# Local Validity for Circular Proofs in Linear Logic with Fixed Points

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## Abstract

Circular (ie. non-wellfounded but regular) proofs have received increasing interest in recent years with the simultaneous development of their applications and meta-theory: infinitary proof theory is now well-established in several proof-theoretical frameworks such as Martin Löf's inductive predicates, linear logic with fixed points, etc. In the setting of non-wellfounded proofs, a validity criterion is necessary to distinguish, among all infinite derivation trees (aka. pre-proofs), those which are logically valid proofs. A standard approach is to consider a pre-proof to be valid if every infinite branch is supported by an infinitely progressing thread.

The paper focuses on circular proofs for MALL with fixed points. Among all representations of valid circular proofs, a new fragment is described, based on a stronger validity criterion. This new criterion is based on a labelling of formulas and proofs, whose validity is purely local. This allows this fragment to be easily handled, while being expressive enough to still contain all circular embeddings of Baelde's $\mu$MALL finite proofs with (co)inductive invariants: in particular deciding validity and computing a certifying labelling can be done efficiently. Moreover the Brotherston-Simpson conjecture holds for this fragment: every labelled representation of a circular proof in the fragment is translated into a standard finitary proof. Finally we explore how to extend these results to a bigger fragment, by relaxing the labelling discipline while retaining (i) the ability to locally certify the validity and (ii) to some extent, the ability to finitize circular proofs.

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$$\dfrac{\vdash \Gamma, S \qquad \vdash S^{\perp}, F[S/X]}{\vdash \Gamma, \nu X.F}\ (\nu_{\mathrm{inv}})$$

**Figure 1** Coinduction rule *à la* Park.

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.X}\ (\mu)}{\vdash \mu X.X}\ (\mu) \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.X, \Gamma}\ (\nu)}{\vdash \nu X.X, \Gamma}\ (\nu)}{\vdash \Gamma}\ (\mathsf{Cut})$$

**Figure 2**

# 1 Introduction

Various logical settings have been introduced to reason about inductive and coinductive statements, both at the level of the logical languages modelling (co)induction (Martin Löf's inductive predicates vs. fixed-point logics, that is $\mu$-calculi) and at the level of the proof-theoretical framework considered (finite proofs with (co)induction *à la* Park [22] vs. infinite proofs with fixed-point/inductive predicate unfoldings) [8, 10, 11, 5, 2, 3]. Moreover, such proof systems have been considered over classical logic [8, 11], intuitionistic logic [12], linear-time or branching-time temporal logic [20, 19, 26, 27, 14, 15, 16] or linear logic [23, 17, 5, 4, 15].

In all those proof systems, the treatment of inductive and coinductive reasoning brings some highly complex proof figures. For instance, in proof systems using (co)induction rules *à la* Park, the rules allowing to derive a coinductive property (or dually to use an inductive hypothesis) have a complex inference of the form of fig. 1 (when presented in the setting of fixed-point logic – here we follow the one-sided sequent tradition of MALL that we will adopt in the rest of the paper). Not only is it difficult to figure out intuitively what is the meaning of this inference, but it is also problematic for at least two additional and more technical reasons: (i) it is hiding a cut rule that *cannot* be eliminated, which is problematic for extending the Curry-Howard correspondence to fixed-points logics, and (ii) it breaks the subformula property, which is problematic for proof search: at each coinduction rule, one has to guess an invariant (in the same way as one has to guess an appropriate induction hypothesis in usual mathematical proofs) which is problematic for automation of proof search.

Infinite (non-wellfounded) proofs have been proposed as an alternative in recent years [8, 10, 11]. By replacing the coinduction rule with simple fixed-point unfoldings and allowing for non-wellfounded branches, those proof systems address the problem of the subformula property for the cut-free systems. The cut-elimination dynamics for inductive-coinductive rules is also much simpler. Among those non-wellfounded proofs, circular, or cyclic proofs, that have infinite but regular derivations trees, have attracted a lot of attention for retaining the simplicity of the inferences of non-wellfounded proof systems but being amenable to a simple finite representation making it possible to have an algorithmic treatment of those proof objects.

However, in those proof systems when considering all possible infinite, non-wellfounded derivations (a. k. a. pre-proofs), it is straightforward to derive any sequent $\Gamma$ (see fig. 2). Such pre-proofs are therefore unsound and one needs to impose a validity criterion to distinguish, among all pre-proofs, those which are logically valid proofs from the unsound ones. This condition will actually reflect the inductive and coinductive nature of our two fixed-point connectives: a standard approach [8, 10, 11, 23, 4] is to consider a pre-proof to be valid if

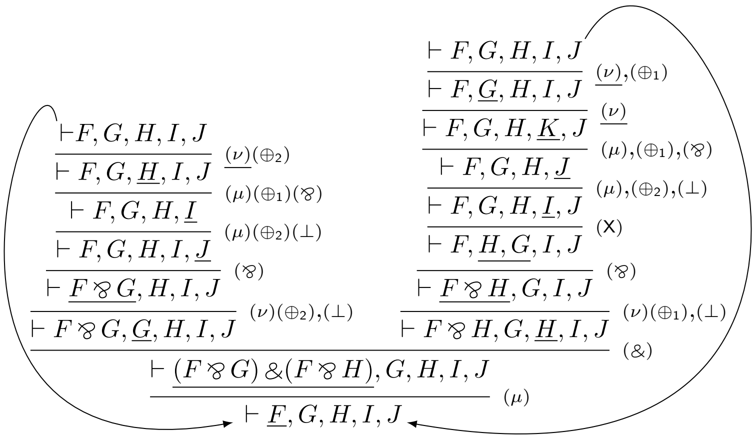



**Figure 3** Proof $\pi_\infty$.

every infinite branch is supported by an infinitely progressing thread. However, doing so, the logical correctness of circular proofs becomes a non-local property, much in the spirit of proof nets correctness criteria [18, 13].

Despite the need for a validity condition, circular proofs have recently received increasing interest with the simultaneous development of their applications and meta-theory: infinitary proof theory is now well-established in several proof-theoretical frameworks such as Martin Löf's inductive predicates, linear logic with fixed-points, *etc.*

This paper is a contribution to two directions in the field of circular proofs:

1. the relationship between finite and circular proofs (at the level of provability and at the level of proofs themselves) and
2. the certification of circular proofs, that is the production of fast and/or small pieces of evidence to support validity of a circular pre-proof.

Comparing finite and infinite proofs is very natural. Informally, it amounts to considering the relative strength of inductive reasoning versus infinite descent: while infinite descent is a very old form of mathematical reasoning which appeared already in Euclid's *Elements* and was systematically investigated by Fermat, making precise its relationship with mathematical induction is still an open question for many proof formalisms. Their equivalence is known as the Brotherston–Simpson conjecture. While it is fairly straightforward to check that infinite descent (circular proofs) prove at least as many statements as inductive reasoning, the converse is complex and remains largely open. Last year, Simpson [24], on the one hand, and Berardi and Tatsuta [6, 7], on the other hand, made progress on this question but only in the framework of Martin Löf's inductive definitions, not in the setting of $\mu$-calculi circular proofs in which invariant extraction is highly complex and known only for some fragments.

We conclude this introduction by considering a typical example of a circular proof with a complex validating thread structure: while this infinite proof has a regular derivation tree, its branches and threads have a complex geometry. The circular (pre-)proof of Figure 3 derives the sequent $\vdash F, G, H, I, J$ where $F = \mu X.(X \parr G) \& (X \parr H)$, $G = \nu X.X \oplus \bot$, $H = \nu X.\bot \oplus X$, $I = \mu Z.((Z \parr J) \oplus \bot)$, $J = \mu X.(K \parr X) \oplus \bot$ and $K = \nu Y.\mu Z.((Z \parr \mu X.(Y \parr X) \oplus \bot) \oplus \bot)$.

This example of a circular derivation happens to be valid (it is a $\mu\mathsf{MALL}^\omega$ proof) but the description of its validating threads is quite complex. Indeed, each infinite branch $\beta$ is validated by exactly one thread (see next section for detailed definitions) going through either $G$, $H$ or $K$ depending on the shape of the branch *at the limit* (infinite branches of this derivations can be described as $\omega$-words on $A = \{l, r\}$ depending on whether the left or right back-edge is taken):

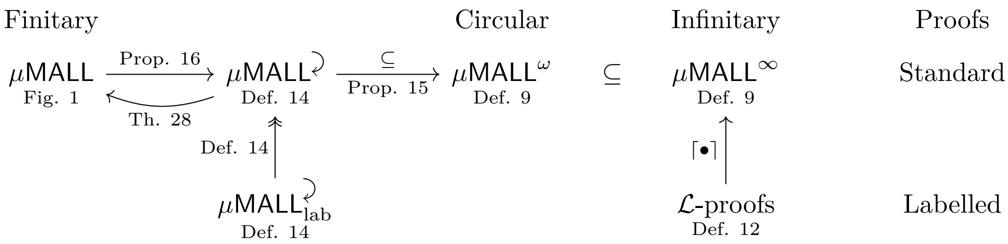



**Figure 4** Relations between the different systems used in the paper.

**(i)** if $\beta$ *ultimately* follows always the left cycle $(A^\star \cdot l^\omega)$, the unfolding of $H$ validates $\beta$;
**(ii)** if $\beta$ *ultimately* follows always the right cycle $(A^\star \cdot r^\omega)$, the unfolding of $G$ validates $\beta$;
**(iii)** if $\beta$ *endlessly switches* between left and right cycles $(A^\star \cdot (r^+ \cdot l^+)^\omega)$, $K$ validates $\beta$.

The description of the thread validating this proof is thus complex. This is reflected in the difficulty to provide a local way to validate this proof and in the lack of a general method for finitizing this into a $\mu$MALL proof: to our knowledge, the usual finitization methods (working only for fragments of $\mu$MALL circular proofs) do not apply here.

**Organization and contributions of the paper.** In section 2, we provide the necessary background on infinitary and circular proof theory of multiplicative additive linear logic with least and greatest fixed points (respectively $\mu$MALL$^\infty$ and $\mu$MALL$^\omega$). Section 3 studies an approach to circular proofs based on labellings of greatest fixed points. We first motivate in section 3.1 such labellings as an alternative way to express the validating threads. Then, in section 3.2 we introduce finite representations of pre-proofs and use such labellings in order to locally certify their validity. Finally, in section 3.3, we turn to alternative characterizations of those circular proofs which can be labelled. The fragment of labellable proofs, while quite constrained (for instance, it does not include the example of Figure 3), is already enough to capture the circular proofs obtained by translation of $\mu$MALL proofs. In section 4, we address the converse: for any labelled derivation tree with back-edges, we provide a corresponding $\mu$MALL proof by generating a (co)inductive invariant based on an inspection of the labelling structure. Therefore, we answer the Brotherston–Simpson conjecture in a restricted fragment. In section 5, we introduce a more permissive labelling strategy that allows to label more proofs (in particular by allowing to loop not only on $(\nu)$ rules but on any rule) and that still ensures validity of the labellable derivations. For this relaxed labelling, we label the example of Figure 3 and show how to finitize it by adapting the method of section 4. Nevertheless, there is not yet a general method applicable to the complete extended labelling fragment. Relations between the various systems considered in the paper are summarized in Figure 4.

## 2 Background on circular proofs

We recall $\mu$MALL$^\infty$ and $\mu$MALL$^\omega$, which are non-wellfounded and circular proof systems, respectively, for an extension of MALL with least and greatest fixed points operators [4, 15].

