Acyclic Colouring of Graphs on Surfaces

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2018

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

An acyclic k-colouring of a graph G is a proper k-colouring of G with no bichromatic cycles. In 1979, Borodin proved that planar graphs are acyclically 5-colourable, an analog of the Four Colour Theorem. Kawarabayashi and Mohar proved in 2010 that "locally" planar graphs are acyclically 7-colourable, an analog of Thomassen's result that "locally" planar graphs are 5-colourable. We say that a graph G is *critical* for (acyclic) k-colouring if G is not (acyclically) k-colourable, but all proper subgraphs of G are. In 1997, Thomassen proved that for every $k \geq 5$ and every surface S, there are only finitely many graphs that embed in S that are critical for k-colouring. Here we prove the analogous result that for all $k \geq 12$ and each surface S, there are finitely many graphs embeddable on S that are critical for acyclic k-colouring. This result implies that there exists a linear time algorithm that, given a surface S and $k \geq 12$, decides whether a graph embedded in S is acyclically k-colourable.

Acknowledgements

First, I would like to thank my supervisor, Luke Postle, for his support and guidance. I would also like to thank him for suggesting the topic of this thesis and for the many opportunities I have had over the last two years because of him.

I would like to thank my readers, Jim Geelen and Peter Nelson, for taking the time to read my thesis.

Additionally, I would like to thank Gary MacGillivray for originally encouraging me to pursue research and for the undergraduate opportunities that gave me the confidence to continue in graduate school.

I would also like to express my gratitude to the Department of Combinatorics and Optimization, the University of Waterloo, the Ontario Ministry of Advanced Education and Skills Development, and the Natural Sciences and Engineering Research Council of Canada.

Finally, I would like to thank my friends and family for their constant support and encouragement. I would specifically like to thank Jesse for everything he does for me and for us. I would not have been able to finish this thesis without him.

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Chapter 1

Introduction

A *proper colouring* ϕ of a graph G is a map $\phi : V(G) \to \mathbb{Z}$ such that for all $e = uv \in E(G)$, we have that $\phi(u) \neq \phi(v)$. In this thesis, all colourings are proper. We say a colouring is *acyclic* if there are no bichromatic cycles in the colouring. If a graph G has a k-colouring, then we say G is *k-colourable*. Similarly, if a graph G has an acyclic k-colouring, then we say G is *acyclically k-colourable*. The *chromatic number* of a graph G, denoted $\chi(G)$, is equal to the least integer k such that G is k-colourable. Similarly, the *acyclic chromatic number* of a graph G, denoted $\chi_a(G)$, is equal to the least integer k such that G is acyclically k-colourable.

Acyclic colouring was introduced by Grünbaum [11] in 1973 when he proved that planar graphs are acyclically 9-colourable and conjectured that planar graphs are acyclically 5-colourable. This conjecture was proved in 1979 by Borodin [6] as follows.

Theorem 1.0.1 (Borodin [6])**.** *Every planar graph is acyclically 5-colourable.*

Grünbaum also showed in [11] that five colours are necessary to acyclically colour a planar graph. Hence, the constant in Theorem 1.0.1 is best possible. Notice that Theorem 1.0.1 could be considered an acyclic analog of the Four Colour Theorem. This answers the question of how many colours are sufficient to acyclically colour a planar graph; however, it would be interesting to know how many colours are sufficient to acyclically colour graphs that embed in other surfaces.

A *surface* is a connected, compact, 2-dimentional manifold without boundary. By the classification theorem of surfaces, every surface S is obtained from the sphere by adding a handles and b crosscaps. The *Euler genus* of S is defined as $2a + b$. For colouring, we have Heawood's well-known theorem from 1890, which says that a graph embedded in a surface S with Euler genus $g > 0$ can be coloured with at most $\lfloor (7+\sqrt{24g+1})/2 \rfloor$ colours. In 1996, Alon, Mohar, and Sanders [3] proved that a graph embedded in a surface S with Euler genus g can be acyclically coloured with at most $100g^{4/7} + 10000$ colours. Notice that this result could be considered an acyclic analog of Heawood's theorem.

Since the problem of determining the maximum chromatic and maximum acyclic chromatic numbers of graphs embedded in a given surface has been solved by Heawood and Alon, Mohar, and Sanders, we look to a more modern approach to colouring graphs on surfaces, initiated by Thomassen in the 1990's.

Thomassen's work in the 1990's included the concept of "locally" planar graphs. We will say a graph G embedded in a surface S is *ρ-locally-planar* if every non-contractible cycle has length at least ρ. In 1993, Thomassen proved that there exists ρ for each surface S such that every ρ-locally-planar graph G embedded in S is 5-colourable [17]. An analog of this theorem for acyclic colouring was proven in 2010 by Kawarabyashi and Mohar [12], as follows.

Theorem 1.0.2 (Kawarabyashi and Mohar, [12])**.** *There exists ρ for each surface S such that every ρ-locally-planar graph G embedded in S is acyclically 7-colourable.*

Thomassen's program from the 1990's also included "critical" graphs, although this concept, in the context of colouring, dates back to the 1950's. We say a graph G is *critical* for (acyclic) k-colouring if G is not (acyclically) k-colourable, but all proper subgraphs of G are. In 1953, Dirac proved that for every $k \geq 7$ and every surface S there are only finitely many graphs that are critical for k-colouring that embed in S [8]. This was improved to $k \geq 6$ by Gallai in 1963 [10] and improved again in 1997 by Thomassen to $k \geq 5$ [18], as follows.

Theorem 1.0.3 (Thomassen, [18])**.** *For every $k \geq 5$ and every surface S there are only finitely many graphs that are critical for k-colouring that embed in S.*

This result actually implies Thomassen's theorem from 1993 that there exists ρ for each surface S such that every ρ-locally-planar graph G embedded in S is 5-colourable. Another consequence of Theorem 1.0.3 is that for every surface S and every $k \geq 5$ there exists a linear time algorithm that decides whether a graph embedded in S is k-colourable.

Now, we are interested to know if there is an acyclic analog of Theorem 1.0.3, for *any* value of k. It is not clear why an equivalent result is possible since vertices of small degree are not as useful when acyclic colouring as they are when colouring. For example, graphs which are critical for k-colouring do not contain vertices with degree less than k. Unfortunately, this is not true for graphs which are critical for acyclic k-colouring. To see this, consider K_n, the complete graph on n vertices, with one edge subdivided once. Call this graph G and let v be the vertex of degree 2 on the subdivided edge and let u and w be the neighbours of v. The only way to colour $G - v$ with $n - 1$ colours is to give u and w the same colour and give all other vertices pairwise distinct colours. Now, we try to colour v in order to get an acyclic $(n - 1)$-colouring of G; however, every colour for v results in a colouring of G with a bichromatic cycle. Since every proper subgraph of G is $(n - 1)$-colourable, we have that G is critical for acyclic $(n - 1)$-colouring.

Despite this challenge, we prove an acyclic analog of Theorem 1.0.3 in this thesis, as follows.

Theorem 1.0.4. *For every $k \geq 12$ and every surface S there are only finitely many graphs that are critical for acyclic k-colouring that embed in S.*

This theorem implies that there exists ρ for each surface S such that every ρ-locally-planar graph G embedded in S is acyclically 12-colourable, a version of Theorem 1.0.2. Theorem 1.0.4 also implies that there exists a linear time algorithm that, given a surface S and $k \geq 12$, decides whether a graph embedded in S is acyclically k-colourable.

In Chapter 2, we start by reviewing the history of acyclic colouring and colouring graphs on surfaces. This is followed by an explanation of how we reduce Theorem 1.0.4 to a problem about planar graphs. Finally, we give an outline for the remainder of the thesis.

Chapter 2

Background

2.1 History

The last 150 years has seen many results on colouring graphs on surfaces and, more recently, on acyclic colouring. In this section, we present a brief history of colouring graphs on surfaces and of acyclic colouring.

2.1.1 Colouring on Surfaces History

The topic of colouring graphs on surfaces arose in 1852 with Francis Guthrie's conjecture that all planar graphs are 4-colourable. The Four Colour Conjecture was left open for over 100 years, until it became known as the Four Colour Theorem in 1977 when Appel and Haken offered a proof [4, 5]. Notice that since there exist planar graphs which are not 3-colourable, we have that the Four Colour Theorem is tight.

During the time when the Four Colour Conjecture was still open, some other results about colouring graphs on surfaces surfaced, including the well-known theorem from Heawood in 1890 which says that a graph embedded in a surface S with Euler genus $g > 0$ can be coloured with at most $\lfloor (7 + \sqrt{24g+1})/2 \rfloor$ colours. In 1968, Ringel and Youngs [16] proved that this bound is tight for every surface except the Klein bottle.

As mentioned in the Introduction, the problem of determining the maximum chromatic numbers of graphs embedded in a given surface has been solved, so at this point we turn to Thomassen's approach to colouring graphs on surfaces from the 1990's. Thomassen's program included "locally" planar graphs and "critical" graphs. Recall that a graph G

embedded in a surface S is ρ-locally-planar if every non-contractible cycle has length at least ρ. Thomassen proved in 1993 that there exists ρ for each surface S such that every ρ-locally-planar graph G embedded in S is 5-colourable [17].

Interestingly, Thomassen's following result about critical graphs from 1997 implies the above locally planar result. Recall that a graph G is critical for k-colouring if G is not k-colourable, but all proper subgraphs of G are. In 1997, Thomassen [18] proved that for every surface S there are only finitely many graphs that are critical for 5-colouring that embed in S. This result improves upon the theorems of Dirac [8] and Gallai [10].

2.1.2 Acyclic History

In 1973, Grünbaum [11] proved that every planar graph is acyclically 9-colourable. He also gave an example of a planar graph that can not be acyclically coloured with four colours. This motivated his conjecture that every planar graph is acyclically 5-colourable. In 1974, Mitchem [14] improved Grünbaum's result by proving that every planar graph is acyclically 8-colourable. This was improved again in 1976 by Kostochka [13] who showed that every planar graph is acyclically 6-colourable. Independently, in 1977, Albertson and Berman [1] proved that every planar graph is acyclically 7-colourable. Grünbaum's conjecture was finally proved in 1979 when Borodin [6] showed that every planar graph is acyclically 5-colourable.

Acyclically colouring planar graphs is still a topic of study; more recent results focus on planar graphs without cycles of certain lengths. However, there has also been progress regarding acyclically colouring graphs in general. Let $\Delta(G)$ denote the maximum degree of the graph G and let $\chi_a(G)$ denote the acyclic chromatic number of G. For $d \in \mathbb{N}$, let $\chi_a(d) = \max\{\chi_a(G) : \Delta(G) = d\}$. In 1991, Alon, McDiarmid, and Reed [2] proved the following:

Theorem 2.1.1 (Alon, McDiarmid, and Reed; [2])**.** $\chi_a(d) = O(d^{4/3})$.

They also proved that there exist graphs such that $\chi_a(d) = \Omega(d^{4/3}/(\log\, d)^{1/3})$; hence, Theorem 2.1.1 is tight up to a factor of $(\log\, d)^{1/3}$.

In terms of acyclically colouring graph on surfaces, we have a result of Alon, Mohar, and Sanders [3] from 1996, as mentioned in the Introduction. They proved that a graph embedded in a surface S with Euler genus g can be acyclically coloured with at most $100g^{4/7} + 10000$ colours. Recall that this result can be seen as an acyclic analog to Heawood's theorem. Alon, Mohar, and Sanders also showed that for $g > 0$ there exist graphs that embed in a surface with Euler genus g whose acyclic chromatic number is at least $\Omega(g^{4/7}/(\log g)^{1/7})$. Thus, their bound is tight up to a factor of $(\log g)^{1/7}$.

Several years later, acyclic colouring joined the modern approach to colouring on surfaces with a result about locally planar graphs. Kawarabayashi and Mohar [12] proved in 2010 that there exists ρ for each surface S such that every ρ-locally-planar graph G embedded in S is acyclically 7-colourable. This result can be seen as an acyclic analog to Thomassen's 1993 result about locally planar graphs.

2.2 Hyperbolic Theory

This section will give a brief introduction to the hyperbolic theory developed by Postle and Thomas [15], and will explain how their results will be applied in this thesis. We refer the reader to [15] for all formal definitions and theorems.

We say a family $\mathcal{F}$ of graphs is *hyperbolic* if there exists a constant $c > 0$ such that if $G \in \mathcal{F}$ is a graph embedded in a surface Σ, then for every closed curve $\gamma : \mathbb{S}^1 \to \Sigma$ that bounds an open disk Δ and intersects G only in vertices, then the number of vertices of G in Δ is at most $c(|\{x \in \mathbb{S}^1 : \gamma(x) \in V(G)\}| - 1)$. This definition has a natural strengthening, as follows. We say a family $\mathcal{F}$ of graphs is *strongly hyperbolic* if $\mathcal{F}$ is hyperbolic and there exists $c' > 0$ such that if $G \in \mathcal{F}$ is a graph embedded in a surface Σ, then for every two closed curves $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \Sigma$ that bound an open annulus Δ and intersect G only in vertices, then the number of vertices of G in Δ is at most $c(|\{x \in \mathbb{S}^1 : \gamma_1(x) \in V(G) \text{ or } \gamma_2(x) \in V(G)\}| - 1)$.

In [15], Postle and Thomas prove a more general version of the following theorem.

Theorem 2.2.1 (Postle and Thomas, [15])**.** *For every strongly hyperbolic family $\mathcal{F}$ of embedded graphs that is closed under curve cutting there exists a constant $\beta > 0$ such that every graph $G \in \mathcal{F}$ embedded in a surface of Euler genus g has at most βg vertices.*

Let $\mathcal{F}$ be the family of graphs which are critical for acyclic k-colouring, where $k \geq 12$. The goal of this thesis is to prove that $|\{G \in \mathcal{F} : G \text{ embeds in } S\}|$ is bounded above for each surface S. However, if we instead prove that $\mathcal{F}$ is strongly hyperbolic, then it follows from Theorem 2.2.1 that $|\{G \in \mathcal{F} : G \text{ embeds in } S\}|$ is bounded above for each surface S. Thus, we focus the remainder of this thesis to proving that $\mathcal{F}$ is strongly hyperbolic.

In order to prove that $\mathcal{F}$ is strongly hyperbolic, we first prove that $\mathcal{F}$ is hyperbolic. This is done by bounding the number of vertices in a plane graph G with outer cycle C with respect to the number of vertices in C, where G is a subgraph of a graph $G' \in \mathcal{F}$. The Main Theorem 5.3.5 of this thesis aims to establish this bound.

2.3 Thesis Outline

The remainder of this thesis is organized as follows. The goal of Chapter 3 is to define the key concept which will allow us to properly discuss the idea of extending an acyclic colouring. In Section 3.1, we start by formalizing some basic definitions regarding colouring and acyclic colouring, and present some graph notation which will be used throughout the thesis. Section 3.2 defines a "mosaic", which is the key concept that will allow us to explain how acyclic colourings can be extended. Section 3.3 rounds out the chapter with a collection of "mosaic" properties which will be used throughout later chapters.

Chapter 4 focuses on a set of extension lemmas, which will give insight into the structure of graphs which are critical for acyclic colouring. Some preliminary definitions are given in Section 4.1, followed by the Extension Lemma 4.2.1 in Section 4.2. Section 4.3 is dedicated to proving the "Fourth Generation" Lemma 4.3.12.

In Chapter 5 we establish a variety of preliminary lemmas and then prove the main result of this thesis. Section 5.1 contains the proofs of the Key Lemma 5.1.2, which uses results from Section 3.3, and the General Structure Lemma 5.1.4, which follows almost immediately from the Extension Lemma 4.2.1. In Section 5.2, we confirm several bounds which are used in the proof of the Main Theorem 5.3.5, which is given in Section 5.3.

The goal of Chapter 6 is to show how the Main Theorem 5.3.5 implies that the family of graphs which are critical for acyclic k-colouring, where $k \geq 12$, is strongly hyperbolic. Section 6.1 aims to prove that this family is hyperbolic, while Section 6.2 shows how the hyperbolic results extend to strongly hyperbolic.

Chapter 3

Mosaics

We begin this chapter with a section containing several basic definitions. This is followed by a section which will define the concept of a "mosaic". Finally, the last section in this chapter contains some basic properties about mosaics.

In this thesis, a graph G is an ordered pair (V, E) where V is a set of vertices and E is a set of 2-element subsets of V called edges. We write $V(G)$ for V and $E(G)$ for E. Also note that in this thesis, we will always use k to denote a natural number. Furthermore, we always use the colours $[k]$ when k-colouring a graph, which isn't always standard, but it will simplify later definitions.

3.1 Initial Definitions

A *colouring* of a graph G is an assignment of labels to the vertices of G such that two adjacent vertices do not receive the same label. A *k-colouring* of a graph G is a colouring that uses labels from $[k]$. In this thesis, we will often want to refer to subgraphs of a graph G which contain only vertices of a certain colour under some colouring of G. Specifically, we care about subgraphs made up of vertices in two fixed colour classes.

Definition 3.1.1. Let G be a graph with a k-colouring ϕ. For each $i \neq j \in [k]$, we denote the graph induced on the vertices that receive colour i or j in ϕ by $G_{ij}(\phi)$. That is, $G_{ij}(\phi) = G[\phi^{-1}(i) \cup \phi^{-1}(j)]$.

Definition 3.1.2. Let G be a graph with a subgraph H. We say a colouring ϕ_H of H *extends* to a colouring ϕ_G of G if $\phi_H(v) = \phi_G(v)$ for all $v \in V(H)$.

Definition 3.1.3. Let G be a graph with a subgraph H. Let ϕ be a k-colouring of G. We say that ϕ' is the *restriction* of ϕ to H if ϕ' is the k-colouring of H where $\phi'(v) = \phi(v)$ for all $v \in V(H)$. Let $\phi_{|H}$ denote the restriction of ϕ to H.

Recall that a colouring is considered *acyclic* if it contains no bichromatic cycles, or equivalently the following definition.

Definition 3.1.4. An *acyclic k-colouring* of a graph G is a k-colouring ϕ where $G_{ij}(\phi)$ is acyclic, for all colours $i \neq j \in [k]$.

The following two definitions give a formal definition of the neighbourhood and second neighbourhood of a vertex.

Definition 3.1.5. Let G be a graph with $u, v \in V(G)$. The distance between u and v, denoted $\mathrm{dist}_G(u, v)$, is the length of a shortest path between u and v in G. If the graph is clear, we drop the subscript and write $\mathrm{dist}(u, v)$.

Definition 3.1.6. Let G be a graph with $v \in V(G)$. The *neighbourhood* of v in G, denoted $N_G(v)$, is the set $\{u \in V(G) : \mathrm{dist}(u, v) = 1\}$. The *second neighbourhood* of v in G, denoted $N^2_G(v)$, is the set $\{u \in V(G) : \mathrm{dist}(u, v) = 2\}$. Note that if the graph is clear from context, we drop the subscript and write $N(v)$ or $N^2(v)$.

We also define the neighbourhood of a set of vertices.

Definition 3.1.7. Let G be a graph with $X \subseteq V(G)$. The *neighbourhood* of X in G, denoted $N_G(X)$, is the set $\{u : u \in N(v) \text{ where } v \in X\} \setminus X$.

The following definitions formally describe some relavent graph operations.

Definition 3.1.8. Let G be a graph with subgraphs A and B. The graph $A \cup B$ has vertex set $V(A) \cup V(B)$ and edge set $E(A) \cup E(B)$. The graph $A \cap B$ has vertex set $V(A) \cap V(B)$ and edge set $E(A) \cap E(B)$.

Definition 3.1.9. Let G be a graph with a subgraph A. The graph G *induced on* A, denoted $G[A]$, has vertex set $V(A)$ and edge set $\{e = uv : u, v \in V(A)\}$. Note that $G[V(A)] = G[A]$.

Definition 3.1.10. Let G be a graph with a subgraph A. The graph $G \setminus A$ has vertex set $V(G) \setminus V(A)$ and edge set $E(G[V(G) \setminus V(A)])$. Note that $G \setminus V(A) = G \setminus A$.

Although this thesis does not address the acyclic list colouring version of our acyclic colouring problem, we do use list colouring in the proof of the Extension Lemma 4.2.1. Thus, we define list colouring as follows.

Definition 3.1.11. Let G be a graph. A *list-assignment* L is a collection of lists $(L(v) \subseteq \mathbb{Z}^+ : v \in V(G))$ where $L(v)$ is non-empty for each $v \in V(G)$. The list-assignment L is a *k-list-assignment* if $|L(v)| \geq k$ for all $v \in V(G)$. An *L-colouring* is a colouring ϕ of G such that $\phi(v) \in L(v)$ for all $v \in V(G)$. We say G is *k-list-colourable* if, for every k-list-assignment L of G, G has an L-colouring.

Definition 3.1.12. Let G be a graph. An *acyclic* L-colouring of G is an acyclic colouring ϕ such that $\phi(v) \in L(v)$ for all $v \in V(G)$. We say G is *acyclic k-list-colourable* if, for every k-list-assignment L of G, G has an acyclic L-colouring.

3.2 Mosaic Motivation and Definitions

In this section, we define the concept of a "mosaic" and describe how mosaics are used to extend acyclic colourings.

We begin with the following definition, which will be used in the definition of a mosaic.

Definition 3.2.1. Let G be a graph and let $\mathcal{P}, \mathcal{P}'$ be partitions of $V(G)$. We say that $\mathcal{P}$ is a *refinement* of $\mathcal{P}'$ if, for each pair $u, v \in V(G)$ that are in the same part of $\mathcal{P}$, we have that u, v are in the same part of $\mathcal{P}'$. Let H be a subgraph of G and let $\mathcal{P}_H$ be a partition of $V(H)$. We say that $\mathcal{P}_H$ is a *refinement* of $\mathcal{P}$ if, for each pair $u, v \in V(H)$ that are in the same part of $\mathcal{P}_H$, we have that u, v are in the same part of $\mathcal{P}$.

Observe that if $\mathcal{P}$ is a refinement of $\mathcal{P}'$, which in turn is a refinement of $\mathcal{P}''$, then $\mathcal{P}$ is a refinement of $\mathcal{P}''$. That is, refinements are transitive.

Let us now define mosaic, as follows.

Definition 3.2.2. A *k-mosaic* M of a graph G is an ordered pair $(\phi, \{\mathcal{P}_{ij} : i \neq j \in [k]\})$ where ϕ is an acyclic k-colouring of G and each $\mathcal{P}_{ij}$ is a partition of $V(G_{ij}(\phi(M))$ such that the partition whose parts are the connected components of $G_{ij}(\phi)$ is a refinement of $\mathcal{P}_{ij}$. That is, if $u, v \in V(G)$ are in a path in $G_{ij}(\phi)$, then u and v are in the same part of $\mathcal{P}_{ij}$. We write $\phi(M)$ or ϕ_M for ϕ and $\mathcal{P}_{ij}(M)$ for $\mathcal{P}_{ij}$.

Let $\mathcal{F}$ be the family of graphs which are critical for acyclic k-colouring, where $k \geq 12$. Recall that in order to prove the main result, we want to bound the number of vertices in a plane graph G with outer cycle C with respect to the number of vertices in C, where G is a subgraph of a graph $G' \in \mathcal{F}$. In the colouring version of this problem, we can determine if a colouring ϕ of $G' \setminus (G \setminus C)$ extends to G' by determining if the colouring $\phi_{|C}$ extends to G. Unfortunately, this reduction does not work for acyclic colouring: We may extend $\phi_{|C}$ to an acyclic colouring ϕ_G of G, but the colouring $\phi_G \cup \phi$ of G' is not necessarily acyclic. This motivates the definition of a mosaic, which is composed of an acyclic colouring and a collection of partitions. The partitions can be used to keep track of paths in $G'_{ij}(\phi)$ for each $i \neq j \in [k]$.

Now that we have the concept of a mosaic formalized, we aim to define precisely what the extension of a mosaic is. This is done using multigraphs, which are defined as follows.

Definition 3.2.3. A *multigraph* H is an ordered pair (V, E) where V is a non-empty set of vertices and E is a multiset of 2-element subsets of V called edges. Two or more edges that have the same endpoints are called *parallel edges*. If $e = uv \in E$ where $u = v$, then e is called a *loop*. The *underlying graph* of a multigraph H is the graph G for which $V(G) = V(H)$ and $uv \in E(G)$ if u and v are joined by at least one edge in H.

Note that a multigraph that does not have parallel edges or loops is a graph.

Definition 3.2.4. A *cycle* in a multigraph H is a loop or a closed walk $v_1e_1v_2e_2 \dots v_ne_nv_1$ where $n \geq 2$, the vertices $v_1, \dots, v_n$ are pairwise distinct, the edges $e_1, \dots, e_n$ are pairwise distinct, and, for all $i \in [n]$, the ends of e_i are v_i and $v_{i+1 \pmod{n}}$. Note that if $n = 2$ then the cycle is a pair of parallel edges.

Definition 3.2.5. A multigraph is *acyclic* if it contains no cycles.

Note that an acyclic multigraph does not contain loops or parallel edges; thus, acyclic multigraphs are graphs.

The following two definitions define the multigraph which will be used in the definitions of mosaic extension.

Definition 3.2.6. Let G be a graph and let $u, v \in V(G)$. If u and v are *identified* to a vertex w, then the resulting graph has vertex set $\{w\} \cup V(G) \setminus \{u, v\}$ and edge set $\{e = wx : yx \in E(G) \text{ where } y \in \{u, v\}\} \cup (E(G) \setminus \{e = yx : y \in \{u, v\}\})$.

Definition 3.2.7. Let G be a graph with a k-colouring ϕ and let H be a subgraph of G with a k-mosaic M_H. Let $i \neq j \in [k]$. Let the (i, j)-*fusion* of M_H in ϕ, denoted $\widetilde{G}_{ij}(\phi, M_H)$, be the multigraph obtained from $G_{ij}(\phi)$ by deleting the edges in $E(H)$ and, for each part $R \in \mathcal{P}_{ij}(M_H)$, identifying the vertices of R to a vertex $\widetilde{R}$. Let $\widetilde{\mathcal{P}}_{ij}(M_H)$ denote the independent set that results from identifying the parts of $\mathcal{P}_{ij}(M_H)$.

There is a natural mapping from vertices and edges in G to vertices and edges in $\widetilde{G}_{ij}(\phi, M_H)$. Each $v \in V(G)$ is mapped to a vertex $\widetilde{v} \in V(\widetilde{G}_{ij}(\phi, M_H))$. If $\widetilde{v} = \widetilde{R} \in \widetilde{\mathcal{P}}_{ij}(M_H)$, then $v \in R \in \mathcal{P}_{ij}(M_H)$. If $\widetilde{v} \notin \widetilde{\mathcal{P}}_{ij}(M_H)$, then we sometimes refer to $\widetilde{v}$ as v for convenience. Each $e \in E(G) \setminus E(H)$ is mapped to an edge $\widetilde{e} \in E(\widetilde{G}_{ij}(\phi, M_H))$. We sometimes refer to $\widetilde{e}$ as e for convenience.

Notice that $\widetilde{G}_{ij}(\phi, M_H)$ is a multigraph since we do not remove multiple edges or loops. Now, we are prepared to define the extension of a mosaic.

Definition 3.2.8. Let G be a graph with a subgraph H. A k-mosaic M_H of H *extends* to a k-colouring ϕ of G if all of the following hold:

1. $\phi_{|H} = \phi(M_H)$, and
2. $\widetilde{G}_{ij}(\phi, M_H)$ is acyclic, for all $i \neq j \in [k]$.

Definition 3.2.9. Let G be a graph with a subgraph H. A k-mosaic M_H of H *extends* to a k-mosaic M_G of G if all of the following hold:

1. $\phi(M_G)_{|H} = \phi(M_H)$,
2. $\mathcal{P}_{ij}(M_H)$ is a refinement of $\mathcal{P}_{ij}(M_G)$, for all $i \neq j \in [k]$, and
3. $\widetilde{G}_{ij}(\phi(M_G), M_H)$ is acyclic, for all $i \neq j \in [k]$.

3.3 Mosaic Properties

In this section, we establish some properties of mosaics which will be used throughout the remainder of the thesis. First, we establish that mosaic extension is transitive.

Proposition 3.3.1. *Let M, M', and M'' be k-mosaics of H, H', and H'', respectively, where $H \subseteq H' \subseteq H''$. If M'' is an extension of M' and M' is an extension of M, then M'' is an extension of M.*

Proof. To prove that M'' is an extension of M we prove, by Definition 3.2.9, that all of the following hold:

1. $\phi(M'')_{|H} = \phi(M)$,

2. for all $i \neq j \in [k]$, $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}(M'')$, and

3. for all $i \neq j \in [k]$, $\widetilde{H''}_{ij}(\phi(M''), M)$ is acyclic.

By Definition 3.2.9, $\phi(M'')_{|H'} = \phi(M')$ and $\phi(M')_{|H} = \phi(M)$. Therefore, $\phi(M'')_{|H} = \phi(M)$ and we have that (1) holds. Since $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}(M')$, which itself is a refinement of $\mathcal{P}_{ij}(M'')$, it follows that $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}(M'')$ for all $i \neq j \in [k]$. Thus, we have that (2) holds.

