leq \ell_u'$.

We know that e is active during I and that $f_e^+(\theta) = x_e'/\ell_u' \leq \nu_e$ for $\theta \in I_u$. Furthermore, there is no queue at the beginning and no queue is building up. Therefore, we have $\ell_v(\phi) = \ell_u(\phi) + \tau_e$. The definition of thin flows with resetting provides $\ell_u' = \ell_v'$ and together with the definition of the extension we obtain

$$\ell_u(\phi + \alpha) + \tau_e = \ell_u(\phi) + \alpha \cdot \ell_u' + \tau_e = \ell_v(\phi) + \alpha \cdot \ell_v' = \ell_v(\phi + \alpha).$$

Hence, the last flow entering e at time $\ell_u(\phi + \alpha)$ leaves the edge at time $\ell_v(\phi + \alpha)$ and since the outflow rate at time $\theta \in I_v$ equals the inflow rate at time $\theta - \tau_e \in I_u$ we get

$$f_e^-(\theta) = f_e^+(\theta - \tau_e) = \frac{x_e'}{\ell_u'} = \frac{x_e'}{\ell_v'}.$$

Furthermore, no flow enters e after $\ell_u(\phi + \alpha)$, and therefore the outflow function is zero after $\ell_v(\phi + \alpha)$.

Case 3: $x_e' > 0$ and ($e \in E^*$ or $x_e'/\nu_e > \ell_u'$).

This means there is either a queue at the beginning and throughout the phase or there is no queue at the beginning but immediately after ϕ a queue will build up. In either case, e is active for all particles in I and $\ell_v' = x_e'/\nu_e$. The inflow rate is $f_e^+(\theta) = x_e'/\ell_u'$ for all I_u, and therefore the amount of flow entering e during this interval is

$$A := x_e'/\ell_u' \cdot (\ell_u(\phi + \alpha) - \ell_u(\phi)) = x_e'/\ell_u' \cdot (\ell_u(\phi) + \alpha \cdot \ell_u' - \ell_u(\phi)) = x_e' \cdot \alpha.$$

Since ϕ leaves e at time $\ell_v(\phi)$ and f_e^- operates at capacity rate the last particle $\phi + \alpha$ leaves e at time

$$\ell_v(\phi) + A/\nu_e = \ell_v(\phi) + \alpha \cdot x_e'/\nu_e = \ell_v(\phi) + \alpha \cdot \ell_v' = \ell_v(\phi + \alpha).$$

Therefore, we have for all $\theta \in I_v$ that

$$f_e^-(\theta) = \nu_e = \frac{x_e'}{\ell_v'}.$$

Since no flow is entering after $\ell_u(\phi + \alpha)$ and particle $\phi + \alpha$ leaves e at time $\ell_v(\phi + \alpha)$ the outflow function is zero afterwards. This completes the proof. ◀

A.5   Proof of Lemma 6

▶ **Lemma 6.** *The α-extension forms a flow over time and the extended ℓ-labels coincide with the earliest arrival times, i.e., satisfy the Bellman equations* (4) *for all $\varphi \in (\phi, \phi + \alpha]$.*

Proof. In order to prove that the α-extension forms a flow over time we have to show that the flow conservation is fulfilled at every $v \in V \setminus \{t\}$, which is true because for all $\theta \in (\ell_v(\phi), \ell_v(\phi + \alpha)]$ it holds that

$$\sum_{e \in \delta^+(v)} f_e^+(\theta) - \sum_{e \in \delta^-(v)} f_e^-(\theta) = \sum_{e \in \delta^+(v)} x_e'/\ell_v' - \sum_{e \in \delta^-(v)} x_e'/\ell_v'$$

$$= \begin{cases} 0 & \text{if } v \in V \setminus (S^+ \cup \{t\}) \\ x_i'/\ell_v' = r_i & \text{if } v = s_i \in S^+. \end{cases}$$

For $\theta > \ell_v(\phi + \alpha)$ all functions as well as the inflow rates are zero, and therefore the flow conservation holds as well.

For the second part we show that the Bellman equations (4) for the earliest arrival times hold. Given an arc $e = uv \in E$, we distinguish between three cases.