▶ **Definition 1.** Given a set of fixed point operators $\mathcal{F} = \{\mu, \nu\}$ and an infinite set of propositional variables $\mathcal{V} = \{X, Y, \dots\}$, $\mu$MALL **pre-formulas** are inductively defined as:

$$A, B ::= \mathbf{0} \mid \top \mid A \oplus B \mid A \,\&\, B \mid \bot \mid \mathbf{1} \mid A \parr B \mid A \otimes B \mid X \mid \sigma X.A \quad \text{with } X \in \mathcal{V} \text{ and } \sigma \in \mathcal{F}.$$

$\sigma \in \mathcal{F}$ binds the variable $X$ in $A$. From there, bound variables, free variables and capture-avoiding substitution are defined in a standard way. The subformula ordering is denoted $\leq$ and $\mathsf{fv}(\bullet)$ denotes free variables. When a pre-formula is closed, we simply call it a **formula**.

$$\frac{}{\vdash \mathbf{1}}\ (\mathbf{1}) \qquad \frac{}{\vdash F, F^{\perp}}\ (\mathsf{Ax}) \qquad \frac{\vdash \Gamma, F \quad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathsf{Cut}) \qquad \frac{\vdash \Gamma, F, G, \Delta}{\vdash \Gamma, G, F, \Delta}\ (\mathsf{X})$$

$$\frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{\vdash F_i, \Gamma}{\vdash F_1 \oplus F_2, \Gamma}\ (\oplus_i) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

$$\frac{}{\vdash \top, \Gamma}\ (\top) \qquad \frac{\vdash F, \Gamma \quad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ (\&) \qquad \frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu)$$

**Figure 5** $\mu\mathsf{MALL}^{\infty}$ inference rules.

Note that negation is not part of the syntax, so that we do not need any positivity condition on fixed-points expressions. We define negation, $(\bullet)^{\perp}$, as a meta-operation on pre-formulas and will use it on formulas.

▶ **Definition 2. Negation**, $(\bullet)^{\perp}$, is the involution on pre-formulas, satisfying: $\mathbf{0}^{\perp} = \top$, $(A \oplus B)^{\perp} = B^{\perp} \,\&\, A^{\perp}$, $\mathbf{1}^{\perp} = \bot$, $(A \otimes B)^{\perp} = B^{\perp} \parr A^{\perp}$, $X^{\perp} = X$, $(\mu X.A)^{\perp} = \nu X.A^{\perp}$.

▶ **Example 3.** The previous definition yields, *e. g.* $(\mu X.X)^{\perp} = (\nu X.X)$ and $(\mu X.\mathbf{1} \oplus X)^{\perp} = (\nu X.X \,\&\, \bot)$, as expected [3]. Note that we also have $(A[B/X])^{\perp} = A^{\perp}[B^{\perp}/X]$.

The reader may find it surprising to define $X^{\perp} = X$, but it is harmless since our proof system only deals with formulas (*i. e.* closed pre-formulas) as examplified right above.

Fixed-points logics come with a notion of subformulas slightly different from usual:

▶ **Definition 4.** The **Fischer-Ladner closure** of a formula $F$, $\mathsf{FL}(F)$, is the least set of formulas such that $F \in \mathsf{FL}(F)$ and, whenever $G \in \mathsf{FL}(F)$, (i) $G_1, G_2 \in \mathsf{FL}(F)$ if $G = G_1 \star G_2$ for any $\star \in \{\oplus, \&, \parr, \otimes\}$; (ii) $B[G/X] \in \mathsf{FL}(F)$ if $G$ is $\mu X.B$ or $\nu X.B$. We say that $G$ is a **FL-subformula** of $F$ if $G \in \mathsf{FL}(F)$.

In this work we choose to present sequents as lists of formulas together with an explicit exchange rule. Another usual choice is to present sequents as multisets of formulas. Yet, our approach takes the viewpoint of structural proof theory in which one is willing not to equate too many proofs. In particular, the sequents as (multi)sets are not relevant from the Curry-Howard perspective, *e. g.* it would equate the proofs denoting the two booleans. Moreover, most proof theoretical observations actually hold when one distinguishes between several *occurrences* of a formula in a sequent, giving the ability to *trace* the provenance of each occurrence. In [4], formula occurrences are localized formulas and the interested reader will check that all the following results hold also in this more explicit approach.

▶ **Definition 5.** A **pre-proof** of $\mu\mathsf{MALL}^{\infty}$ is a possibly infinite tree generated from the inference rules given in fig. 5.

Recall that $\mu\mathsf{MALL}$ [3], on the opposite, is obtained by forming only finite trees and by taking, instead of the $(\nu)$ rule of $\mu\mathsf{MALL}^{\infty}$, the rule with explicit invariant of fig. 1.

When writing sequent proofs, we will often omit exchange rules, using the fact that every inference of def. 5 admits a derivable variant (preserving every correctness criterion considered in the paper) allowing the principal formula of the inference as well as the context (or auxiliary) formulas to be anywhere in the sequent, *e. g.* for the $\parr$ introduction, the derived rule is $\dfrac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, A \parr B, \Delta}\ (\parr)$ . We will use those derived rules when it is not ambiguous with respect to the formula occurrence relation. The following notion of threading function is folklore generally left implicit.

$$\frac{\vdash \Gamma, F \qquad \vdash F^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathsf{Cut}) \qquad \frac{\vdash \Gamma, F, G, \Delta}{\vdash \Gamma, G, F, \Delta}\ (\mathsf{X})$$

$$\frac{\vdash F_1, \Gamma}{\vdash F_1 \oplus F_2, \Gamma}\ (\oplus_1) \qquad \frac{\vdash F_2, \Gamma}{\vdash F_1 \oplus F_2, \Gamma}\ (\oplus_2) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Delta}{\vdash F \otimes G, \Gamma, \Delta}\ (\otimes) \qquad \frac{\vdash F[\mu X.F/X], \Gamma}{\vdash \mu X.F, \Gamma}\ (\mu)$$

$$\frac{\vdash \Gamma}{\vdash \bot, \Gamma}\ (\bot) \qquad \frac{\vdash F, \Gamma \qquad \vdash G, \Gamma}{\vdash F \,\&\, G, \Gamma}\ (\&) \qquad \frac{\vdash F, G, \Gamma}{\vdash F \parr G, \Gamma}\ (\parr) \qquad \frac{\vdash G[\nu X.G/X], \Gamma}{\vdash \nu X.G, \Gamma}\ (\nu)$$

**Figure 6** Threading function.

▶ **Definition 6.** Every rule $r$ of $\mu\mathsf{MALL}^{\infty}$ comes with a **threading function** $\mathfrak{t}(r)$ (see Figure 6) mapping each position of an subformula in a premise to a position of a subformula in the conclusion, except for cut-formulas, by relating the subformula positions of a premise formula $F$ with the corresponding (subformula) positions of the conclusion $F'$, $F$ being the FL-subformula associated to $F'$ by inference $r$; note that in the case of the unfolding of fixed point $F' = \nu X.G$ into $F = G[\nu X.G/X]$ every position of $\nu X.G$ in $F$ is associated to the root position of $F'$ and every position of a subformula in (a copy of) $G$ in $F$ is associated to the corresponding subformula position in $G$ in $F'$. More formally, if $s_1$ is the conclusion and $s_2$ a premise of the same occurrence of rule $r$, then $r$ induces a partial function $\mathfrak{t}(r)\colon \mathsf{Pos}(s_2) \rightharpoonup \mathsf{Pos}(s_1)$, where $\mathsf{Pos}(A_0, \ldots, A_{n-1}) = \{(k,p) \mid 0 \leqslant k < n \text{ and } p \text{ is a position of a subformula in } A_k\}$.

By composing these partial maps we define $\mathfrak{t}(u)$ for any path $u$, mapping positions of subformulas in the top sequent of $u$ to positions of subformulas in its bottom sequent.

▶ **Definition 7.** Let $\gamma = (s_i)_{i \in \omega}$ be (a suffix of) an **infinite branch** in a pre-proof of $\mu\mathsf{MALL}^{\infty}$, that is: the $s_i$ are occurrences of sequents and for all $i$ there is an occurrence of a rule in the preproof which has $s_{i+1}$ as a premise and $s_i$ as conclusion.

A **$\nu$-thread** is the data comprising a $\nu$-formula $\nu X.A$ and a sequence $((s'_i, p_i))_{i<\alpha}$, finite ($\alpha < \omega$) or infinite ($\alpha = \omega$), such that $s'_i$ are sequent occurrences, $p_i$ is the position in $s'_i$ of a subformula equal to $\nu X.A$ and for all $i$, if $i + 1 < \alpha$, there is a rule occurrence $r_i$ which has $s'_i$ and $s'_{i+1}$ as, respectively, conclusion and premise, and such that $p_i$ corresponds to $p_{i+1}$ via the threading function, *i. e.* $p_i = \mathfrak{t}(r_i)(p_{i+1})$. If one of the $p_i$ is the main formula of the conclusion of a $\nu$-rule $r_i$, then the $\nu$-thread is **progressing** at $i$. A $\nu$-thread is **valid** if it is progressing infinitely many times. A $\nu$-thread is in $\gamma$ if $(s'_i)$ is a suffix of $\gamma$.

From now on, we may refer to à $\nu$-thread simply as a thread.

▶ **Definition 8** ($\mathfrak{T}(u)(p)$)**.** If $u$ is a finite path in a $\mu\mathsf{MALL}^{\infty}$ preproof and $p$ a position of subformula in its top sequent then there is a unique thread in $u$, going from $\mathfrak{t}(u)(p)$ up to $p$. This thread is constructed by following the threading relation and is denoted as $\mathfrak{T}(u)(p)$.

▶ **Definition 9.** A $\mu\mathsf{MALL}^{\infty}$ **proof** is a pre-proof in which every infinite branch contains a valid thread. A $\mu\mathsf{MALL}^{\omega}$ **proof** is a **circular** $\mu\mathsf{MALL}^{\infty}$ proof, *i. e.* a regular one, which has a finite number of distinct subtrees.

Since circular $\mu\mathsf{MALL}^{\infty}$ proofs are regular, they can actually be presented as finite trees with back-edges, as exemplified in fig. 3. The main results of the paper rely on such a representation. $\mu\mathsf{MALL}^{\infty}$ proofs enjoy several nice properties, such as cut-elimination:

▶ **Theorem 10** ([4])**.** *Cut-elimination holds for $\mu\mathsf{MALL}^{\infty}$ proofs.*

Thanks to cut-elimination $\mu\mathsf{MALL}^{\infty}$ enjoys the **FL-subformula property**: indeed in a cut-free $\mu\mathsf{MALL}^{\infty}$ proof, premises are always included in FL-closure of conclusion sequents.

# 3 Labelling as validity

## 3.1 $\mathcal{L}$-proofs

In this subsection, we briefly mention an alternative approach to ensure validity of $\mu\mathsf{MALL}^\infty$ pre-proofs, aiming at motivating the tools used in the remainder of this paper (see details in the extended version ). The idea is to witness thread progress by adding labels on some formulas.

▶ **Definition 11** (Labelled formulas)**.** Let $\mathcal{L}$ be an infinite countable set of **atoms** and call **labels** any finite list of atoms. Let $\mathcal{F}^{\mathcal{L}}$ be the set $\{\sigma^L \mid \sigma \in \{\mu, \nu\}, L \in \mathrm{list}(\mathcal{L})\}$. **Labelled formulas**, or $\mathcal{L}$-formulas, are defined as $\mu\mathsf{MALL}$ formulas, by replacing $\mathcal{F}$ with $\mathcal{F}^{\mathcal{L}}$ in the grammar of formulas (def. 1). Negation is lifted to labelled formulas, as $(\mu^L X.A)^\perp = \nu^L X.A^\perp$. We write $\sigma X.A$ for $\sigma^\emptyset X.A$ and standard, unlabelled formulas can thus be seen as labelled formulas where every label is empty. We define a **label-erasing** function $\lceil\bullet\rceil$ that associates to every $\mathcal{L}$-formula $A$ the $\mu\mathsf{MALL}$-formula $\lceil A\rceil$ obtained by erasing every label and satisfying $\lceil\sigma^L X.B\rceil = \sigma X.\lceil B\rceil$.