It remains to prove that (3) holds. Suppose not; that is, suppose $\widetilde{H''}_{ij}(\phi(M''), M)$ is not acyclic for some $i \neq j \in [k]$. Let C be a cycle in $\widetilde{H''}_{ij}(\phi(M''), M)$. Notice that $\widetilde{H'}_{ij}(\phi(M'), M)$ is a subgraph of $\widetilde{H''}_{ij}(\phi(M''), M)$. Since $\widetilde{H'}_{ij}(\phi(M'), M)$ is acyclic, there exists at least one edge $e = wz \in E(C)$ such that e is in $\widetilde{H''}_{ij}(\phi(M''), M)$, but not in $\widetilde{H'}_{ij}(\phi(M'), M)$. Let C' be the subgraph of $\widetilde{H''}_{ij}(\phi(M''), M')$ that results from identifying the components of $C \cap \widetilde{H'}_{ij}(\phi(M'), M)$. Each component of $C \cap \widetilde{H'}_{ij}(\phi(M'), M)$ is incident with at least two edges whose images are in $\widetilde{H''}_{ij}(\phi(M''), M')$. Hence, each vertex $\widetilde{R} \in \widetilde{\mathcal{P}}_{ij}(M')$ has degree at least 2 in $\widetilde{H''}_{ij}(\phi(M''), M')$. Let v be a vertex in $\widetilde{H''}_{ij}(\phi(M''), M') \setminus \widetilde{\mathcal{P}}_{ij}(M')$. Since v has degree at least 2 in $\widetilde{H''}_{ij}(\phi(M''), M)$ and the images of all edges incident with v in $\widetilde{H''}_{ij}(\phi(M''), M)$ are in $\widetilde{H''}_{ij}(\phi(M''), M')$, it follows that v has degree at least 2 in $\widetilde{H''}_{ij}(\phi(M''), M')$. Thus, we have that all vertices in C' have degree at least 2. Hence, it follows that C' contains a cycle. Since C' is a subgraph of $\widetilde{H''}_{ij}(\phi(M''), M')$, we have that $\widetilde{H''}_{ij}(\phi(M''), M')$ is not acyclic, which implies that M' does not extend to M'', a contradiction.

Now conditions (1), (2), and (3) hold; thus, by Definition 3.2.9, it follows that M extends to M''. $\square$

Proposition 3.3.2. *Let G be a graph with a subgraph H. If a k-colouring ϕ of G is an extension of a k-mosaic M_H of H, then ϕ is acyclic.*

Proof. Suppose, towards a contradiction, that ϕ is not an acyclic k-colouring of G. Thus, there exists a cycle C in $G_{ij}(\phi)$. If $E(C) \subseteq E(G) \setminus E(H)$, then C is a cycle in $\widetilde{G}_{ij}(\phi, M_H)$; hence, by Definition 3.2.8(2), we have that M_H does not extend to ϕ, which is a contradiction. If $E(C) \subseteq E(H)$, then C is a cycle in $H_{ij}(\phi(M_H))$; hence, we have that $\phi(M_H)$ is not acyclic, which is a contradiction. Therefore, there exist edges $e, f \in E(C)$ such that $e \in E(G) \setminus E(H)$ and $f \in E(H)$.

Let C' be the subgraph of $\widetilde{G}_{ij}(\phi, M_H)$ that results from identifying the components of $C \cap H$. Each component of $C \cap H$ is incident with at least two edges in $G_{ij}(\phi) \setminus E(H)$;

thus, each $\widetilde{R} \in \widetilde{\mathcal{P}}_{ij}(M_H)$ is incident with at least two edges in $\widetilde{G}_{ij}(\phi, M_H)$. Each vertex in $C \cap (G \setminus H)$ is incident with at least two edges in $G_{ij}(\phi) \setminus E(H)$ and the images of these edges are in $\widetilde{G}_{ij}(\phi, M_H)$; thus, each $\widetilde{v} \in V(\widetilde{G}_{ij}(\phi, M_H)) \setminus \widetilde{\mathcal{P}}_{ij}(M_H)$ has degree at least two. Hence, all vertices in C' have degree at least two and it follows that C' contains a cycle. Since C' is a subgraph of $\widetilde{G}_{ij}(\phi, M_H)$, we have that $\widetilde{G}_{ij}(\phi, M_H)$ is not acyclic. Thus, by Definition 3.2.8, it follows that M_H does not extend to ϕ, which is a contradiction. □

Definition 3.3.3. If G is a graph and ϕ is an acyclic k-colouring of G, then the k-mosaic M *induced* by ϕ is the mosaic where $\phi(M) = \phi$ and $\mathcal{P}_{ij}(M)$ is the partition of $V(G_{ij}(\phi))$ whose parts are the components of $G_{ij}(\phi)$, for each $i \neq j \in [k]$. Let $\mathrm{Mosaic}[\phi]$ denote the mosaic induced by the colouring ϕ.

Proposition 3.3.4. *Let G be a graph. If M is a k-mosaic of G, then* $\mathrm{Mosaic}[\phi(M)]$ *extends to M.*

Proof. Let $M' = \mathrm{Mosaic}[\phi(M)]$. Since $\phi(M) = \phi(M')$, we have that $\phi(M)_{|G} = \phi(M')$. By Definition 3.2.2, it follows that $\mathcal{P}_{ij}(M')$ is a refinement of $\mathcal{P}_{ij}(M)$ for all $i \neq j \in [k]$. Since both M and M' are mosaics of G, we have that $\widetilde{G}_{ij}(\phi(M), M')$ is an independent set; thus $\widetilde{G}_{ij}(\phi(M), M')$ is acyclic. Therefore, by Definition 3.2.9(1), (2), and (3), it follows that M' extends to M. □

Definition 3.3.5. Let G be a graph and let M and M' be two k-mosaics of G such that $\phi(M) = \phi(M')$. The *smallest common coarsening* of $\mathcal{P}_{ij}(M)$ and $\mathcal{P}_{ij}(M')$ is the collection $\{\mathcal{P}_{ij} : i \neq j \in [k]\}$ such that for all $i \neq j \in [k]$: $|\mathcal{P}_{ij}|$ is maximum; and for all u, v that are in the same part of $\mathcal{P}_{ij}(M)$ or $\mathcal{P}_{ij}(M')$, we have that u, v are in the same part of $\mathcal{P}_{ij}$. That is, $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}$ and $\mathcal{P}_{ij}(M')$ is a refinement of $\mathcal{P}_{ij}$, for all $i \neq j \in [k]$.

Definition 3.3.6. Let G be a graph with a subgraph H. Let M be a k-mosaic of H that extends to a k-colouring ϕ of G. We say the k-mosaic M' of G is the *induced extension* of M via ϕ if $\phi(M') = \phi$ and $\mathcal{P}_{ij}(M')$ is the smallest common coarsening of $\mathcal{P}_{ij}(M)$ and $\mathcal{P}_{ij}(\mathrm{Mosaic}[\phi])$, for all $i \neq j \in [k]$. Let $\mathrm{Mosaic}[\phi, M]$ denote the induced extension of M via ϕ.

Proposition 3.3.7. *Let G be a graph with a subgraph H. If a k-mosaic M of H extends to a k-colouring ϕ of G, then M extends to* $\mathrm{Mosaic}[\phi, M]$.

Proof. Let $M' = \mathrm{Mosaic}[\phi, M]$. Since M extends to ϕ, it follows that $\phi(M')_{|H} = \phi_{|H} = \phi(M)$. By Definition 3.3.6, we have that $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}(M')$, for all $i \neq j \in [k]$. Since M extends to ϕ, we have that $\widetilde{G}_{ij}(\phi, M)$ is acyclic, for all $i \neq j \in [k]$; hence, it follows that $\widetilde{G}_{ij}(\phi(M'), M)$ is acyclic, for all $i \neq j \in [k]$. Thus, by Definition 3.2.9, we have that M extends to M'. □

Proposition 3.3.8. *Let G be a graph with a subgraph H. Let M be a k-mosaic of H. If* $\text{Mosaic}[\phi, M]$ *exists for some k-colouring ϕ of G, then M extends to a k-mosaic of G.*

Proof. By Definition 3.3.6, ϕ is an extension of M. By Proposition 3.3.7, it follows that M extends to $\text{Mosaic}[\phi, M]$. Since $\text{Mosaic}[\phi, M]$ is a k-mosaic of G, it follows that M extends to a k-mosaic G. ☐

Proposition 3.3.9. *Let G be a graph with a subgraph H. Let M be a k-mosaic of H. If M extends to a k-mosaic M_G of G, then there exists a k-colouring ϕ of G such that M extends to ϕ.*

Proof. Since M extends to M_G, we have that $\phi(M_G)_{|H} = \phi(M)$ and $\widetilde{G}_{ij}(\phi(M_G), M)$ is acyclic for all $i \neq j \in [k]$. Thus, by Definition 3.2.8, it follows that M extends to $\phi(M_G)$. ☐

Let G be a graph with a subgraph H. We say that a k-mosaic M of H extends to G if M extends to a k-colouring or a k-mosaic of G.

Corollary 3.3.10. *Let G be a graph with a subgraph H. Let M be a k-mosaic of H. If M extends to G, then Mosaic$[\phi, M]$ exists for some acyclic k-colouring ϕ of G.*

Proof. The result follows from Proposition 3.3.7. ☐

Proposition 3.3.11. *Let G be a graph with a k-mosaic M_G. Let G' be a subgraph of G and let H be a subgraph of G' with a k-mosaic M_H. If M_H extends to M_G, then* $\text{Mosaic}[\phi(M_G)_{|G'}, M_H]$ *extends to M_G.*

Proof. Suppose not. Let $M = \text{Mosaic}[\phi(M_G)_{|G'}, M_H]$. M is a k-mosaic of G' whose acyclic k-colouring is defined to be $\phi(M_G)_{|G'}$. Since every component of $G'_{ij}(\phi(M))$ is contained in a component of $G_{ij}(\phi(M_G))$, we have that $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}(M_G)$. Thus, Definition 3.2.9(1) and (2) hold for M extending to M_G. Since M_G is not an extension of M, it now follows by Definition 3.2.9(3) that there exists a cycle C in $\widetilde{G}_{ij}(\phi(M_G), M)$, for some $i \neq j \in [k]$.

Let $\{\widetilde{R}_1, \ldots, \widetilde{R}_p\}$ be the set of vertices of $\widetilde{\mathcal{P}}_{ij}(M)$ that are in $V(C)$. Note that $p \geq 1$ since $\phi(M_G)$ is acyclic. Since $\widetilde{\mathcal{P}}_{ij}(M)$ is an independent set, $\widetilde{R}_q$ is incident with two edges $\widetilde{e}_q, \widetilde{f}_q$ whose preimages e_q, f_q are in $E(G) \setminus E(G')$, for each $q \in \{1, \ldots, p\}$. Thus, for each $q \in \{1, \ldots, p\}$, we have that e_q is incident with some vertex $x_q \in V(G')$ and f_q is incident with some vertex $y_q \in V(G')$ such that x_q, y_q are in the same part $R_q \in \mathcal{P}_{ij}(M)$. Let $\widetilde{x}_q, \widetilde{y}_q$ be the images of x_q and y_q in $\widetilde{G'}_{ij}(\phi(M), M_H)$ for each $q \in \{1, \ldots, p\}$.

Claim 3.3.12. *There exists an $\widetilde{x}_q, \widetilde{y}_q$-path P_q in $\widetilde{G'}_{ij}(\phi(M), M_H)$, for each $q \in \{1, \dots, p\}$.*

Proof. Suppose not. Thus, it follows that $\widetilde{G'}_{ij}(\phi(M), M_H)$ is not connected. If two vertices u, v are in the same component of $\widetilde{G'}_{ij}(\phi(M), M_H)$, then it follows that u, v are in the same part of $\mathcal{P}_{ij}(M)$. Suppose the vertices of two components X and Y are in the same part R of $\mathcal{P}_{ij}(M)$. Let $R_X = R \cap V(X)$ and $R_Y = R \cap V(Y)$. Let $\mathcal{P}_{ij} = (\mathcal{P}_{ij}(M) \setminus \{R\}) \cup \{R_X, R_Y\}$. If u, v are in the same part of $\mathcal{P}_{ij}(M_H)$, then u, v are identified to the same vertex in $\widetilde{G'}_{ij}(\phi(M), M_H)$; thus, u, v are in the same part of $\mathcal{P}_{ij}$. If u, v are in the same part of $\mathcal{P}_{ij}(\text{Mosaic}[\phi(M)])$, then there is a u, v-path in $G'_{ij}(\phi(M))$; hence, there is a u, v-path in $\widetilde{G'}_{ij}(\phi(M), M_H)$, which implies that u, v are in the same part in $\mathcal{P}_{ij}$. Therefore, since $|\mathcal{P}_{ij}| > |\mathcal{P}_{ij}(M)|$, it follows that $\mathcal{P}_{ij}(M)$ is not the smallest common coarsening of $\mathcal{P}_{ij}(M_H)$ and $\mathcal{P}_{ij}(\text{Mosaic}[\phi(M)])$, which is a contradiction. $\square$

Since $\widetilde{G'}_{ij}(\phi(M), M_H)$ is a subgraph of $\widetilde{G}_{ij}(\phi(M_G), M_H)$, it follows that P_q is a path in $\widetilde{G}_{ij}(\phi(M_G), M_H)$, for each $q \in \{1, \dots, p\}$. Thus, $G[(V(C) \setminus V(\widetilde{\mathcal{P}}_{ij}(M))) \cup V(P_1) \cup \dots \cup V(P_p)]$ is a subgraph of $\widetilde{G}_{ij}(\phi(M_G), M_H)$ where each vertex has degree at least 2. Hence, this subgraph contains a cycle, which implies that $\widetilde{G}_{ij}(\phi(M_G), M_H)$ is not acyclic, a contradiction. $\square$

Proposition 3.3.13. *Let G be a graph with a subgraph H. If an acyclic k-colouring ϕ of H extends to an acyclic k-colouring ϕ' of G, then* Mosaic$[\phi]$ *extends to* Mosaic$[\phi']$.

Proof. Let $M = \text{Mosaic}[\phi]$ and $M' = \text{Mosaic}[\phi']$ and suppose, towards a contradiction, that M does not extend to M'. Since ϕ extends to ϕ', it follows that $\phi(M')_{|H} = \phi'_{|H} = \phi = \phi(M)$. Since H is a subgraph of G, it follows that $H_{ij}(\phi)$ is a subgraph of $G_{ij}(\phi')$, for all $i \neq j \in [k]$. Thus, if two vertices $u, v \in V(H)$ are in the same component of $H_{ij}(\phi)$, then u and v are in the same component of $G_{ij}(\phi')$. Hence, we have that $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}(M')$. Thus, Definition 3.2.9(1) and (2) hold for M extending to M'.

Since M does not extend to M', it now follows by Definition 3.2.9(3) that there exists a cycle C in $\widetilde{G}_{ij}(\phi(M'), M)$ for some $i \neq j \in [k]$. Let $\{\widetilde{R}_1, \dots, \widetilde{R}_p\}$ be the set of vertices of $\widetilde{\mathcal{P}}_{ij}(M)$ that are in $V(C)$. Since $\widetilde{\mathcal{P}}_{ij}(M)$ is an independent set, $\widetilde{R}_q$ is incident with two edges $\widetilde{e}_q, \widetilde{f}_q \in V(C)$ such that their preimages e_q, f_q are in $E(G) \setminus E(H)$, for each $q \in \{1, \dots, p\}$. Thus, for each $q \in \{1, \dots, p\}$, we have that e_q is incident with some vertex $x_q \in V(H)$ and f_q is incident with some vertex $y_q \in V(H)$ such that x_q, y_q are in the same part $R_q \in \mathcal{P}_{ij}(M)$. Since M is the mosaic induced by ϕ, it follows that there exists an x_q, y_q-path P_q in $H_{ij}(\phi(M))$, for each $q \in \{1, \dots, p\}$.

Since $H_{ij}(\phi(M))$ is a subgraph of $G_{ij}(\phi(M'))$, it follows that P_q is a path in $G_{ij}(\phi(M'))$, for each $q \in \{1, \dots, p\}$. Let $C' = C \cap (G \setminus H) + e_1 P_1 f_1 + \dots + e_p P_p f_p$. Since vertices that are in the same component of $G_{ij}(\phi(M'))$ are in the same part of $\mathcal{P}_{ij}(M')$ and the parts of $\mathcal{P}_{ij}(M')$ are disjoint, it follows that two distinct paths in $\{P_q : q \in \{1, \dots, p\}\}$ are disjoint. Thus, we have that C' is a cycle in $G_{ij}(\phi(M'))$. Since $G_{ij}(\phi(M'))$ is not acyclic, it follows that $\phi(M') = \phi'$ is not an acyclic k-colouring, a contradiction. $\square$

Definition 3.3.14. Let G be a graph with a k-mosaic M and let H be a subgraph of G. We say a k-mosaic M' is the *restriction* of M to H if $\phi(M') = \phi(M)_{|H}$ and, for all $i \neq j \in [k]$, $\mathcal{P}_{ij}(M') = \{P \cap V(H) : P \in \mathcal{P}_{ij}(M)\}$.

Proposition 3.3.15. *Let G be a graph with a k-mosaic M. Let G' be a subgraph of G with a k-mosaic M' such that M' extends to M. If H is a subgraph of G, then the restriction of M' to $H \cap G'$ extends to the restriction of M to H.*

Proof. Let M'_H be the restriction of M' to $H \cap G'$ and let M_H be the restriction of M to H. Notice that $\phi(M'_H) = \phi(M')_{|(H \cap G')}$ and $\phi(M_H) = \phi(M)_{|H}$.

Suppose M'_H does not extend to M_H. Since M' extends to M, it follows that $\phi(M)_{|G'} = \phi(M')$; thus, we have that $\phi(M_H)_{|G'} = (\phi(M)_{|H})_{|G'} = \phi(M)_{|(G' \cap H)} = (\phi(M)_{|G'})_{|(H \cap G')} = \phi(M')_{|(H \cap G')} = \phi(M'_H)$. Hence, Definition 3.2.9(1) holds for M'_H extending to M_H.

Since M' extends to M, it follows that $\mathcal{P}_{ij}(M')$ is a refinement of $\mathcal{P}_{ij}(M)$. By Definition 3.3.14, we have that $\mathcal{P}_{ij}(M'_H) = \{P \cap V(H \cap G') : P \in \mathcal{P}_{ij}(M')\}$ and $\mathcal{P}_{ij}(M_H) = \{P \cap V(H) : P \in \mathcal{P}_{ij}(M)\}$. Thus, $\mathcal{P}_{ij}(M'_H)$ is a refinement of $\mathcal{P}_{ij}(M_H)$. Hence, Definition 3.2.9(2) holds for M'_H extending to M_H.

Since M'_H does not extend to M_H, it now follows by Definition 3.2.9(3) that there exists a cycle C in $\widetilde{H}_{ij}(\phi(M_H), M'_H)$ for some $i \neq j \in [k]$. Let $\{\widetilde{R}_1, \dots, \widetilde{R}_p\}$ be the set of vertices of $\widetilde{\mathcal{P}}_{ij}(M'_H)$ that are in $V(C)$. Notice that $V(C) \setminus \{\widetilde{R}_1, \dots, \widetilde{R}_p\}$ is a subset of $V(G) \setminus V(G')$.

By definition of $\mathcal{P}_{ij}(M'_H)$, it follows that R_q is a subset of some part R'_q of $\mathcal{P}_{ij}(M')$, for all $q \in \{1, \dots, p\}$. Hence, if a vertex v is adjacent to $\widetilde{R}_q$ in $\widetilde{H}_{ij}(\phi(M_H), M'_H)$ for some $q \in \{1, \dots, p\}$, then v is adjacent to $\widetilde{R}'_q$ in $\widetilde{G}_{ij}(\phi(M), M')$. Thus, $(V(C) \setminus \{\widetilde{R}_1, \dots, \widetilde{R}_p\}) \cup \{\widetilde{R}'_1, \dots, \widetilde{R}'_p\}$ induces a cycle in $\widetilde{G}_{ij}(\phi(M), M')$. Hence, $\widetilde{G}_{ij}(\phi(M), M')$ is not acyclic. Thus, M' does not extend to M, a contradiction. $\square$

Proposition 3.3.16. *Let G be a graph with subgraphs A and B such that $G = A \cup B$. Let M_A be a k-mosaic of A and let $M_{A \cap B}$ be the restriction of M_A to $A \cap B$. If $M_{A \cap B}$ extends to B, then M_A extends to G.*

Proof. Since $M_{A\cap B}$ extends to B, it follows from Proposition 3.3.9 that there exists an acyclic k-colouring ϕ_B of B such that $M_{A\cap B}$ extends to ϕ_B. By Definition 3.2.8(1), we have that $(\phi_B)_{|(A\cap B)} = \phi(M_{A\cap B})$. By Definition 3.3.14, we have that $\phi(M_{A\cap B}) = \phi(M_A)_{|(A\cap B)}$. Hence, $(\phi_B)_{|(A\cap B)} = \phi(M_A)_{|(A\cap B)}$. Therefore, $\phi(M_A) \cup \phi_B$ is a well-defined k-colouring of G. Let $\phi = \phi(M_A) \cup \phi_B$.

Claim 3.3.17. *M_A extends to ϕ.*

Proof. Suppose, towards a contradiction, that M_A does not extend to ϕ. Since $\phi_{|A} = \phi(M_A)$, it follows that Definition 3.2.8(1) holds for M_A extending to ϕ. Since M_A does not extend to ϕ, it follows by Definition 3.2.8(2) that $\widetilde{G}_{ij}(\phi, M_A)$ contains a cycle C for some $i \neq j \in [k]$. Since ϕ_B is acyclic, it follows that C contains at least one vertex $\widetilde{R} \in \widetilde{\mathcal{P}}_{ij}(M_A)$.

Let $\{\widetilde{R}_1, \ldots, \widetilde{R}_p\}$ be the set of vertices of $\widetilde{\mathcal{P}}_{ij}(M_A)$ that are in $V(C)$. Since $\mathcal{P}_{ij}(M_{A\cap B}) = \{P \cap V(A\cap B) : P \in \mathcal{P}_{ij}(M_A)\}$, it follows that some part R'_q of $\mathcal{P}_{ij}(M_{A\cap B})$ is a subset of R_q for all $q \in \{1, \ldots, p\}$. Notice that $\widetilde{R}'_1, \ldots, \widetilde{R}'_p$ are vertices in $\widetilde{B}_{ij}(\phi_B, M_{A\cap B})$. Additionally, notice that $V(C) \cap V(G \setminus A)$ is a subset of $V(\widetilde{B}_{ij}(\phi_B, M_{A\cap B}))$.

Let e be an edge in $\widetilde{G}_{ij}(\phi, M_A)$. By definition, the preimage of e is in $E(B)$. Thus, both endpoints of e are in $V(B)$. Hence, it follows that e is an edge in $\widetilde{B}_{ij}(\phi_B, M_{A\cap B})$. Thus, we have that $(V(C) \setminus \{\widetilde{R}_1, \ldots, \widetilde{R}_p\}) \cup \{\widetilde{R}'_1, \ldots, \widetilde{R}'_p\}$ induces a cycle in $\widetilde{B}_{ij}(\phi_B, M_{A\cap B})$. Since $\widetilde{B}_{ij}(\phi_B, M_{A\cap B})$ is not acyclic, it follows from Definition 3.2.8(2) that $M_{A\cap B}$ does not extend to ϕ_B, which is a contradiction. □

Since M_A extends to ϕ by Claim 3.3.17, it follows that M_A extends to G. □

Chapter 4

Canvases

In this chapter, we prove a collection of extension lemmas. These lemmas will be used in Chapter 5 to better understand the structure of graphs which are critical for acyclic k-colouring. Speficially, we aim to identify the structure of plane subgraphs of graphs which are critical for acyclic k-colouring. Therefore, the extension lemmas in this chapter deal with plane graphs.

4.1 Canvas Motivation and Definitions

In this short section, we establish a few definitions which will be used in the extension lemmas of this chapter.

Definition 4.1.1. Let G be a plane graph with a cycle C. The *interior* of C, denoted $\text{int}(C)$, is the set of vertices contained in the interior of the disk bounded by C. Let $G\langle C\rangle = G[C \cup \text{ int}(C)]$.

Since most results in this chapter and in Chapter 5 deal with a graph G and a connected subgraph H, we find it convenient to define the pair of a graph and a subgraph, as follows.

Definition 4.1.2. A *canvas* $\Gamma = (G, H)$ is a plane graph G and a connected subgraph H of G.

The following structure definitions are needed for the extension lemmas.

Definition 4.1.3. A *bichord* of a canvas $\Gamma = (G, C)$, where C is the outer cycle of G, is a path $P = uvw$ where $v \in V(G) \setminus V(C)$ and $u \neq w \in V(C)$ such that $\mathrm{dist}_C(u, w) \geq 2$. We say P is a *dividing* bichord if $\mathrm{dist}_C(u, w) \geq 3$.

Definition 4.1.4. A *bipod* of a canvas $\Gamma = (G, C)$, where C is the outer cycle of G, is a vertex $v \in V(G) \setminus V(C)$ such that v is in at least one bichord.

Definition 4.1.5. Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and let $v \in V(G) \setminus V(C)$. Recall $N_C(v) = N(v) \cap V(C)$ and let $\widetilde{N}_C^2(v) = \{u \in V(C) : u \in N(N(v) \setminus N_C(v))\}$. Let $\mathrm{feet}(v) = N_C(v) \cup \widetilde{N}_C^2(v)$. We refer to the vertices in $\mathrm{feet}(v)$ as the *feet* of v.

Definition 4.1.6. An *r-double-pod* of a canvas $\Gamma = (G, C)$, where C is the outer cycle of G, is a vertex $v \in V(G) \setminus V(C)$ where $|\mathrm{feet}(v)| = r$.

Definition 4.1.7. Let v be an r-double-pod of a canvas $\Gamma = (G, C)$ where C is the outer cycle of G. Since $\mathrm{feet}(v) = N_C(v) \cup \widetilde{N}_C^2(v)$, there exists, for each $u \in \mathrm{feet}(v)$, a (v, u)-path P_u of the form vu or vwu where $w \in N(v) \setminus N_C(v)$, in G. Fix such a path P_u for each $u \in \mathrm{feet}(v)$ and let $\mathrm{legs}(v) = \{P_u : u \in \mathrm{feet}(v)\}$. Notice that $|\mathrm{legs}(v)| = r$.

4.2 Extension Lemmas

In this section, we prove the Extension Lemma 4.2.1 and deduce two corollaries from it.

Lemma 4.2.1 (Extension Lemma)**.** *Given a canvas $\Gamma = (G, C)$, where C is the outer cycle of G, and a k-mosaic M of C, we have that M extends to G unless there exists at least one of the following:*

(i) a chord uv of C, or

(ii) a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$, or

(iii) an r-double-pod v of Γ where $|\{\phi_M(u) : u \in \mathrm{feet}(v)\}| \geq k - 6$.

Proof. Suppose, towards a contradiction, that there does not exist:

(i) a chord uv of C, or

(ii) a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$, or

(iii) an r-double-pod v of Γ where $|\{\phi_M(u) : u \in \text{feet}(v)\}| \geq k-6$,

and M does not extend to G. Let L be a k-list-assignment of G such that, for each vertex $v \in V(G)\backslash V(C)$, $L(v) = [k]\backslash\{\phi_M(u) : u \in \text{feet}(v)\}$.

Since (iii) does not exist, it follows that $|L(v)| \geq k - |\{\phi_M(u) : u \in \text{feet}(v)\}| \geq k-(k-7) = 7$, and we have that there exists an acyclic L-colouring ϕ' of $V(G)\backslash V(C)$ by [7]. Since the two k-colourings ϕ' and $\phi(M)$ are disjoint, it follows that $\phi' \cup \phi(M)$ defines a k-colouring of G. Let $\phi'' = \phi' \cup \phi(M)$.

Claim 4.2.2. *The colouring ϕ'' is an acyclic k-colouring of G.*

Proof. Suppose, towards a contradiction, that ϕ'' is not acyclic. That is, there exists a cycle C' in $G_{ij}(\phi'')$ for some $i \neq j \in [k]$. Since ϕ' and $\phi(M)$ are both acyclic, we have that C' contains both a vertex in C and a vertex in $V(G)\backslash V(C)$. Thus, there exists an edge $e = uv \in E(C')$ where $v \in V(G)\backslash V(C)$ and $u \in V(C)$. Let $w \neq u$ be the vertex such that $wv \in E(C')$. Notice that $\phi_M(u) = \phi''(w)$. This implies that u is not adjacent to w. If $w \in V(C)$, then uvw is a bichord of Γ where $\phi_M(u) = \phi_M(w)$, which is a contradiction. Thus, $w \notin V(C)$. Therefore, by the definition of L, we have that $\phi_M(u) \notin L(w)$ and thus, $\phi'(w) \neq \phi_M(u)$. Since $\phi'(w) = \phi''(w) = \phi''(u) = \phi_M(u)$, we have a contradiction. □

Let $\{\mathcal{P}_{ij} : i \neq j \in [k]\}$ be a collection of partitions of $V(G)$ such that each $\mathcal{P}_{ij}$ is the smallest common coarsening of $\mathcal{P}_{ij}(M)$ and $\mathcal{P}_{ij}(\text{Mosaic}[\phi''])$.

Claim 4.2.3. *The partition whose parts are the connected components of $G_{ij}(\phi'')$ is a refinement of $\mathcal{P}_{ij}$, for each $i \neq j \in [k]$.*

Proof. For each $i \neq j \in [k]$, the partition $\mathcal{P}_{ij}(\text{Mosaic}[\phi''])$ is exactly the partition whose parts are the connected components of $G_{ij}(\phi'')$, by Definition 3.3.3. By Definition 3.3.5, each $\mathcal{P}_{ij}(\text{Mosaic}[\phi''])$ is a refinement of $\mathcal{P}_{ij}$. □

By Claims 4.2.2 and 4.2.3, it follows that ϕ'' and $\{\mathcal{P}_{ij} : i \neq j \in [k]\}$ define a k-mosaic of G. Let M' denote this k-mosaic.

Since $\phi(M') = \phi'' = \phi(M) \cup \phi'$, it follows that $\phi(M')_{|C} = \phi(M)$; thus, Definition 3.2.9(1) holds for M extending to M'. Since $\mathcal{P}_{ij}(M') = \mathcal{P}_{ij}$ is the smallest common coarsening of $\mathcal{P}_{ij}(M)$ and $\mathcal{P}_{ij}(\text{Mosaic}[\phi''])$, it follows from Definition 3.3.5 that $\mathcal{P}_{ij}(M)$ is a refinement of $\mathcal{P}_{ij}(M')$, for all $i \neq j \in [k]$. Hence, Definition 3.2.9(2) holds for M extending to M'.

