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The natural brackets on couples of vector fields and 1-forms

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Abstract: All natural bilinear operators transforming pairs of couples of vector fields and 1-forms into couples of vector fields and 1-forms are found. All natural bilinear operators as above satisfying the Leibniz rule are extracted. All natural Lie algebra brackets on couples of vector fields and 1-forms are collected.

Key words: Natural operator, vector field, 1-form, Leibniz rule

1. Introduction

Let $\mathcal{M}f_m$ be the category of m-dimensional $\mathcal{C}^\infty$ manifolds and their embeddings.

The "doubled" tangent bundle $T \oplus T^*$ over $\mathcal{M}f_m$ is of great interest because of the seminal papers, where it is proved that it has the natural inner product, and the Courant bracket, see, e.g., [1, 4, 5].

If $m \geq 2$, we classify all $\mathcal{M}f_m$-natural bilinear operators

$$A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$$

transforming pairs of couples $X^i \oplus \omega^i \in \mathcal{X}(M) \oplus \Omega^1(M)$ $(i = 1, 2)$ of vector fields and 1-forms on m-manifolds M into couples $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in \mathcal{X}(M) \oplus \Omega^1(M)$ of vector fields and 1-forms on M.

In particular, we get that if $m \geq 2$ then any $\mathcal{M}f_m$-natural skew-symmetric bilinear operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ coincides with the Courant bracket up to three real constants; see Corollary 3.3.

If $m \geq 2$, we find all $\mathcal{M}f_m$-natural bilinear operators $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ satisfying the Leibniz rule

$$A(X, A(Y, Z)) = A(A(X, Y), Z) + A(Y, A(X, Z))$$

for any $X, Y, Z \in \mathcal{X}(M) \oplus \Omega^1(M)$ and $M \in obj(\mathcal{M}f_m)$.

If $m \geq 2$, we also find all $\mathcal{M}f_m$-natural Lie algebra brackets $[-, -]$ on $\mathcal{X}(M) \oplus \Omega^1(M)$, i.e. all $\mathcal{M}f_m$-natural skew-symmetric bilinear operators $A = [-, -]$ as above satisfying the Leibniz rule.

Some linear natural operators on vector fields, forms, and some other tensor fields have been studied in many papers; see [2, 3, 7, 8], etc.

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From now on, (x^i) $(i = 1, ..., m)$ denote the usual coordinates on $\mathbb{R}^m$ and $\partial_i = \frac{\partial}{\partial x^i}$ are the canonical vector fields on $\mathbb{R}^m$.

All manifolds are assumed to be Hausdorff, second countable, finite dimensional, without boundary, and smooth (of class $\mathcal{C}^\infty$). Maps between manifolds are assumed to be $\mathcal{C}^\infty$.

2. The basic notions

The notion of natural operators is rather well known. In the present note we need the following particular definitions of natural operators.

Definition 2.1 *A bilinear $\mathcal{M}f_m$-natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ is a $\mathcal{M}f_m$-invariant family of bilinear operators*

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M) \oplus \Omega^1(M)$$

for m-dimensional manifolds M, where $\mathcal{X}(M)$ is the space of vector fields on M and $\Omega^1(M)$ is the space of 1-forms on M. The $\mathcal{M}f_m$-invariance of A means that if $(X^1 \oplus \omega^1, X^2 \oplus \omega^2) \in (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M))$ and $(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2) \in (\mathcal{X}(\overline{M}) \oplus \Omega^1(\overline{M})) \times (\mathcal{X}(\overline{M}) \oplus \Omega^1(\overline{M}))$ are φ-related by an $\mathcal{M}f_m$-map $\varphi : M \to \overline{M}$ (i.e. $\overline{X}^i \circ \varphi = T\varphi \circ X^i$ and $\overline{\omega}^i \circ \varphi = T^*\varphi \circ \omega^i$ for $i = 1, 2$), then so are $A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)$ and $A(\overline{X}^1 \oplus \overline{\omega}^1, \overline{X}^2 \oplus \overline{\omega}^2)$.

Definition 2.2 *A bilinear $\mathcal{M}f_m$-natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$ is a $\mathcal{M}f_m$-invariant family of bilinear operators*

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \mathcal{X}(M)$$

for m-manifolds M.

Definition 2.3 *A bilinear $\mathcal{M}f_m$-natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T^*$ is a $\mathcal{M}f_m$-invariant family of bilinear operators*

$$A : (\mathcal{X}(M) \oplus \Omega^1(M)) \times (\mathcal{X}(M) \oplus \Omega^1(M)) \to \Omega^1(M)$$

for m-manifolds M.