The standard $\mu\mathsf{MALL}^\infty$ proof system is adapted, to handle labels, by updating $(\mathsf{Ax})$ and $(\nu)$ as $\dfrac{A \perp B}{\vdash A, B}\,(\mathsf{Ax}')$ $\dfrac{\vdash A[\nu^{L,a}X.A], \Gamma}{\vdash \nu^L X.A, \Gamma}\,(\nu_{\mathsf{b}}(a))$ where (i) $A, B$ are said to be **orthogonal**, written $A \perp B$, when $\lceil A\rceil = \lceil B\rceil^\perp$ and (ii) in $(\nu_{\mathsf{b}}(a))$, $a$ must be a fresh label name, *i. e.* $a$ does not appear free in the conclusion sequent of $(\nu_{\mathsf{b}}(a))$ (in particular, $a \notin L$). Since we are in a one-sided framework, only labels on $\nu$ operators are relevant. Therefore, from now on, formulas have non-empty labels only on $\nu$ and require, for the cut inference, that all labels of cut formulas are empty. **$\mathcal{L}$-pre-proofs** are, as in def. 5, possibly infinite derivations using $\mathcal{L}$-formulas, and the validity condition is expressed in terms of labels:

▶ **Definition 12** ($\mathcal{L}$-proof)**.** An **$\mathcal{L}$-proof** is an $\mathcal{L}$-pre-proof such that for every infinite branch $\gamma = (s_i)_{i\in\omega}$, there exists a sequence $(\nu^{L_i}X.G_i)_{i\in\omega}$ and a strictly increasing function $\epsilon$ on natural numbers such that for every $i \in \omega$, (i) the formula $\nu^{L_i}X.G_i$ is principal in $s_{\epsilon(i)}$ (ii) $\lceil\nu^{L_i}X.G_i\rceil = \lceil\nu^{L_{i+1}}X.G_{i+1}\rceil$ and (iii) $L_{i+1} = (L_i, a_i)$ for some $a_i \in \mathcal{L}$.

Note that the label-erasing function $\lceil\bullet\rceil$ is easily lifted to sequents and $\mathcal{L}$-pre-proofs. And if $\pi$ is an $\mathcal{L}$-proof, then $\lceil\pi\rceil$ is a $\mu\mathsf{MALL}^\infty$ proof.

## 3.2 Finite representations of circular $\mathcal{L}$-proofs.

We now turn our attention to finite representations of (circular) $\mathcal{L}$-proofs. Immediately a difficulty occurs in comparison to non-labelled proofs: whereas an infinite non-labelled proof may happen to be regular, a valid $\mathcal{L}$-proof cannot be circular, for, along every infinite branch, the sets of labels will grow endlessly. To form circular proofs with labels, some atoms must be forgotten when going bottom-up.

We introduce two more rules: $(\circlearrowleft(a))$ and $(\mathsf{LW})$. The first one allows to forget one atom, just before recreating it by means of a back-edge to an already encountered $\nu$-rule. The other one allows to forget any atom that will not be used to validate the proof. It is used to synchronise the different labels in a sequent before travelling through a back-edge.

- labelled back-edge: $\dfrac{}{\vdash \nu^{L,a}X.A, \Gamma}\,(\circlearrowleft(a))$ with the constraint that it must be the source of a back-edge to the conclusion of a $\dfrac{\vdash A[\nu^{L,a}X.A], \Gamma}{\vdash \nu^L X.A, \Gamma}\,(\nu_{\mathsf{b}}(a))$ below $(\circlearrowleft(a))$.

- labelled weakening: $\dfrac{\vdash \Gamma, B[\nu^{L} X.A], \Delta}{\vdash \Gamma, B[\nu^{L,a} X.A], \Delta}$ (LW)

▶ **Definition 13** ($\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$)**.** $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ denotes the finite derivations of $\mathcal{L}$-sequents built from the rules in fig. 5 by replacing $(\nu)$ by $(\nu_{\flat}(a))$, $(\circlearrowleft(a))$, (LW), such that (i) the root sequent has empty labels and (ii) in every two $(\nu_{\flat}(a))$ and $(\nu_{\flat}(b))$ occurring in the proofs, $a \neq b$.

The label-erasing function $\lceil \bullet \rceil$ lifts to a translation from $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ to the finite representations of $\mu\mathsf{MALL}^{\omega}$ *pre-proofs*. Every rule of the labelled $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ proof is sent by $\lceil \bullet \rceil$ to a valid rule of unlabelled $\mu\mathsf{MALL}^{\infty}$, except for the (LW) rule, which can safely be removed:

$$\frac{\vdash \Gamma, B[\nu^{L} X.A], \Delta}{\vdash \Gamma, B[\nu^{L,a} X.A], \Delta}\ (\mathsf{LW}) \quad \text{becomes useless} \quad \frac{\vdash \lceil\Gamma\rceil, \lceil B\rceil \left[\nu X. \lceil A\rceil\right], \lceil\Delta\rceil}{\vdash \lceil\Gamma\rceil, \lceil B\rceil \left[\nu X. \lceil A\rceil\right], \lceil\Delta\rceil} \tag{1}$$

Since $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ proofs are finite, label-erasing and unfolding give rise to $\mu\mathsf{MALL}^{\omega}$ pre-proofs:

▶ **Definition 14** ($\mu\mathsf{MALL}^{\circlearrowleft}$)**.** We denote as $\mu\mathsf{MALL}^{\circlearrowleft}$ the set of circular pre-proofs that are obtained from $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ by label-erasing and total unfolding.

▶ **Proposition 15** ($\mu\mathsf{MALL}^{\circlearrowleft} \subseteq \mu\mathsf{MALL}^{\omega}$)**.** *Every pre-proof of $\mu\mathsf{MALL}^{\omega}$ that is the image of a proof in $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ by label-erasing and total unfolding satisfies thread validity.*

**Proof sketch (details are in appendix A, p. 19).** Consider a pre-proof $\lceil\pi\rceil$ in $\mu\mathsf{MALL}^{\circlearrowleft}$ which is the image of an $\mathcal{L}$-proof $\pi$ in $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$. We want to prove that every infinite branch $b$ in $\lceil\pi\rceil$ is contains a valid thread (see def. 7). Let $b_0$ be the corresponding infinite $\mathcal{L}$-branch in $\pi$. Notice that there is a sequent $S_0$ which is the lowest back-edge target crossed infinitely often by $b_0$. Besides, $S_0$ is the conclusion of a $(\nu_{\flat}(a))$ rule, which unfolds some $\nu^{L} X.A$.

We decompose $b_0$, with root $r$ ; $S_0$ conclusion of $(\nu_{\flat}(a))$ and $\nu^{L} X.A$ at position $p_0$ in $S_0$ ; for any $i \geq 1$, $S_i$ conclusion of a back-edge $(\circlearrowleft(a))$ with $\nu^{L,a} X.A$ at position $p_0$ in $S_i$ . Then we notice that $\mathfrak{T}(u_i)(p_0)$ is a thread $(S_0, p_0) \xrightarrow{*} (S_i, p_0)$ which is progressing, as its source is the principal conclusion of the rule $(\nu_{\flat}(a))$. By gluing the $\mathfrak{T}(u_i)(p_0)$ and then erasing labels, we get a valid thread of $b$ in $\lceil\pi\rceil$. ◀

$$\begin{array}{c} \overline{S_i}^{(\circlearrowleft(a))} \\ u_i \uparrow \\ \overline{S_0}^{(\nu_{\flat}(a))} \\ u_0 \uparrow \\ r \end{array}$$

▶ **Proposition 16.** *$\mu$MALL proofs can be translated to $\mu\mathsf{MALL}^{\circlearrowleft}$.*

**Proof.** The target of the usual translation [15] $\mu\mathsf{MALL} \to \mu\mathsf{MALL}^{\omega}$ is included in $\mu\mathsf{MALL}^{\circlearrowleft}$. The key case of this translation is shown in appendix A. ◀

Observe that a proof in $\mu\mathsf{MALL}^{\circlearrowleft}$ is not, in general, the translation of a $\mu$MALL proof.

## 3.3 Two alternative characterizations of $\mu\mathsf{MALL}^{\circlearrowleft}$

In the two following sections, we give two characterizations of $\mu\mathsf{MALL}^{\circlearrowleft}$ through validating sets (def. 20) and through a threading criterion over back-edges (def. 24).

▶ **Definition 17.** Given a directed graph $G = (V, E)$ and a set $S \subseteq V$, the set of vertices from which $S$ is accessible is denoted as $S{\uparrow} := \{v \in V s.t. \exists s \in S, v \to^{*} s\}$. Similarly $S{\downarrow}$ is the set of vertices accessible from $S$.

▶ **Definition 18** ($G_\pi$)**.** For a finite representation $\pi$ of a $\mu\mathsf{MALL}^\omega$ pre-proof, the graph $G_\pi$ is *s. t.* (i) its **vertices** are all positions of $\nu$-formulas in all occurrences of sequents in $\pi$, plus the vertex $\bot$: $V_\pi := \left\{ (v,i,p) \text{ such that } \begin{array}{l} (i)\ v \text{ position of a sequent } \Gamma \text{ in } \pi \\ (ii)\ i \text{ position of a formula } A \text{ in } \Gamma \\ (iii)\ p \text{ position of a } \nu\text{-subformula in } A \end{array} \right\} \uplus \{\bot\}$;
(ii) its **edges** go from a position in a formula to the position that comes from it in the sequent just below, as induced by the threading function of def. 6, or to the extra vertex $\bot$ if it is a cut formula. In case this is a conclusion formula, there is no outgoing edge.

▶ **Definition 19** ($G_r, S_r, T_r$)**.** Let $\pi$ be a finite representation of a $\mu\mathsf{MALL}^\omega$ pre-proof and $(r)$ an occurrence of a $(\nu)$-rule. We define the subgraph $G_r = (V_r, E_r)$ of $G_\pi$ and $S_r, T_r \subseteq V_r$ st:

- **vertices $V_r$** are the extra vertex $\bot$ plus all positions that are in the conclusion of this rule and in all above sequents, that is all sequents from which the conclusion of $(r)$ can be reached, in the sense of def. 17;
- **edges $E_r$** are all edges of $G_\pi$ between those vertices minus the edges of $G_\pi$ that are induced by the back-edges of $\pi$ targetting the conclusion of $(r)$, if there are some.
- **$S_r \subseteq V_r$** is the set of all positions of the principal formulas of the sources sequents of the back-edges targetting the conclusion of $(r)$;
- **$T_r \subseteq V_r$** is the set of all positions of all subformulas of the conclusion of $(r)$ except for the very position of its principal formula, plus the extra vertex $\bot$.

▶ **Definition 20.** Let $(r)$ be an occurrence of a $(\nu)$-rule in a pre-proof $\pi$ of $\mu\mathsf{MALL}^\omega$. A **validating set** for $(r)$ is a set $L \subseteq V_\pi$ such that $L = L{\downarrow}$ and $S_r \subseteq L \subseteq (V_r \setminus T_r)$.

▶ **Proposition 21.** *Let $(r)$ be an occurrence of a $(\nu)$-rule of a pre-proof $\pi$ of $\mu\mathsf{MALL}^\omega$. There exists a validating set for $(r)$ iff $T_r$ is not accessible from $S_r$ in $G_r$ iff $S_r{\downarrow} \subseteq V_r \setminus (T_r{\uparrow})$.*
*In this case, $S_r{\downarrow}$ is the smallest validating set of $(r)$ and $V_r \setminus (T_r{\uparrow})$ is the biggest one.*

**Proof.** It is based on the fact that the complement of a downward-closed set is upward-closed. We then get the inclusions : $S_r \subseteq S_r{\downarrow} \subseteq L{\downarrow} = L \subseteq V_r \setminus (T_r{\uparrow}) \subseteq V_r \setminus T_r$. ◀

The following proposition gives an alternative criterion for $\mu\mathsf{MALL}^{\circlearrowleft}$ (see app. A, p. 19):

▶ **Proposition 22.** *A finite representation $\pi$ of a $\mu\mathsf{MALL}^\omega$ pre-proof is a representation of a $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ proof iff every occurrence of a $\nu$-rule of $\pi$ has a validating set.*

▶ **Proposition 23.** *Checking validity of a $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ pre-proof is decidable. Membership in $\mu\mathsf{MALL}^{\circlearrowleft}$ can be decided in a time quadratic in the size of the (circular) pre-proof.*

**Proof.** The former is immediate. The latter reduces to checking accessibility in a graph for each back-edge target, which can be done in quadratic time. ◀

▶ **Definition 24.** A finite representation of a $\mu\mathsf{MALL}^\omega$ pre-proof finite representation is **strongly valid** when:

**(i)** every back-edge targets the conclusion of a $(\nu)$ rule and

**(ii)** if an occurrence $(r')$ of $\dfrac{\vdash A[\nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}\,(\nu)$ is the target of a back-edge, coming from an occurrence $(r)$ of $\overline{\vdash \nu X.A, \Gamma}^{\,\circlearrowleft}$ then every path $t$ starting from the principal formula $\nu X.A$ of the conclusion of $(r)$, following the thread function (potentially through several back-edges, but never on or below the occurrence $(r')$ of $(\nu)$), ends on the principal formula $\nu X.A$ of the conclusion of $(r')$.