Since M does not extend to G, it follows that M does not extend to M' and, thus, by Definition 3.2.9(3), we have that $\widetilde{G}_{ij}(\phi(M'), M)$ contains a cycle C' for some $i \neq j \in [k]$.

Since $\widetilde{\mathcal{P}}_{ij}(M)$ is an independent set, there exists at least one path P that is a subgraph of C' with end points $\widetilde{R}_1, \widetilde{R}_2 \in \widetilde{\mathcal{P}}_{ij}(M)$ where $E(P) \subseteq E(G) \setminus E(C)$ and $V(P) \setminus \{\widetilde{R}_1, \widetilde{R}_2\} \subseteq V(G) \setminus V(C)$. Note that $\widetilde{R}_1, \widetilde{R}_2$ are not necessarily distinct. If P is a single edge e, then e is an edge not in C that is incident with two vertices of C; that is, e is a chord of C. Since C has no chords by (i), it follows that P has length at least 2.

If P has length exactly 2, then $P = \widetilde{R}_1 v \widetilde{R}_2$ for some $v \in V(G) \setminus V(C)$. Thus, v is adjacent (in G) to $x, y \in V(C)$. Since G is simple, $x \neq y$; hence, xvy is a bichord of Γ where $\phi_M(x) = \phi_M(y)$, contradicting (ii). Therefore, P has length at least 3.

Let $P = \widetilde{R}_1 v_1 v_2 \dots v_\ell \widetilde{R}_2$. Since v_1 is adjacent to $\widetilde{R}_1$ in P, it follows that v_1 is adjacent (in G) to some vertex $x \in V(C)$ where x is in the part $R_1 \in \mathcal{P}_{ij}(M)$. Since C' is a subgraph of $\widetilde{G}_{ij}(\phi(M'), M)$ and $\text{dist}_{C'}(x, v_2) = 2$, we have that $\phi_{M'}(x) = \phi_{M'}(v_2)$. However, since v_2 is in the second neighbourhood of x, it follows by the definition of L that $\phi_M(x) \notin L(v_2)$. Thus, we have that $\phi_{M'}(x) = \phi_M(x) \neq \phi'(v_2) = \phi_{M'}(v_2)$, which is a contradiction. □

Corollary 4.2.4. *If G is a plane graph with outer cycle C where C is a triangle and $k \geq 10$, then every k-mosaic of C extends to G.*

Proof. Let $\Gamma = (G, C)$ be a canvas. Notice that C is the outer cycle of G. Let M be a k-mosaic of C and suppose, towards a contradiction, that M does not extend to G. By Lemma 4.2.1, there exists at least one of the following:

(i) a chord uv of C, or

(ii) a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$, or

(iii) a r-double-pod v of Γ where $|\{\phi_M(u) : u \in \text{feet}(v)\}| \geq k - 6$.

Since C is a triangle, it follows that C does not have a chord and, thus, (i) does not exist. Furthermore, since C is a triangle, we have that the three vertices of C have pairwise distinct colours in $\phi(M)$. Therefore, every bichord uvw of Γ has $\phi_M(u) \neq \phi_M(w)$ and, hence, (ii) does not exist. Since $k \geq 10$, we have that $k - 6 \geq 4 > |V(C)| = 3$; thus, C does not have a $(k-6)$-double-pod and, hence, (iii) does not exist. Therefore, (i), (ii), and (iii) do not exist, which is a contradiction. □

Corollary 4.2.5. *Let G be a plane graph with outer 4-cycle C, where C has no chords, and let $k \geq 11$. If M is a k-mosaic of C and there does not exist $v \in int(C)$ such that v is adjacent to $u, w \in V(C)$ where $\phi_M(u) = \phi_M(w)$, then M extends to G.*

Proof. Let $\Gamma = (G, C)$ be a canvas. Notice that C is the outer cycle of G. Let M be a k-mosaic of C and suppose that there does not exist $v \in \text{int}(C)$ such that v is adjacent to $u, w \in V(C)$ where $\phi_M(u) = \phi_M(w)$. Suppose, towards a contradiction, that M does not extend to G. By Lemma 4.2.1, there exists at least one of the following:

(i) a chord uv of C, or

(ii) a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$, or

(iii) a r-double-pod v of Γ where $|\{\phi_M(u) : u \in \text{feet}(v)\}| \geq k - 6$.

Since we are given that C has no chords, it follows that (i) does not exist. Furthermore, since there does not exist $v \in \text{int}(C)$ such that v is adjacent to $u, w \in V(C)$ where $\phi_M(u) = \phi_M(w)$, we have that every bichord uvw of Γ has $\phi_M(u) \neq \phi_M(w)$; hence, (ii) does not exist. Since $k \geq 11$, we have that $k - 6 \geq 5 > |V(C)| = 4$; thus, C does not have a $(k-6)$-double-pod and, hence, (iii) does not exist. Therefore, (i), (ii), and (iii) do not exist, which is a contradiction. □

4.3 Generation Lemmas

In this section, we prove the "Fourth Generation" Lemma 4.3.12, which is used to prove the Main Theorem 5.3.5. However, the proof of the "Fourth Generation" Lemma first requires a few additional results and definitions.

Lemma 4.3.1 (Unique Bichord Lemma)**.** *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| \geq 7$. Let v be a bipod of Γ. If v is not in a dividing bichord, then it is in a unique bichord.*

Proof. Suppose not. Let $C = v_0 v_1 \ldots v_{t-1}$ where $t \geq 7$. Let xvy be a bichord of Γ containing v. Since xvy is not a dividing bichord, it follows that $\text{dist}_C(x, y) = 2$. Without loss of generality, let $x = v_0$ and $y = v_2$. Since v is in at least 2 bichords, it has at least one more neighbour in C, call it z, where v and z are in a bichord of Γ. (Note that z is necessarily distinct from x and y.) If $\text{dist}_C(x, z) \geq 3$, then xvz is a dividing bichord. Similarly, if $\text{dist}_C(y, z) \geq 3$, then yvz is a dividing bichord. Thus, $\text{dist}_C(x, z) \leq 2$ and $\text{dist}_C(y, z) \leq 2$. Hence, we have that $z \in \{v_{t-2}, v_{t-1}, v_1, v_3, v_4\}$. Since $t \geq 7$, it follows that $\text{dist}_C(v_{t-2}, y) \geq 3$ and $\text{dist}_C(v_{t-1}, y) \geq 3$. Similarly, since $t \geq 7$, we have that $\text{dist}_C(x, v_3) \geq 3$ and $\text{dist}_C(x, v_4) \geq 3$. Hence, $z = v_1$. Since $\text{dist}_C(z, y) = 1$, it follows by the definition of a bichord that zvy is not a bichord. Similarly, since $\text{dist}_C(x, z) = 1$, we have that xvz is not a bichord. Thus, it follows that v has another neighbour in C, call it w. (Note that w is necessarily distinct from x, y, and z.) If $\text{dist}_C(x, w) \geq 3$, then xvw is a dividing bichord. Similarly, if $\text{dist}_C(y, w) \geq 3$, then yvw is a dividing bichord. Thus, $\text{dist}_C(x, w) \leq 2$ and $\text{dist}_C(y, w) \leq 2$. Hence, we have that $w \in \{v_{t-2}, v_{t-1}, v_3, v_4\}$. As determined earlier, we have that $\text{dist}_C(v_{t-2}, y), \text{dist}_C(v_{t-1}, y), \text{dist}_C(x, v_3), \text{dist}_C(x, v_4) \geq 3$. Thus, it follows that xvw or yvw is a dividing bichord, which is a contradiction. □

Definition 4.3.2. Let $B(\Gamma)$ denote the set of bipods of the canvas $\Gamma = (G, C)$, where C is the outer cycle of G, that are in a unique, non-dividing bichord.

Lemma 4.3.3. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| \geq 5$. Let $B \subseteq B(\Gamma)$ and let E_C denote the set of chords of C. The graph $G[V(C) \cup B] \setminus (E(G[B]) \cup E_C)$ has exactly one interior face of degree at least 5.*

Proof. Suppose not. Let $\Gamma = (G, C)$ with $B \subseteq B(\Gamma)$ be a counterexample with $|V(G)|$ minimized and, subject to that, $|B|$ minimized. Let $G' = G[V(C) \cup B] \setminus (E(G[B]) \cup E_C)$. If $|B| = 0$, then $G' = G[V(C)] \setminus E_C = C$. Thus, there is only one interior face and it has degree equal to $|V(C)| \geq 5$, a contradiction. Hence, we may assume that $|B| > 0$.

Let $|B| = k$ and let uvw be a bichord of Γ such that $v \in B$. Since $v \in B$, we have that $\text{dist}_C(u, w) = 2$. Let $x \in V(C)$ such that $ux, xw \in E(C)$. Notice that $G' - v$ has exactly one face of degree at least 5 by minimality.

First suppose $\deg_{G'}(v) = 3$. Since $v \in B(\Gamma)$, it follows that v is adjacent to x. Hence, v is incident with three faces, two of which are triangles that are incident with the outer face. Let F be the face incident with v that is also incident with u and w and let C_F be the cycle that bounds F. The graph $G' - v$ contains the cycle $(C_F \setminus v) \cup uxw$ which bounds a face F' of $G' - v$. Notice that $\deg_{G'-v}(F') = \deg_{G'}(F)$. If $\deg_{G'-v}(F') \geq 5$, then F' is the only face of $G' - v$ with degree at least 5; thus, F is the only face of G' with degree at least 5. If $\deg_{G'-v}(F') < 5$, then $\deg_{G'}(F) < 5$. Let F^* be the only face of $G' - v$ with degree at least 5. Since all faces of G' are faces in $G' - v$, except those incident with v, it follows that F^* is the only face in G' with degree at least 5.

Now suppose that $\deg_{G'}(v) = 2$. Hence, the bichord uvw is incident with two interior faces of G', call them F_1 and F_2. Let C_i be the cycle that bounds F_i for each $i \in \{1, 2\}$. Let $C' = C_1 \cup C_2 \setminus v$ and let F' be the face bounded by C' in $G' - v$. Notice that $\deg_{G'-v}(F') = \deg_{G'}(F_1) + \deg_{G'}(F_2) - 4$. Without loss of generality, say F_1 is in the interior of the cycle $C^* = uvwxu$. If $\deg_{G'}(F_1) \geq 5$, then there exists a path from u to w in the interior of C^* with length at least 3. Therefore, there is at least one face $F^* \neq F_1$ in the interior of C^* with degree at least 5. Since $\deg_{G'-v}(F') \geq 5$, it follows that $G' - v$ has at least two faces of degree at least 5, which is a contradiction. Thus, we have that

$\deg_{G'}(F_1) = 4$. If $\deg_{G'}(F_2) = 4$, then $\deg_{G'-v}(F') = 4$. Let F^* be the only face of $G' - v$ with degree at least 5. Since all faces of G' are faces in $G' - v$, except those incident with v, it follows that F^* is the only face in G' with degree at least 5. Now consider the case where $\deg_{G'}(F_2) \geq 5$. In this case, $\deg_{G'-v}(F') \geq 5$. Thus, F' is the only face of degree at least 5 in $G' - v$. Hence, it follows that F_2 is the only face of degree at least 5 in G'.

Therefore, G' has exactly one face of degree at least 5, which is a contradiction. □

Definition 4.3.4. Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| \geq 5$. Let $B \subseteq B(\Gamma)$ and let E_C be the set of chords of C. By Lemma 4.3.3, there exists a unique interior face F with degree at least 5 of $G[V(C) \cup B] \setminus (E(G[B]) \cup E_C)$. Let C' be the cycle that bounds F. Let $G' = G\langle C' \rangle$ and let $\Gamma' = (G', C')$. We say that Γ' is the *relaxation* of Γ with respect to B, denoted $R(\Gamma, B)$.

We may think of a canvas and its relaxation as being different generations. If Γ is a canvas and $\Gamma' = R(\Gamma, B(\Gamma))$, we may think of Γ' as being the generation below Γ.

Definition 4.3.5. Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G. If $u, w \in V(C)$ and $\mathrm{dist}_C(u, w) = 2$ and $|X| = |\{v \in B(\Gamma) : \{u, w\} \subseteq N(v) \cap V(C)\}| \geq 1$, then we say X is the *bundle* on u, w. If $|X| < 3$, then we say X is a *thin bundle*. If $|X| \geq 3$, then we say X is a *thick bundle*.

Proposition 4.3.6. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and let ϕ be an acyclic k-colouring of G. If B is a bundle on $u, w \in V(C)$ and $\phi(u) = \phi(w)$, then $\phi(b_1) \neq \phi(b_2)$ for all $b_1 \neq b_2 \in B$.*

Proof. Suppose, towards a contradiction, that $\phi(b_1) = \phi(b_2)$ for some $b_1 \neq b_2 \in B$. Notice that ub_1wb_2u is a cycle in $G_{ij}(\phi)$. Thus, we have that ϕ is not an acyclic colouring, which is a contradiction. □

Proposition 4.3.7. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| = n \geq 5$. Let $B \subseteq B(\Gamma)$ and let $\Gamma' = (G', C') = R(\Gamma, B)$. Let $V(C) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each $i \in \{0, 1, \ldots, n-1\}$, either $u_i \in V(C')$ and there is no bundle on u_{i-1}, u_{i+1}, or $u_i \notin V(C')$ and there exists a unique vertex in the bundle on u_{i-1}, u_{i+1} that is in $V(C')$.*

Proof. Suppose not. Let $H = G[V(C) \cup B] \setminus (E(G[B]) \cup E_C)$. If $B = \emptyset$, then $v \in V(C')$ for all $v \in V(C)$ and there are no bundles in G'.

Let $B_0 \subseteq B(\Gamma)$ such that $\Gamma' = (G', C') = R(\Gamma, B_0)$ is a counterexample with $|B_0|$ minimized. Since Γ' is a counterexample, we have, for some $i \in \{0, 1, \ldots, n-1\}$, that either $u_i \in V(C')$ and there is a bundle on u_{i-1}, u_{i+1}, or $u_i \notin V(C')$ and there is not a unique vertex in the bundle on u_{i-1}, u_{i+1} that is in $V(C')$.

Suppose, towards a contradiction, that $u_i \in V(C')$ and there is a bundle B_i on u_{i-1}, u_{i+1}. Thus, we have that $bu_{i-1}u_iu_{i+1}b$ is a 4-cycle for each $b \in B_i$. Hence, it follows that u_i is incident with the outer face and inner faces of degree at most 4 in H. Since C' bounds a face of H with degree at least 5, we have that $u_i \notin V(C')$, a contradiction.

Now, it follows that $u_i \notin V(C')$ and there is not a unique vertex in the bundle on u_{i-1}, u_{i+1} that is in $V(C')$. If there is no bundle on u_{i-1}, u_{i+1}, then $u_i \in V(C')$, which is a contradiction. Thus, we have that there is a bundle B_i on u_{i-1}, u_{i+1}. Let $b \in B_i$. Let $\Gamma'' = (G'', C'') = R(\Gamma, B \setminus \{b\})$. By minimality, we have that $u_i \in V(C'')$ and there is no bundle on u_{i-1}, u_{i+1}, or $u_i \notin V(C'')$ and there is a unique vertex in the bundle on u_{i-1}, u_{i+1} that is in $V(C'')$. If $u_i \in V(C'')$ and there is no bundle on u_{i-1}, u_{i+1}, then b is the only bipod in B_i. Thus, b is incident with the face of degree at least 5 in H. Hence, $b \in V(C')$, which is a contradiction. Now suppose $u_i \notin V(C'')$ and there is a unique vertex in the bundle on u_{i-1}, u_{i+1} that is in $V(C'')$. If $b \notin V(C')$, then $C' = C''$ and it follows that there is a unique vertex in the bundle on u_{i-1}, u_{i+1} that is in $V(C')$, a contradiction. Thus, $b \in V(C')$. Suppose $b' \in B_i$ is in $V(C')$ as well. Without loss of generality, b is in the interior of the cycle $C_i = b'u_{i-1}u_iu_{i+1}b'$. Thus, it follows that $V(C')$ is in $G\langle C_i \rangle$. Since each vertex in B_i is adjacent to u_{i-1} and u_{i+1}, it follows that the faces in H that are interior to C_i have degree at most 4. Since, C' bounds a face of degree at least 5, we have a contradiction. □

Proposition 4.3.8. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| \geq 5$. If $\Gamma' = (G', C') = R(\Gamma, B)$ where $B \subseteq B(\Gamma)$, then each vertex in $V(C')$ is either a bipod in $B(\Gamma)$ or a vertex in $V(C)$ and $|V(C')| = |V(C)|$.*

Proof. Since C' is a cycle in $G[V(C) \cup B]$, it follows that $V(C') \subseteq V(C) \cup B$. Since $V(C) \cap B = \emptyset$, we have that each vertex in $V(C')$ is either in $B \subseteq B(\Gamma)$ or in $V(C)$. Now it follows from Proposition 4.3.7 that $|V(C')| = |V(C)|$. □

Proposition 4.3.9. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| = n \geq 5$. Let $\Gamma' = (G', C') = R(\Gamma, B(\Gamma))$. Let $V(C) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V(C') = \{u'_0, u'_1, \ldots, u'_{n-1}\}$ such that u'_i is in the bundle on u_{i-1}, u_{i+1} or is equal to u_i for all $i \in \{0, 1, \ldots, n-1\}$. If $u'_i \in B(\Gamma)$, then u'_i is adjacent to $u'_{i-1} = u_{i-1}$ and $u'_{i+1} = u_{i+1}$. Equivalently, if $u'_i \in B(\Gamma)$, then $u'_{i-1}, u'_{i+1} \notin B(\Gamma)$.*

Proof. By the definition of the cycle C', we have that u'_i is adjacent to u'_{i-1} and u'_{i+1}. Since $u'_i \in B(\Gamma)$, it follows that $u'_{i-1}u'_iu'_{i+1}$ is a bichord of Γ. Thus, we have that $u'_{i-1}, u'_{i+1} \in$

$V(C)$. By Proposition 4.3.8, it follows that $u_{i-1}, u_{i+1} \notin B(\Gamma)$. Hence, by the definition of $V(C')$, we have that $u'_{i-1} = u_{i-1}$ and $u'_{i+1} = u_{i+1}$. □

Proposition 4.3.10. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| = n \geq 5$. Let $\Gamma' = (G', C') = R(\Gamma, B(\Gamma))$. Let $V(C) = \{u_0, u_1, \ldots, u_{n-1}\}$ and $V(C') = \{u'_0, u'_1, \ldots, u'_{n-1}\}$ such that u'_i is in the bundle on u_{i-1}, u_{i+1} or is equal to u_i for all $i \in \{0, 1, \ldots, n-1\}$. If $u'_i \in B(\Gamma)$, then $u'_i \neq u_i$.*

Proof. Suppose not. If $u'_i = u_i$, then $u'_i \in V(C)$. Thus, by Proposition 4.3.8, it follows that $u'_i \notin B(\Gamma)$, a contradiction. □

Definition 4.3.11. Let $\Gamma = (G, C)$ be a canvas with a bichord uvw and let ϕ be a colouring of C. We say uvw is *monochromatic* in ϕ if $\phi(u) = \phi(w)$.

Lemma 4.3.12 ("Fourth Generation" Lemma)**.** *Let $\Gamma_0 = (G_0, C_0)$ be a canvas where C_0 is the outer cycle of G_0 and $|V(C_0)| \geq 5$. Let $\Gamma_i = (G_i, C_i) = R(\Gamma_{i-1}, B(\Gamma_{i-1}))$ for each $i \in \{1, 2, 3\}$. If all of the following hold for all $i \in \{0, 1, 2, 3\}$:*

(i) C_i has no chords,

(ii) every bipod v of Γ_i is such that $v \in B(\Gamma_i)$,

(iii) Γ_i has no 6-double-pod,

and a k-mosaic M of C_0 extends to $G_0[V(C_0) \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)]$, then M extends to G_0.

Proof. Suppose not. Let $H = G_0[V(C_0) \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)]$ and $H_2 = G_0[V(C_0) \cup B(\Gamma_0) \cup B(\Gamma_1)]$ and $H_1 = G_0[V(C_0) \cup B(\Gamma_0)]$.

Claim 4.3.13. *$\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ are pairwise not equal.*

Proof. Since M is a k-mosaic of C_0 that extends to H, it follows that M also extends to a k-mosaic M_1 of H_1 and a k-mosaic M_2 of H_2 and a k-mosaic M_3 of H. Notice that M_1, M_2, and M_3 do not extend to G_0 by Proposition 3.3.1. Let M'_i be the restriction of M_i to C_i for each $i \in \{1, 2, 3\}$. By the converse of Proposition 3.3.16, since M_i does not extend to G_0, it follows that M'_i does not extend to G_i, for each $i \in \{1, 2, 3\}$. Since (i)-(iii) do not hold for $i \in \{1, 2, 3\}$ by assumption, it follows by the Extension Lemma 4.2.1 that, for each $i \in \{1, 2, 3\}$, Γ_i has at least one bichord uvw that is monochromatic in $\phi(M'_i)$ where $v \in B(\Gamma_i)$. Thus, we have that $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ are pairwise not equal. □

Let $\mathcal{M}$ be the set of k-mosaics of C_0 that extend to H, but not to G_0. Since M is a k-mosaic of C_0 that extends to H, but not to G_0, it follows that $M \in \mathcal{M}$; thus, we have that $\mathcal{M}$ is non-empty. Let ϕ be a k-colouring of H such that M_0 extends to ϕ, for some $M_0 \in \mathcal{M}$, and the number of bichords of Γ_3 that are monochromatic in ϕ is minimum.

Let $C_0 = u_{0,0}u_{0,1}\dots u_{0,n-1}$ and let $C_i = u_{i,0}u_{i,1}\dots u_{i,n-1}$ such that $u_{i,j}$ is in the bundle on $u_{i-1,j-1}, u_{i-1,j+1}$ or $u_{i,j} = u_{i-1,j}$, for each $i \in \{1,2,3\}$ and $j \in \{0,1,\dots,n-1\}$.[1]

Since (i)-(iii) do not hold for $i = 3$ by assumption, it follows by the Extension Lemma 4.2.1 that Γ_3 has at least one bichord uvw that is monochromatic in ϕ where $v \in B(\Gamma_3)$. Since $v \in B(\Gamma_3)$, we have that at least one of u, w is in $B(\Gamma_2)$. Without loss of generality, say $u \in B(\Gamma_2)$. Notice that $w \in V(C_0) \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)$. Let $u = u_{3,x}$ for some $x \in \{0,\dots,n-1\}$. Without loss of generality, say $w = u_{3,x+2}$. Since $v \in B(\Gamma_3)$, it follows that $u, w \notin B(\Gamma_3)$. Since $u_{3,x} \in B(\Gamma_2)$, we have that $u_{3,x}$ is in the bundle on $u_{2,x-1}, u_{2,x+1}$. Additionally, since $u_{3,x} \in B(\Gamma_2)$, we have that $u_{3,x} \neq u_{i,x}$ for all $i \in \{0,1,2\}$ by Proposition 4.3.10. Notice that $w = u_{3,x+2}$ is not necessarily distinct from $u_{0,x+2}$, $u_{1,x+2}$, and $u_{2,x+2}$.

Let $B_{i,j}$ denote the set of vertices in the bundle on $u_{i-1,j-1}$ and $u_{i-1,j+1}$, for each $i \in \{1,2,3\}$ and $j \in \{0,1,\dots,n-1\}$. If there is no such bundle, then we let $B_{i,j} = \emptyset$. Notice that if $u_{i,j} \neq u_{i-1,j}$, then $u_{i,j} \in B_{i,j}$.

Claim 4.3.14. *Let $p \in \{1,2,3\}$ and $q \in \{0,1,\dots,n-1\}$. Let $c_1, c_2, \dots, c_5 \in [k]$. If $u_{p,q} \in B(\Gamma_{p-1})$ and $\phi(u_{p-1,q-1}) \neq \phi(u_{p-1,q+1})$, then there exists $c \in [k] \setminus (\{\phi(u_{p,q-2}), \phi(u_{p,q+2})\} \cup \{c_1, c_2, \dots, c_5\})$ such that there exists a k-colouring ϕ' of H_p, where $\phi'(u_{p,q}) = c$ and $\phi'(v) = \phi(v)$ for all $v \in V(H_p) \setminus u_{p,q}$, that is an extension of M_0.*

Proof. Notice that, since $u_{p,q} \in B(\Gamma_{p-1})$, it follows that $u_{p,q} \in V(C_p)$. In H_p, the vertex $u_{p,q}$ has degree 2 or 3. If $\deg_{H_p}(u_{p,q}) = 3$, then $u_{p,q}$ is adjacent to $u_{p-1,q}$ or a vertex $b \in B_{p,q}$. Let $v = b$ if $u_{p,q}$ is adjacent to $b \in B_{p,q}$, and let $v = u_{p-1,q}$ otherwise. Notice that v is adjacent to $u_{p-1,q-1}$ and $u_{p-1,q+1}$, hence, we have that $\phi(v) \neq \phi(u_{p-1,q-1}), \phi(u_{p-1,q+1})$. Since $k \geq 12$, it follows that there exists $c \in [k] \setminus (\{\phi(u_{p,q-2}), \phi(u_{p-1,q-1}), \phi(v), \phi(u_{p,q}), \phi(u_{p-1,q+1}), \phi(u_{p,q+2})\} \cup \{c_1, c_2, \dots, c_5\})$. Let ϕ' be a k-colouring of H_p such that $\phi'(u_{p,q}) = c$ and $\phi'(v) = \phi(v)$ for all $v \in V(H_p) \setminus u_{p,q}$. Let H' denote H_p.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $u_{p,q} \notin V(C_0)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H'}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H'}_{ij}(\phi, M_0)$; thus, we have that $u_{p,q} \in V(C)$. Let w_1, w_2 be the neighbours of $u_{p,q}$ in C. For

[1] Indices are taken modulo n here and in the remainder of the proof of Lemma 4.3.12.

each $t \in \{1,2\}$, w_t is either in $\{u_{p-1,q-1}, u_{p-1,q+1}, v\}$ or is equal to $\widetilde{R}$ where at least one of $u_{p-1,q-1}, u_{p-1,q+1}, v$ is in $R \in \mathcal{P}_{ij}(M_0)$. Since $\phi'(u_{p,q}) \neq \phi'(u_{p-1,q-1}), \phi'(u_{p-1,q+1}), \phi'(v)$ and $\phi'(u_{p-1,q-1}), \phi'(u_{p-1,q+1}), \phi'(v)$ are pairwise not equal, it follows that at most one of w_1, w_2 is in $\widetilde{H'}_{ij}(\phi', M_0)$, which is a contradiction. $\square$

Claim 4.3.15. $\phi(u_{2,x-1}) = \phi(u_{2,x+1})$.

Proof. Suppose, towards a contradiction, that $\phi(u_{2,x-1}) \neq \phi(u_{2,x+1})$. Thus, by Claim 4.3.14, there exists $c \in [k] \setminus \{\phi(u_{3,x-2}), \phi(u_{3,x+2})\}$ such that there exists a k-colouring ϕ' of H, where $\phi'(u_{3,x}) = c$ and $\phi'(v) = \phi(v)$ for all $v \in V(H) \setminus u_{3,x}$, that is an extension of M_0. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. $\square$

Claim 4.3.16. *Let $p \in \{1,2,3\}$ and $q \in \{0,1,\ldots,n-1\}$. Let $|B_{p,q}| \geq 2$ and $y, z \in B_{p,q}$. If $\phi(u_{p-1,q-1}) = \phi(u_{p-1,q+1})$, then M_0 extends to the k-colouring ϕ' of H_p, where $\phi'(y) = \phi(z)$, $\phi'(z) = \phi(y)$, and $\phi'(v) = \phi(v)$ for all $v \in V(H_p) \setminus \{y,z\}$.*

Proof. Let H' denote H_p. If two vertices in $B_{p,q} \cup \{u_{p-1,q}\}$ have the same colour in ϕ, then those two vertices are in a bichromatic 4-cycle with $u_{p-1,q-1}$ and $u_{p-1,q+1}$, contradicting the assumption that ϕ is acyclic. Thus, since $\phi(u_{p-1,q-1}) = \phi(u_{p-1,q+1})$, we have that all vertices in $B_{p,q} \cup \{u_{p-1,q}\}$ have pairwise distinct colours in ϕ. Additionally, all vertices in $B_{p,q} \cup \{u_{p-1,q}\}$ have pairwise distinct colours in ϕ'.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $y, z \notin V(C_0)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H'}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H'}_{ij}(\phi, M_0)$; thus, we have that at least one of y, z is in $V(C)$. Without loss of generality, suppose $y \in V(C)$. If $z \in V(C)$, then $\phi'(z), \phi'(y) \in \{i,j\}$; thus, we have that $\phi(z), \phi(y) \in \{i,j\}$, which implies that C is a cycle in $\widetilde{H'}_{ij}(\phi, M_0)$, a contradiction. Hence, it follows that $z \notin V(C)$. Let $P = C - y$ and let w_1, w_2 be the neighbours of y in C. Notice that P is a path in $\widetilde{H'}_{ij}(\phi, M_0)$. Also, notice that $N(y) \cap V(H') \subseteq \{u_{p-1,q-1}, u_{p-1,q+1}, u_{p-1,q}\} \cup B_{p,q}$. Since $\phi'(y) \notin \{\phi'(z), \phi'(u_{p-1,q-1}), \phi'(u_{p-1,q+1}), \phi'(u_{p-1,q})\} \cup \{\phi'(b) : b \in B_{p,q}\}$ and all colours in $\{\phi'(z), \phi'(u_{p-1,q-1}), \phi'(u_{p-1,q+1}), \phi'(u_{p-1,q})\} \cup \{\phi'(b) : b \in B_{p,q}\}$ are distinct except for $\phi'(u_{p-1,q-1}) = \phi'(u_{p-1,q+1})$, it follows that $u_{p-1,q-1} = w_1$ or $u_{p-1,q-1} \in w_1 = \widetilde{R}_1$ where $R_1 \in \mathcal{P}_{ij}(M_0)$, and $u_{p-1,q+1} = w_2$ or $u_{p-1,q+1} \in w_2 = \widetilde{R}_2$ where $R_2 \in \mathcal{P}_{ij}(M_0)$. Since $\phi'(y) \in \{i,j\}$, we have that $\phi(z) \in \{i,j\}$; hence, $z \in \widetilde{H'}_{ij}(\phi, M_0)$. Since z is adjacent to $u_{p-1,q-1}$ and $u_{p-1,q+1}$, it follows that $P + w_1 z w_2$ is a cycle in $\widetilde{H'}_{ij}(\phi, M_0)$, which is a contradiction. $\square$

Claim 4.3.17. *The bundle $B_{3,x}$ is a thin bundle.*

Proof. Suppose, towards a contradiction, that $B_{3,x}$ is a thick bundle. Let $B_{3,x} = \{b_1, b_2, \dots, b_\ell\}$ where $b_1 = u_{3,x}$. Since $B_{3,x}$ is thick, it follows that $\ell \geq 3$. Since $\phi(u_{2,x-1}) = \phi(u_{2,x+1})$ by Claim 4.3.15, we have that $\phi(b_i) \neq \phi(b_j)$ for all $i \neq j \in [\ell]$.