Case 1: $e \in E \setminus E_\phi'$.
Since α satisfies equation (7) it is satisfied for all $\xi \in (0, \alpha]$ and hence,

$$\ell_v(\phi + \xi) = \ell_v(\phi) + \xi \cdot \ell_v' \overset{(7)}{\leq} \ell_u(\phi) + \xi \cdot \ell_u' + \tau_e \leq T_e(\ell_u(\phi) + \xi \cdot \ell_u') = T_e(\ell_u(\phi + \xi)).$$

Case 2: $e \in E_\phi' \setminus E_\phi^*$ and $\ell_u' \geq x_e'/\nu_e$.
Since e is active we have $\ell_v(\phi) = T_e(\ell_u(\phi)) = \ell_u(\phi) + \tau_e$ and (TF3) implies $\ell_v' \leq \ell_u'$. There is no queue building up, which means $z_e(\ell_u(\phi + \xi) + \tau_e) = 0$ for all $\xi \in (0, \alpha]$. Combining these yields

$$\ell_v(\phi + \xi) = \ell_v(\phi) + \xi \cdot \ell_v' \overset{(TF3)}{\leq} \ell_u(\phi) + \tau_e + \xi \cdot \ell_u' = \ell_u(\phi + \xi) + \tau_e = T_e(\ell_u(\phi + \xi)).$$

Case 3: $e \in E_\phi^*$ or ($e \in E_\phi'$ and $\ell_u' < x_e'/\nu_e$).
Again, e is active, which means $\ell_v(\phi) = T_e(\ell_u(\phi)) = \ell_u(\phi) + \tau_e + z_e(\ell_u(\phi) + \tau_e)/\nu_e$. Additionally, $e \in E_\phi^*$ or $x_e'/\ell_u' \leq \nu_e$ together with the thin flow condition (TF3) implies $\ell_v' \leq x_e'/\nu_e$. Since $f_e^+(\ell_u(\phi)) - \nu_e = x_e'/\ell_u' - \nu_e > 0$, equation (1) implies $z_e'(\ell_u(\phi) + \tau_e) = $

$f_e^+(\ell_u(\phi)) - \nu_e = x_e'/\ell_u' - \nu_e$. Rearranging gives us, $x_e'/\nu_e = z_e'(\ell_u(\phi) + \tau_e) \cdot \ell_u'/\nu_e + \ell_u'$. Hence, for all $\xi \in (0, \alpha]$ we obtain with (TF3) that

$$
\begin{aligned}
\ell_v(\phi + \xi) &= \ell_v(\phi) + \xi \cdot \ell_v' \\
&\leq \ell_v(\phi) + \xi \cdot x_e'/\nu_e \\
&= \ell_u(\phi) + \tau_e + z_e(\ell_u(\phi) + \tau_e)/\nu_e + \xi \cdot (z'(\ell_u(\phi) + \tau_e) \cdot \ell_u'/\nu_e + \ell_u') \\
&= \ell_u(\phi + \xi) + \tau_e + z_e(\ell_u(\phi) + \tau_e + \xi \cdot \ell_u')/\nu_e \\
&= T_e(\ell_u(\phi + \xi)).
\end{aligned}
$$

This shows that there is no arc with an exit time earlier than the earliest arrival time, and therefore the left hand side of the Bellman equations is always smaller or equal to the right hand side. It remains to show that the equations hold with equality. For a source s_i we have $x_i' = f_i(\phi) = r_i \cdot T_i'(\phi)$, and therefore

$$
\ell_{s_i}(\phi + \xi) = \ell_{s_i}(\phi) + \xi \cdot \ell_{s_i}' = T_i(\phi) + \xi \cdot x_i'/r_i = T_i(\phi) + \xi \cdot T_i'(\phi) = T_i(\phi + \xi)
$$

for all $\xi \in (0, \alpha]$. Hence, entering the network at a specific source is always a fastest option to reach it. For every node $v \in V \backslash S^+$ there is at least one arc $e \in E'$ with $\ell_v' = \rho(\ell_u', x_e')$ in the thin flow due to (TF3). No matter if this arc belongs to Case 2 or Case 3 the corresponding equation holds with equality, which shows for all $\xi \in (0, \alpha]$ that

$$
\ell_v(\phi + \xi) = \min_{e = uv \in E} T_e(\ell_u(\phi + \xi)).
$$

This completes the proof. ◀

A.6 Bound on node labels of thin flows with resetting

▶ **Lemma 17.** *For every thin flow with resetting (x', ℓ') in G we have $\ell_v' \leq 1/\sigma$ for all $v \in V$.*

Proof. It holds that $\ell_{s_i}' = x_i'/r_i$ and for $v \in V \backslash S^+$ the ℓ_v' labels are equal to ℓ_u' or x_e'/ν_e for some incoming arc $e = uv$. It follows that all ℓ' labels in the original graph are bounded from above by

$$
\max \left\{ \left(\max_{i=1,\dots,n} x_i'/r_i \right), \left(\max_{e \in E} x_e'/\nu_e \right) \right\} \leq \max \{ 1/r_{\min}, 1/\nu_{\min} \} = 1/\sigma.
$$

Note that all x_i' and x_e' are bounded by 1 from above since the flow value of x' is 1. ◀