Remark 2.4 *By the multilinear Peetre theorem, see [6], any $\mathcal{M}f_m$-natural bilinear operator A (as above) is of finite order. It means that there is a finite number r such that we have the following implication*

$$(j^r_x X_i = j^r_x \overline{X}_i, j^r_x \omega_i = j^r_x \overline{\omega}_i, i = 1, 2) \Rightarrow A(X_1 \oplus \omega_1, X_2 \oplus \omega_2)_{|x} = A(\overline{X}_1 \oplus \overline{\omega}_1, \overline{X}_2 \oplus \overline{\omega}_2)_{|x}$$

Remark 2.5 *We say that an operator A is regular if it transforms smoothly parametrized families of objects into smoothly parametrized families. One can show that bilinear $\mathcal{M}f_m$-natural operators are regular because of the Peetre theorem.*

Definition 2.6 *A $\mathcal{M}f_m$-natural operator $B : T \oplus T^{(0,0)} \rightsquigarrow T^*$ is a $\mathcal{M}f_m$-invariant family of regular (not necessarily bilinear) operators*

$$B : \mathcal{X}(M) \oplus \mathcal{C}^\infty(M) \to \Omega^1(M)$$

for m-manifolds M, where $\mathcal{C}^\infty(M)$ is the space of smooth maps $M \to \mathbb{R}$.

The most interesting example of a bilinear $\mathcal{M}f_m$-natural operator $A:(T\oplus T^*)\times(T\oplus T^*)\rightsquigarrow T\oplus T^*$ is the famous Courant bracket $[-,-]_C$ presented below.

Example 2.7 *On the vector bundle $TM\oplus T^*M$ there exist canonical symmetric and skew-symmetric pairings*

$$< X^1\oplus\omega^1, X^2\oplus\omega^2 >_\pm = \frac{1}{2}(< X^2,\omega^1 > \pm < X^1,\omega^2 >)$$

for any $X^1\oplus\omega^1, X^2\oplus\omega^2\in\mathcal{X}(M)\oplus\Omega^1(M)$, where $<-,->: TM\times_M T^*M\to\mathbb{R}$ is the usual canonical pairing. Further, a bracket (Courant bracket) is given by

$$[X^1\oplus\omega^1, X^2\oplus\omega^2]_C = [X^1,X^2]\oplus(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1+d< X^1\oplus\omega^1, X^2\oplus\omega^2 >_-)$$

for any $X^1\oplus\omega^1, X^2\oplus\omega^2\in\mathcal{X}(M)\oplus\Omega^1(M)$, where $\mathcal{L}$ denotes the usual Lie derivative, d denotes the usual differentiation, and $[-,-]$ denotes the usual bracket on vector fields.

Definition 2.8 *A $\mathcal{M}f_m$-natural bilinear operator A in the sense of Definition 2.1 satisfies the Leibniz rule if*

$$A(X,A(Y,Z)) = A(A(X,Y),Z)+A(Y,A(X,Z))$$

for any $X,Y,Z\in\mathcal{X}(M)\oplus\Omega^1(M)$.

The Courant bracket is skew-symmetric bilinear but does not satisfy the Jacobi identity.

3. The main results

The main results of the present note are the following classification theorems.

Theorem 3.1 *If $m\ge 2$, any bilinear $\mathcal{M}f_m$-natural operator $A:(T\oplus T^*)\times(T\oplus T^*)\rightsquigarrow T\oplus T^*$ is of the form*

$$A(\rho^1,\rho^2) = a[X^1,X^2]\oplus(b_1\mathcal{L}_{X^2}\omega^1+b_2\mathcal{L}_{X^1}\omega^2+b_3 d<\rho^1,\rho^2>_+ + b_4 d<\rho^1,\rho^2>_-)$$

for (uniquely determined by A) real numbers a,b_1,b_2,b_3,b_4, where $\rho^i = X^i\oplus\omega^i$ for $i=1,2$ and where $<-,->_+$ and $<-,->_-$ are as in Example 2.7.