Since $\ell \geq 3$, there exists $c \in \{\phi(b_1), \dots, \phi(b_\ell)\} \setminus \{\phi(u_{3,x-2}), \phi(u_{3,x+2})\}$. Let $y \in \{2, \dots, \ell\}$ be such that $\phi(b_y) = c$. By Claim 4.3.16, we have that there exists a k-colouring ϕ' of H where $\phi'(b_y) = \phi(u_{3,x})$ and $\phi'(u_{3,x}) = \phi(b_y)$ and $\phi'(v) = \phi(v)$ for all $v \in H \setminus \{u_{3,x}, b_y\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. $\square$

Claim 4.3.18. *If $w \in B(\Gamma_2)$, then $\phi(u_{2,x+1}) = \phi(u_{2,x+3})$ and $B_{3,x+2}$ is a thin bundle.*

Proof. First we prove that $\phi(u_{2,x+1}) = \phi(u_{2,x+3})$. Suppose, towards a contradiction, that $\phi(u_{2,x-1}) \neq \phi(u_{2,x+1})$. Thus, by Claim 4.3.14, there exists $c \in [k] \setminus \{\phi(u_{3,x-2}), \phi(u_{3,x+2})\}$ such that M_0 extends to a k-colouring ϕ' of H, where $\phi'(u_{3,x}) = c$ and $\phi'(v) = \phi(v)$ for all $v \in V(H) \setminus u_{3,x}$. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ.

Now we prove that $B_{3,x+2}$ is a thin bundle. Suppose, towards a contradiction, that $B_{3,x+2}$ is a thick bundle. Let $B_{3,x+2} = \{b_1, b_2, \dots, b_\ell\}$ where $b_1 = u_{3,x+2}$. Since $B_{3,x+2}$ is thick, it follows that $\ell \geq 3$. Since $\phi(u_{2,x+1}) = \phi(u_{2,x+3})$ from above, we have that $\phi(b_i) \neq \phi(b_j)$ for all $i \neq j \in [\ell]$. Since $\ell \geq 3$, there exists $c \in \{\phi(b_1), \dots, \phi(b_\ell)\} \setminus \{\phi(u_{3,x}), \phi(u_{3,x+4})\}$. Let $y \in \{2, \dots, \ell\}$ be such that $\phi(b_y) = c$. By Claim 4.3.16, we have that there exists a k-colouring ϕ' of H where $\phi'(b_y) = \phi(u_{3,x+2})$ and $\phi'(u_{3,x+2}) = \phi(b_y)$ and $\phi'(v) = \phi(v)$ for all $v \in H \setminus \{u_{3,x+2}, b_y\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x+2}$, then $r \in \{u_{3,x}, u_{3,x+4}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. $\square$

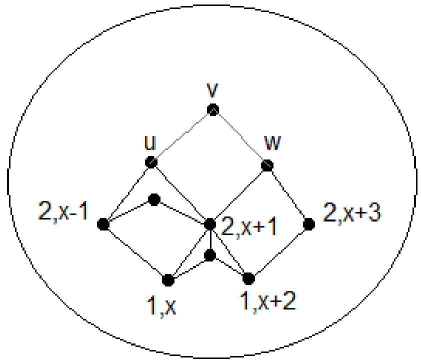

Figure 4.1: A possible configuration of the vertices of interest in Claim 4.3.19.

Claim 4.3.19. $u_{2,x+1} \notin B(\Gamma_1)$.

Proof. Suppose, towards a contradiction that $u_{2,x+1} \in B(\Gamma_1)$. Note that Figure 4.1 shows an approximate configuration of the vertices of interest here. Since $u_{2,x+1}$ is adjacent to w, it follows that $w \in B(\Gamma_2) \cup B(\Gamma_0) \cup V(C_0)$. If $w \in B(\Gamma_2)$, then it follows from Claim 4.3.18 that $\phi(u_{2,x+1}) = \phi(u_{2,x+3})$ and $B_{3,x+2}$ is a thin bundle. Notice that if $w \in B(\Gamma_0) \cup V(C_0)$, then there is no bundle on $u_{2,x+1}$ and $u_{2,x+3}$.

Let $\Phi = \{\phi(z) : z \in N_H(u_{3,x}) \cup N_H(u_{2,x+1})\}$. Notice that $N_H(u_{3,x}) \subseteq \{u_{2,x-1}, u_{2,x+1}, u_{1,x}\} \cup B_{3,x}$. By planarity, we have that $|N_H(u_{3,x}) \cap (B_{3,x} \cup \{u_{1,x}\})| \leq 1$. Notice that $N_H(u_{2,x+1}) \subseteq \{u_{1,x}, u_{1,x+2}, u_{0,x+1}\} \cup B_{3,x} \cup B_{3,x+2} \cup B_{2,x+1}$. By planarity, we have that $|N_H(u_{2,x+1}) \cap (B_{2,x+1} \cup \{u_{0,x+1}\})| \leq 1$. Since $B_{3,x}$ is a thin bundle and $B_{3,x+2}$ is a thin bundle, if it exists, we have that $|B_{3,x} \cup B_{3,x+2}| \leq 4$. Since $\phi(u) = \phi(w)$, it follows that $|\Phi| \leq 8$.

Since $k \geq 12$, we have that there exists $c \in [k] \setminus \Phi$. Let ϕ' be a k-colouring of H such that $\phi'(u) = c$ and $\phi'(v) = \phi(v)$ for all $v \in V(H) \setminus \{u\}$.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $u \notin V(C_0)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that $u \in V(C)$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let P be the component of C' in $H_{ij}(\phi')$ that contains u. Recall that $N_H(u_{3,x}) \subseteq \{u_{2,x-1}, u_{2,x+1}, u_{1,x}\} \cup B_{3,x}$. By planarity, we have that $|N_H(u_{3,x}) \cap (B_{3,x} \cup \{u_{1,x}\})| \leq 1$. Since $u_{3,x} \notin V(C_0)$, it follows that two vertices in $N_H(u_{3,x})$ are the neighbours of $u_{3,x}$ in P.

Note that each vertex in $B_{3,x} \cup \{u_{1,x}\}$ is adjacent to $u_{2,x-1}$ and $u_{2,x+1}$. Hence, $\phi'(u') \neq \phi'(u_{2,x-1}), \phi'(u_{2,x+1})$ for all $u' \in B_{3,x} \cup \{u_{1,x}\}$. Since $\phi'(u) \neq \phi'(u'), \phi'(u_{2,x-1}), \phi'(u_{2,x+1})$ for all $u' \in B_{3,x} \cup \{u_{1,x}\}$, it follows that $u_{2,x-1}$ and $u_{2,x+1}$ are the neighbours of u in P. Since $u_{2,x+1} \notin V(C_0)$, we have that two neighbours of $u_{2,x+1}$ are in P.

Recall that $N_H(u_{2,x+1}) \subseteq \{u_{1,x}, u_{1,x+2}, u_{0,x+1}\} \cup B_{3,x} \cup B_{3,x+2} \cup B_{2,x+1}$ and say $z \neq u_{3,x} \in N_H(u_{2,x+1})$ is a neighbour of $u_{2,x+1}$ in P. Since $\phi'(z) = \phi(z) \notin \Phi$, it follows that $\phi'(z) \neq \phi'(u)$. Thus, we have that P is not in $H_{ij}(\phi')$, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(v) = \phi(v)$ for all $v \in B(\Gamma_2) \setminus \{u\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. □

Since $u_{3,x} \in B(\Gamma_2)$ and $u_{2,x+1} \notin B(\Gamma_1)$, it follows that $u_{2,x-1} \in B(\Gamma_1)$. Also, note that $u_{2,x+1} \in B(\Gamma_0) \cup V(C_0)$.

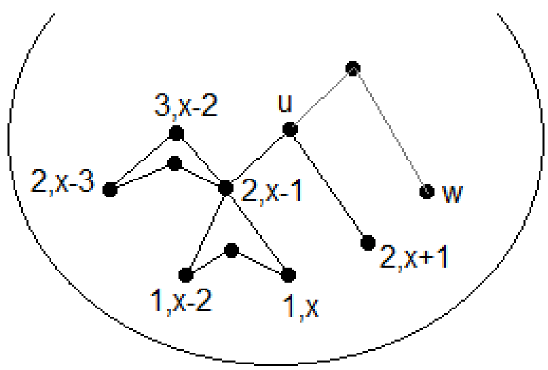

Figure 4.2: A possible configuration of the vertices of interest in Claims 4.3.20 and 4.3.21.

Claim 4.3.20. *The bundle $B_{3,x-2}$ is a thick bundle.*

Proof. Suppose, towards a contradiction, that $B_{3,x-2}$ is a thin bundle or $B_{3,x-2} = \emptyset$. Note that Figure 4.2 shows an approximate configuration of the vertices of interest here. Let $\Phi = \{\phi(z) : z \in N_H(u_{3,x}) \cup N_H(u_{2,x-1})\}$. Notice that $N_H(u_{3,x}) \subseteq \{u_{2,x-1}, u_{2,x+1}, u_{1,x}\} \cup B_{3,x}$. By planarity, we have that $|N_H(u_{3,x}) \cap (B_{3,x} \cup \{u_{1,x}\})| \leq 1$. Also, notice that $N_H(u_{2,x-1}) \subseteq \{u_{1,x}, u_{1,x-2}, u_{0,x-1}\} \cup B_{3,x} \cup B_{3,x-2} \cup B_{2,x-1}$. By planarity, we have that $|N_H(u_{2,x-1}) \cap (B_{2,x-1} \cup \{u_{0,x-1}\})| \leq 1$. Since $B_{3,x}$ is a thin bundle and $B_{3,x-2}$ is a thin bundle, if it exists, we have that $|B_{3,x} \cup B_{3,x-2}| \leq 4$. Thus, it follows that $|\Phi| \leq 9$.

Since $k \geq 12$, we have that there exists $c \in [k] \setminus \Phi$. Let ϕ' be a k-colouring of H such that $\phi'(u) = c$ and $\phi'(v) = \phi(v)$ for all $v \in V(H) \setminus \{u\}$.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $u \notin V(C_0)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that $u \in V(C)$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let P be the component of C' in $H_{ij}(\phi')$ that contains u. Recall that $N_H(u_{3,x}) \subseteq \{u_{2,x-1}, u_{2,x+1}, u_{1,x}\} \cup B_{3,x}$. By planarity, we have that $|N_H(u_{3,x}) \cap (B_{3,x} \cup \{u_{1,x}\})| \leq 1$. Since $u_{3,x} \notin V(C_0)$, it follows that two vertices in $N_H(u_{3,x})$ are the neighbours of $u_{3,x}$ in P.

Note that each vertex in $B_{3,x} \cup \{u_{1,x}\}$ is adjacent to $u_{2,x-1}$ and $u_{2,x+1}$. Hence, $\phi'(u') \neq \phi'(u_{2,x-1}), \phi'(u_{2,x+1})$ for all $u' \in B_{3,x} \cup \{u_{1,x}\}$. Since $\phi'(u) \neq \phi'(u'), \phi'(u_{2,x-1}), \phi'(u_{2,x+1})$ for all $u' \in B_{3,x} \cup \{u_{1,x}\}$, it follows that $u_{2,x-1}$ and $u_{2,x+1}$ are the neighbours of u in P. Since $u_{2,x-1} \notin V(C_0)$, we have that two neighbours of $u_{2,x-1}$ are in P.

Recall that $N_H(u_{2,x-1}) \subseteq \{u_{1,x}, u_{1,x-2}, u_{0,x-1}\} \cup B_{3,x} \cup B_{3,x-2} \cup B_{2,x-1}$ and say $z \neq u_{3,x} \in N_H(u_{2,x-1})$ is a neighbour of $u_{2,x-1}$ in P. Since $\phi'(z) = \phi(z) \in \Phi$, it follows that $\phi'(z) \neq \phi'(u)$. Thus, we have that P is not in $H_{ij}(\phi')$, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(v) = \phi(v)$ for all $v \in B(\Gamma_2) \setminus \{u\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. □

Claim 4.3.21. $\phi(u_{2,x-1}) = \phi(u_{2,x-3})$.

Proof. Suppose, towards a contradiction, that $\phi(u_{2,x-1}) \neq \phi(u_{2,x-3})$. Note that Figure 4.2 shows an approximate configuration of the vertices of interest here. Let $\Phi = \{\phi(z) : z \in N_H(u_{3,x}) \cup N_{H_2}(u_{2,x-1})\}$. Notice that $N_H(u_{3,x}) \subseteq \{u_{2,x-1}, u_{2,x+1}, u_{1,x}\} \cup B_{3,x}$. By planarity, we have that $|N_H(u_{3,x}) \cap (B_{3,x} \cup \{u_{1,x}\})| \leq 1$. Also, notice that $N_{H_2}(u_{2,x-1}) \subseteq$

$\{u_{1,x}, u_{1,x-2}, u_{0,x-1}\} \cup B_{2,x-1}$. By planarity, we have that $|N_H(u_{2,x-1}) \cap (B_{2,x-1} \cup \{u_{0,x-1}\})| \leq 1$. Thus, it follows that $|\Phi| \leq 6$.

Since $k \geq 12$, we have that there exists $c \in [k] \setminus \Phi$. Let ϕ' be a k-colouring of H such that $\phi'(u) = c$ and $\phi'(v) = \phi(v)$ for all $v \in V(H) \setminus \{u\}$.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $u \notin V(C_0)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that $u \in V(C)$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let P be the component of C' in $H_{ij}(\phi')$ that contains u. Recall that $N_H(u_{3,x}) \subseteq \{u_{2,x-1}, u_{2,x+1}, u_{1,x}\} \cup B_{3,x}$. By planarity, we have that $|N_H(u_{3,x}) \cap (B_{3,x} \cup \{u_{1,x}\})| \leq 1$. Since $u_{3,x} \notin V(C_0)$, it follows that two vertices in $N_H(u_{3,x})$ are the neighbours of $u_{3,x}$ in P.

Note that each vertex in $B_{3,x} \cup \{u_{1,x}\}$ is adjacent to $u_{2,x-1}$ and $u_{2,x+1}$. Hence, $\phi'(u') \neq \phi'(u_{2,x-1}), \phi'(u_{2,x+1})$ for all $u' \in B_{3,x} \cup \{u_{1,x}\}$. Since $\phi'(u) \neq \phi'(u'), \phi'(u_{2,x-1}), \phi'(u_{2,x+1})$ for all $u' \in B_{3,x} \cup \{u_{1,x}\}$, it follows that $u_{2,x-1}$ and $u_{2,x+1}$ are the neighbours of u in P. Since $u_{2,x-1} \notin V(C_0)$, we have that at least two neighbours of $u_{2,x-1}$ are in P.

Recall that $N_H(u_{2,x-1}) \subseteq \{u_{1,x}, u_{1,x-2}, u_{0,x-1}\} \cup B_{3,x} \cup B_{3,x-2} \cup B_{2,x-1}$ and say $z \neq u_{3,x} \in N_H(u_{2,x-1})$ is a neighbour of $u_{2,x-1}$ in P. Thus, $\phi'(z) = \phi'(u)$. If $z \in N_{H_2}(u_{2,x-1}) \cup B_{3,x}$, then $\phi'(z) = \phi(z) \in \Phi$, and it follows that $\phi'(z) \neq \phi'(u)$, a contradiction. Hence, we have that $z \in B_{3,x-2}$. Since $B_{3,x-2} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-3}, u_{2,x-1}, u_{1,x-2}$ are in P. For each $b \in B_{3,x-2} \cup \{u_{1,x-2}\}$, since b is adjacent to $u_{2,x-3}$ and $u_{2,x-1}$, and $\phi(b) = \phi'(b)$ and $\phi(u_{2,x-3}) = \phi'(u_{2,x-3})$ and $\phi(u_{2,x-1}) = \phi'(u_{2,x-1})$, we have that $\phi'(b) \neq \phi'(u_{2,x-3}), \phi'(u_{2,x-1})$ and $\phi'(u_{2,x-3}) \neq \phi'(u_{2,x-1})$. Hence, we have that P is not in $H_{ij}(\phi')$, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(v) = \phi(v)$ for all $v \in B(\Gamma_2) \setminus \{u\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. □

Claim 4.3.22. *The bundle $B_{2,x-1}$ is a thin bundle.*

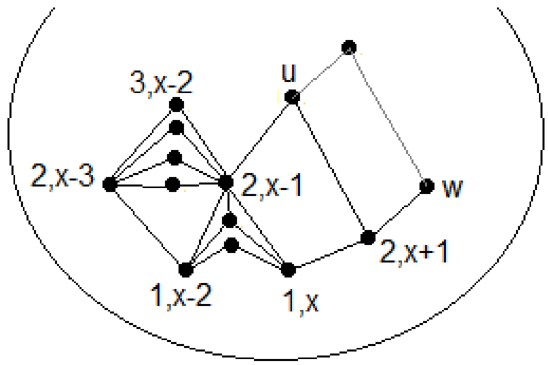

Figure 4.3: A possible configuration of the vertices of interest in Claim 4.3.22.

Proof. Suppose, towards a contradiction, that $B_{2,x-1}$ is a thick bundle. Note that Figure 4.3 shows an approximate configuration of the vertices of interest here. We will construct a new k-colouring ϕ' of H. If $\phi(u_{1,x-2}) = \phi(u_{1,x})$, then, since $|B_{2,x-1}| \geq 3$, it follows that there exists $y \in B_{2,x-1}$ such that $\phi(y) \neq \phi(u_{2,x-3}), \phi(u_{2,x+1})$. In this case, let $\phi'(y) = \phi(u_{2,x-1})$ and $\phi'(u_{2,x-1}) = \phi(y)$. If $\phi(u_{1,x-2}) \neq \phi(u_{1,x})$, then let $c_1, c_2, c_3 \in [k] \setminus \{\phi(u_{1,x-2}), \phi(u_{1,x}), \phi(u_{0,x-1}), \phi(u_{2,x-3}), \phi(u_{2,x+1}), \phi(u_{3,x-2})\}$ such that c_1, c_2, c_3 are pairwise distinct. Note that c_1, c_2, c_3 exist since $k \geq 12 \geq 9$. Let $\phi'(u_{2,x-1}) = c_1$ and for each $b \in B_{2,x-1} \setminus \{u_{2,x-1}\}$, let $\phi'(b) \in \{c_1, c_2, c_3\}$ such that adjacent vertices have distinct colours.

Notice that, for each $b \in B_{3,x-2}$, we have that $N_H(b) \subseteq B_{3,x-2} \cup \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Also, note that $\phi(u_{2,x-3}) \neq \phi'(u_{2,x-1})$. Let $c_4, c_5, c_6 \in [k] \setminus \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2}), \phi(u_{3,x-4})\}$ such that c_4, c_5, c_6 are distinct. Note that c_4, c_5, c_6 exist since $k \geq 12 \geq 7$. Let $\phi'(u_{3,x-2}) = c_4$ and, for each $b \in B_{3,x-2} \setminus \{u_{3,x-2}\}$, let $\phi'(b) \in \{c_4, c_5, c_6\}$ such that adjacent vertices have distinct colours.

Similarly, for each $b \in B_{3,x}$, we have that $N_H(b) \subseteq B_{3,x} \cup \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Also, note that $\phi(u_{2,x+1}) \neq \phi'(u_{2,x-1})$. Let $c, c' \in [k] \setminus \{\phi'(u_{2,x-1}), \phi(u_{2,x+1}), \phi(u_{1,x}), \phi(w), c_4\}$ such that c, c' are distinct. Note that c, c' exist since $k \geq 12 \geq 7$. Let $\phi'(u) = c$, and if there exists $u' \neq u \in B_{3,x}$, then let $\phi'(u') = c'$. Let $\phi'(v) = \phi(v)$ for all v in H that have not yet been assigned a colour under ϕ'.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $v \notin V(C_0)$ for all v where $\phi'(v) \neq \phi(v)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to M', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that C contains at least one of the following:

(i) a vertex in $B_{3,x}$,

(ii) a vertex in $B_{3,x-2}$,

(iii) a vertex in $B_{2,x-1}$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let $v \in V(C)$ such that $v \in B_{3,x} \cup B_{2,x-1} \cup B_{3,x-2}$ and let P be the component of C' in $H_{ij}(\phi')$ that contains v. Note that $v \notin V(C_0)$.

Subclaim 4.3.23. *The vertex v is not in $B_{3,x}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x}$. Notice that $N_H(B_{3,x}) \subseteq \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Since $B_{3,x} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-1}, u_{2,x+1}, u_{1,x}$ are in P. Since $u_{1,x}$ is adjacent to $u_{2,x+1}$ and $\phi(u_{1,x}) = \phi'(u_{1,x})$ and $\phi(u_{2,x+1}) = \phi'(u_{2,x+1})$, we have that $\phi'(u_{1,x}) \neq \phi'(u_{2,x+1})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x}), \phi'(u_{2,x+1})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi'(u_{2,x-1}), \phi(u_{2,x+1}), \phi(u_{1,x})\}$ for all $b \in B_{3,x}$. Hence, we have that $\phi'(v), \phi'(u_{2,x-1}), \phi'(u_{2,x+1}), \phi'(u_{1,x})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. $\square$

Subclaim 4.3.24. *The vertex v is not in $B_{3,x-2}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x-2}$. Notice that $N_H(B_{3,x-2}) \subseteq \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $B_{3,x-2} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-3}, u_{2,x-1}, u_{1,x-2}$ are in P. Since $u_{1,x-2}$ is adjacent to $u_{2,x-3}$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{2,x-3}) = \phi'(u_{2,x-3})$, we have that $\phi'(u_{1,x-2}) \neq \phi'(u_{2,x-3})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x-2}), \phi'(u_{2,x-3})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2})\}$ for all $b \in B_{3,x-2}$. Hence, we have that $\phi'(v)$, $\phi'(u_{2,x-3})$, $\phi'(u_{2,x-1})$, $\phi'(u_{1,x-2})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. $\square$

By Subclaims 4.3.23 and 4.3.24, it follows that $v' \notin V(C)$ for all $v' \in B_{3,x} \cup B_{3,x-2}$; thus, we have that $v \in B_{2,x-1}$. Notice that $N_H(B_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{3,x} \cup B_{3,x-2}$. Since vertices of C are not in $B_{3,x} \cup B_{3,x-2}$ and $B_{2,x-1} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{1,x-2}, u_{1,x}, u_{0,x-1}$ are in P.

Since $u_{0,x-1}$ is adjacent to $u_{1,x-2}$ and $u_{1,x}$, and $\phi(u_{0,x-1}) = \phi'(u_{0,x-1})$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{1,x}) = \phi'(u_{1,x})$, we have that $\phi'(u_{0,x-1}) \neq \phi'(u_{1,x-2})$, $\phi'(u_{1,x})$. If $\phi'(u_{1,x-2}) \neq \phi'(u_{1,x})$, then $\phi'(b) \in \{c_1, c_2, c_3\}$; thus, $\phi'(b) \neq \phi'(u_{0,x-1}), \phi'(u_{1,x-2}), \phi'(u_{1,x})$ for all $b \in$

$B_{2,x-1}$. Hence, we have that $\phi'(v)$, $\phi'(u_{0,x-1})$, $\phi'(u_{1,x-2})$, $\phi'(u_{1,x})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. Therefore, we have that $\phi'(u_{1,x-2}) = \phi'(u_{1,x})$ and, thus, $\phi(u_{1,x-2}) = \phi(u_{1,x})$.

Thus, the colours of the vertices in $B_{2,x-1} \cup \{u_{0,x-1}\}$ in ϕ are pairwise distinct and not equal to $\phi(u_{1,x-2})$ or $\phi(u_{1,x})$ and the colours of the vertices in $B_{2,x-1} \cup \{u_{0,x-1}\}$ in ϕ' are pairwise distinct and not equal to $\phi'(u_{1,x-2})$ or $\phi'(u_{1,x})$. Hence, it follows that P contains $u_{1,x-2}, u_{1,x}$ and at most one vertex in $B_{2,x-1}$. Thus, we have that $u_{1,x-2}, u_{1,x}$ are the neighbours of v in P. Since $u_{2,x-1} \in B(\Gamma_1)$, it follows that at most one of $u_{1,x-2}, u_{1,x}$ is in $V(C_0)$. Hence, we have that $\widetilde{u}_{1,x-2}\widetilde{v}\widetilde{u}_{1,x}$ is a subpath of C in $\widetilde{H}_{ij}(\phi', M_0)$. Let P' be the other $(u_{1,x-2}, u_{1,x})$-path in C and notice that $\widetilde{b} \notin V(P')$ for all $b \in B_{3,x} \cup B_{3,x-2}$ by Subclaims 4.3.23 and 4.3.24. By the construction of ϕ', we have that there exists a vertex $b \in B_{2,x-1}$ such that $\phi'(v) = \phi(b)$. Thus, it follows that $P' + \widetilde{u}_{1,x-2}\widetilde{b}\widetilde{u}_{1,x}$ is a cycle in $\widetilde{H}_{ij}(\phi, M_0)$. Thus, we have that M_0 does not extend to ϕ, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(u_{3,x-2}) \neq \phi'(u_{3,x-4})$, $\phi'(u_{3,x})$ and $\phi'(v) = \phi(v)$ for all $v \in V(C_3) \setminus \{u, u_{3,x-2}\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Similarly, if pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x-2}$, then $r \in \{u_{3,x-4}, u_{3,x}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. $\square$

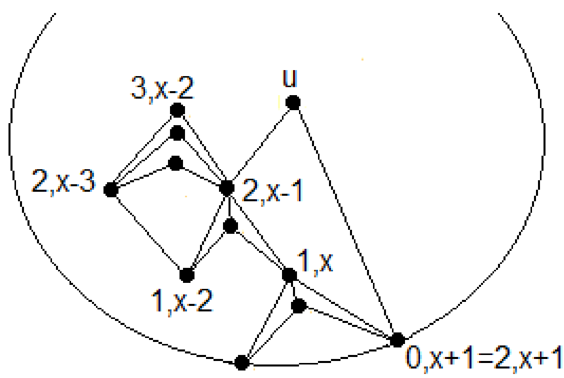

Figure 4.4: A possible configuration of the vertices of interest in Claim 4.3.25.

Claim 4.3.25. $u_{1,x} \notin B(\Gamma_0)$.

Proof. Suppose, towards a contradiction, that $u_{1,x} \in B(\Gamma_0)$. Note that Figure 4.4 shows an approximate configuration of the vertices of interest here. Since $u_{1,x}$ is adjacent to $u_{2,x+1}$ and, by Claim 4.3.19, $u_{2,x+1} \notin B(\Gamma_1)$, it follows that $u_{2,x+1} \in V(C_0)$. That is, $u_{2,x+1} = u_{0,x+1}$. Thus, there is no bundle on $u_{1,x}$ and $u_{1,x+2}$, and we have that $B_{2,x+1} = \emptyset$. We will construct a new k-colouring ϕ' of H.

Let $\Phi = \{\phi(z) : z \in N_{H_2}(u_{2,x-1}) \cup N_H(u_{1,x}) \cup \{u_{2,x-3}, u_{3,x-2}\}\}$. Notice that $N_{H_2}(u_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{2,x-1}$. By planarity, we have that $|N_{H_2}(u_{2,x-1}) \cap (B_{2,x-1} \cup \{u_{0,x-1}\})| \leq 1$. Also, notice that $N_H(u_{1,x}) \subseteq \{u_{0,x-1}, u_{0,x}, u_{0,x+1}\} \cup B_{1,x} \cup B_{2,x-1} \cup B_{2,x+1}$. By planarity, we have that $|N_H(u_{1,x}) \cap (B_{1,x} \cup \{u_{0,x}\})| \leq 1$. Since $B_{2,x-1}$ is a thin bundle and $B_{2,x+1} = \emptyset$, we have that $|B_{2,x-1} \cup B_{2,x+1}| \leq 2$. Thus, it follows that $|\Phi| \leq 10$. Since $k \geq 12$, there exists $c \in [k] \setminus \Phi$. Let $\phi'(u_{2,x-1}) = c$.