▶ **Proposition 25.** *A finite representation $\pi$ of a $\mu\mathsf{MALL}^{\omega}$ pre-proof is strongly valid iff every $\nu$-rule of $\pi$ has a validating set iff it is the representation of a $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ proof.*

**Proof.** See proof in appendix B, p. 21. ◀

## 4 On Brotherston-Simpson's conjecture: finitizing circular proofs

The aim of this section is to prove a converse of prop. 16: *Every provable sequent of $\mu\mathsf{MALL}^{\circlearrowleft}$ is provable in $\mu\mathsf{MALL}$.*

Let us consider a $\mu\mathsf{MALL}^{\circlearrowleft}$ proof $\pi$. Up to renaming of bound variables, we can assume that all $(\nu_b)$ rules are labelled by distinct labels. For every two labels $a$ and $b$ occurring in $\pi$, we say that $a \leqslant b$ whenever $(\nu_b(a))$ is under $(\nu_b(b))$. This order is well-founded because finite.

▶ **Definition 26.** For every rule $\dfrac{\vdash A[\nu^{V,a}X.A], \Gamma}{\vdash \nu^V X.A, \Gamma}\,(\nu_b(a))$ we define $\Gamma_{(a)}$ to be $\Gamma$.

We now define (i) for each atom $a$ a sequent $\Gamma_a$ formed of non-labelled formulas; (ii) for each formula $A$ (with labels) occurring in the proof, a formula $[\![A]\!]$ without labels:

▶ **Definition 27.** We define by mutual induction: (1) $\Gamma_a := [\![\Gamma_{(a)}]\!]$.
(2) $H_\emptyset[F] := F$ and $H_{V,a}[F] := \otimes\Gamma_a^\perp \oplus H_V[F]$. (*i. e.* $H_V[F]$ is isomorphic to $\left(\bigoplus_{a\in V} \otimes\Gamma_a^\perp\right) \oplus F$.)
(3) By induction on formula $A$ $[\![A]\!]$ is: (i) $[\![\nu^V X.A]\!] := \nu X.H_V[[\![A]\!]]$ (ii) it is homomorphic on other connectives: $[\![X]\!] := X$, $[\![\mathbf{1}]\!] := \mathbf{1}$, $[\![\mu X.A]\!] := \mu X.[\![A]\!]$, $[\![A \otimes B]\!] := [\![A]\!] \otimes [\![B]\!]$, *etc.*
(3) $[\![\cdot]\!]$ is lifted from formulas to sequences of formulas, pointwise.

This is well-founded because since any two distinct $\nu_b$ rules wear distinct variables the only $\Gamma_b$ that are needed in the computation of $\Gamma_a$ are those with $b < a$. Note that $[\![A]\!] = A$ as soon as $A$ has no label variable. We can now state and prove the finitization theorem:

▶ **Theorem 28.** *Every provable sequent of $\mu\mathsf{MALL}^{\circlearrowleft}$ is provable in $\mu\mathsf{MALL}$.*

**Proof.** Let $\pi$ be a $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ proof and replace, everywhere, each formula $A$ by $[\![A]\!]$. All rules in this (almost) new derivation are now valid instances of $\mu\mathsf{MALL}$ rules, except for $(\nu_b)$, $(\mathsf{LW})$ and $(\circlearrowleft)$ rules. Actually, images of these rules by sequent translation $[\![\cdot]\!]$ are derivable in $\mu\mathsf{MALL}$ as shown in fig. 7 (a), (b) and (c) for $(\circlearrowleft)$, $(\mathsf{LW})$ and $(\nu_b)$, respectively.

Replacing each instance of a $(\nu_b)$, $(\mathsf{LW})$ or $(\circlearrowleft)$ rule in $\pi$ by its derived version, we get a fully valid proof of $\mu\mathsf{MALL}$. If the conclusion of the original $\mu\mathsf{MALL}^{\circlearrowleft}$ proof was $\vdash \Gamma$ then what we get is a proof in $\mu\mathsf{MALL}$ of $\vdash [\![\Gamma]\!]$, *i. e.* the conclusion of the original $\mu\mathsf{MALL}^{\circlearrowleft}$ proof, if $\Gamma$ contains no label variable. ◀

## 5 Relaxing the labelling of proofs

In this section, we discuss a possible extension of the labelling defined in section 3, in order to capture more proofs retaining (i) the ability to locally certify the validity and (ii) to some extent, the ability to finitize circular proofs. In order to motivate this extension, we shall consider a simpler example than the one in fig. 3 ($\pi_\infty$).

Let $D$ be an arbitrary formula. Lists of $D$ can be represented as proofs of $L_0 := \mu X.\mathbf{1} \oplus (D \otimes X)$ and it is possible to encode in $\mu\mathsf{MALL}^{\omega}$ the function taking two lists and

(a)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{$(\otimes)$ (Id)}
\UnaryInfC{$\vdash \otimes\Gamma_a^{\perp}, \Gamma_a$}
\RightLabel{$(\oplus_1)$}
\UnaryInfC{$\vdash H_{V,a}\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right], \Gamma_a$}
\RightLabel{$(\nu)$}
\UnaryInfC{$\vdash \llbracket \nu^{V,a}X.A\rrbracket, \Gamma_a$}
\end{prooftree}
$$

(b)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$\vdash H_V\left[\llbracket A[\nu^{V}X.A]\rrbracket\right], H_V\left[\llbracket A[\nu^{V}X.A]\rrbracket\right]$}
\RightLabel{$(\oplus_2)$}
\UnaryInfC{$\vdash H_{V,a}\left[\llbracket A[\nu^{V}X.A]\rrbracket\right], H_V\left[\llbracket A[\nu^{V}X.A]\rrbracket\right]$}
\RightLabel{$(\mu)$}
\UnaryInfC{$\vdash H_{V,a}\left[\llbracket A[\nu^{V}X.A]\rrbracket\right], \llbracket \nu^{V}X.A\rrbracket^{\perp}$}
\RightLabel{$(\nu^{0}_{\mathrm{inv}})$}
\UnaryInfC{$\vdash \llbracket \nu^{V,a}X.A\rrbracket, \llbracket \nu^{V}X.A\rrbracket^{\perp}$}
\doubleLine
\RightLabel{$\llbracket \mathsf{B}\rrbracket$}
\UnaryInfC{$\vdash \llbracket B[\nu^{V,a}X.A]\rrbracket, \llbracket B[\nu^{V}X.A]\rrbracket^{\perp}$}
\AxiomC{$\vdash \llbracket B[\nu^{V}X.A]\rrbracket, \Gamma$}
\RightLabel{(Cut)}
\BinaryInfC{$\vdash \llbracket B[\nu^{V,a}X.A]\rrbracket, \Gamma$}
\end{prooftree}
$$

(c)

$$
\begin{prooftree}
\AxiomC{$\vdash \llbracket A[\nu^{V,a}X.A]\rrbracket, \Gamma_a$}
\RightLabel{$(\oplus_2)^{|V|}$}
\UnaryInfC{$\vdash H_V\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right], \Gamma_a$}
\RightLabel{$(\parr)$}
\UnaryInfC{$\vdash H_V\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right], \parr\Gamma_a$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$\vdash H_V\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right], H_V\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right]$}
\RightLabel{$(\&)$}
\BinaryInfC{$\vdash H_V\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right], H_{V,a}\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right]$}
\RightLabel{$(\mu)$}
\UnaryInfC{$\vdash H_V\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right], \llbracket \nu^{V,a}X.A\rrbracket^{\perp}$}
\AxiomC{}
\RightLabel{$(\otimes)$ (Id)}
\UnaryInfC{$\vdash \otimes\Gamma_a^{\perp}, \Gamma_a$}
\RightLabel{$(\oplus_1)$}
\UnaryInfC{$\vdash H_{V,a}\left[\llbracket A[\nu^{V,a}X.A]\rrbracket\right], \Gamma_a$}
\RightLabel{$(\nu)$}
\UnaryInfC{$\vdash \llbracket \nu^{V,a}X.A\rrbracket, \Gamma_a$}
\RightLabel{$(\nu_{\mathrm{inv}})$}
\BinaryInfC{$\vdash \llbracket \nu^{V}X.A\rrbracket, \Gamma_a$}
\end{prooftree}
$$

**Figure 7** Derivability of (a) $\llbracket(\circlearrowleft)\rrbracket$ rule; (b) $\llbracket(\mathsf{LW})\rrbracket$ rule and (c) $\llbracket(\nu_{\mathsf{b}})\rrbracket$ rule.

(a)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$D \vdash D$}
\AxiomC{(1)}
\RightLabel{$(\circlearrowleft)$}
\UnaryInfC{$L, L \vdash T$}
\RightLabel{$(\parr)$ $(\otimes)$}
\BinaryInfC{$D \otimes L, L \vdash D \otimes T$}
\RightLabel{$(\nu)$}
\UnaryInfC{$L, L \vdash D \otimes T$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$D \vdash D$}
\AxiomC{(1)}
\RightLabel{$(\circlearrowleft)$}
\UnaryInfC{$L, L \vdash T$}
\RightLabel{$(\parr)$ $(\otimes)$}
\BinaryInfC{$L, D \otimes L \vdash D \otimes T$}
\RightLabel{$(\nu)$}
\UnaryInfC{$L, L \vdash D \otimes T$}
\RightLabel{$(\mu)$, $(\&)$}
\BinaryInfC{$L, L \vdash T \quad (1)$}
\end{prooftree}
$$

(b)

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$D \vdash D$}
\AxiomC{(1)}
\RightLabel{$(\circlearrowleft(a))$}
\UnaryInfC{$L^{a+}, L \vdash T$}
\RightLabel{$(\parr)$ $(\otimes)$}
\BinaryInfC{$D \otimes L^{a+}, L \vdash D \otimes T$}
\RightLabel{$(\nu(a))$}
\UnaryInfC{$L^{a-}, L \vdash D \otimes T$}
\RightLabel{$(\mathsf{LW}(b-))$}
\UnaryInfC{$L^{a-}, L^{b-} \vdash D \otimes T$}
\AxiomC{}
\RightLabel{(Id)}
\UnaryInfC{$D \vdash D$}
\AxiomC{(2)}
\RightLabel{$(\circlearrowleft(b))$}
\UnaryInfC{$L^{a-}, L^{b+} \vdash T$}
\RightLabel{$(\parr)$ $(\otimes)$}
\BinaryInfC{$L^{a-}, D \otimes L^{b+} \vdash D \otimes T$}
\RightLabel{$(\nu(b))$}
\UnaryInfC{$L^{a-}, L^{b-} \vdash D \otimes T$}
\RightLabel{$(\mu)$ $(\&)$}
\BinaryInfC{$L^{a-}, L^{b-} \vdash T \quad (2)$}
\RightLabel{$(\mathsf{Rec}(b))$}
\UnaryInfC{$L^{a-}, L \vdash T \quad (1)$}
\RightLabel{$(\mathsf{Rec}(a))$}
\UnaryInfC{$L, L \vdash T$}
\end{prooftree}
$$

**Figure 8** (a) Interleaving example; (b) Interleaving example labelled.
Corresponding sources and targets of back-edges are denoted by parenthesized numbers.

computing the tree of all their possible interleaving, as a proof with conclusion[1] $L_0, L_0 \vdash T_0$, where $T_0 \coloneqq \mu X.L_0 \oplus ((D \otimes X) \,\&\, (D \otimes X))$. By replacing $L_0$ and $T_0$ with $L \coloneqq \mu X.D \otimes X$ and $T \coloneqq \mu X.(D \otimes X) \,\&\, (D \otimes X)$, we get a example equally interesting and more readable, which we present in fig. 8. In this interleaving function, every recursive call leaves one of the two arguments untouched and makes the other one decrease. This guarantees that the tree of recursive calls is well-founded. Difficulties, however, arises from the fact that it is not necessarily always the same argument that will decrease.