Theorem 3.2 *If $m\ge 2$, any bilinear $\mathcal{M}f_m$-natural operator $A:(T\oplus T^*)\times(T\oplus T^*)\rightsquigarrow T\oplus T^*$ satisfying the Leibniz rule is the constant multiple of the one of the following four operators:*

$$\begin{aligned}
&A_1(\rho^1,\rho^2) = [X^1,X^2]\oplus 0\\
&A_2(\rho^1,\rho^2) = [X^1,X^2]\oplus(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1)\\
&A_3(\rho^1,\rho^2)_3 = [X^1,X^2]\oplus\mathcal{L}_{X^1}\omega^2\\
&A_4(\rho^1,\rho^2) = [X^1,X^2]\oplus(\mathcal{L}_{X^1}\omega^2-\mathcal{L}_{X^2}\omega^1+d< X^2,\omega^1 >)
\end{aligned}$$

where $\rho^1 = X^1\oplus\omega^1$ and $\rho^2 = X^2\oplus\omega^2$.

From Theorem 3.1 we obtain immediately

Corollary 3.3 *If $m \geq 2$, any skew-symmetric bilinear $\mathcal{M}f_m$-natural operator $A : (T \otimes T^*) \oplus (T \otimes T^*) \rightsquigarrow T \oplus T^*$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b(\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) + cd < X^1 \oplus \omega^1, X^2 \oplus \omega^2 >_-)$$

for (uniquely determined by A) real numbers a, b, c.

Roughly speaking, Corollary 3.3 says that any skew-symmetric bilinear $\mathcal{M}f_m$-natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ coincides with the Courant bracket up to three real constants.

From Theorem 3.2 and Corollary 3.3 it follows immediately

Corollary 3.4 *If $dim(M) \geq 2$, any $\mathcal{M}f_m$-natural Lie algebra bracket on $\mathcal{X}(M) \oplus \Omega^1(M)$ is the constant multiple of the one of the following two Lie algebra brackets:*

$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_1 = [X^1, X^2] \oplus 0 \ ,$$
$$[X^1 \oplus \omega^1, X^2 \oplus \omega^2]_2 = [X^1, X^2] \oplus (\mathcal{L}_{X^1}\omega^2 - \mathcal{L}_{X^2}\omega^1) \ .$$

The rest of the paper is dedicated to proving the results mentioned above.

4. The natural operators in the sense of Definition 2.2

In this section we prove the following:

Proposition 4.1 *If $m \geq 2$, any bilinear $\mathcal{M}f_m$-natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$ is of the form*

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2]$$

for a (uniquely determined by A) real number a.

Proof Consider a bilinear $\mathcal{M}f_m$-natural operator $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$. Clearly, A is determined by the values

$$< A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|o}, \eta > \in \mathbb{R}$$

for all $X^i \oplus \omega^i \in \mathcal{X}(\mathbb{R}^m) \oplus \Omega^1(\mathbb{R}^m)$, $\eta \in T_0^*\mathbb{R}^m$, $i = 1, 2$. Moreover, by the invariance and the regularity of A and the Frobenius theorem we may additionally assume that $X^1 = \partial_1$ and $\eta = d_0x^1$. In other words, A is determined by the values

$$< A(\partial_1 \oplus \omega^1, X \oplus \omega^2)_{|o}, d_0x^1 > \in \mathbb{R}$$

for all $X \in \mathcal{X}(\mathbb{R}^m)$, $\omega^i \in \Omega^1(\mathbb{R}^m)$, $i = 1, 2$. Using the invariance of A with respect to the homotheties and the bilinearity of A we have the homogeneity condition

$$< A(\partial_1 \oplus t(\frac{1}{t}id)_*\omega^1, t(\frac{1}{t}id)_*X \oplus t(\frac{1}{t}id)_*\omega^2)_{|o}, d_0x^1 >= t < A(\partial_1 \oplus \omega^1, X \oplus \omega^2)_{|o}, d_0x^1 > \ .$$

Thus, by the homogeneous function theorem, since A is of finite order and regular, the value $< A(\partial_1 \oplus \omega^1, X \oplus \omega^2)_{|o}, d_0x^1 >$ depends on j_0^1X only. Then A is determined by the values

$$< A(\partial_1 \oplus 0, (\sum_{k=1}^{m} a^k\partial_k + \sum_{i,j=1}^{m} b_i^j x^i\partial_j) \oplus 0)_{|o}, d_0x^1 >$$

for all $a^k, b_i^j \in \mathbb{R}$, $i,j,k = 1,...,m$. Then, by the invariance of A with respect to the diffeomorphisms $(t_1x^1, t_2x^2, ..., t_mx^m)$, $t_l \in \mathbb{R}_+$, $l = 1,...,m$, and by the bilinearity of A, we may assume that $a^k = 0$ for $k = 1,...,m$ and $b_i^j = 0$ for $i,j = 1,...,m$ with $i \neq j$, that is, A is determined by the values $< A(\partial_1 \oplus 0, x^i\partial_i \oplus 0)_{|0}, d_0x^1 > \in \mathbb{R}$, $i = 1,...,m$, and then A is determined by the values