Notice that, for each $b \in B_{3,x-2}$, we have that $N_H(b) \subseteq B_{3,x-2} \cup \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $\phi(u_{2,x-3}) = \phi(u_{2,x-1})$ by Claim 4.3.21, it follows that the colours of the vertices in $B_{3,x-2} \cup \{u_{1,x-2}\}$ in ϕ are pairwise distinct. Also, note that $\phi(u_{2,x-3}) \neq \phi'(u_{2,x-1})$. Let $c_1, c_2, c_3 \in [k] \setminus \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2})\}$ such that $c_1 = \phi(u_{3,x-2})$ and c_1, c_2, c_3 are distinct. Note that c_1, c_2, c_3 exist since $k \geq 12 \geq 6$. For each $b \in B_{3,x-2} \setminus \{u_{3,x-2}\}$, let $\phi'(b) \in \{c_1, c_2, c_3\}$ such that adjacent vertices have distinct colours.

Similarly, for each $b \in B_{3,x}$, we have that $N_H(b) \subseteq B_{3,x} \cup \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Also, note that $\phi(u_{2,x+1}) \neq \phi'(u_{2,x-1})$. Let $c, c' \in [k] \setminus \{\phi'(u_{2,x-1}), \phi(u_{2,x+1}), \phi(u_{1,x}), \phi(w), c_1\}$ such that c, c' are distinct. Note that c, c' exists since $k \geq 12 \geq 7$. Let $\phi'(u) = c$, and if there exists $u' \neq u \in B_{3,x}$, then let $\phi'(u') = c'$. Let $\phi'(v) = \phi(v)$ for all v in H that have not yet been assigned a colour under ϕ'.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $v \notin V(C_0)$ for all v where $\phi'(v) \neq \phi(v)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that C contains at least one of the following:

(i) a vertex in $B_{3,x}$,

(ii) a vertex in $B_{3,x-2} \setminus \{u_{3,x-2}\}$,

(iii) $u_{2,x-1}$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let $v \in V(C)$ such that

$v \in B_{3,x} \cup \{u_{2,x-1}\} \cup B_{3,x-2}$ and let P be the component of C' in $H_{ij}(\phi')$ that contains v. Note that $v \notin V(C_0)$.

Subclaim 4.3.26. *The vertex v is not in $B_{3,x}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x}$. Notice that $N_H(B_{3,x}) \subseteq \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Since $B_{3,x} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-1}, u_{2,x+1}, u_{1,x}$ are in P. Since $u_{1,x}$ is adjacent to $u_{2,x+1}$ and $\phi(u_{1,x}) = \phi'(u_{1,x})$ and $\phi(u_{2,x+1}) = \phi'(u_{2,x+1})$, we have that $\phi'(u_{1,x}) \neq \phi'(u_{2,x+1})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x}), \phi'(u_{2,x+1})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi'(u_{2,x-1}), \phi(u_{2,x+1}), \phi(u_{1,x})\}$ for all $b \in B_{3,x}$. Hence, we have that $\phi'(v), \phi'(u_{2,x-1}), \phi'(u_{2,x+1}), \phi'(u_{1,x})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. $\square$

Subclaim 4.3.27. *The vertex v is not in $B_{3,x-2}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x-2}$. Notice that $N_H(B_{3,x-2}) \subseteq \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $B_{3,x-2} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-3}, u_{2,x-1}, u_{1,x-2}$ are in P. Since $u_{1,x-2}$ is adjacent to $u_{2,x-3}$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{2,x-3}) = \phi'(u_{2,x-3})$, we have that $\phi'(u_{1,x-2}) \neq \phi'(u_{2,x-3})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x-2}), \phi'(u_{2,x-3})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2})\}$ for all $b \in B_{3,x-2}$. Hence, we have that $\phi'(v)$, $\phi'(u_{2,x-3})$, $\phi'(u_{2,x-1})$, $\phi'(u_{1,x-2})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. $\square$

By Subclaims 4.3.26 and 4.3.27, it follows that $v' \notin V(C)$ for all $v' \in B_{3,x} \cup B_{3,x-2}$; thus, we have that $v = u_{2,x-1}$. Notice that $N_H(B_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{3,x} \cup B_{3,x-2}$. Since vertices of C are not in $B_{3,x} \cup B_{3,x-2}$ and $B_{2,x-1} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{1,x-2}, u_{1,x}, u_{0,x-1}$ are in P.

Since $u_{0,x-1}$ is adjacent to $u_{1,x-2}$ and $u_{1,x}$, and $\phi(u_{0,x-1}) = \phi'(u_{0,x-1})$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{1,x}) = \phi'(u_{1,x})$, we have that $\phi'(u_{0,x-1}) \neq \phi'(u_{1,x-2}), \phi'(u_{1,x})$. If $\phi'(u_{1,x-2}) \neq \phi'(u_{1,x})$, then $\phi'(u_{2,x-1})$, $\phi'(u_{0,x-1})$, $\phi'(u_{1,x-2})$, $\phi'(u_{1,x})$ are distinct; hence, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. Thus, $\phi'(u_{1,x-2}) = \phi'(u_{1,x})$ and it follows that $u_{1,x-2}$ and $u_{1,x}$ are the neighbours of $u_{2,x-1}$ in P. Since $u_{1,x} \notin V(C_0)$, we have that two neighbours of $u_{1,x}$ are in P.

Recall that $N_H(u_{1,x}) \subseteq \{u_{0,x-1}, u_{0,x}, u_{0,x+1}\} \cup B_{1,x} \cup B_{2,x-1} \cup B_{2,x+1}$ and say $z \neq u_{2,x-1} \in N_H(u_{1,x})$ is a neighbour of $u_{1,x}$ in P. Since $\phi'(z) = \phi(z) \in \Phi$, it follows that $\phi'(z) \neq \phi'(v)$. Thus, we have that P is not in $H_{ij}(\phi')$, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(v) = \phi(v)$ for all $v \in V(C_3) \setminus \{u\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. □

Since $u_{2,x-1} \in B(\Gamma_1)$ and $u_{1,x} \notin B(\Gamma_0)$, it follows that $u_{1,x-2} \in B(\Gamma_0)$. Also, note that $u_{1,x} \in V(C_0)$.

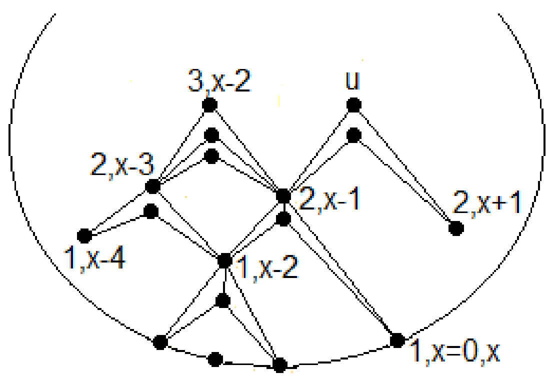

Figure 4.5: A possible configuration of the vertices of interest in Claim 4.3.28.

Claim 4.3.28. *$|B_{2,x-3}| \geq 5$.*

Proof. Suppose not. Note that Figure 4.5 shows an approximate configuration of the vertices of interest here. Since $u_{1,x-2} \in B(\Gamma_0)$ and $u_{1,x-2}$ is adjacent to $u_{2,x-3}$, it follows that $u_{2,x-3} \in B(\Gamma_1) \cup V(C_0)$. Thus, either $u_{2,x-3} \in B(\Gamma_1)$ and $|B_{2,x-3}| \leq 4$ or $u_{2,x-3} \in V(C_0)$ and $B_{2,x-3} = \emptyset$. We will construct a new k-colouring ϕ' of H.

Let $\Phi = \{\phi(z) : z \in N_{H_2}(u_{2,x-1}) \cup (N_H(u_{1,x-2}) \setminus \{u_{2,x-1}\}) \cup \{u_{3,x-2}\}\}$. Notice that $N_{H_2}(u_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{2,x-1}$. By planarity, we have that $|N_{H_2}(u_{2,x-1}) \cap (B_{2,x-1} \cup \{u_{0,x-1}\})| \leq 1$. Also, notice that $N_H(u_{1,x-2}) \setminus \{u_{2,x-1}\} \subseteq \{u_{0,x-3}, u_{0,x-1}, u_{0,x-2}\} \cup B_{1,x-2} \cup B_{2,x-3} \cup (B_{2,x-1} \setminus \{u_{2,x-1}\})$. By planarity, we have that $|N_H(u_{1,x-2}) \cap (B_{1,x-2} \cup \{u_{0,x-2}\})| \leq 1$. Since $B_{2,x-1}$ is a thin bundle and $|B_{2,x-3}| \leq 4$, we have that $|(B_{2,x-1} \setminus \{u_{2,x-1}\}) \cup B_{2,x-3}| \leq 5$. Thus, it follows that $|\Phi| \leq 11$. Since $k \geq 12$, there exists $c \in [k] \setminus \Phi$. Let $\phi'(u_{2,x-1}) = c$.

Notice that, for each $b \in B_{3,x-2}$, we have that $N_H(b) \subseteq B_{3,x-2} \cup \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $\phi(u_{2,x-3}) = \phi(u_{2,x-1})$, it follows that the colours of the vertices in $B_{3,x-2} \cup \{u_{1,x-2}\}$

in ϕ are pairwise distinct. Also, note that $\phi(u_{2,x-3}) \neq \phi'(u_{2,x-1})$. Let $c_1, c_2, c_3 \in [k] \setminus \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2})\}$ such that $c_1 = \phi(u_{3,x-2})$ and c_1, c_2, c_3 are distinct. Note that c_1, c_2, c_3 exist since $k \geq 12 \geq 6$. For each $b \in B_{3,x-2} \setminus \{u_{3,x-2}\}$, let $\phi'(b) \in \{c_1, c_2, c_3\}$ such that adjacent vertices have distinct colours.

Similarly, for each $b \in B_{3,x}$, we have that $N_H(b) \subseteq B_{3,x} \cup \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Let $\Phi' = \{\phi'(u_{2,x-1}), c_1, c_2, c_3\} \cup \{\phi(z) : z \in N_{H_2}(u_{2,x-1}) \cup \{u_{2,x+1}, w\}\}$. Recall that $N_{H_2}(u_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{2,x-1}$ and $|N_{H_2}(u_{2,x-1}) \cap (B_{2,x-1} \cup \{u_{0,x-1}\})| \leq 1$. Thus, it follows that $|\Phi'| \leq 9$. Since $k \geq 12$, there exist $c_4, c_5 \in [k] \setminus \Phi'$ where c_4, c_5 are distinct. Let $\phi'(u) = c_4$, and if there exists $u' \neq u \in B_{3,x}$, then let $\phi'(u') = c_5$. Let $\phi'(v) = \phi(v)$ for all v in H that have not yet been assigned a colour under ϕ'.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $v \notin V(C_0)$ for all v where $\phi'(v) \neq \phi(v)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) hold for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that C contains at least one of the following:

(i) a vertex in $B_{3,x}$,

(ii) a vertex in $B_{3,x-2} \setminus \{u_{3,x-2}\}$,

(iii) $u_{2,x-1}$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let $v \in V(C)$ such that $v \in B_{3,x} \cup \{u_{2,x-1}\} \cup B_{3,x-2}$ and let P be the component of C' in $H_{ij}(\phi')$ that contains v. Note that $v \notin V(C_0)$.

Subclaim 4.3.29. *The vertex v is not in $B_{3,x}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x}$. Notice that $N_H(B_{3,x}) \subseteq \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Since $B_{3,x} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-1}, u_{2,x+1}, u_{1,x}$ are in P. Since $u_{1,x}$ is adjacent to $u_{2,x+1}$, and $\phi(u_{1,x}) = \phi'(u_{1,x})$ and $\phi(u_{2,x+1}) = \phi'(u_{2,x+1})$, we have that $\phi'(u_{1,x}) \neq \phi'(u_{2,x+1})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x})$ and $\phi'(v) \notin \{\phi'(u_{2,x-1}), \phi(u_{2,x+1}), \phi(u_{1,x})\}$ for all $v \in B_{3,x}$. Furthermore, we have that $\phi'(u) \neq \phi'(u')$ if $u' \neq u \in B_{3,x}$. Thus, it follows that $u_{2,x-1}$ and $u_{2,x+1}$ are the neighbours of v in P and $\phi'(u_{2,x-1}) = \phi'(u_{2,x+1})$. Since $u_{2,x-1} \notin V(C_0)$, it follows that at least one neighbour of $u_{2,x-1}$, other than v, is in P.

Notice that $N_H(u_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{2,x-1} \cup B_{3,x} \cup B_{3,x-2}$. Say $z \neq v \in N_H(u_{2,x-1})$ is a neighbour of $u_{2,x-1}$ in P. Recall that $\phi'(b) \in \{c_1, c_2, c_3\}$ for all $b \in B_{3,x-2}$ and the colours of the vertices in $B_{3,x}$ are pairwise distinct. Thus, we have that $\phi'(z) \in \Phi' \cup \{c_4, c_5\}$ and it follows that $\phi'(z) \neq \phi'(v)$. Thus, we have that P is not in $H_{ij}(\phi')$, which is a contradiction. $\square$

Subclaim 4.3.30. *The vertex v is not in $B_{3,x-2}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x-2}$. Notice that $N_H(B_{3,x-2}) \subseteq \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $B_{3,x-2} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-3}$, $u_{2,x-1}$, $u_{1,x-2}$ are in P. Since $u_{1,x-2}$ is adjacent to $u_{2,x-3}$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{2,x-3}) = \phi'(u_{2,x-3})$, we have that $\phi'(u_{1,x-2}) \neq \phi'(u_{2,x-3})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x-2}), \phi'(u_{2,x-3})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2})\}$ for all $b \in B_{3,x-2}$. Hence, we have that $\phi'(v)$, $\phi'(u_{2,x-3})$, $\phi'(u_{2,x-1})$, $\phi'(u_{1,x-2})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. $\square$

By Subclaims 4.3.29 and 4.3.30, it follows that $v' \notin V(C)$ for all $v' \in B_{3,x} \cup B_{3,x-2}$; thus, we have that $v = u_{2,x-1}$. Notice that $N_H(B_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{3,x} \cup B_{3,x-2}$. Since vertices of C are not in $B_{3,x} \cup B_{3,x-2}$ and $B_{2,x-1} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{1,x-2}, u_{1,x}, u_{0,x-1}$ are in P.

Since $u_{0,x-1}$ is adjacent to $u_{1,x-2}$ and $u_{1,x}$, and $\phi(u_{0,x-1}) = \phi'(u_{0,x-1})$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{1,x}) = \phi'(u_{1,x})$, we have that $\phi'(u_{0,x-1}) \neq \phi'(u_{1,x-2}), \phi'(u_{1,x})$. If $\phi'(u_{1,x-2}) \neq \phi'(u_{1,x})$, then $\phi'(u_{2,x-1})$, $\phi'(u_{0,x-1})$, $\phi'(u_{1,x-2})$, $\phi'(u_{1,x})$ are distinct; hence, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. Thus, $\phi'(u_{1,x-2}) = \phi'(u_{1,x})$ and it follows that $u_{1,x-2}$ and $u_{1,x}$ are the neighbours of $u_{2,x-1}$ in P. Since $u_{1,x-2} \notin V(C_0)$, we have that at least one neighbour of $u_{1,x-2}$, other than v, is in P.

Recall that $N_H(u_{1,x-2}) \subseteq \{u_{0,x-3}, u_{0,x-2}, u_{0,x-1}\} \cup B_{1,x-2} \cup B_{2,x-1} \cup B_{2,x-3}$ and say $z \neq u_{2,x-1} \in N_H(u_{1,x-2})$ is a neighbour of $u_{1,x-2}$ in P. Since $\phi'(z) = \phi(z) \in \Phi$, it follows that $\phi'(z) \neq \phi'(v)$. Thus, we have that P is not in $H_{ij}(\phi')$, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(v) = \phi(v)$ for all $v \in V(C_3) \setminus \{u\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. $\square$

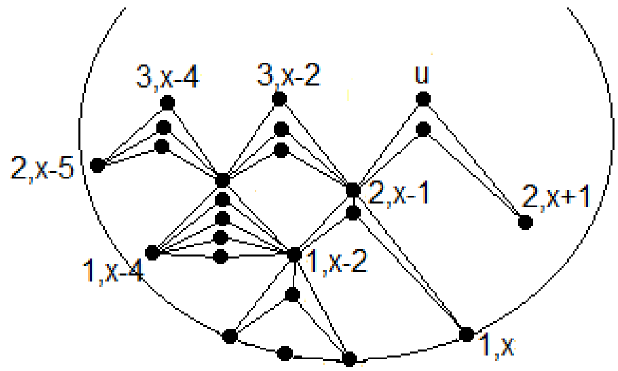

Figure 4.6: A possible configuration of the vertices of interest in Claim 4.3.31.

Claim 4.3.31. $\phi(u_{1,x-4}) = \phi(u_{1,x-2})$.

Proof. Suppose not. Note that Figure 4.6 shows an approximate configuration of the vertices of interest here. We will construct a new k-colouring ϕ' of H.

Let $\Phi = \{\phi(z) : z \in N_{H_2}(u_{2,x-1}) \cup N_{H_1}(u_{1,x-2}) \cup \{u_{2,x+1}, u_{2,x-3}, u_{3,x-2}\}\}$. Notice that $N_{H_2}(u_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x},\ u_{0,x-1}\} \cup B_{2,x-1}$. By planarity, we have that $|N_{H_2}(u_{2,x-1}) \cap (B_{2,x-1} \cup \{u_{0,x-1}\})| \leq 1$. Also, notice that $N_{H_1}(u_{1,x-2}) \subseteq \{u_{0,x-3},\ u_{0,x-1},\ u_{0,x-2}\} \cup B_{1,x-2}$. By planarity, we have that $|N_H(u_{1,x-2}) \cap (B_{1,x-2} \cup \{u_{0,x-2}\})| \leq 1$. Thus, it follows that $|\Phi| \leq 9$. Since $k \geq 12$, there exists $c \in [k] \setminus \Phi$. Let $\phi'(u_{2,x-1}) = c$.

Notice that, for each $b \in B_{3,x-2}$, we have that $N_H(b) \subseteq B_{3,x-2} \cup \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $\phi(u_{2,x-3}) = \phi(u_{2,x-1})$, it follows that the colours of the vertices in $B_{3,x-2} \cup \{u_{1,x-2}\}$ in ϕ are pairwise distinct. Also, note that $\phi(u_{2,x-3}) \neq \phi'(u_{2,x-1})$. Let $c_1, c_2, c_3 \in [k] \setminus \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2})\}$ such that $c_1 = \phi(u_{3,x-2})$ and c_1, c_2, c_3 are distinct. Note that c_1, c_2, c_3 exist since $k \geq 12 \geq 6$. For each $b \in B_{3,x-2} \setminus \{u_{3,x-2}\}$, let $\phi'(b) \in \{c_1, c_2, c_3\}$ such that adjacent vertices have distinct colours.

Similarly, for each $b \in B_{3,x}$, we have that $N_H(b) \subseteq B_{3,x} \cup \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Also, note that $\phi(u_{2,x+1}) \neq \phi'(u_{2,x-1})$. Let $c_4, c_5 \in [k] \setminus \{\phi'(u_{2,x-1}), \phi(u_{2,x+1}), \phi(u_{1,x}), \phi(w), c_1\}$ such that c_4, c_5 are distinct. Note that c_4, c_5 exist since $k \geq 12 \geq 7$. Let $\phi'(u) = c_4$, and if there exists $u' \neq u \in B_{3,x}$, then let $\phi'(u') = c_5$. Let $\phi'(v) = \phi(v)$ for all v in H that have not yet been assigned a colour under ϕ'.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $v \notin V(C_0)$ for all v where $\phi'(v) \neq \phi(v)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that C contains at least one of the following:

(i) a vertex in $B_{3,x}$,

(ii) a vertex in $B_{3,x-2} \setminus \{u_{3,x-2}\}$,

(iii) $u_{2,x-1}$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let $v \in V(C)$ such that $v \in B_{3,x} \cup \{u_{2,x-1}\} \cup B_{3,x-2}$ and let P be the component of C' in $H_{ij}(\phi')$ that contains v. Note that $v \notin V(C_0)$.

Subclaim 4.3.32. *The vertex v is not in $B_{3,x}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x}$. Notice that $N_H(B_{3,x}) \subseteq \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Since $B_{3,x} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-1}, u_{2,x+1}, u_{1,x}$ are in P. Since $u_{1,x}$ is adjacent to $u_{2,x+1}$ and $\phi(u_{1,x}) = \phi'(u_{1,x})$ and $\phi(u_{2,x+1}) = \phi'(u_{2,x+1})$, we have that $\phi'(u_{1,x}) \neq \phi'(u_{2,x+1})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x}), \phi'(u_{2,x+1})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi'(u_{2,x-1}), \phi(u_{2,x+1}), \phi(u_{1,x})\}$ for all $b \in B_{3,x}$. Hence, we have that $\phi'(v), \phi'(u_{2,x-1}), \phi'(u_{2,x+1}), \phi'(u_{1,x})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. □

Subclaim 4.3.33. *The vertex v is not in $B_{3,x-2}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x-2}$. Notice that $N_H(B_{3,x-2}) \subseteq \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $B_{3,x-2} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-3}, u_{2,x-1}, u_{1,x-2}$ are in P. Since $u_{1,x-2}$ is adjacent to $u_{2,x-3}$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{2,x-3}) = \phi'(u_{2,x-3})$, we have that $\phi'(u_{1,x-2}) \neq \phi'(u_{2,x-3})$. By the construction of ϕ', we have that $\phi'(u_{2,x-1}) \neq \phi'(u_{1,x-2}), \phi'(u_{2,x-3})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi(u_{2,x-3}), \phi'(u_{2,x-1}), \phi(u_{1,x-2})\}$ for all $b \in B_{3,x-2}$. Hence, we have that $\phi'(v)$, $\phi'(u_{2,x-3})$, $\phi'(u_{2,x-1})$, $\phi'(u_{1,x-2})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. □

By Subclaims 4.3.32 and 4.3.33, it follows that $v' \notin V(C)$ for all $v' \in B_{3,x} \cup B_{3,x-2}$; thus, we have that $v = u_{2,x-1}$. Notice that $N_H(B_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x}, u_{0,x-1}\} \cup B_{3,x} \cup B_{3,x-2}$. Since vertices of C are not in $B_{3,x} \cup B_{3,x-2}$ and $B_{2,x-1} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{1,x-2}, u_{1,x}, u_{0,x-1}$ are in P.

Since $u_{0,x-1}$ is adjacent to $u_{1,x-2}$ and $u_{1,x}$, and $\phi(u_{0,x-1}) = \phi'(u_{0,x-1})$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{1,x}) = \phi'(u_{1,x})$, we have that $\phi'(u_{0,x-1}) \neq \phi'(u_{1,x-2}), \phi'(u_{1,x})$. If $\phi'(u_{1,x-2}) \neq \phi'(u_{1,x})$, then $\phi'(u_{2,x-1})$, $\phi'(u_{0,x-1})$, $\phi'(u_{1,x-2})$, $\phi'(u_{1,x})$ are distinct; hence, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. Thus, $\phi'(u_{1,x-2}) = \phi'(u_{1,x})$ and it follows that $u_{1,x-2}$ and $u_{1,x}$ are the neighbours of $u_{2,x-1}$ in P. Since $u_{1,x-2} \notin V(C_0)$, we have that at least one neighbour of $u_{1,x-2}$, other than v, is in P.

Recall that $N_H(u_{1,x-2}) \subseteq \{u_{0,x-3}, u_{0,x-2}, u_{0,x-1}\} \cup B_{1,x-2} \cup B_{2,x-1} \cup B_{2,x-3}$ and say $z \neq u_{2,x-1} \in N_H(u_{1,x-2})$ is a neighbour of $u_{1,x-2}$ in P. Since $\phi'(u_{1,x-4}) \neq \phi'(u_{1,x-2})$, it follows that $z \notin B_{2,x-3} \setminus \{u_{2,x-3}\}$. Thus, we have that $\phi'(z) = \phi(z) \in \Phi$. Since $\phi'(z) \in \Phi$, it follows that $\phi'(z) \neq \phi'(v)$. Thus, we have that P is not in $H_{ij}(\phi')$, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(v) = \phi(v)$ for all $v \in V(C_3) \setminus \{u\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. $\square$

Since $u_{1,x-2} \in B(\Gamma_0)$, it follows that $u_{2,x-3} \in B(\Gamma_1)$; thus, we have that $u_{3,x-4} \in B(\Gamma_2)$ or $u_{3,x-4} = u_{1,x-4}$. Note that Figure 4.6 shows an approximate configuration of the vertices of interest here as well.

We will now construct a new k-colouring ϕ' of H. Since $\phi(u_{1,x-4}) = \phi(u_{1,x-2})$, it follows that the colours of the vertices in $B_{2,x-3} \cup \{u_{0,x-3}\}$ in ϕ are pairwise distinct. Since $|B_{2,x-3}| \geq 5$, we have that there exists $y \in B_{2,x-3}$ such that $\phi(y) \neq \phi(u_{2,x-1})$, $\phi(u_{2,x-5})$, $\phi(u_{3,x-4})$, $\phi(u_{3,x-2})$. Let $\phi'(u_{2,x-3}) = \phi(y)$ and $\phi'(y) = \phi(u_{2,x-3})$.

Notice that $N_H(b) \subseteq B_{3,x-4} \cup \{u_{2,x-5}, u_{2,x-3}, u_{1,x-4}\}$ for all $b \in B_{3,x-4}$. Note that $\phi'(u_{2,x-3}) \neq \phi(u_{2,x-5})$. Let $c_1, c_2, c_3 \in [k] \setminus \{\phi(u_{2,x-5}), \phi(u_{1,x-4}), \phi'(u_{2,x-3}), \phi(u_{3,x-6}), \phi(u_{3,x-2})\}$ such that c_1, c_2, c_3 are pairwise distinct. Note that c_1, c_2, c_3 exist since $k \geq 12 \geq 8$. Let $\phi'(u_{3,x-4}) = c_1$ and, for each $b \in B_{3,x-4} \setminus \{u_{3,x-4}\}$, let $\phi'(b) \in \{c_1, c_2, c_3\}$ such that adjacent vertices have distinct colours.

Also notice that, for each $b \in B_{3,x-2}$, we have that $N_H(b) \subseteq B_{3,x-2} \cup \{u_{2,x-1}, u_{2,x-3}, u_{1,x-2}\}$. Since $\phi(u_{2,x-3}) = \phi(u_{2,x-1})$, it follows that the colours of the vertices in $B_{3,x-2} \cup \{u_{1,x-2}\}$ in ϕ are pairwise distinct. Note that $\phi'(u_{2,x-3}) \neq \phi(u_{2,x-1})$. Let $c_4, c_5, c_6 \in [k] \setminus \{\phi'(u_{2,x-3}), \phi(u_{2,x-1}), \phi(u_{1,x-2})\}$ such that $c_4 = \phi(u_{3,x-2})$ and c_4, c_5, c_6 are distinct. Note that c_4, c_5, c_6 exist since $k \geq 12 \geq 6$. For each $b \in B_{3,x-2} \setminus \{u_{3,x-2}\}$, let $\phi'(b) \in \{c_4, c_5, c_6\}$ such that adjacent vertices have distinct colours.

Similarly, for each $b \in B_{3,x}$, we have that $N_H(b) \subseteq B_{3,x} \cup \{u_{2,x+1}, u_{2,x-1}, u_{1,x}\}$. Let $\Phi' = \{c_4, c_5, c_6\} \cup \{\phi(z) : z \in N_{H_2}(u_{2,x-1}) \cup \{u_{2,x-1}, u_{2,x+1}, w\}\}$. Recall that $N_{H_2}(u_{2,x-1}) \subseteq$

$\{u_{1,x-2}, u_{1,x},\ u_{0,x-1}\} \cup B_{2,x-1}$ and $|N_{H_2}(u_{2,x-1}) \cap (B_{2,x-1} \cup \{u_{0,x-1}\})| \leq 1$. Thus, it follows that $|\Phi'| \leq 9$. Since $k \geq 12$, there exist $c_7, c_8 \in [k] \setminus \Phi'$ where c_7, c_8 are distinct. Let $\phi'(u) = c_7$, and if there exists $u' \neq u \in B_{3,x}$, then let $\phi'(u') = c_8$. Let $\phi'(v) = \phi(v)$ for all v in H that have not yet been assigned a colour under ϕ'.

Suppose, towards a contradiction, that M_0 does not extend to ϕ'. Since $v \notin V(C_0)$ for all v where $\phi'(v) \neq \phi(v)$, it follows that $\phi'_{|C_0} = \phi_{|C_0} = \phi(M_0)$. Hence, Definition 3.2.8(1) holds for M_0 extending to ϕ'. Since M_0 does not extend to ϕ', it follows by Definition 3.2.8(2) that $\widetilde{H}_{ij}(\phi', M_0)$ contains a cycle C for some $i \neq j \in [k]$. Since M_0 extends to ϕ, it follows that C is not a cycle in $\widetilde{H}_{ij}(\phi, M_0)$; thus, we have that C contains at least one of the following:

(i) a vertex in $B_{3,x}$,

(ii) a vertex in $B_{3,x-2} \setminus \{u_{3,x-2}\}$,

(iii) a vertex in $B_{3,x-4}$,

(iv) a vertex in $B_{2,x-3}$.

Notice that the cycle C in $\widetilde{H}_{ij}(\phi', M_0)$ is equivalent to a subgraph C' in $H_{ij}(\phi')$ where C' is a cycle or a collection of paths with endpoints in $V(C_0)$. Let $v \in V(C)$ such that $v \in B_{3,x} \cup B_{3,x-2} \cup B_{3,x-4} \cup B_{2,x-3}$ and let P be the component of C' in $H_{ij}(\phi')$ that contains v. Note that $v \notin V(C_0)$.