More formally: every infinite branch in the preproof above has two interesting threads, going through the $L$ formulas. In every branch going infinitely often to the left (resp. to the right), the thread going through the left $L$ (resp. the right $L$) will be validating. That preproof is thus a valid $\mu\mathsf{MALL}^{\omega}$ proof. However, our previous labelling method cannot be applied here for two reasons:

1. in our previous setting, labelled pre-proof have the property that one can know which thread will validate a branch, just by knowing the lowest target of back-edge that is visited infinitely often by the branch. This is not the case here, because the two back-edges, while inducing different validating threads, have the same target;
2. in our previous setting, back-edges must target $(\nu)$ rules, which is not the case here.

Both difficulties have, in fact, the same origin, namely that in our previous setting the $(\nu)$ rule has two roles: being the target of a back-edge and ensuring thread progression. Both difficulties also have the same solution: dissociating these two roles. We therefore introduce, in def. 29, a new rule $(\mathsf{Rec})$, whose only effect is to allow its premise to be the target of a back-edge, and to introduce a new label. Since $(\mathsf{Rec})$ is disentangled from greatest fixed point

[1] In the following, we write $A \multimap B$ for $A^{\perp} \parr B$, and $\Gamma \vdash \Delta$ for $\vdash \Gamma^{\perp}, \Delta$; exchange rules are left implicit.

unfolding, the labelling must account for the progression of a thread. That is why every atomic label is now given in one of two modes: a passive mode ($a-$) and an active one ($a+$). Only an unfolding by a $(\nu)$ can turn a $-$ into a $+$.

Let us now turn back to our introductory example: $\pi_\infty$. For that example, simply separating the introduction of back-edges and the coinductive progress is not enough. Indeed, since targets of back-edges do not require to unfold a $\nu$, there is *a priori* no reason to require that the sequents contains some $\nu$-formula. While this is slightly hidden in the merge example, $\pi_\infty$ gives a clear example of that and suggests that the $(\mathsf{Rec})$ inference should have the ability to add labels *deeply* in the sequent, *i. e.* not only on the topmost $\nu$ fixed-points, but also to greatest fixed points occurring under some other connectives. The same remark applies to the back-edge rule since its conclusion sequents have the same structure as those of $(\mathsf{Rec})$.

Driven by these observations, we now define a new labelling of circular preproofs and prove its correctness with respect to thread-validity.

▶ **Definition 29** (Extended labelling). Labelled formulas are built on the same grammar as previously, except that labels are lists of **signed variables**, that is of pairs of a variable and a symbol in $\{+,-\}$. Derivations are built with $\mu$MALL inferences plus the following rules:

$$\frac{\vdash \nu^{L} X.A, \Gamma}{\vdash \nu^{L,a-} X.A, \Gamma}\,(\mathsf{LW}(a-)) \quad \frac{\vdash \nu^{L,a-,L'} X.A, \Gamma}{\vdash \nu^{L,a+,L'} X.A, \Gamma}\,(\mathsf{LW}(a+)) \quad \frac{\vdash A[\nu^{a_1+,\dots,a_n+} X.A], \Gamma}{\vdash \nu^{a_1-,\dots,a_n-} X.A, \Gamma}\,(\nu) \quad \frac{\vdash \Gamma[\nu^{L,a-} X.A]}{\vdash \Gamma[\nu^{L} X.A]}\,(\mathsf{Rec}(a)) \quad \frac{}{\vdash \Gamma[\nu^{L,a+} X.A]}\,(\circlearrowleft(a))$$

and the constraints that:

- a cut-formula cannot contain a non-empty label;
- all $(\mathsf{Rec})$ rules must wear distinct variables;
- every $(\mathsf{Rec}(a))$ rule must have at least one occurrence of "$a-$" in its premise;
- each $\dfrac{}{\vdash \Gamma[\nu^{L,a+} X.A]}\,(\circlearrowleft(a))$ rule is connected to the premise of a $\dfrac{\vdash \Gamma[\nu^{L,a-} X.A]}{\vdash B[\nu^{L} X.A], \Gamma}\,(\mathsf{Rec}(a))$ via a back-edge. This implies in particular that this $(\circlearrowleft(a))$ must be above this $(\mathsf{Rec}(a))$ and that the premise of this $(\mathsf{Rec}(a))$ must be the same sequent as the conclusion of this $(\circlearrowleft(a))$ except for the change of sign of $a$, at every of its occurrences in the sequent.

▶ **Proposition 30** (Soundness of labelling). *If $\pi$ is an extended labelled circular representation then $\lceil \pi \rceil$ is a circular representation of a* valid $\mu$MALL$^\omega$ proof.

**Proof.** See proof in appendix C, p. 21. ◀

We now label our two examples with this new system. We will show that, while it is quite straightforward for the interleaving, it requires to unfold one back-edge of $\pi_\infty$.

$\pi_\infty$ is presented labelled according to the extended labelling of fig. 9a. We make $K$ apparent as a subformula of $I$ and $J$ respectively by decomposing:

$$I = I'[K] \qquad J = J'[K] \qquad J'[Y] \coloneqq \mu X.((Y \parr X) \oplus \bot) \qquad I'[Y] \coloneqq \mu Z.((Z \parr J'[Y]) \oplus \bot).$$

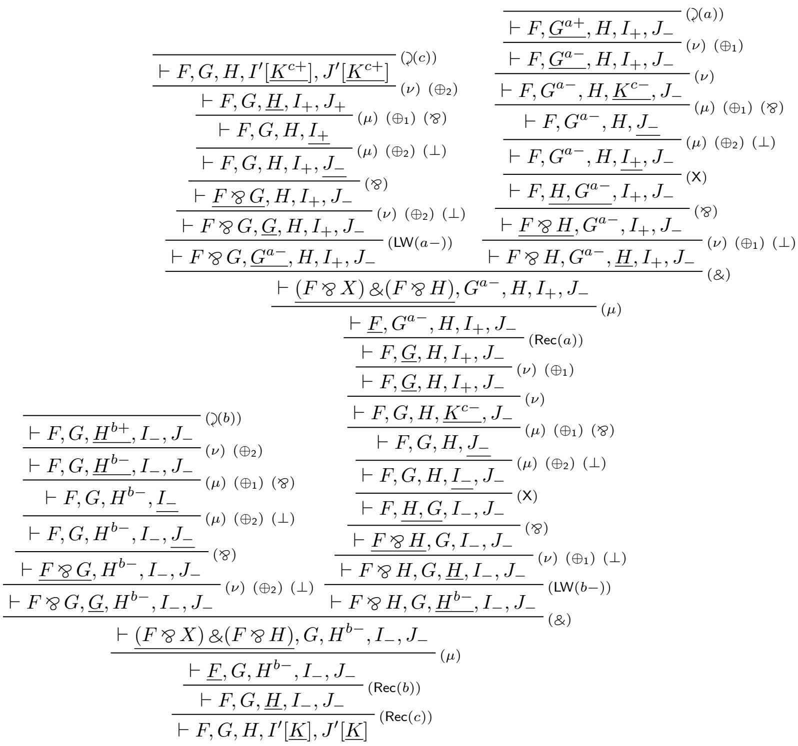
Then we first did one step of unfolding on the right back-edge, and we took advantage of the two new facilites of the extended labelling:

1. we added three $(\mathsf{Rec})$ rules, corresponding to the three ways for a branch of $\pi_\infty$ to be valid, as summarized in the following array.

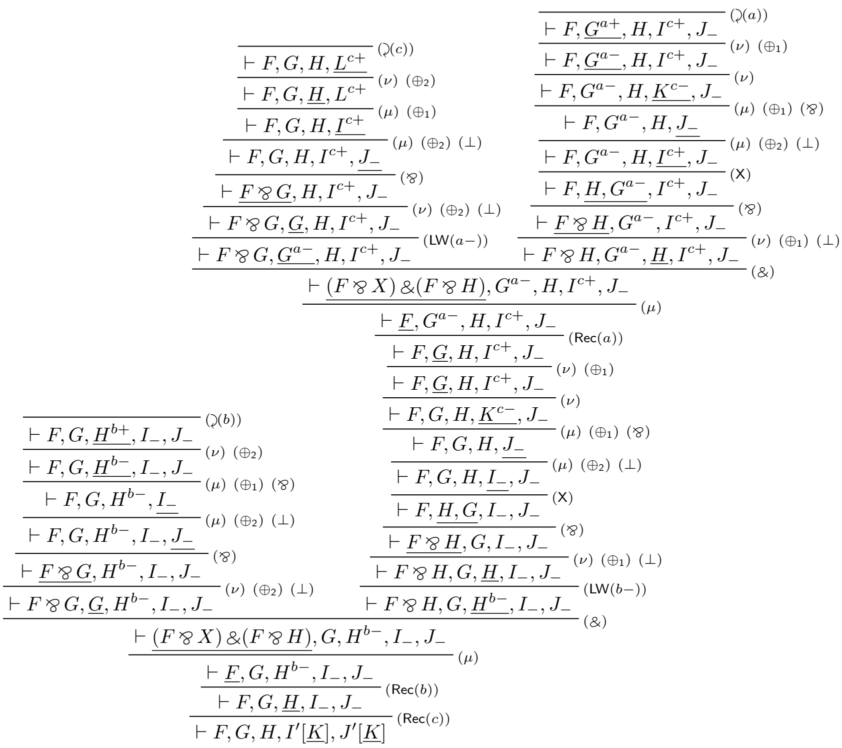
| Shape of the branch | $A^\star \cdot l^\omega$ | $A^\star \cdot r^\omega$ | $l^\star \cdot (r^+ \cdot l^+)^\omega$ |
|---|---|---|---|
| Lowest $(\mathsf{Rec})$ visited $\infty^{\text{ly}}$ | $b$ | $a$ | $c$ |
| Validating $\nu$-formula | $H$ | $G$ | $K$ |

2. and so, we labelled the three formulas $H$, $G$ and $K$ at each corresponding $(\mathsf{Rec})$, using for $K$ the ability to label several occurrences at a time, and to label deeply $\nu$-subformulas.

This indeed forms a correct labelling of $\pi_\infty$ according to the extended labelling, hence ensuring their thread-validity.



**(a)** Labelling of $\pi_\infty$



**(b)** Finitization of $\pi_\infty$. Brackets $[\![\bullet]\!]_{\mathrm{e}}$ shoud be put around every formula and rule name. They were omitted only for the sake of readability.

**Figure 9** We use the following abbreviations: $I_- = I'[K^{c-}]$, $I_+ = I'[K^{c+}]$, $J_- = J'[K^{c-}]$ and $J_+ = J'[K^{c+}]$.