Subclaim 4.3.34. *The vertex v is not in $B_{3,x}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x}$. Notice that $N_H(B_{3,x}) \subseteq \{u_{2,x+1}, u_{2,x-1},\ u_{1,x}\}$. Since $B_{3,x} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-1}, u_{2,x+1}, u_{1,x}$ are in P. Since $u_{1,x}$ is adjacent to $u_{2,x+1}$ and $u_{2,x-1}$, and $\phi(u_{1,x}) = \phi'(u_{1,x})$ and $\phi(u_{2,x+1}) = \phi'(u_{2,x+1})$ and $\phi(u_{2,x-1}) = \phi'(u_{2,x-1})$, we have that $\phi'(u_{1,x}) \neq \phi'(u_{2,x+1}), \phi'(u_{2,x-1})$. By the construction of ϕ', we have that $\phi'(v) \notin \{\phi'(u_{2,x-1}),\ \phi(u_{2,x+1}),\ \phi(u_{1,x})\}$ for all $v \in B_{3,x}$. Furthermore, we have that $\phi'(u) \neq \phi'(u')$ if $u' \neq u \in B_{3,x}$. Thus, it follows that $u_{2,x-1}$ and $u_{2,x+1}$ are the neighbours of v in P and $\phi'(u_{2,x-1}) = \phi'(u_{2,x+1})$. Since $u_{2,x-1} \notin V(C_0)$, it follows that at least one neighbour of $u_{2,x-1}$, other than v, is in P.

Notice that $N_H(u_{2,x-1}) \subseteq \{u_{1,x-2}, u_{1,x},\ u_{0,x-1}\} \cup B_{2,x-1} \cup B_{3,x} \cup B_{3,x-2}$. Say $z \neq v \in N_H(u_{2,x-1})$ is a neighbour of $u_{2,x-1}$ in P. Recall that $\phi'(b) \in \{c_4, c_5, c_6\}$ for all $b \in B_{3,x-2}$ and the colours of the vertices in $B_{3,x}$ are pairwise distinct. Thus, we have that $\phi'(z) \in \Phi' \cup \{c_7, c_8\}$ and it follows that $\phi'(z) \neq \phi'(v)$. Thus, we have that P is not in $H_{ij}(\phi')$, which is a contradiction. $\square$

Subclaim 4.3.35. *The vertex v is not in $B_{3,x-2}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x-2}$. Notice that $N_H(B_{3,x-2}) \subseteq \{u_{2,x-1},\ u_{2,x-3},\ u_{1,x-2}\}$. Since $B_{3,x-2} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-3}$, $u_{2,x-1}$, $u_{1,x-2}$ are in P. Since $u_{1,x-2}$ is adjacent to $u_{2,x-1}$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$ and $\phi(u_{2,x-1}) = \phi'(u_{2,x-1})$, we have that $\phi'(u_{1,x-2}) \neq \phi'(u_{2,x-1})$. By the construction of ϕ', we have that $\phi'(u_{2,x-3}) \neq \phi'(u_{1,x-2}), \phi'(u_{2,x-1})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi'(u_{2,x-3}), \phi(u_{2,x-1}), \phi(u_{1,x-2})\}$ for all $b \in B_{3,x-2}$. Hence, we have that $\phi'(v)$, $\phi'(u_{2,x-3})$, $\phi'(u_{2,x-1})$, $\phi'(u_{1,x-2})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. □

Subclaim 4.3.36. *The vertex v is not in $B_{3,x-4}$.*

Proof. Suppose, towards a contradiction, that $v \in B_{3,x-4}$. Notice that $N_H(B_{3,x-4}) \subseteq \{u_{2,x-3},\ u_{2,x-5},\ u_{1,x-4}\}$. Since $B_{3,x-4} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{2,x-5}$, $u_{2,x-3}$, $u_{1,x-4}$ are in P. Since $u_{1,x-4}$ is adjacent to $u_{2,x-5}$ and $\phi(u_{1,x-4}) = \phi'(u_{1,x-4})$ and $\phi(u_{2,x-5}) = \phi'(u_{2,x-5})$, we have that $\phi'(u_{1,x-4}) \neq \phi'(u_{2,x-5})$. By the construction of ϕ', we have that $\phi'(u_{2,x-3}) \neq \phi'(u_{1,x-4}), \phi'(u_{2,x-5})$. It also follows from the construction of ϕ' that $\phi'(b) \notin \{\phi'(u_{2,x-3}), \phi(u_{2,x-5}), \phi(u_{1,x-4})\}$ for all $b \in B_{3,x-4}$. Hence, we have that $\phi'(v)$, $\phi'(u_{2,x-3})$, $\phi'(u_{2,x-5})$, $\phi'(u_{1,x-4})$ are pairwise distinct. Thus, it follows that P is not in $H_{ij}(\phi')$, which is a contradiction. □

By Subclaims 4.3.34, 4.3.35, and 4.3.36, it follows that $v' \notin V(C)$ for all $v' \in B_{3,x} \cup B_{3,x-2} \cup B_{3,x-4}$; thus, we have that $v \in B_{2,x-3}$. Notice that $N_H(B_{2,x-3}) \subseteq \{u_{1,x-4}, u_{1,x-2}, u_{0,x-3}\} \cup B_{3,x-2} \cup B_{3,x-4}$. Since vertices of C are not in $B_{3,x-2} \cup B_{3,x-4}$ and $B_{2,x-3} \cap V(C_0) = \emptyset$, it follows that at least two of $u_{1,x-4}, u_{1,x-2}, u_{0,x-3}$ are in P.

Since $u_{0,x-3}$ is adjacent to $u_{1,x-4}$ and $u_{1,x-2}$, and $\phi(u_{0,x-3}) = \phi'(u_{0,x-3})$ and $\phi(u_{1,x-4}) = \phi'(u_{1,x-4})$ and $\phi(u_{1,x-2}) = \phi'(u_{1,x-2})$, we have that $\phi'(u_{0,x-3}) \neq \phi'(u_{1,x-4}), \phi'(u_{1,x-2})$. Since $\phi(u_{1,x-4}) = \phi(u_{1,x-2})$ by Claim 4.3.31, we have that $\phi'(u_{1,x-4}) = \phi(u_{1,x-4}) = \phi(u_{1,x-2}) = \phi'(u_{1,x-2})$. Thus, it follows that the colours of the vertices in $B_{2,x-3} \cup \{u_{0,x-3}\}$ in ϕ are pairwise distinct and not equal to $\phi(u_{1,x-4})$ or $\phi(u_{1,x-2})$. Additionally, we have that the colours of the vertices in $B_{2,x-3} \cup \{u_{0,x-3}\}$ in ϕ' are pairwise distinct and not equal to $\phi'(u_{1,x-4})$ or $\phi'(u_{1,x-2})$. Hence, it follows that P contains $u_{1,x-4}, u_{1,x-2}$ and at most one vertex in $B_{2,x-3}$. Thus, we have that $u_{1,x-4}, u_{1,x-2}$ are the neighbours of v in P. Since $u_{1,x-2} \notin V(C_0)$, it follows that $\widetilde{u}_{1,x-4}\widetilde{v}\widetilde{u}_{1,x-2}$ is a subpath of C in $\widetilde{H}_{ij}(\phi', M_0)$. Let P' be the other $(u_{1,x-4}, u_{1,x-2})$-path in C and notice that $\widetilde{b} \notin P'$ for all $b \in B_{3,x-2} \cup B_{3,x-4}$ by Subclaims 4.3.35 and 4.3.36. By the construction of ϕ', we have that there exists a vertex

$b \in B_{2,x-3}$ such that $\phi'(v) = \phi(b)$. Thus, it follows that $P' + \widetilde{u}_{1,x-4}\widetilde{b}\widetilde{u}_{1,x-2}$ is a cycle in $\widetilde{H}_{ij}(\phi, M_0)$. Thus, we have that M_0 does not extend to ϕ, which is a contradiction.

Therefore, it follows that there exists a k-colouring ϕ' of H where $\phi'(u) \neq \phi'(u_{3,x-2})$, $\phi'(u_{3,x+2})$ and $\phi'(u_{3,x-4}) \neq \phi'(u_{3,x-6})$, $\phi'(u_{3,x-2})$ and $\phi'(v) = \phi(v)$ for all $v \in V(C_3) \setminus \{u\}$ such that M_0 extends to ϕ'. If pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x}$, then $r \in \{u_{3,x-2}, u_{3,x+2}\}$. Similarly, if pqr is a bichord of Γ_3 where $q \in B(\Gamma_3)$ and $p = u_{3,x-4}$, then $r \in \{u_{3,x-6}, u_{3,x-2}\}$. Hence, if a bichord is monochromatic in ϕ', then it is monochromatic in ϕ. Since v is in a bichord of Γ_3 that is monochromatic in ϕ, but is not monochromatic in ϕ', it follows that ϕ' has fewer monochromatic bichords of Γ_3 than ϕ, which contradicts the minimality of ϕ. $\square$

Chapter 5

Critical Canvases

In this chapter, we prove the Main Theorem 5.3.5. In order to do that, we first determine some structure in graphs which are critical for acyclic k-colouring. This is followed by a collection of calculations which will also be used to prove the main result.

5.1 General Structure

In this section, we prove the Key Lemma 5.1.2 and the General Structure Lemma 5.1.4.

Definition 5.1.1. Let G be a graph with a proper subgraph H. The graph G is *H-critical for acyclic k-colouring* if, for all proper subgraphs G' of G that contain H, there exists a k-mosaic of H which extends to G', but not to G.

Lemma 5.1.2 (Key Lemma)**.** *Let G be a graph with a subgraph H where G is H-critical for acyclic k-colouring. If $G = A \cup B$ where $H \subseteq A$ and $B \neq A \cap B$, then B is $(A \cap B)$-critical for acyclic k-colouring.*

Proof. Suppose not. Thus, there exists a proper subgraph S of B where $A \cap B$ is a subgraph of S such that every k-mosaic of $A \cap B$ that extends to S, also extends to B. Let $T = S \cup A$.

Since G is H-critical for acyclic k-colouring, there exists a k-mosaic M_H of H which extends to a k-mosaic M_T of T, but not to G. Let $M_A = \text{Mosaic}[\phi(M_T)_{|A}, M_H]$ and let $M_{A \cap B}$ be the restriction of M_A to $A \cap B$. Let M_S be the restriction of M_T to B. Since T has a k-mosaic M_T and A is a subgraph of T and H is a subgraph of A with a k-mosaic M_H where M_H extends to M_T, we have that T, A, and H satisfy the conditions of G, G', and H in Proposition 3.3.11. Thus, by Proposition 3.3.11, it follows that M_A extends to M_T. Since A is a subgraph of T and the k-mosaic M_A of A extends to the k-mosaic M_T of T and B is a subgraph of T, we have that T, A, and B satisfy the conditions of G, G', and H in Proposition 3.3.15. Hence, by Proposition 3.3.15, it follows that $M_{A\cap B}$ extends to M_S. Thus, since every k-mosaic of $A \cap B$ that extends to S also extends to B, we have that $M_{A\cap B}$ extends to B.

Since $M_{A\cap B}$ extends to B, we have that G, A, and B satisfy the conditions of G, A, and B in Proposition 3.3.16. Thus, by Proposition 3.3.16, it follows that M_A extends to G. Now, since M_A extends to G and M_H extends to M_A, we have by Proposition 3.3.1 that M_H extends to G, which is a contradiction. □

Definition 5.1.3. We say a canvas $\Gamma = (G, H)$ is *k-critical* if G is H-critical for acyclic k-colouring.

Lemma 5.1.4 (General Structure Lemma)**.** *If a canvas $\Gamma = (G, C)$ where C is the outer cycle of G is k-critical for $k \geq 12$, then there exists at least one of the following:*

(i) a chord of C, or

(ii) a bichord of Γ, or

(iii) a 6-double-pod of Γ.

Proof. Since G is C-critical for acyclic k-colouring, there exists a k-mosaic M of C that does not extend to G. By the Extension Lemma 4.2.1, there exists either *(a)* a chord uv of C, *(b)* a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$, or *(c)* an r-double-pod v of Γ where $|\{\phi_M(u) : u \in \text{feet}(v)\}| \geq k - 6$. In the case of *(a)*, it follows that G contains a chord of C; thus, *(i)* holds. In the case of *(b)*, it follows that Γ contains a bichord; thus, *(ii)* holds. In the case of *(c)*, it follows that Γ contains an r-double-pod where $r \geq k - 6$ and, since $k \geq 12$, we have that Γ contains a 6-double-pod; thus, *(iii)* holds. □

Theorem 5.1.5. *If $k \geq 10$ is an integer, then there does not exist a plane graph G with outer triangle C such that G is C-critical for acyclic k-colouring.*

Proof. Suppose, towards a contradiction, that there exists a plane graph G with outer cycle C, where C is a triangle, and G is C-critical for acyclic k-colouring for some $k \geq 10$. Since G is C-critical for acyclic k-colouring, $G \neq C$. Hence, for every proper subgraph H of G where $C \subseteq H$, there exists a k-mosaic of C which extends to H, but not to G. Therefore, there exists a k-mosaic M of C which does not extend to G, contradicting Corollary 4.2.4. □

Theorem 5.1.6. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and C is a 4-cycle. Let $k \geq 11$. If Γ is k-critical, then $|V(G) \setminus V(C)| \leq k-2$ and all vertices in $V(G) \setminus V(C)$ are bipods of Γ.*

Proof. Suppose not. Let M be a k-mosaic of C that does not extend to G. Let $C = u_1u_2u_3u_4u_1$. If C has a chord, say u_1u_3, then $G\langle u_1u_2u_3u_1\rangle = u_1u_2u_3u_1$ and $G\langle u_1u_4u_3u_1\rangle = u_1u_4u_3u_1$ by Theorem 5.1.5, and we have that $|V(G) \setminus V(C)| = 0$, a contradiction. Thus, we may assume that C has no chords.

By Corollary 4.2.5, there exists a vertex $v \in \text{int}(C)$ such that v is adjacent to $u, w \in V(C)$ where $\phi_M(u) = \phi_M(w)$. Since $\phi(M)$ is proper and acyclic, there is at most one pair of vertices of C that have the same colour in $\phi(M)$. Without loss of generality, say that $\phi_M(u_1) = \phi_M(u_3) = k$. Thus, there exists a vertex in $V(G) \setminus V(C)$ that is adjacent to u_1 and u_3. Let A be the (non-empty) set of vertices in $V(G) \setminus V(C)$ that are adjacent to u_1 and u_3.

Claim 5.1.7. *The size of A is at most $k-2$.*

Proof. Suppose not. Let $v \in A$. Since G is C-critical for acyclic k-colouring, there exists a k-mosaic M_C of C which extends to a k-mosaic M' of $G-v$, but not to G. Notice that there are at least k vertices adjacent to u_1 and u_3 in $G-v$: the $k-2$ vertices of A, and the other vertices u_2, u_4 of C. Let $A' = (A \backslash \{v\}) \cup \{u_2, u_4\}$. Recall that $\phi_M(u_1) = \phi_M(u_3) = k$; thus, $\phi_{M'}(u_1) = \phi_{M'}(u_3) = k$. For all vertices $a \in A'$, we have that $\phi_{M'}(a) \neq \phi_{M'}(u_1)$; hence, the colours of the vertices of A' in $\phi(M')$ are in $[k-1]$. By the Pigeonhole Principle, there exist vertices $x, y \in A'$ such that $\phi_{M'}(x) = \phi_{M'}(y)$. Now $u_1xu_3yu_1$ is a cycle in $G_{ij}(\phi(M'))$, a contradiction. $\square$

By Theorem 5.1.5, the interiors of all triangles in G are empty. That is, for every triangle T in G, $G\langle T\rangle = T$. Therefore, if G is a triangulation, then $V(C) \cup A$ are the only vertices of G and we have $|V(G) \setminus V(C)| \leq k-2$. Thus, we may assume that there exists a 4-cycle $C' = u_1xu_3yu_1$ where $x, y \in A$ such that $\text{int}(C')$ is non-empty and $\text{int}(C') \cap A = \emptyset$. If C' has a chord, then it is a triangulation and therefore the interior of C' is empty; hence, C' has no chords.

Notice that $G = (G \backslash \text{int}(C')) \cup G\langle C'\rangle)$ and $C \subseteq G \backslash \text{int}(C')$. Furthermore, $(G \backslash \text{int}(C')) \cap G\langle C'\rangle = C'$. Thus, by Lemma 5.1.2, $G\langle C'\rangle$ is C'-critical for acyclic k-colouring. Let M be a k-mosaic of $G \setminus \text{int}(C')$ that does not extend to G and let M' be the restriction of M to C'. Note that $u_1, u_3 \in V(C')$, and $\phi_{M'}(u_1) = \phi_M(u_1) = k = \phi_M(u_3) = \phi_{M'}(u_3)$. Furthermore, since $\phi(M)$ is proper and acyclic, the only pair of vertices of C' that have the same colour in $\phi(M)$ (and thus in $\phi(M')$) is (u_1, u_3). By Corollary 4.2.5, there exists a vertex $v \in \operatorname{int}(C')$ that is adjacent to u_1 and u_3. By the definition of A, we have that $v \in A$; thus, $\{v\} \subseteq \operatorname{int}(C') \cap A$, which implies that $\operatorname{int}(C') \cap A \neq \emptyset$, a contradiction. □

5.2 Calculations

In this section, we establish some bounds which will be used in the proof of the Main Theorem 5.3.5.

Lemma 5.2.1. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| \geq 5$ and G contains a chord uv of C. Let C_1 and C_2 be the cycles that bound the two inner faces of $C + uv$. Let $G_i = G\langle C_i \rangle$ for each $i \in \{1, 2\}$. Let $k \geq 1$ and let $z = 36k$. If all of the following hold for each $i \in \{1, 2\}$:*

(i) if $|V(C_i)| = 3$, then $|V(G_i) \setminus V(C_i)| = 0$;

(ii) if $|V(C_i)| = 4$, then $|V(G_i) \setminus V(C_i)| \leq k$;

(iii) if $|V(C_i)| \geq 5$, then $|V(G_i) \setminus V(C_i)| \leq |V(C_i)| - \gamma_i$ for some $5 - \varepsilon z \geq \gamma_i \geq 4.8 + \varepsilon z$;

then $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma - \varepsilon y$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, where $y = 12k$.

Proof. If both C_1 and C_2 are 3- or 4-cycles, then $|V(C)| \leq 6$ and we have that

$$\begin{aligned} \varepsilon|V(G) \setminus V(C)| &= \varepsilon(|V(G_1) \setminus V(C_1)| + |V(G_2) \setminus V(C_2)|) \\ &\leq 2\varepsilon k. \end{aligned}$$

Since $z - y \geq 2k$, we have that $2\varepsilon k \leq \varepsilon z - \varepsilon y$. Since $5 - \varepsilon z \geq \gamma$, it follows that $\varepsilon z \leq 5 - \gamma$. Thus,

$$\begin{aligned} \varepsilon|V(G) \setminus V(C)| &\leq 5 - \gamma - \varepsilon y \\ &\leq |V(C)| - \gamma - \varepsilon y, \end{aligned}$$

for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, as desired.

If one of C_1, C_2 is a 3- or 4-cycle, say C_1, and the other, say C_2, has length at least 5, then

$$\begin{aligned} \varepsilon|V(G) \setminus V(C)| &= \varepsilon(|V(G_1) \setminus V(C_1)| + |V(G_2) \setminus V(C_2)|) \\ &\leq \varepsilon k + |V(C_2)| - \gamma_2. \end{aligned}$$

Since $|V(C_1)|+|V(C_2)| = |V(C)|+2$ and $|V(C_1)| \geq 3$, it follows that $|V(C_2)| \leq |V(C)|-1$. Since $z - y \geq k$, we have that $\varepsilon k \leq \varepsilon z - \varepsilon y$. Thus,

$$\varepsilon|V(G) \setminus V(C)| \leq \varepsilon z - \varepsilon y + |V(C)| - 1 - \gamma_2.$$

Since $5 - \varepsilon z \geq 4.8 + \varepsilon z$, it follows that $0.2 \geq 2\varepsilon z$. Thus, we have that $0.1 \geq \varepsilon z$. Since $0.1 - 1 \leq 0$, it follows that $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma_2 - \varepsilon y$. Thus, we have that $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma - \varepsilon y$, for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, as desired.

Otherwise, if $|V(C_1)|, |V(C_2)| \geq 5$, then

$$\begin{aligned}
\varepsilon|V(G) \setminus V(C)| &= \varepsilon(|V(G_1) \setminus V(C_1)| + |V(G_2) \setminus V(C_2)|) \\
&\leq |V(C_1)| + |V(C_2)| - \gamma_1 - \gamma_2 \\
&= |V(C)| + 2 - \gamma_1 - \gamma_2.
\end{aligned}$$

Since $2 - \gamma_1 \leq 2 - (4.8 + \varepsilon z) \leq -\varepsilon y$, we find that $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma - \varepsilon y$, for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, as desired. □

Lemma 5.2.2. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| \geq 5$ and Γ contains a dividing bichord uvw. Let C_1 and C_2 be the cycles that bound the two inner faces of $C + uvw$. Let $G_i = G\langle C_i \rangle$ for each $i \in \{1, 2\}$. Let $k \geq 1$ and let $z = 36k$. If, for each $i \in \{1, 2\}$, we have that $|V(G_i) \setminus V(C_i)| \leq |V(C_i)| - \gamma_i$ for some $5 - \varepsilon z \geq \gamma_i \geq 4.8 + \varepsilon z$, then $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma - \varepsilon y$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, where $y = 12k$.*

Proof. Since $|V(G_i) \setminus V(C_i)| \leq |V(C_i)| - \gamma_i$ for some $5 - \varepsilon z \geq \gamma_i \geq 4.8 + \varepsilon z$, for each $i \in \{1, 2\}$, it follows that:

$$\begin{aligned}
\varepsilon|V(G) \setminus V(C)| &= \varepsilon(|V(G_1) \setminus V(C_1)| + |V(G_2) \setminus V(C_2)| + 1) \\
&\leq |V(C_1)| + |V(C_2)| - \gamma_1 - \gamma_2 + \varepsilon \\
&= |V(C)| + 4 - \gamma_1 - \gamma_2 + \varepsilon.
\end{aligned}$$

Since $\gamma_1 \geq 4.8 - \varepsilon z$ and $\varepsilon \leq \varepsilon z \leq 0.1$, we have that $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| + 4 - 4.8 - \varepsilon z - \gamma_2 + 0.1$. Since $4.1 - 4.8 \leq 0$, it follows that $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \varepsilon y - \gamma_2$. Thus, we have that $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma - \varepsilon y$, for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, as desired. □

Lemma 5.2.3. *Let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G and $|V(C)| \geq 5$ and Γ contains a 6-double-pod v such that G has no chords of C and Γ has no dividing bichords. Let $C_1, C_2, \ldots, C_6$ be the cycles that bound the six inner faces of $C \cup \text{legs}(v)$. Let $G_i = G\langle C_i \rangle$ for each $i \in \{1, 2, \ldots, 6\}$. Let $k \geq 1$ and let $z = 36k$. If all of the following hold for each $i \in \{1, 2, \ldots, 6\}$:*

(i) if $|V(C_i)| = 3$, then $|V(G_i) \setminus V(C_i)| = 0$;

(ii) if $|V(C_i)| = 4$, then $|V(G_i) \setminus V(C_i)| \leq k$;

(iii) if $|V(C_i)| \geq 5$, then $|V(G_i) \setminus V(C_i)| \leq |V(C_i)| - \gamma_i$ for some $5 - \varepsilon z \geq \gamma_i \geq 4.8 + \varepsilon z$;

then $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma - \varepsilon y$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, where $y = 36k$.

Proof. First note that, since Γ has a 6-double-pod, we have that $|V(C)| \geq 6$. If C_i is a 3- or 4-cycle for all $i \in \{1, 2, \ldots, 6\}$, then for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$:

$$\begin{aligned} \varepsilon|V(G) \setminus V(C)| &\leq \varepsilon \sum_{i=1}^{6} (|V(G_i) \setminus V(C_i)|) + 7\varepsilon \\ &\leq 6\varepsilon k + 7\varepsilon. \end{aligned}$$

Since $2z - y = 36k \geq 6k + 7$, it follows that $6\varepsilon k + 7\varepsilon \leq 2\varepsilon z - \varepsilon y$. Since $5 - \varepsilon z \geq \gamma$, we have that $\varepsilon z \leq 5 - \gamma$. Thus,

$$\varepsilon|V(G) \setminus V(C)| \leq 5 - \gamma + \varepsilon z - \varepsilon y.$$

Since $|V(C)| \geq 6$, we have that $|V(C)| - 1 \geq 5$. Recall that $\varepsilon z \leq 0.1$. Thus,

$$\begin{aligned} \varepsilon|V(G) \setminus V(C)| &\leq |V(C)| - 1 - \gamma + 0.1 - \varepsilon y \\ &\leq |V(C)| - \gamma - \varepsilon y, \end{aligned}$$

as desired.

Now suppose at least one of $C_1, \ldots, C_6$ has at least 5 vertices. Let t denote the number of $C_1, \ldots, C_6$ that are 3- or 4-cycles. Since not all of $C_1, \ldots, C_6$ are 3- or 4-cycles, we have that $6 - t \geq 1$. Without loss of generality, suppose $|V(C_i)| \geq 5$ for $i \in \{1, \ldots, 6 - t\}$.

Claim 5.2.4. *$\sum_{i=1}^{6-t}(|V(C_i)| - 4) \leq |V(C)| - t$.*

Proof. For each $i \in \{1, \ldots, 6 - t\}$, we have that C_i has at most 4 edges that are not in $E(C)$. Thus, it follows that $\sum_{i=1}^{6-t}(|V(C_i)| - 4) \leq \sum_{i=1}^{6-t} |E(C_i) \cap E(C)|$. Notice that $|E(C_i) \cap E(C)| \geq 1$ for all $i \in \{6 - t, \ldots, 6\}$. Since $E(C_1) \cap E(C), \ldots, E(C_6) \cap E(C)$ are pairwise disjoint, it follows that $\sum_{i=1}^{6-t} |E(C_i) \cap E(C)| \leq |E(C)| - t = |V(C)| - t$. Thus, we have that $\sum_{i=1}^{6-t}(|V(C_i)| - 4) \leq |V(C)| - t$, as desired. $\square$

Thus,

$$\begin{aligned}
\varepsilon|V(G) \setminus V(C)| &\leq \varepsilon \sum_{i=1}^{6}(|V(G_i) \setminus V(C_i)|) + 7\varepsilon \\
&\leq \varepsilon \sum_{i=1}^{6-t}(|V(C_i)| - \gamma_i) + t\varepsilon k + 7\varepsilon \\
&\leq \sum_{i=1}^{6-t} |V(C_i)| - \sum_{i=1}^{6-t}(\gamma_i) + t\varepsilon k + 7\varepsilon \\
&\leq \sum_{i=1}^{6-t}(|V(C_i)| - 4) + 4(6-t) - \sum_{i=1}^{6-t}(\gamma_i) + t\varepsilon k + 7\varepsilon.
\end{aligned}$$

By Claim 5.2.4, we have that $\sum_{i=1}^{6-t}(|V(C_i)| - 4) \leq |V(C)| - t$. Since $z = 36k$, it follows that $tk + 7 \leq z$. Thus,

$$\begin{aligned}
\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - t + 24 - 4t - \sum_{i=1}^{6-t}(\gamma_i) + \varepsilon z \\
\leq |V(C)| - \gamma_1 + 24 - 5t - \sum_{i=2}^{6-t}(\gamma_i) + \varepsilon z.
\end{aligned}$$

Since $\gamma_i \geq 4.8 + \varepsilon z$ for all $i \in \{1, \ldots, 6\}$, it follows that $\sum_{i=2}^{6-t}(\gamma_i) \geq (5-t)(4.8 + \varepsilon z) = 24 - 4.8t + 5\varepsilon z - t\varepsilon z$. Thus,

$$\begin{aligned}
\varepsilon|V(G) \setminus V(C)| &\leq |V(C)| - \gamma_1 + 24 - 5t - (24 - 4.8t + 5\varepsilon z - t\varepsilon z) + \varepsilon z \\
&\leq |V(C)| - \gamma_1 - \varepsilon z - 0.2t - 3\varepsilon z + t\varepsilon z.
\end{aligned}$$

Recall that $\varepsilon z \leq 0.1$. Hence, we have that $0.2t \geq t\varepsilon z$, which implies that $-0.2t + t\varepsilon z \leq 0$. Since $-3\varepsilon z \leq 0$ and $z = y$, it follows that

$$\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma_1 - \varepsilon y.$$

Hence, we have that $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma - \varepsilon y$, for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, as desired. □

5.3 Proving the Main Result

In this section, we prove the Main Theorem 5.3.5.

Definition 5.3.1. Let $\Gamma = (G, C)$ be a canvas with a bichord uvw where $v \in B(\Gamma)$. We say u and w are the *parents* of v.

Definition 5.3.2. Let $\Gamma_0 = (G_0, C_0)$ be a canvas where C_0 is the outer cycle of G_0 and $|V(C_0)| \geq 5$. Let $\Gamma_i = (G_i, C_i) = R(\Gamma_{i-1}, B(\Gamma_{i-1}))$ for each $i \in \{1, 2, 3\}$. Let $X \subseteq V(C_i)$ for some $i \in \{0, 1, 2, 3\}$. We say that A_i is the set of *ancestors* of X if $A_0 = X$ and $A_j = A_{j-1} \cup \{a : a \text{ is a parent of } a' \text{ where } a' \in A_{j-1}\}$ for all $j \in \{1, \ldots, i\}$.