(a) $$\dfrac{\vdash \nu X.\llbracket A\rrbracket_{\mathrm{e}}[\Gamma^\perp \oplus X], \Delta}{\vdash \Gamma^\perp \oplus \nu X.\llbracket A\rrbracket_{\mathrm{e}}[\Gamma^\perp\oplus X],\Delta}\ (\oplus_2)$$

(b) $$\dfrac{}{\vdash \llbracket \nu^{\Gamma+} X.A\rrbracket_{\mathrm{e}}, \Gamma}\ (\oplus_1)\ (\otimes)^\star,\ (\mathsf{Id})$$

(c) $$\dfrac{\dfrac{}{\vdash \llbracket A\rrbracket_{\mathrm{e}}[\Gamma^\perp\oplus \nu X.\llbracket A\rrbracket_{\mathrm{e}}[X]], \mu X.\llbracket A^\perp\rrbracket[X]}\ (\mu)\ [\llbracket \mathsf{A}\rrbracket_{\mathrm{e}}]\ (\oplus_2)\ (\mathsf{Id}) \qquad \nu X.\llbracket A\rrbracket[X],\Delta}{\vdash \nu X.\llbracket A\rrbracket_{\mathrm{e}}[\Gamma^\perp\oplus X],\Delta}\ (\nu_{\mathrm{inv}})$$

(d) $$\dfrac{\dfrac{\dfrac{}{\vdash \llbracket A\rrbracket_{\mathrm{e}}[\llbracket\nu^{\Gamma+}X.A\rrbracket_{\mathrm{e}}], \mu X.\llbracket A^\perp\rrbracket_{\mathrm{e}}[X \,\&\, C]}\ (\mu)\ (\mathsf{Id}) \qquad \dfrac{\dfrac{}{\vdash \llbracket \nu^{\Gamma-}X.A\rrbracket_{\mathrm{e}}, \mu X.\llbracket A^\perp\rrbracket_{\mathrm{e}}[X \,\&\, C]}\ (\mathsf{Id}) \qquad \dfrac{\vdash \llbracket\nu^{\Gamma-}X.A\rrbracket_{\mathrm{e}},\Gamma}{\vdash \llbracket\nu^{\Gamma-}X.A\rrbracket_{\mathrm{e}}, C}\ (\parr)^\star}{\vdash \llbracket\nu^{\Gamma-}X.A\rrbracket_{\mathrm{e}}, (\mu X.\llbracket A^\perp\rrbracket_{\mathrm{e}}[X \,\&\, C]) \,\&\, C}\ (\&)}{\vdash \llbracket A\rrbracket_{\mathrm{e}}[\llbracket\nu^{\Gamma+}X.A\rrbracket_{\mathrm{e}}], (\mu X.\llbracket A^\perp\rrbracket_{\mathrm{e}}[X \,\&\, C]) \,\&\, C}\ (\mathsf{Cut}) \qquad \dfrac{}{\vdash \llbracket\nu^{\Gamma+}X.A\rrbracket_{\mathrm{e}},\Gamma}\ (\oplus_1)\ (\otimes)^\star,\ (\mathsf{Id})}{\vdash \nu X.\llbracket A\rrbracket_{\mathrm{e}},\Gamma}\ (\nu_{\mathrm{inv}})$$

**Figure 10** Derivability of a. $\llbracket(\mathsf{LW}(\Gamma+))\rrbracket_{\mathrm{e}}$ b. $\llbracket(\circlearrowleft(\Gamma))\rrbracket_{\mathrm{e}}$ c. $\llbracket(\mathsf{LW}(\Gamma-))\rrbracket_{\mathrm{e}}$ & d. $\llbracket(\mathsf{Rec}'(\Gamma))\rrbracket_{\mathrm{e}}$ with $C = \parr\,\Gamma$.

## 5.1 Extended finitization

As for the case of our previous labelling, we will rely on the labelled presentation of these proofs in order to finitize them. Observe already that the $(\mathsf{Rec})$ rule, as introduced in def. 29 is never really used in all its power because (i) in both examples above, no $\nu$-formula wears more than one variable and (ii) except for the labelling of $K$ in $\pi_\infty$, $(\mathsf{Rec})$ is used only in the particular form $\dfrac{\vdash \nu^{a-}X.A,\Gamma}{\vdash \nu X.A,\Gamma}\,(\mathsf{Rec}'(a))$ in which only one occurrence of $\nu X.A$ is labelled and this occurrence is a formula of the sequent and not a strict subformula.

We show now how to finitize any labelled representation which verify those two restrictions. As this is the case of fig. 8, it gives a finitization for fig. 8. We will then show how to extend this method in an *ad hoc* way to finitize entirely $\pi_\infty$ (fig. 3) from the labelling of fig. 9a.

As before, it is enough, in order to turn a labelled formula into an unlabelled one, to translate the $\nu$ connectives, leaving all other connectives untouched. For any unlabelled context $\Gamma$, we define the following unlabelled formulas:

$$\llbracket\nu^{\Gamma-}X.A[X]\rrbracket_{\mathrm{e}} \coloneqq \nu X.\llbracket A\rrbracket_{\mathrm{e}}[\otimes\Gamma^\perp \oplus X] \qquad \llbracket\nu^{\Gamma+}X.A[X]\rrbracket_{\mathrm{e}} \coloneqq \otimes\Gamma^\perp \oplus \llbracket\nu^{\Gamma-}X.A[X]\rrbracket_{\mathrm{e}}$$

so the following rules are derivable: (See full derivations on fig. 10, p. 14.)

$$\dfrac{\vdash \llbracket\nu X.A\rrbracket_{\mathrm{e}},\Delta}{\vdash \llbracket\nu^{\Gamma-}X.A\rrbracket_{\mathrm{e}},\Delta}\,\llbracket(\mathsf{LW}(\Gamma-))\rrbracket_{\mathrm{e}} \qquad \dfrac{\vdash \llbracket\nu^{\Gamma-}X.A\rrbracket_{\mathrm{e}},\Delta}{\vdash \llbracket\nu^{\Gamma+}X.A\rrbracket_{\mathrm{e}},\Delta}\,\llbracket(\mathsf{LW}(\Gamma+))\rrbracket_{\mathrm{e}}$$

$$\dfrac{\vdash \llbracket\nu^{\Gamma-}X.A\rrbracket_{\mathrm{e}},\Gamma}{\vdash \llbracket\nu X.A\rrbracket_{\mathrm{e}},\Gamma}\,\llbracket(\mathsf{Rec}'(\Gamma))\rrbracket_{\mathrm{e}} \qquad \dfrac{}{\vdash \llbracket\nu^{\Gamma+}X.A\rrbracket_{\mathrm{e}},\Gamma}\,\llbracket(\circlearrowleft(\Gamma))\rrbracket_{\mathrm{e}}$$

Remark moreover that $\dfrac{\vdash \llbracket A[\nu^{\Gamma+}X.A[X]]\rrbracket_{\mathrm{e}},\Delta}{\vdash \llbracket\nu^{\Gamma-}X.A[X]\rrbracket_{\mathrm{e}},\Delta}\,(\nu)$ is the usual $(\nu)$ rule.

These allow to translate any labelled proof verifying the constraints (i) and (ii) stated at the beginning of sec. 5.1 into a $\mu$MALL finitary proof, by choosing, for every label variable, the context $\Gamma$ corresponding to its $(\mathsf{Rec})$ rule.

These works almost as well for finitizing $\pi_\infty$ based on the labelling of fig. 9a: it allows to expand everything concerning the variables $a$ and $b$. It cannot however be applied as it is to expand the variable $c$, for which conditions (ii) is not verified. We can anyway finitize $\pi_\infty$, but at the cost of a somewhat *ad hoc* translation:

$$\llbracket C\rrbracket_{\mathrm{e}} \coloneqq F \parr G \parr H \qquad \llbracket K^{c-}\rrbracket_{\mathrm{e}} \coloneqq \nu Y.\mu\_.((C^\perp \oplus (I'[Y] \parr J'[Y])) \oplus \bot)$$

$$I^{c+} \coloneqq \llbracket I_+\rrbracket_{\mathrm{e}} = \llbracket I'[K^{c+}]\rrbracket_{\mathrm{e}} \coloneqq \mu\_.((C^\perp \oplus (I'[\llbracket K^{c-}\rrbracket_{\mathrm{e}}] \parr J'[\llbracket K^{c-}\rrbracket_{\mathrm{e}}])) \oplus \bot)$$

$$L^{c+} \coloneqq \llbracket I'[K^{c+}] \parr J'[K^{c+}]\rrbracket_{\mathrm{e}} \coloneqq C^\perp \oplus (I'[\llbracket K^{c-}\rrbracket_{\mathrm{e}}] \parr J'[\llbracket K^{c-}\rrbracket_{\mathrm{e}}])$$

The analysis leading to this choice of formulas is detailed in appendix D, p. 22. It allows to make finitary the derivation of fig. 9b, by expanding every formula as explained above, and by replacing every rule dealing with labels with an appropriate derivation, while leaving untouched the structure of rules not dealing with labels.

## 6 Conclusion

**Summary of the contributions.** In this paper, we contributed to the theory of circular proofs for $\mu$MALL in two directions: (i) identifying fragments of circular proofs for which local conditions account for the validity of circular proof objects (in contrast to the global nature of thread conditions) and (ii) designing methods for translating circular proofs to finitary proofs (with explicit (co)induction rules). To do so, we introduced and studied several labelling systems, for circular proofs, or, more precisely, finite representation thereof, and made the following contributions:

**(i)** First, we investigated how such labellings ensure validity of a labellable proof, turning a global and complex problem into a local and simpler one. Indeed, validity-checking is far from trivial in circular proof-theory for fixed-point logics, the best known bound for this problem being PSPACE. We provide two labellings, a simple and fairly restricted labelling discipline which forces back-edges to target $(\nu)$-inferences and a more liberal one for which we only know that it ensures thread-validity.

**(ii)** Second, we provided evidence on the usability of such labellings as a helpful guide in the generation of (co)inductive invariants which are necessary to translate a circular proof in a finitary proof system with (co)induction rules *à la* Park. We provided a full finitization method in a fairly restricted labelling system which contains at least all the translations of $\mu$MALL proofs. However, this fragment is too constrained to treat standard examples that we discuss in the paper, and which contain most of the difficulties in finitizing circular proofs, namely: (i) interleaving of fixed-points and (ii) interleaving of back-edges resulting in various choices of a valid thread to support a branch.

**Related and future works.** We discuss related works as well as perspectives for pursuing this work along the above-mentioned directions:

**Labelling and local certification** is the basis of our approach. The idea of labelling $\mu$-formulas to gather information on fixed-points unfoldings is naturally not new, already to be found in fixed-point approximation methods (see [14] for instance). The closest work in this direction is Stirling's annotated proofs [25] and the application Afshari and Leigh [1] made of such proofs in obtaining completeness for the modal $\mu$-calculus. Our labelling system works quite differently since only fixed-point operators are labelled while, in Stirling's annotated proofs, every formula is labelled and labels are transmitted to immediate subformulas with a label extension on greatest fixed-points. Despite their difference, the relationships of those systems should be investigated further (in particular the role of the annotation restriction rule of Stirling's system, def. 4 of [25]).

A less immediately connected topic is the connection between size-change termination (SCT) [21] and thread validity in $\mu$-calculi: connections between those fields are not yet well understood despite early investigations by Dax *et al.*[14] for instance. More than a connection, this looks like an interplay: size-change termination is originally shown decidable by using Büchi automata and size-change graphs can be used to show validity of circular proofs [14]. There seems to be connections with our labelling system too.

In addition to investigating more closely those connections, we have several directions for improving our labelled proof system. The first task is to lift the results of section 3 to the extended labelling system. Indeed, for the more restricted fragment and given a circular proof presented as a graph with back-edges, we provided a method to effectively check that one can assign labels. It is therefore natural to expect extending these results to the relaxed framework. Another point we plan to investigate is whether every circular $\mu$MALL proof can be labelled. Even though this can look paradoxical given the complexity of checking validity of circular proofs, one should keep in mind that it might well be the case that, in order to label a circular proof presented as a tree with back-edges, one has to unfold some of the back-edges, or possibly pick a different finite representation of the proof which may result in a space blow up. Related to this question is the connection of our labelling methods with size-change termination methods. Indeed, in designing the extended labelling, one gets closer to the kind of constructions one finds in SCT-based approaches: this should be investigated further since it may also be a key for our finitization objective. Note that the previous two directions would lead to a solution to the Brotherston-Simpson conjecture.