Proposition 5.3.3. *Let $\Gamma_0 = (G_0, C_0)$ be a canvas where C_0 is the outer cycle of G_0 and $|V(C_0)| \geq 5$. Let $\Gamma_i = (G_i, C_i) = R(\Gamma_{i-1}, B(\Gamma_{i-1}))$ for each $i \in \{1, 2, 3\}$. Let $X \subseteq V(C_i)$ for some $i \in \{0, 1, 2, 3\}$ and let A_i be the ancestors of X. It follows that $|A_i| \leq |X|(i + 2)(i + 1)/2$.*

Proof. If $i = 0$, then $A_0 = X$ and we have that $|A_0| = |X| = |X|(0 + 2)(0 + 1)/2$, as desired. If $i = 1$, then at most all of the vertices in X are in $B(\Gamma_0)$ and their parents are distinct; thus, $|A_1| \leq 3|X| = |X|(1 + 2)(1 + 1)/2$, as desired. Suppose $i \in \{2, 3\}$. Let $v \in A_i$ and suppose v has two parents u, w in A_i where each of u, w have two parents in A_i. Since uvw is not a dividing bichord, it follows that $\text{dist}(u, w) = 2$; thus, u and w have a parent in common. Hence, we have that u and w have at most three parents x, y, z. If x, y, z all have parents in A_i, then without loss of generality, x and y have a parent in common, and y and z have a parent in common. Thus, we have that x, y, z have at most four parents. Hence, if $i = 2$, then $|A_2| \leq 6|X| = |X|(2 + 2)(2 + 1)/2$, as desired. If $i = 3$, then $|A_3| \leq 10|X| = |X|(3 + 2)(3 + 1)/2$, as desired. $\square$

Proposition 5.3.4. *Let $k \geq 12$ and let $\Gamma = (G, C)$ be a canvas where C is the outer cycle of G such that G is C-critical for acyclic k-colouring. The maximum size of a bundle B on $u, w \in V(C)$ is $k - 1$.*

Proof. Since B is a bundle on u, w, it follows that $\text{dist}_C(u, w) = 2$. Let x be the vertex that is adjacent to both u and w in C. Let $v \in B$ such that all vertices in $B \setminus \{v\}$ are in the interior of the cycle $C' = vuxwv$. Let $G' = G\langle C' \rangle$. Since $G = (G \setminus \text{int}(C')) \cup G'$ and $C' \subseteq (G \setminus \text{int}(C'))$ and $G' \neq (G \setminus \text{int}(C')) \cap G'$, it follows by the Key Lemma 5.1.2 that G' is C'-critical for acyclic k-colouring. By Theorem 5.1.6, we have that $|V(G') \setminus V(C')| \leq k - 2$. Since $V(G') \setminus V(C') = B \setminus \{v\}$, it follows that $|B| \leq k - 1$. $\square$

Theorem 5.3.5 (Main Theorem)**.** *For each $k \geq 12$, there exists $\varepsilon = \varepsilon(k) > 0$ such that if a canvas $\Gamma = (G, C)$ where C is the outer cycle of the plane graph G is k-critical and $|V(C)| \geq 5$, then $\varepsilon|V(G) \setminus V(C)| \leq |V(C)| - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$ where $z = 36k$.*

Proof. Suppose not. Let $\Gamma_0 = (G_0, C_0)$ where C_0 is the outer cycle of G_0 be a counterexample with $|V(G_0)| + |E(G_0)|$ minimized. Thus, we have that G_0 is C_0-critical for acyclic k-colouring and $|V(C_0)| \geq 5$. Let $\Gamma_i = (G_i, C_i) = R(\Gamma_{i-1}, B(\Gamma_{i-1}))$ for each $i \in \{1, 2, 3\}$.

Claim 5.3.6. *G_i does not contain a chord of C_i, for each $i \in \{0, 1, 2, 3\}$.*

Proof. Suppose, towards a contradiction, that G_i does contain a chord of C_i. Let u and v be the endpoints of the chord. Let A_i be the set of ancestors of $\{u, v\}$. By Proposition 5.3.3, it follows that $|A_i| \leq |X|(i + 2)(i + 1)/2 \leq 2(20)/2 = 20$ and $|A_i \setminus V(C_0)| \leq |X|((i - 1) + 2)((i - 1) + 1)/2 \leq 2(12)/2 = 12$.

Let $\Gamma'_0 = (G'_0, C'_0) = \Gamma_0$. For each $j = 1, \ldots, i$, let $\Gamma'_j = (G'_j, C'_j) = R(\Gamma'_{j-1}, B)$ where $B = B(\Gamma'_{j-1}) \cap A_i$.

Let $C_{i,1}$ and $C_{i,2}$ be the cycles that bound the two inner faces of $C'_i + uv$. Let $G_{i,j} = G'_i\langle C_{i,j}\rangle$ for each $j \in \{1, 2\}$. Since $G'_i = (G'_i \setminus \mathrm{int}(C_{i,j})) \cup G_{i,j}$ and $C'_i \subseteq (G'_i \setminus \mathrm{int}(C_{i,j}))$ and $G_{i,j} \neq (G'_i \setminus \mathrm{int}(C_{i,j})) \cap G_{i,j}$, it follows by the Key Lemma 5.1.2 that $G_{i,j}$ is $C_{i,j}$-critical for acyclic k-colouring, for each $j \in \{1, 2\}$.

For each $j \in \{1, 2\}$, if $C_{i,j}$ is a 3-cycle, then by Theorem 5.1.5 we have that $|V(G_{i,j}) \setminus V(C_{i,j})| = 0$. If $C_{i,j}$ is a 4-cycle, then by Theorem 5.1.6 we have that $|V(G_{i,j}) \setminus V(C_{i,j})| \leq k$. Otherwise $|V(C_{i,j})| \geq 5$, and since Γ_0 is a minimum counterexample we have that $\varepsilon|V(G_{i,j}) \setminus V(C_{i,j})| \leq |V(C_{i,j})| - \gamma_{i,j}$, for some $5 - \varepsilon z \geq \gamma_{i,j} \geq 4.8 + \varepsilon z$.

Thus, by Lemma 5.2.1, it follows that $\varepsilon|V(G'_i) \setminus V(C'_i)| \leq |V(C'_i)| - \gamma_i - \varepsilon y$ for some $5 - \varepsilon z \geq \gamma_i \geq 4.8 + \varepsilon z$, where $y = 12k$. By Proposition 4.3.8, we have that $|V(C'_i)| = |V(C_0)|$. Notice that each vertex $a \in A_i$ is either in $V(C_0)$ or in $B(\Gamma_j)$ for some $j \in \{0, \ldots, i\}$. Thus, it follows that a is in a bundle B_a, for all $a \in A_i \setminus V(C_0)$. By Proposition 5.3.4, we have that $|B_a| \leq k - 1$ for all $a \in A_i$. Let $B = \bigcup_{a \in A_i}(B_a)$. That is, B is the union of the ancestors in $A_i \setminus V(C_0)$ and their bundles. Thus, we have that $|B| \leq |A_i \setminus V(C_0)|(k-1) \leq 12(k-1)$. Let $y' = 12(k-1)$. By the construction of Γ'_i, we have that $(V(G_0) \setminus V(C_0)) \setminus (V(G'_i) \setminus V(C'_i)) = B$. Thus, it follows that $|V(G_0) \setminus V(C_0)| - |V(G'_i) \setminus V(C'_i)| \leq y'$. Notice that $y' \leq y$; hence, we have that $y' - y \leq 0$. Therefore,

$$
\begin{aligned}
\varepsilon|V(G_0) \setminus V(C_0)| &\leq \varepsilon|V(G'_i) \setminus V(C'_i)| + \varepsilon y' \\
&\leq |V(C'_i)| - \gamma_i - \varepsilon y + \varepsilon y' \\
&\leq |V(C_0)| - \gamma_i + \varepsilon(y' - y) \\
&\leq |V(C_0)| - \gamma_i.
\end{aligned}
$$

Thus, we have that $\varepsilon|V(G_0) \setminus V(C_0)| \leq |V(C_0)| - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, which contradicts the assumption that Γ_0 is a counterexample. $\square$

Claim 5.3.7. Γ_i *does not contain a dividing bichord, for each* $i \in \{0,1,2,3\}$.

Proof. Suppose, towards a contradiction, that Γ_i does contain a dividing bichord, uvw. Let A_i be the set of ancestors of $\{u,w\}$. By Proposition 5.3.3, it follows that $|A_i| \leq |X|(i+2)(i+1)/2 \leq 2(20)/2 = 20$ and $|A_i \setminus V(C_0)| \leq |X|((i-1)+2)((i-1)+1)/2 \leq 2(12)/2 = 12$.

Let $\Gamma'_0 = (G'_0, C'_0) = \Gamma_0$. For each $j = 1,\dots,i$, let $\Gamma'_j = (G'_j, C'_j) = R(\Gamma'_{j-1}, B)$ where $B = B(\Gamma'_{j-1}) \cap A_i$.

Let $C_{i,1}$ and $C_{i,2}$ be the cycles that bound the two inner faces of $C'_i + uvw$. Let $G_{i,j} = G'_i\langle C_{i,j}\rangle$ for each $j \in \{1,2\}$. Since $G'_i = (G'_i \setminus \text{int}(C_{i,j})) \cup G_{i,j}$ and $C'_i \subseteq (G'_i \setminus \text{int}(C_{i,j}))$ and $G_{i,j} \neq (G'_i \setminus \text{int}(C_{i,j})) \cap G_{i,j}$, it follows by the Key Lemma 5.1.2 that $G_{i,j}$ is $C_{i,j}$-critical for acyclic k-colouring, for each $j \in \{1,2\}$.

Since uvw is a dividing bichord, it follows that $\text{dist}_{C'_i}(u,w) \geq 3$; thus, we have that $|V(C_{i,1})|, |V(C_{i,2})| \geq 5$. Hence, since Γ_0 is a minimum counterexample, we have that $\varepsilon|V(G_{i,j}) \setminus V(C_{i,j})| \leq |V(C_{i,j})| - \gamma_{i,j}$, for some $5 - \varepsilon z \geq \gamma_{i,j} \geq 4.8 + \varepsilon z$.

Thus, by Lemma 5.2.2, it follows that $\varepsilon|V(G'_i) \setminus V(C'_i)| \leq |V(C'_i)| - \gamma_i - \varepsilon y$ for some $5 - \varepsilon z \geq \gamma_i \geq 4.8 + \varepsilon z$, where $y = 12k$. By Proposition 4.3.8, we have that $|V(C'_i)| = |V(C_0)|$. Notice that each vertex $a \in A_i$ is either in $V(C_0)$ or in $B(\Gamma_j)$ for some $j \in \{0,\dots,i\}$. Thus, it follows that a is in a bundle B_a, for all $a \in A_i \setminus V(C_0)$. By Proposition 5.3.4, we have that $|B_a| \leq k-1$ for all $a \in A_i$. Let $B = \bigcup_{a \in A_i}(B_a)$. That is, B is the union of the ancestors in $A_i \setminus V(C_0)$ and their bundles. Thus, we have that $|B| \leq |A_i \setminus V(C_0)|(k-1) \leq 12(k-1)$. Let $y' = 12(k-1)$. By the construction of Γ'_i, we have that $(V(G_0) \setminus V(C_0)) \setminus (V(G'_i) \setminus V(C'_i)) = B$. Thus, it follows that $|V(G_0) \setminus V(C_0)| - |V(G'_i) \setminus V(C'_i)| \leq y'$. Notice that $y' \leq y$; hence, we have that $y' - y \leq 0$. Therefore,

$$\begin{aligned}
\varepsilon|V(G_0) \setminus V(C_0)| &\leq \varepsilon|V(G'_i) \setminus V(C'_i)| + \varepsilon y' \\
&\leq |V(C'_i)| - \gamma_i - \varepsilon y + \varepsilon y' \\
&\leq |V(C_0)| - \gamma_i + \varepsilon(y' - y) \\
&\leq |V(C_0)| - \gamma_i.
\end{aligned}$$

Thus, we have that $\varepsilon|V(G_0) \setminus V(C_0)| \leq |V(C_0)| - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, which contradicts the assumption that Γ_0 is a counterexample. ☐

Claim 5.3.8. Γ_i *does not contain a 6-double-pod, for each* $i \in \{0,1,2,3\}$.

Proof. Suppose, towards a contradiction, that Γ_i does contain a 6-double-pod, v. Let A_i be the set of ancestors of $\{u : u \in \text{feet}(v)\}$. By Proposition 5.3.3, it follows that

$|A_i| \leq |X|(i+2)(i+1)/2 \leq 6(20)/2 = 60$ and $|A_i \setminus V(C_0)| \leq |X|((i-1)+2)((i-1)+1)/2 \leq 6(12)/2 = 36$.

Let $\Gamma'_0 = (G'_0, C'_0) = \Gamma_0$. For each $j = 1, \ldots, i$, let $\Gamma'_j = (G'_j, C'_j) = R(\Gamma'_{j-1}, B)$ where $B = B(\Gamma'_{j-1}) \cap A_i$.

Let $C_{i,1}, C_{i,2}, \ldots, C_{i,6}$ be the cycles that bound the six inner faces of $C'_i \cup \text{legs}(v)$. Let $G_{i,j} = G'_i\langle C_{i,j}\rangle$ for each $j \in \{1, 2, \ldots, 6\}$. Since $G'_i = (G'_i \setminus \text{int}(C_{i,j})) \cup G_{i,j}$ and $C'_i \subseteq (G'_i \setminus \text{int}(C_{i,j}))$ and $G_{i,j} \neq (G'_i \setminus \text{int}(C_{i,j})) \cap G_{i,j}$, it follows by the Key Lemma 5.1.2 that $G_{i,j}$ is $C_{i,j}$-critical for acyclic k-colouring, for each $j \in \{1, 2, \ldots, 6\}$.

For each $j \in \{1, 2, \ldots, 6\}$, if $C_{i,j}$ is a 3-cycle, then by Theorem 5.1.5 we have that $|V(G_{i,j}) \setminus V(C_{i,j})| = 0$. If $C_{i,j}$ is a 4-cycle, then by Theorem 5.1.6 we have that $|V(G_{i,j}) \setminus V(C_{i,j})| \leq k$. Otherwise $|V(C_{i,j})| \geq 5$, and since Γ_0 is a minimum counterexample we have that $\varepsilon|V(G_{i,j}) \setminus V(C_{i,j})| \leq |V(C_{i,j})| - \gamma_{i,j}$, for some $5 - \varepsilon z \geq \gamma_{i,j} \geq 4.8 + \varepsilon z$.

Thus, by Lemma 5.2.3, it follows that $\varepsilon|V(G'_i) \setminus V(C'_i)| \leq |V(C'_i)| - \gamma_i - \varepsilon y$ for some $5 - \varepsilon z \geq \gamma_i \geq 4.8 + \varepsilon z$, where $y = 36k + 36$. By Proposition 4.3.8, we have that $|V(C'_i)| = |V(C_0)|$. Notice that each vertex $a \in A_i$ is either in $V(C_0)$ or in $B(\Gamma_j)$ for some $j \in \{0, \ldots, i\}$. Thus, it follows that a is in a bundle B_a, for all $a \in A_i \setminus V(C_0)$. By Proposition 5.3.4, we have that $|B_a| \leq k - 1$ for all $a \in A_i$. Let $B = \bigcup_{a \in A_i}(B_a)$. That is, B is the union of the ancestors in $A_i \setminus V(C_0)$ and their bundles. Thus, we have that $|B| \leq |A_i \setminus V(C_0)|(k-1) \leq 36(k-1)$. Let $y' = 36(k-1)$. By the construction of Γ'_i, we have that $(V(G_0) \setminus V(C_0)) \setminus (V(G'_i) \setminus V(C'_i)) = B$. Thus, it follows that $|V(G_0) \setminus V(C_0)| - |V(G'_i) \setminus V(C'_i)| \leq y'$. Notice that $y' \leq y$; hence, we have that $y' - y \leq 0$. Therefore,

$$\begin{aligned}
\varepsilon|V(G_0) \setminus V(C_0)| &\leq \varepsilon|V(G'_i) \setminus V(C'_i)| + \varepsilon y' \\
&\leq |V(C'_i)| - \gamma_i - \varepsilon y + \varepsilon y' \\
&\leq |V(C_0)| - \gamma_i + \varepsilon(y' - y) \\
&\leq |V(C_0)| - \gamma_i.
\end{aligned}$$

Thus, we have that $\varepsilon|V(G_0) \setminus V(C_0)| \leq |V(C_0)| - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, which contradicts the assumption that Γ_0 is a counterexample. $\square$

Claim 5.3.9. *Γ_i does not contain a non-unique, non-dividing bichord, for each $i \in \{0, 1, 2, 3\}$.*

Proof. Suppose not. That is, there exists $i \in \{0, 1, 2, 3\}$ such that Γ_i contains a non-dividing bichord uvw where $v \notin B(\Gamma_i)$. Since uvw is not a dividing bichord of Γ_i, we have that $\text{dist}_{C_i}(u, w) = 2$. Since $v \notin B(\Gamma_i)$, there exists at least one other neighbour of v in C_i. By the Unique Bichord Lemma 4.3.1, it follows that $|V(C_i)| \leq 6$. Since $|V(C_i)| = |V(C_0)|$, we have that $|V(C_i)| \geq 5$.

Let $u_1, u_2, \ldots, u_t$ be the neighbours of v in C_i. Notice $3 \leq t \leq 6$. If $|V(C_i)| = 6$ and $t = 6$, then $u_1, \ldots, u_t$ are precisely the vertices of C_i; hence, it follows that $u_1 v u_4$ is a dividing bichord, which is a contradiction. Thus, it follows that $t \leq 5$.

Let A_i be the set of ancestors of $\{u_1, \ldots, u_t\}$. By Proposition 5.3.3, it follows that $|A_i| \leq |X|(i+2)(i+1)/2 \leq 5(20)/2 = 50$ and $|A_i \setminus V(C_0)| \leq |X|((i-1)+2)((i-1)+1)/2 \leq 5(12)/2 = 30$.

Let $\Gamma'_0 = (G'_0, C'_0) = \Gamma_0$. For each $j = 1, \ldots, i$, let $\Gamma'_j = (G'_j, C'_j) = R(\Gamma'_{j-1}, B)$ where $B = B(\Gamma'_{j-1}) \cap A_i$.

Let $C_{i,1}, \ldots, C_{i,t}$ be the cycles that bound the t inner faces of $C'_i + vu_1 + \cdots + vu_t$. Let $G_{i,j} = G'_i\langle C_{i,j}\rangle$ for each $j \in \{1, \ldots, t\}$. Since $G'_i = (G'_i \setminus \mathrm{int}(C_{i,j})) \cup G_{i,j}$ and $C'_i \subseteq (G'_i \setminus \mathrm{int}(C_{i,j}))$ and $G_{i,j} \neq (G'_i \setminus \mathrm{int}(C_{i,j})) \cap G_{i,j}$, it follows by the Key Lemma 5.1.2 that $G_{i,j}$ is $C_{i,j}$-critical for acyclic k-colouring, for each $j \in \{1, \ldots, t\}$.

Since v is not in a dividing bichord, it follows that $3 \leq |V(C_{i,1})|, \ldots, |V(C_{i,t})| \leq 4$. Thus, for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$:

$$\begin{aligned}
\varepsilon|V(G'_i) \setminus V(C'_i)| &\leq \varepsilon \sum_{j=1}^{t} (|V(G_{i,j}) \setminus V(C_{i,j})|) + \varepsilon \\
&\leq t\varepsilon k + \varepsilon \\
&\leq \varepsilon z - \varepsilon y \\
&\leq 5 - \gamma - \varepsilon y \\
&\leq |V(C'_i)| - \gamma - \varepsilon y,
\end{aligned}$$

where $y = 30k$.

By Proposition 4.3.8, we have that $|V(C'_i)| = |V(C_0)|$. Notice that each vertex $a \in A_i$ is either in $V(C_0)$ or in $B(\Gamma_j)$ for some $j \in \{0, \ldots, i\}$. Thus, it follows that a is in a bundle B_a, for all $a \in A_i \setminus V(C_0)$. By Proposition 5.3.4, we have that $|B_a| \leq k - 1$ for all $a \in A_i$. Let $B = \bigcup_{a \in A_i} (B_a)$. That is, B is the union of the ancestors in $A_i \setminus V(C_0)$ and their bundles. Thus, we have that $|B| \leq |A_i \setminus V(C_0)|(k-1) \leq 30(k-1)$. Let $y' = 30(k-1)$. By the construction of Γ'_i, we have that $(V(G_0) \setminus V(C_0)) \setminus (V(G'_i) \setminus V(C'_i)) = B$. Thus, it follows that $|V(G_0) \setminus V(C_0)| - |V(G'_i) \setminus V(C'_i)| \leq y'$. Notice that $y' \leq y$; hence, we have that $y' - y \leq 0$. Therefore,

$$\begin{aligned}
\varepsilon|V(G_0) \setminus V(C_0)| &\leq \varepsilon|V(G'_i) \setminus V(C'_i)| + \varepsilon y' \\
&\leq |V(C'_i)| - \gamma - \varepsilon y + \varepsilon y' \\
&\leq |V(C_0)| - \gamma_i + \varepsilon(y' - y) \\
&\leq |V(C_0)| - \gamma_i.
\end{aligned}$$

Thus, we have that $\varepsilon|V(G_0) \setminus V(C_0)| \le |V(C_0)| - \gamma$ for some $5 - \varepsilon z \ge \gamma \ge 4.8 + \varepsilon z$, which contradicts the assumption that Γ_0 is a counterexample. $\square$

By Claims 5.3.7 and 5.3.9, it follows that, for all $i \in \{0, 1, 2, 3\}$, if Γ_i contains a bichord uvw, then $v \in B(\Gamma_i)$. For all $i \in \{0, 1, 2, 3\}$, we have that C_i has no chords by Claim 5.3.6 and Γ_i has no 6-double-pods by Claim 5.3.8.

Claim 5.3.10. Γ_0 *does not contain a bichord.*

Proof. Let M be a k-mosaic of C_0 that extends to $G_0[V(C_0) \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)]$. Thus, by Lemma 4.3.12, we have that M extends to G_0. Since G_0 is C_0-critical for acyclic k-colouring, it follows that $G_0 = G_0[V(C_0) \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)]$. Hence, we have that $|V(G_0)| = |V(C_0)| + |B(\Gamma_0)| + |B(\Gamma_1)| + |B(\Gamma_2)| \le (3k+1)|V(C_0)|$. Thus, it follows that $|V(G_0) \setminus V(C_0)| \le 3k|V(C_0)|$. Let $5 - \varepsilon z \ge \gamma \ge 4.8 + \varepsilon z$. Since $z \ge 15k$, we have that $5 - 15k\varepsilon \ge \gamma$; thus, it follows that $\varepsilon \le \frac{1}{3k} - \frac{\gamma}{15k}$. Note that since $\frac{\gamma}{5} < 1$, it follows that $\frac{1}{3k} - \frac{\gamma}{15k} > 0$. Since $\varepsilon \le \frac{1}{3k} - \frac{\gamma}{15k}$, we have that $3k\varepsilon \le 1 - \frac{\gamma}{5}$. Also, note that since $|V(C_0)| \ge 5$, we have that $-\frac{|V(C_0)|}{5} \le -1$. Thus,

$$
\begin{aligned}
\varepsilon|V(G_0) \setminus V(C_0)| &\le 3k\varepsilon|V(C_0)| \\
&\le |V(C_0)| - \frac{\gamma|V(C_0)|}{5} \\
&\le |V(C_0)| - \gamma,
\end{aligned}
$$

which contradicts the assumption that Γ_0 is a counterexample. $\square$

By Claims 5.3.6, 5.3.10, and 5.3.8, we have that G_0 does not contain a chord of C_0 and Γ_0 does not contain a bichord or a 6-double-pod. Thus, by the converse of the General Structure Lemma 5.1.4, it follows that Γ_0 is not k-critical, which is a contradiction. $\square$

Chapter 6

Extending the Main Result

In this chapter we show that the Main Theorem 5.3.5 implies that the family of graphs which are critical for acyclic k-colouring, where $k \geq 12$, is hyperbolic and strongly hyperbolic.

6.1 Hyperbolic

In this section, we prove that the family of graphs which are critical for acyclic k-colouring, where $k \geq 12$, is hyperbolic.

Theorem 6.1.1. *For each $k \geq 12$, there exists $c > 1$ such that if G is plane and S is a non-empty independent set of G whose vertices are incident with the outer face of G and G is S-critical for acyclic k-colouring, then $|V(G)| \leq c(|V(S)| - 1)$.*

Proof. Suppose not. Let $\Gamma_0 = (G_0, S_0)$ be a counterexample where $|E(G_0)| + |V(G_0)|$ is minimized.

Claim 6.1.2. *S_0 does not contain a cut vertex.*

Proof. Suppose, towards a contradiction, that $v \in V(S_0)$ is a cut vertex. Let H_1 and H_2 be the two components of $G_0 - v$ and let $G_i = G_0[V(H_i) \cup \{v\}]$, for each $i \in \{1, 2\}$. Let $S_i = S_0 \cap V(G_i)$, for each $i \in \{1, 2\}$. For each $i \in \{1, 2\}$, it follows from the Key Lemma 5.1.2 that G_i is S_i-critical for acyclic k-colouring.

Since Γ_0 is a minimal counterexample, it follows that $|V(G_i)| \leq c(|V(S_i)| - 1)$ for all $i \in \{1, 2\}$. Thus, we have that $|V(G_1)| + |V(G_2)| \leq c(|V(S_1)| + |V(S_2)| - 2)$. Notice that

$|V(G_1)| + |V(G_2)| = |V(G_0)| + 1$ and $|V(S_1)| + |V(S_2)| = |V(S_0)| + 1$. Now, it follows that $|V(G_0)| + 1 \leq c(|V(S_0)| - 1)$. Thus, we have that $|V(G_0)| \leq c(|V(S_0)| - 1)$, which contradicts the minimality of Γ_0. $\square$

Let $v_0, \ldots, v_{n-1}$ be the cyclic order of the vertices of S_0 around the outer face of G_0. By Claim 6.1.2, we have that each vertex of S_0 appears once in $v_0, \ldots, v_{n-1}$.

For each $i \in \{0, \ldots, n-1\}$, add vertices u_i, w_i and the path $v_i u_i w_i v_{i+1}$ to G_0.[1] Embed these paths in the outer face of G_0 and let G' denote the resulting graph. Let C_0 be the union of these paths; that is, let $C_0 = \bigcup_{i=0}^{n-1}(v_i u_i w_i v_{i+1})$. Notice that C_0 is the outer cycle of G' and $|V(C_0)| = 3|V(S_0)|$. Let $X = \{u_i, w_i : i \in \{0, \ldots, n-1\}\}$. Notice that $G' \setminus X = G_0$.

Claim 6.1.3. *G' is C_0-critical for acyclic k-colouring.*

Proof. Suppose not. Thus, there exists a proper subgraph H of G', where $H \supseteq C_0$, such that every k-mosaic of C_0 that extends to H, also extends to G'. Since G_0 is S_0-critical for acyclic k-colouring and $H \supseteq C_0 \supseteq S_0$, we have that there exists a k-mosaic M_S of S_0 that extends to $H \setminus X$, but not G_0. Let ϕ be a k-colouring of C_0 where $\phi(v) = \phi_{M_S}(v)$ for all $v \in V(S_0)$ and $\phi(u_i) \neq \phi(w_i) \in [k] \setminus \{\phi(v_i), \phi(v_{i+1})\}$ for all $i \in \{0, \ldots, n-1\}$.

Subclaim 6.1.4. *M_S extends to ϕ.*

Proof. Suppose not. Notice that $\phi_{|S_0} = \phi(M_S)$; thus, Definition 3.2.8(1) holds for M_S extending to ϕ. Since M_S does not extend to ϕ, it follows from Definition 3.2.8(2) that $\widetilde{G'}_{ij}(\phi, M_S)$ contains a cycle C' for some $i \neq j \in [k]$. Since S_0 is an independent set in G', it follows that C' contains at least one vertex in X. Without loss of generality, say $u_1 \in V(C')$.

Notice that the cycle C' in $\widetilde{G'}_{ij}(\phi, M_S)$ is equivalent to a subgraph C'' in $G'_{ij}(\phi)$ where C'' is a cycle or a collection of paths with endpoints in $V(S_0)$. Let P be the component of C'' in $G'_{ij}(\phi)$ that contains u_1. Notice that $N_{G'}(u_1) = \{v_1, w_1\}$; thus it follows that v_1, w_1 are the neighbours of u_1 in P. Notice, by the construction of ϕ, we have that $\phi(v_1), \phi(u_1), \phi(w_1)$ are pairwise distinct. Thus, it follows that P is not in $G'_{ij}(\phi)$, which is a contradiction. $\square$

Let $M_C = \text{Mosaic}[\phi, M_S]$. Note that M_C exists by Proposition 3.3.7.

Subclaim 6.1.5. *M_S is the restriction of M_C to S_0.*

[1] Note that here and in the remainder of this proof, indices are taken mod n.

Proof. Let $u \neq v \in V(S_0)$. Since M_S extends to M_C, we have that $\mathcal{P}_{ij}(M_S)$ is a refinement of $\mathcal{P}_{ij}(M_C)$ for all $i \neq j \in [k]$. Thus, for all $i \neq j \in [k]$, if u, v are in the same part of $\mathcal{P}_{ij}(M_S)$, then u, v are in the same part of $\mathcal{P}_{ij}(M_C)$. Notice that there is no (u, v)-path in $(C_0)_{ij}(\phi)$. Since, for all $i \neq j \in [k]$, we have that $\mathcal{P}_{ij}(M_C)$ is the smallest common coarsening of $\mathcal{P}_{ij}(M_S)$ and $\mathcal{P}_{ij}(\text{Mosaic}[\phi])$, it follows that if u, v are not in the same part of $\mathcal{P}_{ij}(M_S)$, then u, v are not in the same part of $\mathcal{P}_{ij}(M_C)$. Thus, we have that $\mathcal{P}_{ij}(M_S) = \{P \cap V(S_0) : P \in \mathcal{P}_{ij}(M_C)\}$, for all $i \neq j \in [k]$. Now, since $\phi(M_S) = \phi(M_C)_{|S_0}$, it follows that M_S is the restriction of M_C to S_0. □

Subclaim 6.1.6. *M_C extends to H.*

Proof. By Subclaim 6.1.5, we have that M_S is the restriction of M_C to S_0. Notice that $G' = C_0 \cup (H \setminus X)$ and $C_0 \cap (H \setminus X) = S_0$. Since M_S extends to $H \setminus X$, it now follows from Proposition 3.3.16 that M_C extends to H. □

Subclaim 6.1.7. *M_C does not extend to G'.*

Proof. Suppose, towards a contradiction, that M_C extends to a k-mosaic M of G'. By Subclaim 6.1.5, we have that M_S is the restriction of M_C to S_0. Notice that G_0 and C_0 are subgraphs of G' and $G_0 \cap C_0 = S_0$. Thus, we have that M_S is the restriction of M_C to $G_0 \cap C_0$. Let M' be the restriction of M to G_0. Now, by Proposition 3.3.15, it follows that M_S extends to M'. Thus, we have that M_S extends to G_0, which is a contradiction. □

By Subclaims 6.1.6 and 6.1.7, we have that the k-mosaic M_C of C_0 extends to H, but not to G', which is a contradiction. □

Notice that $|V(G') \setminus V(C_0)| = |V(G_0)| - |V(S_0)|$. Since $|V(C_0)| = 3|V(S_0)|$, if $|V(C_0)| < 5$, then we have that $|V(C_0)| = 3$. Thus, by Theorem 5.1.5, it follows that G' is not C_0-critical for acyclic k-colouring, which contradicts Claim 6.1.3. Thus, it follows that $|V(C_0)| \geq 5$.