**Finitization of circular proofs** has been recently a very active topic with much research effort on solving Brotherston-Simpson's conjecture. The following recent contributions were made in the setting of Martin-Löf's inductive definitions: firstly, Berardi and Tatsuta proved [6] that, in general, the equivalence is false by providing a counter-example inspired by the *Hydra* paradox. Secondly, Simpson [24] on the one hand and Berardi and Tatsuta [7] on the other hand provided a positive answer in the restricted frameworks when the proof system contains arithmetics. While Simpson used tools from reverse mathematics and internalized circular proofs in $\mathsf{ACA}_0$, a fragment of second-order arithmetic with a comprehension axiom on arithmetical statements, Tatsuta and Berardi proved an equivalent result by a direct proof translation relying on an arithmetical version of the Ramsey and Podelsky-Rybalchenko theorems. A very natural question for future work is to extend the still *ad hoc* finitization method presented in the last section to the whole fragment of relaxed labelled proofs.

**Circular proof search** triggered interest compared to proof system with explicit inductive invariants (lacking subformula property). This has actually been turned to practice by Brotherston and collaborators [9]. We wish to investigate the potential use of labellings in circular proof-search. Indeed, there are several different labellings for a given finite derivation with back-edges where the labels are weakened. Prop. 21 characterizes least and greatest validating sets: those extremal validating sets correspond to different strategies in placing the labels, which have different properties with respect to the ability to form back-edges or to validate the proof that one may exploit in proof-search.

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## A Proofs of section 3

▶ **Lemma 31.** *Let $b$ be an infinite branch in a finite, circular representation, i.e. an infinite ascending path from the root of a tree with back-edges. There is a vertex $s$ in the tree, i.e. an occurrence of sequent in the representation, which is the lowest one infinitely appearing on $b$. Moreover, this vertex / occurrence of sequent is the target of a back-edge.*

**Proof.** This comes only for the tree-with-back-edges structure and does not rely on the proof structure. The crucial fact to notice is that in a tree, if $S$ is a non empty, finite set of vertices that is connected for the relation of comparability, *i.e.* if $\forall v, v' \in S, v \leqslant v'$ or $v' \leqslant v$, then $S$ has a minimum. This is proved by induction on the cardinal of $S$. Take then for $S$ the set of vertices appearing infinitely on the branch $b$, and you get a vertex $v$, which is the desired vertex. In particular, when $v$ is accessed in $b$ from another infinitely appearing vertex, it has to be via a back-edge. ◀

▶ **Lemma 32** (Follow-up of labels). *If $u$ is a path in a labelled circular representation, if $u$ does not cross the rule $(\nu_b(a))$, and if $p$ is a position in the target sequent of $u$ (its top sequent) that is labelled with $a$, then $\mathfrak{t}(u)(p)$ is defined and is a position labelled with $a$ in the source sequent of $u$ (its bottom sequent).*

**Proof.** This is quite straightforward, by induction on the length of $u$, and by looking at the first (or the last) rule crossed by $u$. We use notably the fact that, when the induced thread $\mathfrak{T}(u)(p)$ is followed top-down, the label $a$ cannot be erased because we do not cross $(\mathrm{Rec}(a))$ and the thread cannot reach a cut-formula because cut-formulas do not contain labels. ◀

▶ **Proposition 15.** *Every pre-proof of $\mu\mathsf{MALL}^\omega$ that is the image of a proof in $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ by label-erasing and total unfolding satisfies thread validity.*

**Proof.** Suppose $\pi$ is a labelled circular representation.

- Let $\lceil\pi\rceil$ be its erasure. $\lceil\pi\rceil$ is thus a circular representation of a $\mu\mathsf{MALL}^\omega$ preproof.
- Suppose $b$ an infinite branch of $\lceil\pi\rceil$, that is an infinite ascending path in the tree-with-back-edges $\lceil\pi\rceil$, starting from the root.
- Let $b_0$ be the corresponding infinite branch in $\pi$.
- Le $S_0$ be the occurrence of sequent in $\pi$ which is the lowest back-edge target infinitely often crossed by $b_0$ (lemma 31). Being the target of some back-edge(s), $S_0$ is the conclusion of a $(\nu_b(a))$ rule, which unfolds some $\nu X.A$.
- This implies that $b_0$ is of the form $b_0 = r \xrightarrow[u_0]{*} S_0 \xrightarrow[u_1]{*} S_1 \xrightarrow[\text{be}]{} S_0 \xrightarrow[u_2]{*} S_2 \xrightarrow[\text{be}]{} S_0 \cdots$ where $r$ is the root of $\pi$ and where the $u_i$s do not cross $S_0$ except at their sources.
- Let $p_0 = (0, \epsilon)$ be the position of the principal formula $\nu X.A$ in $S_0$.
- Remark that, because of the existence of back-edges from every $S_{i+1}$ to $S_0$, all $S_i$s are identical sequents, except for the fact that $a$ does not appear in $S_0$ whereas it appears at the only position $p_0$ in $S_{i+1}$.
- Now remark that for $i \geqslant 1$: $\mathfrak{T}(u_i)(p_0)$ is a $\nu$-thread in $u_i$, its target is $p_0$ in $S_i$, which is labelled with $a$, in the occurrence of sequent just above $S_0$, *i. e.* in the premise of $\nu_b(a)$, it goes through a position labelled with $a$ (lemma 32), hence a position of $\nu X.A$ in the unfolding $A[\nu X.A]$, therefore, according to the definition of $\mathfrak{T}$, as described on Figure 6, p. 6, the source of $\mathfrak{T}(u_i)(p_0)$ is again the position $p_0$ of the main formula $\nu X.A$ in $S_0$. To sum up: $\mathfrak{T}(u_i)(p_0)$ is a thread $(S_0, p_0) \xrightarrow[\mathfrak{T}(u_1)(p_1)]{*} (S_1, p_0)$, and it is progressing, because its source is the principal conclusion of the rule $(\nu_b(a))$.
- By glueing the $\mathfrak{T}(u_i)(p_0)$ together, we get an infinite thread

  $$(S_0, p_0) \xrightarrow[\mathfrak{T}(u_1)(p_0)]{*} (S_1, p_0) \xrightarrow[\text{be}]{} (S_0, p_0) \xrightarrow[\mathfrak{T}(u_2)(p_0)]{*} (S_2, p_0) \xrightarrow[\text{be}]{} (S_0, p_0) \cdots$$

  This thread is valid because every $\mathfrak{T}(u_i)(p_0)$ is progressing. And it is indeed a thread of $b_0 = r \xrightarrow[u_0]{*} S_0 \xrightarrow[u_1]{*} S_1 \xrightarrow[\text{be}]{} S_0 \xrightarrow[u_2]{*} S_2 \xrightarrow[\text{be}]{} S_0 \cdots$ Hence $b_0$ is valid, what was to be demonstrated. ◀

▶ **Proposition 16.** *$\mu\mathsf{MALL}$ proofs can be translated to $\mu\mathsf{MALL}^{\circlearrowleft}$.*

**Proof.** The target of the usual translation $\mu\mathsf{MALL} \to \mu\mathsf{MALL}^\omega$ is included in $\mu\mathsf{MALL}^{\circlearrowleft}$. See key case of the translation on figure 11. ◀

$$\dfrac{\vdash A[B], B^{\perp} \quad \vdash B, \Gamma}{\vdash \nu X.A, \Gamma}\,\nu_{\text{inv}} \qquad \equiv \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \nu^{a} X.A, B^{\perp}}\,\curvearrowleft(a)}{\vdash A[\nu^{a} X.A], A[B]^{\perp}}\,[A] \quad \vdash A[B], B^{\perp}}{\vdash A[\nu^{a} X.A], B^{\perp}}\,\text{cut}}{\vdash \nu X.A, B^{\perp}}\,\nu_b(a) \qquad \vdash B, \Gamma}{\vdash \nu X.A, \Gamma}\,\text{cut}$$

**Figure 11** translation $\mu\mathsf{MALL} \to \mu\mathsf{MALL}^{\curvearrowleft}_{\text{lab}}$.

▶ **Proposition 22.** *A finite representation $\pi$ of a $\mu\mathsf{MALL}^{\omega}$ pre-proof is a representation of a $\mu\mathsf{MALL}^{\curvearrowleft}_{\text{lab}}$ proof iff every occurrence of a $\nu$-rule of $\pi$ has a validating set.*

**Proof.** Let us assume that every $\nu$ rule of $\pi$ has a validating set. There is a finite number of $\nu$ rules in the representation; we choose a we label them with distinct variables $a_1, \dots, a_n$, in a way such that if the $\nu$ rule labelled by $a_i$ is below the rule labelled by $a_j$ in the representation then $i \leqslant j$. We denote by $L_i$ a validating set for $\nu(a_i)$. We then do the following for each $i$, going from 1 to $n$: for each occurrence of $\nu$-formula $\nu^{V} X.A$ that is at a position belonging to $L_i$, add the variable $a_i$ to $V$, that is replace this occurrence of $\nu^{V} X.A$ with $\nu^{V,a_i} X.A$. By doing this it may happen that we break the validity of some rules of the representation: because $L_i$, although downward closed, is in general not upward closed, so we may end with the following situation:
$\dfrac{\vdash A, C[\nu^{V} X.D] \quad \vdash A, C[\nu^{V} X.D]}{\vdash A \,\&\, B, C[\nu^{V} X.D]}\,\&$ becoming $\dfrac{\vdash A, C[\nu^{V,a} X.D] \quad \vdash B, C[\nu^{V} X.D]}{\vdash A \,\&\, B, C[\nu^{V,a} X.D]}\,\&$ which is not anymore a valid rule. We then patch this by adding as many (LW) rules as needed on the premises:

$$\dfrac{\vdash A, C[\nu^{V,a} X.D] \quad \dfrac{\vdash B, C[\nu^{V} X.D]}{\vdash B, C[\nu^{V,a} X.D]}\,(\mathsf{LW})}{\vdash A \,\&\, B, C[\nu^{V,a} X.D]}\,\&$$

Similarly it may happen that the source of a back-edge get a bigger labelling than the target of this back-edge; we patch this by adding (LW) rules under the source sequent of the back-edge. When this operation has been done for every $i$, from 1 to $n$, we obtain a validly labelled proof of $\mu\mathsf{MALL}^{\curvearrowleft}_{\text{lab}}$.

Conversely, let $\pi_0$ be a $\mu\mathsf{MALL}^{\curvearrowleft}_{\text{lab}}$ representation such that $\pi = |\pi_0|$. Up to renaming, we can assume that all $(\nu_b)$ rules of $\pi_0$ are labelled with distinct variables. For every $(\nu)$ rule occurrence in $\pi$, consider the corresponding $(\nu_b(a))$ rule in $\pi_0$ and let $L_a$ be the set of all occurrences of $\nu$-formulas in $\pi_0$ that carry the variable $a$ in their labelling. The constraints on the labelling of $\mu\mathsf{MALL}^{\curvearrowleft}_{\text{lab}}$ proof precisely get $L_a$ to be a validating set for the considered occurrence of $(\nu_b)$ in $\pi$. ◀

## B Details and proofs for section 3.3

We illustrate the construction of the edges of the graph defined in definition 18 with the the following examples in which we have indexed the apparent $\nu$-formulas by numbers representing vertices of the graph:

$\dfrac{\vdash \nu_1 X.X, \nu_2 X.X \quad \vdash \mathbf{1} \oplus \nu_3 X.X}{\vdash \nu_4 X.X \otimes (\mathbf{1} \oplus \nu_5 X.X), \nu_6 X.X} \otimes$ induces edges $1 \to 4, 2 \to 6, 3 \to 5$,
$\dfrac{\vdash \nu_1 X.X, (\mathbf{1} \oplus \nu_2 X.X), \nu_3 X.X}{\vdash \nu_4 X.X \parr (\mathbf{1} \oplus \nu_5 X.X), \nu_6 X.X} \parr$ induces edges $1 \to 4, 2 \to 5, 3 \to 6$ and
$\dfrac{\vdash (\nu_4 Y.(\nu_5 X.(\nu_6 Y.X) \otimes X)) \otimes \nu_7 X.(\nu_8 Y.X) \otimes X, \nu_9 X.X}{\vdash \nu_1 X.(\nu_2 Y.X) \otimes X, \nu_3 X.X} \nu$ induces edges $4 \to 2, 6 \to 2, 8 \to 2, 5 \to 1, 7 \to 1, 9 \to 3$. Moreover, if the conclusion of this last rule is the target of a back-edge whose source is $\vdash \nu_{10} X.(\nu_{11} Y.X) \otimes X, \nu_{12} X.X$ then this back-edge also induces edges $1 \to 10, 2 \to 11, 3 \to 12$.

In the case of a cut formula, the formula has no corresponding formula in the conclusion sequent and in this case it induces an outgoing edge, pointing to the extra vertex $\perp$:
$\dfrac{\vdash \nu_2 X.X \quad \vdash \mu X.X, \nu_3 X.X}{\vdash \nu_1 X.X} \text{cut}$ induces edges $2 \to \perp, 3 \to 1$.

▶ **Proposition 25.** *A finite representation $\pi$ of a $\mu$MALL$^\omega$ pre-proof is strongly valid iff every $\nu$-rule of $\pi$ has a validating set iff it is the representation of a $\mu\mathsf{MALL}^{\circlearrowleft}_{\mathrm{lab}}$ proof.*

**Proof.** The second equivalence is prop. 22, so that we need to check the first one:

Let us assume that $\pi$ has a validating set. Let us consider one occurrence $\dfrac{\vdash A[\nu X.A], \Gamma}{\vdash \nu X.A, \Gamma}$ of a $\nu$-rule in $\pi$ and a path $u$ in the subgraph above this $\nu$-rule, going down, from the source of a back-edge targetting this $\nu$-rule, to the $\nu$-rule itself, ending by this $\nu$-rule. $u$ has then premise and conclusion equals to $\vdash \nu X.A, \Gamma$.

Let us denote by $L$ a validating set of this $(\nu)$-rule occurrence, and let us denote by $t$ the maximal thread going down in $u$ starting from the main $\nu X.A$ in its premise. This occurrence of $\nu X.A$ is in $L$, because $L$ is a validating set. Then, because $L$ is downward closed, all vertices of $t$ are in $L$. Therefore the lowest vertex of $t$, which is a position in the $\vdash \nu X.A, \Gamma$ conclusion of the considered $\nu$-rule, or $\perp$, is also in $L$. But in this last sequent occurrence, the only position that is in $L$ is the one of the main $\nu X.A$, which is consequently the end point of $t$.

Conversely, let us consider an occurrence of a $(\nu)$-rule in $\pi$, whose conclusion has the form $\vdash \nu X.A, \Gamma$, and let us assume that it has no validating set. It is, by prop. 21, equivalent to say that there is a path $t$ such that:

- $t$ stays above the considered occurrence of $(\nu)$-rule;
- $t$ goes down from the source $\nu X.A, \Gamma$ of a back-edge targetting the $(\nu)$-rule we consider, to the conclusion $\nu X.A, \Gamma$ of this $(\nu)$-rule;
- $t$ starts from the main $\nu X.A$ of its premise;
- $t$ ends either on a cut-formula or on a position that is not the principal $\nu X.A$.

$u$ therefore violates strong validity (def. 24). ◀

## C Details and proofs for section 5

Remember that this proposition is about the extended labelling of def. 29:

▶ **Proposition 30.** *If $\pi$ is an extended labelled circular representation then $\lceil \pi \rceil$ is a circular representation of a* valid $\mu$MALL$^\omega$ proof.

**Proof.** First remark that lemma 32, as it is stated on p. 19, still holds for this extended labelling. The proof is the same as before, bearing in mind to replace every mention of $(\nu_b(a))$ with $(\mathsf{Rec}(a))$. As for the previous labelling, the proof of this proposition crucially rely on it.

Suppose $\pi$ is a labelled circular representation. Let $\lceil\pi\rceil$ be its erasure. $\lceil\pi\rceil$ is thus a circular representation of a $\mu\mathsf{MALL}^\omega$ preproof. Suppose $b$ an infinite branch of $\lceil\pi\rceil$, that is an infinite ascending path in the tree-with-back-edges $\lceil\pi\rceil$, starting from the root. Let $b_0$ be the corresponding infinite branch in $\pi$. Le $S_0$ be the occurrence of sequent in $\pi$ which is the lowest back-edge target infinitely often crossed by $b_0$. Being the target of some back-edge(s), $S_0$ is the premise of a $(\mathrm{Rec}(a))$ rule, for some variable $a$.

This implies that $b_0$ is of the form $b_0 = r \underset{u_0}{\overset{*}{\to}} S_0 \underset{u_1}{\overset{*}{\to}} S_1 \underset{\mathrm{be}}{\to} S_0 \underset{u_2}{\overset{*}{\to}} S_2 \underset{\mathrm{be}}{\to} S_0 \cdots$ where $r$ is the root of $\pi$ and where the $u_i$ do not cross $S_0$ except at their sources.

Remark that the positions labelled by $a$ are the same in all $S_i$, as there are back-edges from every $S_{i+1}$ to $S_0$. The difference, however, is that these positions are labelled with $a-$ in $S_0$ and with $a+$ in every $S_{i+1}$. Let $P_0$ be the set of those positions. $P_0$ is finite and non empty. Now we would like, as in the proof of prop. 15, to construct an infinite thread along $b_0$. However, because $P_0$ may contain more than one element, we cannot know by advance, for each $S_i$, which $p \in P_0$ will support an infinite thread. Thus, we will use Kőnig's lemma to show the existence of such a thread. Let $T_0$ be the tree whose vertices are the pairs $(i, p)$ where $1 \leqslant i < \omega$ and $p \in P_0$, whose roots are the vertices of the form $(1, p)$ and where, for $i > 1$, the father of $(i, p)$ is[2] $(i - 1, \mathfrak{t}(u_i)(p))$. Here we have to prove that $\mathfrak{t}(u_i)(p)$ is defined and that it belongs to $P_0$ for every $i$ and $p \in P_0$. This is ensured by lemma 32 thanks to the labels.

Remark that every edge in $T_0$ induces a progressing thread. Indeed, for $i \geqslant 1$ and $p \in P_0$:

- $\mathfrak{T}(u_i)(p)$ is a $\nu$-thread in $u_i$,
- its target is $p$ in $S_i$, which is labelled with $a+$
- and its source is $p$ in $S_0$, which is labelled with $a-$.

An examination of the rules that may compose $u_i$ shows that the only way for that to be true is that $\mathfrak{T}(u_i)(p)$ is progressing. Now $T_0$ is an infinite tree with a finite number of roots and an arity bounded by $\mathbf{Card}(P_0)$, hence, by Kőnig's lemma, it has an infinite branch $(1, p_1) \leftarrow (2, p_2) \leftarrow (3, p_3) \cdots$.

This infinite branch induces in turn an infinite thread

$$(S_0, p_0) \xrightarrow[\mathfrak{T}(u_1)(p_1)]{*} (S_1, p_1) \underset{\mathrm{be}}{\to} (S_0, p_1) \xrightarrow[\mathfrak{T}(u_2)(p_2)]{*} (S_2, p_2) \underset{\mathrm{be}}{\to} (S_0, p_2) \cdots$$

This thread is valid because every $\mathfrak{T}(u_i)(p_i)$ is progressing. And it is indeed a thread of $b_0 = r \underset{u_0}{\overset{*}{\to}} S_0 \underset{u_1}{\overset{*}{\to}} S_1 \underset{\mathrm{be}}{\to} S_0 \underset{u_2}{\overset{*}{\to}} S_2 \underset{\mathrm{be}}{\to} S_0 \cdots$ Hence $b_0$ is valid, what was to be demonstrated. ◀

## D Details of finitization for $\pi_\infty$

To finitize $\pi_\infty$ we try to apply the same method as for the example (8) p. 11, by expanding every labelled formula to a non-labelled one and expanding the rules that need it to match these transform. This works perfectly for $H$ and $G$, which appear respectively as formulas of the premises $(\mathrm{Rec}(b))$ and $(\mathrm{Rec}(a))$. But the situation is more delicate for $K$ for which we have to face a double difficulty: in the premise of $(\mathrm{Rec}(c))$, $K$ is not a formula of the sequent but a subformula, and it appears in two different formulas.

Let us try to transform this situation into one that would fit our method. First we would like to have only one formula containing $K$ instead of the two $I$ and $J$. Unfortunately, none of them can be unlabelled without breaking the labelling. Fortunately the solution to that is easy: $I, J$ is simply equivalent to $L \coloneqq I \parr J$.

[2] Recall that $\mathfrak{t}(u)$ and $\mathfrak{T}(u)$ are defined in defs. 6 and 8, p. 6.

Now we would like $I \parr J$ to be a $\nu$-formula that we could label. We already made use, in the previous example, of the isomorphism $A[\nu X.B[A[X]]] \simeq \nu X.A[B[X]] \quad (*)$

to turn an almost-$\nu$-formula into a real one. Let us apply that again.

The formula $L = I \parr J$ is equal to $L'[K]$ where $L'[Y] := I'[Y] \parr J'[Y]$, that is: $L = L'[\nu Y.I'[Y]]$. In order to apply an isomorphism of the form $(*)$ we would like $I'[Y]$ to be of the form $M'[L'[Y]]$ for a given $M'$. This is unfortunately not the case as $I'[Y]$ is a subformula of $L'[Y]$. However, a careful examination of the flow of $I$, $J$ and $K$ along the loops of $\pi_\infty$ makes apparent the fact that

$$I'[Y] = \mu Z.((Z \parr J'[Y]) \oplus \bot) \simeq \mu\_.((I'[Y] \parr J'[Y]) \oplus \bot) = M'[L'[Y]]$$

where $M'[Y]$ is defined to be $\mu\_.(Y \oplus \bot)$, in which we use the notation $\mu\_.A$ to denote a $\mu X.A$ with $X$ not appearing free in $A$. This degenerate $\mu$ binder could be removed to simplify the formulas involved in the finitisation, but we keep it to stay as close as possible to the original structure of $I$, trying to preserve its head connective.

When we stick all that together we get $L = I \parr J \simeq L'[\nu Y.M'[L'[Y]]] \simeq \nu Y.L'[M'[Y]]$ which is a $\nu$-formula that we know, when labelled, how to expand into an unlabelled formula. If we stopped here our analysis, we would then define:

$$C := F \parr G \parr H \qquad L^{c-} := \nu Y.L'[M'[C^\perp \oplus Y]] \qquad L^{c+} := C^\perp \oplus L^{c-}.$$

However we will do yet a bit more work in order to get the structure of $L^{c-}$ closer to $L$'s one.

Indeed the isomorphism $(*)$ can be used in the other direction:

$$\nu Y.L'[M'[C^\perp \oplus Y]] \simeq L'[\nu Y.M'[C^\perp \oplus L'[Y]]] = I'[\nu Y.M'[C^\perp \oplus L'[Y]]] \parr J'[\nu Y.M'[C^\perp \oplus L'[Y]]].$$

This, finally, leads us to define: $C := F \parr G \parr H \quad K^{c-} := \nu Y.M'[C^\perp \oplus L'[Y]]$ which allows to expand $I'[K^{c-}]$ and $J'[K^{c-}]$. On the other hand, this is not sufficient to define an expansion of $K^{c+}$, and we still need an *ad hoc* treatment for formulas containing it:

$$\text{“}I'[K^{c+}]\text{”} := I^{c+} := M'[C^\perp \oplus L'[K^{c-}]] \qquad \text{“}I'[K^{c+}] \parr J'[K^{c+}]\text{”} := L^{c+} := C^\perp \oplus L'[K^{c-}]$$

With these expansions of labelled formulas into unlabelled formulas, we can finitize the derivation of fig. 9a into the very close derivation of fig. 9b, on which the rules dealing with labelling can be expanded into $\mu$MALL derivations.