Since $|V(C_0)| \geq 5$, we have by Theorem 5.3.5 that $\varepsilon|V(G') \setminus V(C_0)| \leq |V(C_0)| - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$ where $z = 36k$. Since $|V(C_0)| = 3|V(S_0)|$ and $|V(G') \setminus V(C_0)| = |V(G_0)| - |V(S_0)|$, it follows that $\varepsilon(|V(G_0)| - |V(S_0)|) \leq 3|V(S_0)| - \gamma$. Thus, we have that $|V(G_0)| \leq (3 + \varepsilon)|V(S_0)|/\varepsilon - \gamma/\varepsilon$. Since $\gamma \geq 4.8$ and $\varepsilon \leq 0.1$, it follows that $|V(G_0)| \leq \frac{3+\varepsilon}{\varepsilon}(|V(S_0)| - 1)$. Let $c \geq \frac{3+\varepsilon}{\varepsilon}$ and now it follows that Γ_0 is not a counterexample, which is a contradiction. □

Proposition 6.1.8. *A graph G is critical for acyclic k-colouring if and only if G is $\emptyset$-critical for acyclic k-colouring.*

Proof. If G is critical for acyclic k-colouring, then there exists an acyclic k-colouring of every proper subgraph H of G, but there does not exist an acyclic k-colouring of G. Thus, it follows that the mosaic of $\emptyset$ extends to a k-colouring of H for every proper subgraph H of G, but the mosaic of $\emptyset$ does not extend to a k-colouring of G. Thus, it follows that G is $\emptyset$-critical for acyclic k-colouring.

If G is $\emptyset$-critical for acyclic k-colouring, then for each proper subgraph H of G there exists a k-mosaic of $\emptyset$ that extends to H, but not to G. Since there is only one mosaic of $\emptyset$, we have that there exists an acyclic k-colouring of H for all proper subgraphs H of G, but there does not exist an acyclic k-colouring of G. Thus, it follows that G is critical for acyclic k-colouring. $\square$

Theorem 6.1.9. *The family $\mathcal{F}$ of graphs which are critical for acyclic k-colouring, where $k \geq 12$, is hyperbolic.*

Proof. Let $k \geq 12$ and let G be a graph that is critical for acyclic k-colouring, where G is embedded on a surface Σ with Euler genus g. Let $\gamma : \mathbb{S}^1 \to \Sigma$ be a closed curve that bounds an open disk Δ and intersects G only in vertices. We may assume that Δ includes at least one vertex of G; otherwise there is nothing to show. Let S be the set of vertices of G intersected by γ. Let X be the set of vertices drawn in Δ (not including S). Let $B = G[S \cup X]$ and let $A = G \setminus X$. Notice that $G = A \cup B$ and $\emptyset \subseteq A$ and $B \neq A \cap B = S$. By Proposition 6.1.8, it follows that G is $\emptyset$-critical for acyclic k-colouring. Thus, by the Key Lemma 5.1.2, it follows that B is S-critical for acyclic k-colouring. Hence, by Theorem 6.1.1, we have that $|V(B)| \leq c(|V(S)|-1)$ for some $c > 1$ depending on k. Thus, it follows that $\mathcal{F}$ is hyperbolic. $\square$

6.2 Strongly Hyperbolic

In this section, we show that the family of graphs which are critical for acyclic k-colouring, where $k \geq 12$, is strongly hyperbolic. In order to do this, we need to redefine a few definitions for plane graphs bounded by two cycles, rather than one. We also need two cycle versions of several lemmas and theorems from Chapters 4 and 5.

Definition 6.2.1. Let G be a plane graph with two cycles, C and C', where without loss of generality, $V(C) \subseteq \text{int}(C')$. The *interior* of $C \cup C'$, denoted $\text{int}(C \cup C')$, is the set of vertices contained in the interior of the annulus bounded by $C \cup C'$. Let $G\langle C \cup C' \rangle = G[C \cup C' \cup \text{ int}(C \cup C')]$.

Definition 6.2.2. A *bichord* of a canvas $\Gamma = (G, C \cup C')$, where C and C' are the two cycles that bound G, is a path $P = uvw$ where $v \in V(G) \setminus V(C \cup C')$ and $u \neq w \in V(C \cup C')$ such that $\text{dist}_{C \cup C'}(u, w) \geq 2$. We say P is a *dividing* bichord if $\text{dist}_{C \cup C'}(u, w) \geq 3$ or $\text{dist}_{C \cup C'}(u, w) = 2$ and, without loss of generality, C' is drawn in the face of degree 4 induced by $C \cup P$.

Definition 6.2.3. A *bipod* of a canvas $\Gamma = (G, C \cup C')$, where C and C' are the two cycles that bound G, is a vertex $v \in V(G) \setminus V(C \cup C')$ such that v is in at least one bichord.

Definition 6.2.4. Let $\Gamma = (G, H)$ be a canvas and let $v \in V(G) \setminus V(H)$. Recall $N_H(v) = N(v) \cap V(H)$ and let $\widetilde{N}_H^2(v) = \{u \in V(H) : u \in N(N(v) \setminus N_H(v))\}$. Let $\text{feet}(v) = N_H(v) \cup \widetilde{N}_H^2(v)$. We refer to the vertices in $\text{feet}(v)$ as the *feet* of v.

Definition 6.2.5. An *r-double-pod* of a canvas $\Gamma = (G, H)$ is a vertex $v \in V(G) \setminus V(H)$ where $|\text{feet}(v)| = r$.

Definition 6.2.6. Let v be an r-double-pod of a canvas $\Gamma = (G, H)$. Since $\text{feet}(v) = N_H(v) \cup N_H^2(v)$, there exists, for each $u \in \text{feet}(v)$, a (v, u)-path P_u of the form vu or vwu where $w \in N(v) \setminus N_H(v)$, in G. Fix such a path P_u for each $u \in \text{feet}(v)$ and let $\text{legs}(v) = \{P_u : u \in \text{feet}(v)\}$. Notice that $|\text{legs}(v)| = r$.

Lemma 6.2.7 (Two Cycle Extension Lemma)**.** *Let $\Gamma = (G, C \cup C')$ be a canvas where C and C' are the cycles that bound G. Given a k-mosaic M of C, we have that M extends to G unless there exists at least one of the following:*

(i) a chord of $C \cup C'$, or

(ii) a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$, or

(iii) an r-double-pod v of Γ where $|\{\phi_M(u) : u \in \text{feet}(v)\}| \geq k - 6$.

The proof of the Two Cycle Extension Lemma follows almost identically to the proof of the Extension Lemma 4.2.1.

Corollary 6.2.8. *If $\Gamma = (G, C \cup C')$ is a canvas where C and C' are the cycles that bound G, and $|V(C)|, |V(C')| = 3$, and $\text{dist}(C, C') > 4$, we have that every k-mosaic M of C extends to G.*

Corollary 6.2.9. *Let $\Gamma = (G, C \cup C')$ be a canvas where C and C' are the cycles that bound G, $|V(C)| = 4$, $|V(C')| = 3$, and $\text{dist}(C, C') > 4$. Given a k-mosaic M of C, we have that M extends to G, unless there exists a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$ and $u, w \in V(C)$.*

Corollary 6.2.10. *Let $\Gamma = (G, C \cup C')$ be a canvas where C and C' are the cycles that bound G, $|V(C)| = 4$, $|V(C')| = 4$, and* $\text{dist}(C, C') > 4$. *Given a k-mosaic M of C, we have that M extends to G, unless there exists a bichord uvw of Γ where $\phi_M(u) = \phi_M(w)$ and either $u, w \in V(C)$ or $u, w \in V(C')$.*

Lemma 6.2.11 (Two Cycle Unique Bichord Lemma)**.** *Let $\Gamma = (G, C \cup C')$ be a canvas, where C and C' are the two cycles that bound G and $|V(C)|, |V(C')| \geq 7$. Let v be a bipod of Γ. If v is not in a dividing bichord, then it is in a unique bichord.*

Notice that if v is not in a dividing bichord of a canvas $\Gamma = (G, C \cup C')$, where C and C' are the two cycles that bound G, then either $N_{C \cup C'}(v) \subseteq V(C)$ or $N_{C \cup C'}(v) \subseteq V(C')$. Thus, we have that Lemma 6.2.11 follows from Lemma 4.3.1.

Definition 6.2.12. Let $B(\Gamma)$ denote the set of bipods of the canvas $\Gamma = (G, C \cup C')$, where C and C' are the two cycles that bound G, that are in a unique, non-dividing bichord.

Lemma 6.2.13. *Let $\Gamma = (G, C \cup C')$ be a canvas, where C and C' are the two cycles that bound G, and $|V(C)| \geq 4$, $|V(C')| \geq 5$. Without loss of generality, say $V(C) \subseteq int(C')$. Let $B \subseteq B(\Gamma)$ and let E_C denote the set of chords of $C \cup C'$. The graph $G[V(C \cup C') \cup B] \setminus (E(G[B]) \cup E_C)$ has exactly one interior face bounded by two cycles C_1 and C_2 where $|V(C_1 \cup C_2)| \geq 9$.*

The proof of Lemma 6.2.13 uses Lemma 4.3.3 twice.

Definition 6.2.14. Let $\Gamma = (G, C \cup C')$ be a canvas where G is bounded by the cycles C and C', and $|V(C)| \geq 4$, $|V(C')| \geq 5$. Without loss of generality, let $V(C) \subseteq \text{int}(C')$. Let $B \subseteq B(\Gamma)$ and let E_C denote the set of chords of $C \cup C'$. By Lemma 6.2.13, there exists a unique interior face of $G[V(C \cup C') \cup B] \setminus (E(G[B]) \cup E_C)$ bounded by two cycles C_1, C_2 where $|V(C_1 \cup C_2)| \geq 9$. Let $G' = G\langle C_1 \cup C_2 \rangle$ and let $\Gamma' = (G', C_1 \cup C_2)$. We say that Γ' is the *relaxation* of Γ with respect to B, denoted $R(\Gamma, B)$.

Just as in Chapter 4, we may think of a canvas and its relaxation as being different generations. If Γ is a canvas and $\Gamma' = R(\Gamma, B(\Gamma))$, we may think of Γ' as being the generation below Γ. The remaining definitions and propositions that lead up to the "Fourth Generation" Lemma 4.3.12 have natural two cycle versions.

Lemma 6.2.15 (Two Cycle "Fourth Generation" Lemma)**.** *Let $\Gamma_0 = (G_0, C_0 \cup C_0')$ be a canvas, where C_0 and C_0' are the cycles that bound G_0, and $|V(C_0)| \geq 4$, $|V(C_0')| \geq 5$, and dist$(C_0, C_0') > 10$. Let $\Gamma_i = (G_i, C_i \cup C_i') = R(\Gamma_{i-1}, B(\Gamma_{i-1}))$ for each $i \in \{1, 2, 3\}$. If all of the following hold for all $i \in \{0, 1, 2, 3\}$:*

(i) $C_i \cup C'_i$ *has no chords,*

(ii) *every bipod* v *of* Γ_i *is such that* $v \in B(\Gamma_i)$*,*

(iii) Γ_i *has no 6-double-pod,*

and a k*-mosaic* M *of* $C_0 \cup C'_0$ *extends to* $G_0[V(C_0 \cup C'_0) \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)]$*, then* M *extends to* G_0*.*

The proof of Lemma 6.2.15 follows almost identically to the proof of Lemma 4.3.12.

Lemma 6.2.16 (Two Cycle General Structure Lemma)**.** *If a canvas* $\Gamma = (G, C \cup C')$*, where* C *and* C' *are the cycles that bound* G*, is* k*-critical for* $k \geq 12$*, then there exists at least one of the following:*

(i) *a chord of* $C \cup C'$*, or*

(ii) *a bichord of* Γ*, or*

(iii) *a 6-double-pod of* Γ*.*

Notice that the Two Cycle General Structure Lemma 6.2.16 follows from the Two Cycle Extension Lemma 6.2.7, just as the General Structure Lemma 5.1.4 follows from the Extension Lemma 4.2.1.

Theorem 6.2.17. *If* $k \geq 12$*, then there does not exist a canvas* $\Gamma = (G, C \cup C')$ *where* C *and* C' *are the cycles that bound* G*,* $|V(C)|, |V(C')| = 3$*, and* $dist(C, C') > 4$ *such that* Γ *is* k*-critical.*

The proof of Theorem 6.2.17 follows from Corollary 6.2.8, just as Theorem 5.1.5 follows from Corollary 4.2.4.

Theorem 6.2.18. *Let* $\Gamma = (G, C \cup C')$ *be a canvas, where* C *and* C' *are the cycles that bound* G*,* $|V(C)| = 4$ *and* $|V(C')| \leq 4$*, and* $dist(C, C') > 4$*. If* Γ *is* k*-critical where* $k \geq 12$*, then* $|V(G) \setminus V(C \cup C')| \leq 2k - 4$ *and each vertex in* $V(G) \setminus V(C \cup C')$ *is a bipod* v *of* Γ *where* $N(v) \subseteq V(C)$ *or* $N(v) \subseteq V(C')$*.*

The proof of Theorem 6.2.18 is similar to the proof of Theorem 5.1.6 and uses Corollaries 6.2.9 and 6.2.10.

Theorem 6.2.19. *Let $\Gamma = (G, C \cup C')$ be a canvas, where C and C' are the cycles that bound G, and $|V(C)|, |V(C')| \leq 4$. If Γ is k-critical where $k \geq 12$, then $|V(G) \backslash V(C \cup C')| \leq |V(C \cup C')| + 8 + 3\varepsilon - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$ where $z = 36k$.*

The proof of Theorem 6.2.19 uses a claim, similar to Claim 6.2.21 below, and Theorems 6.2.18 and 6.2.17.

Theorem 6.2.20. *For each $k \geq 12$, there exists $\varepsilon = \varepsilon(k) > 0$ such that if a canvas $\Gamma = (G, C_1 \cup C_2)$ where C_1 and C_2 are the cycles that bound G and G is $(C_1 \cup C_2)$-critical for acyclic k-colouring and $|V(C_1)| \geq 5$, then $\varepsilon|V(G) \setminus V(C_1 \cup C_2)| \leq |V(C_1)| + |V(C_2)| + 20 + 9\varepsilon - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$ where $z = 36k$.*

Proof Sketch. Suppose not. Let $\Gamma_0 = (G_0, C_0 \cup C_0')$, where G_0 is bounded by the cycles C_0 and C_0', be a counterexample with $|V(G_0)| + |E(G_0)|$ minimized. Thus, we have that G_0 is $(C_0 \cup C_0')$-critical for acyclic k-colouring and at least one of $|V(C_0)|, |V(C_0')|$ is at least 5.

Claim 6.2.21. $\text{dist}(C_0, C_0') > 10$.

Proof Sketch. Suppose, towards a contradiction, that $\text{dist}(C_0, C_0') \leq 10$. Let $P = v_1, v_2, \ldots, v_n$ be a path from C_0 to C_0' such that $|V(P)| = n \leq 11$. Since G_0 is plane and P is a (C_0, C_0')-path, there are two (local) well-defined sides of P. Let E_L (E_R) denote the set of edges incident with P on the left (right).

Let G_0' be the graph obtained from G_0 by making a copy of P, called P', and making the edges of E_R incident with P' instead of P. Let $P' = v_1', v_2', \ldots, v_n'$ where v_x' is the copy of v_x for each $x \in [n]$. Notice that G_0' has an outer cycle, call it C.

Since G_0 is $(C_0 \cup C_0')$-critical for acyclic k-colouring, it follows that every proper subgraph H of G_0 where $(C_0 \cup C_0') \subseteq H$, there exists a k-mosaic of $(C_0 \cup C_0')$ that extends to H, but not to G_0. Notice that every subgraph H' of G_0' corresponds to a subgraph H of G_0 (by identifying P and P'). For each subgraph H' of G_0' where $C \subseteq H'$, let H be the corresponding subgraph of G. Let M be the k-mosaic of $(C_0 \cup C_0')$ that extends to a k-colouring ϕ_H of H, but not to G_0.

Now we define a k-colouring ϕ of C. Let $\phi(u) = \phi_M(u)$ for all $u \in V(C_0 \cup C_0')$. Let $\phi(v_x) = \phi_H(v_x)$ and $\phi(v_x') = \phi_H(v_x)$ for all $x \in [n]$. Let $\{\mathcal{P}_{ij} : i \neq j \in [k]\}$ be a collection of partitions of $V(G_0')$ where each $\mathcal{P}_{ij}$ is the smallest common coarsening of $\mathcal{P}_{ij}(M)$ and $\mathcal{P}_{ij}(\text{Mosaic}[\phi])$ such that v_x, v_x' are in the same part of $\mathcal{P}_{ij}$ for all $x \in [n]$. Let M_C be the k-mosaic of C defined by ϕ and $\{\mathcal{P}_{ij} : i \neq j \in [k]\}$.

Subclaim 6.2.22. *M_C extends to H'.*

Subclaim 6.2.23. *M_C does not extend to G'_0.*

By Subclaims 6.2.22 and 6.2.23, it follows that for all proper subgraphs H' of G'_0 where $C \subseteq H'$, we can find a k-mosaic of C which extends to H', but not to G'_0. Thus, it follows that G'_0 is C-critical for acyclic k-colouring.

By Theorem 5.3.5, it follows that $\varepsilon|V(G'_0) \setminus V(C)| \leq |V(C)| - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$ where $z = 36k$. Since $|V(C)| = |V(C_0)| + |V(C'_0)| + 20$ and $|V(G'_0)| = |V(G_0)| + 11$, we have that

$$\varepsilon(|V(G_0)| + 11 - |V(C_0 \cup C'_0)| - 20) \leq |V(C_0 \cup C'_0)| + 20 - \gamma.$$

Thus, it follows that

$$\varepsilon|V(G_0) \setminus V(C_0 \cup C'_0)| \leq |V(C_0 \cup C'_0)| + 20 + 9\varepsilon - \gamma.$$

Hence, we have that Γ_0 is not a counterexample, which is a contradiction. □

Let $\Gamma_i = (G_i, C_i \cup C'_i) = R(\Gamma_{i-1}, B(\Gamma_{i-1}))$ for each $i \in \{1, 2, 3\}$. Since $\text{dist}(C_0, C'_0) > 10$, it follows that $\text{dist}(C_i, C'_i) > 10 - 2i$ for each $i \in \{1, 2, 3\}$.

Thus, we have that all of the following hold for all $i \in \{1, 2, 3\}$:

- If G_i contains a chord uv of $C_i \cup C'_i$, then $u, v \in V(C_i)$ or $u, v \in V(C'_i)$.
- If Γ_i contains a bichord uvw, then $u, w \in V(C_i)$ or $u, w \in V(C'_i)$.
- If Γ_i contains a 6-double-pod v, then $u \in V(C_i)$ for all $u \in \text{feet}(v)$ or $u \in V(C'_i)$ for all $u \in \text{feet}(v)$.

Claim 6.2.24. *G_i does not contain a chord of $C_i \cup C'_i$, for each $i \in \{0, 1, 2, 3\}$.*

Proof Sketch. Suppose not. Without loss of generality, say G_i contains a chord uv of C_i. Let A_i be the set of ancestors of $\{u, v\}$. By Proposition 5.3.3, it follows that $|A_i| \leq |X|(i+2)(i+1)/2 \leq 2(20)/2 = 20$ and $|A_i \setminus V(C_0)| \leq |X|((i-1)+2)((i-1)+1)/2 \leq 2(12)/2 = 12$.

Let $\Gamma'_0 = (G'_0, C^1_0 \cup C^2_0) = \Gamma_0$ where $C^1_0 = C_0$ and $C^2_0 = C'_0$. For each $j = 1, \ldots, i$, let $\Gamma'_j = (G'_j, C^1_j \cup C^2_j) = R(\Gamma'_{j-1}, B)$ where $B = B(\Gamma'_{j-1}) \cap A_i$. Without loss of generality, say C^1_i is the outer cycle of G'_i.

Let $C_{i,1}$ and $C_{i,2}$ be the cycles that bound the two inner faces of $C^1_i + uv$. Let $G_{i,j} = G'_i\langle C_{i,j}\rangle$ for each $j \in \{1, 2\}$. Notice that either $G_{i,1}$ or $G_{i,2}$ contains C^2_i. Without loss of generality, say C^2_i is a cycle in $G_{i,2}$.

Since $G'_i = (G'_i \setminus \mathrm{int}(C_{i,1})) \cup G_{i,1}$ and $(C^1_i \cup C^2_i) \subseteq (G'_i \setminus \mathrm{int}(C_{i,1}))$ and $G_{i,1} \neq (G'_i \setminus \mathrm{int}(C_{i,1})) \cap G_{i,1}$, it follows by the Key Lemma 5.1.2 that $G_{i,1}$ is $C_{i,1}$-critical for acyclic k-colouring. Since $G'_i = (G'_i \setminus \mathrm{int}(C_{i,2} \cup C^2_i)) \cup G_{i,2}$ and $(C^1_i \cup C^2_i) \subseteq (G'_i \setminus \mathrm{int}(C_{i,2} \cup C^2_i))$ and $G_{i,2} \neq (G'_i \setminus \mathrm{int}(C_{i,2} \cup C^2_i)) \cap G_{i,2}$, it follows by the Key Lemma 5.1.2 that $G_{i,2}$ is $(C_{i,2} \cup C^2_i)$-critical for acyclic k-colouring.

If $C_{i,1}$ is a 3-cycle, then by Theorem 5.1.5 we have that $|V(G_{i,1}) \setminus V(C_{i,1})| = 0$. If $C_{i,1}$ is a 4-cycle, then by Theorem 5.1.6 we have that $|V(G_{i,1}) \setminus V(C_{i,1})| \leq k$. Otherwise $|V(C_{i,1})| \geq 5$, and by Theorem 5.3.5 we have that $\varepsilon|V(G_{i,1}) \setminus V(C_{i,1})| \leq |V(C_{i,1})| - \gamma$, for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$.

If $|V(C^2_i)|, |V(C_{i,2})| \leq 4$, then it follows from Theorem 6.2.19 that $|V(G_{i,2}) \setminus V(C_{i,2} \cup C^2_i)| \leq |V(C_{i,2} \cup C^2_i)| + 8 + 3\varepsilon - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$. Otherwise, if one of $|V(C_{i,2})|, |V(C^2_i)|$ is at least 5, then since Γ_0 is a minimum counterexample, we have that $\varepsilon|V(G_{i,2}) \setminus V(C_{i,2} \cup C^2_i)| \leq |V(C_{i,2})| + |V(C^2_i)| + 20 + 9\varepsilon - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$.

The rest of the proof follows similarly to the proof of Claim 5.3.6 and uses calculations similar to those found in the proof of Lemma 5.2.1. In the end, we find that $\varepsilon|V(G_0) \setminus V(C_0 \cup C'_0)| \leq |V(C_0)| + |V(C'_0)| + 20 + 9\varepsilon - \gamma$ for some $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$, which contradicts the assumption that Γ_0 is a counterexample. □

Claim 6.2.25. Γ_i *does not contain a dividing bichord, for each* $i \in \{0, 1, 2, 3\}$.

Claim 6.2.26. Γ_i *does not contain a 6-double-pod, for each* $i \in \{0, 1, 2, 3\}$.

Claim 6.2.27. Γ_i *does not contain a non-unique, non-dividing bichord, for each* $i \in \{0, 1, 2, 3\}$.

The proofs of Claims 6.2.25, 6.2.26, and 6.2.27 follow similarly to the proof of Claim 6.2.24. In each proof, we start by supposing the claim is not true. Next, we define the correct relaxation Γ'_i of Γ_0. After that, Γ'_i is divided into smaller k-critical canvases using the bichord or double-pod that it is assumed to have. All of these canvases have one outer cycle, except for one, in which the graph is bounded by two cycles. This is why the addition of $20 + 9\varepsilon$ does not compound in each calculation. The remainder of the proofs, including the calculations, follow similarly to the proofs of Claims 5.3.7, 5.3.8, 5.3.9 and Lemmas 5.2.2, 5.2.3.

By Claims 6.2.25 and 6.2.27, it follows that, for all $i \in \{0, 1, 2, 3\}$, if Γ_i contains a bichord uvw, then $v \in B(\Gamma_i)$. For all $i \in \{0, 1, 2, 3\}$, we have that $C_i \cup C'_i$ has no chords by Claim 6.2.24 and Γ_i has no 6-double-pods by Claim 6.2.26.

Claim 6.2.28. Γ_0 *does not contain a bichord.*

Proof Sketch. Let M be a k-mosaic of $C_0 \cup C_0'$ that extends to $G_0[V(C_0 \cup C_0') \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)]$. Thus, by Lemma 6.2.15, we have that M extends to G_0. Since G_0 is $(C_0 \cup C_0')$-critical for acyclic k-colouring, it follows that $G_0 = G_0[V(C_0 \cup C_0') \cup B(\Gamma_0) \cup B(\Gamma_1) \cup B(\Gamma_2)]$. Hence we have that $|V(G_0)| = |V(C_0 \cup C_0')| + |B(\Gamma_0)| + |B(\Gamma_1)| + |B(\Gamma_2)| \leq (3k+1)|V(C_0 \cup C_0')|$. Thus, it follows that $|V(G_0) \setminus V(C_0 \cup C_0')| \leq 3k|V(C_0 \cup C_0')|$. Let $5 - \varepsilon z \geq \gamma \geq 4.8 + \varepsilon z$. Since $z \geq 15k$, we have that $5 - 15k\varepsilon \geq \gamma$; thus, it follows that $\varepsilon \leq \frac{1}{3k} - \frac{\gamma}{15k}$. Note that since $\frac{\gamma}{5} < 1$, it follows that $\frac{1}{3k} - \frac{\gamma}{15k} > 0$. Since $\varepsilon \leq \frac{1}{3k} - \frac{\gamma}{15k}$, we have that $3k\varepsilon \leq 1 - \frac{\gamma}{5}$. Also, note that since $|V(C_0 \cup C_0')| \geq 5$, we have that $-\frac{|V(C_0 \cup C_0')|}{5} \leq -1$. Thus,

$$\begin{aligned}
\varepsilon|V(G_0) \setminus V(C_0 \cup C_0')| &\leq 3k\varepsilon|V(C_0 \cup C_0')| \\
&\leq |V(C_0 \cup C_0')| - \frac{\gamma|V(C_0 \cup C_0')|}{5} \\
&\leq |V(C_0 \cup C_0')| - \gamma,
\end{aligned}$$

which contradicts the assumption that Γ_0 is a counterexample. □

By Claims 6.2.24, 6.2.28, and 6.2.26, we have that G_0 does not contain a chord of $C_0 \cup C_0'$ and Γ_0 does not contain a bichord or a 6-double-pod. Thus, by the converse of the Two Cycle General Structure Lemma 6.2.16, it follows that Γ_0 is not k-critical, which is a contradiction. □

Theorem 6.2.29. *For each $k \geq 12$, there exists $c > 1$ such that if G is plane and S is a non-empty independent set of G whose vertices are incident with at most two faces of G and G is S-critical for acyclic k-colouring, then $|V(G)| \leq c(|V(S)| - 1)$.*

The proof of Theorem 6.2.29 follows similarly to the proof of Theorem 6.1.1, but relies on Theorem 6.2.20 instead of Theorem 5.3.5. Additionally, in the proof of Theorem 6.2.29, we add vertices to create two cycles that bound the graph, rather than just one.

Theorem 6.2.30. *The family $\mathcal{F}$ of graphs which are critical for acyclic k-colouring, where $k \geq 12$, is strongly hyperbolic.*

The proof of Theorem 6.2.30 follows similarly to the proof of Theorem 6.1.9, but relies on Theorem 6.2.29 instead of Theorem 6.1.1.

Let us recall that we set out to prove Theorem 1.0.4, which says that, for each $k \geq 12$ and each surface S, there are finitely many graphs that are critical for acyclic k-colouring that embed in S.

Proof of Theorem 1.0.4. This follows from Theorem 6.2.30 and Theorem 2.2.1, which is Theorem 1.3 in [15]. □

Theorem 6.2.31. *For each $k \geq 12$ and each surface S, there exists a linear time algorithm that decides whether a graph embedded in S is acyclically k-colourable.*

Proof. Given $k \geq 12$ and a surface S, we have by Theorem 1.0.4 that there are finitely many graphs that embed in S which are critical for acyclic k-colouring. Let L be a list of these graphs and notice that L can be generated in constant time since k and S are fixed. By a result from Eppstein [9], we know that subgraph testing can be done in linear time for graphs that embed in a fixed surface. Therefore, there exists an algorithm which checks if a graph G embedded in S contains a graph in L as a subgraph in linear time. If the algorithm finds that G does not contain a graph in L as a subgraph, then G is acyclically k-colourable. □

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