$$< A(\partial_1 \oplus 0, x^1\partial_1 \oplus 0)_{|0}, d_0x^1 > \in \mathbb{R} \text{ and } A(\partial_1 \oplus 0, X \oplus 0)_{|0} \in T_0\mathbb{R}^m$$

for all $X \in \mathcal{X}(\mathbb{R}^{m-1})$ (depending on $x^2, ..., x^m$). Further by the regularity of A we may assume that $X_{|0} \neq 0$, and then (by the invariance of A with respect to local diffeomorphisms of the form $id_{\mathbb{R}} \times \psi(x^2,...,x^m)$ and the Frobenius theorem) we may assume $X = \partial_2$. Using the bilinearity and the invariance of A with respect to the homotheties one can easily see that $A(\partial_1 \oplus 0, \partial_2 \oplus 0)_{|0} = 0$. Consequently, A is determined by the value

$$< A(\partial_1 \oplus 0, x^1\partial_1 \oplus 0)_{|0}, d_0x^1 > \in \mathbb{R} ,$$

i.e. the vector space of all bilinear $\mathcal{M}f_m$-natural operators $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T$ is not more than 1-dimensional. On the other hand, we have the bilinear $\mathcal{M}f_m$-natural operator A_o (in question) given by $A_o(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = [X^1, X^2]$. The proof of Proposition 4.1 is complete. □

5. On natural operators in the sense of Definition 2.6

In this section we prove the following:

Lemma 5.1 *Let $B : T \oplus T^{(0,0)} \rightsquigarrow T^*$ be a $\mathcal{M}f_m$-natural operator satisfying*

$$B(tX \oplus f) = t^2B(X \oplus f) = B(X \oplus t^2f),$$
$$B(X \oplus (f + f_1)) = B(X \oplus f) + B(X \oplus f_1) .$$

If $m \geq 2$, then B is of the form

$$B(X \oplus f) = \lambda d(XXf) ,$$

for a (uniquely determined by B) real number λ, where d is the usual differentiation.

Proof By the classical Petree theorem (since B is linear in f), B is of finite order in f, i.e. for any m-manifold M, any point $x \in M$ and any vector field $X \in \mathcal{X}(M)$ there is a natural number r such that for any $f, \overline{f} \in \mathcal{C}^\infty(M)$ from $j_x^rf = j_x^r\overline{f}$ it follows $B(X,f)_{|x} = B(X,\overline{f})_{|x}$. Clearly, B is determined by the values $< B(X \oplus f)_{|0}, v > \in \mathbb{R}$ for $X \in \mathcal{X}(\mathbb{R}^m)$, $f \in \mathcal{C}^\infty(M)$, $v \in T_0\mathbb{R}^m$. By the regularity of B and $m \geq 2$, we may assume $X_{|0}$ and v are linearly independent, and then by the invariance of B and the Frobenius theorem, we may assume $X = \partial_1$, and $v = \partial_{2|0}$, i.e. B is determined by the values

$$< B(\partial_1 \oplus f)_{|0}, \partial_{2|0} > \in \mathbb{R}$$

for $f \in \mathcal{C}^\infty(\mathbb{R}^m)$. Since B is of finite order in f, we may assume, f is polynomial. Now, by the invariance of B with respect to the diffeomorphisms $(t_1x^1, ..., t_mx^m)$, $t_l \in \mathbb{R}_+$, $l = 1,...,m$ and the conditions of B, we derive that $< B(\partial_1 \oplus f)_{|0}, \partial_{2|0} >$ is determined by $< B(\partial_1 \oplus (x^1)^2x^2)_{|0}, \partial_{2|0} >$. Consequently, the vector space of all such operators B is of dimension not more than 1. On the other hand, we have an $\mathcal{M}f_m$-operator B_o in question given by $B_o(X \oplus f) = d(XXf)$. Lemma 5.1 is complete. □

6. The natural operators in the sense of Definition 2.3

In this section we prove the following:

Proposition 6.1 *Let $A : (T\oplus T^*)\times(T\oplus T^*) \rightsquigarrow T^*$ be a bilinear $\mathcal{M}f_m$-natural operator. If $m \geq 2$, then A is the linear combination with real coefficients of the bilinear $\mathcal{M}f_m$-natural operators $A^{<j>} : (T\oplus T^*)\times(T\oplus T^*) \rightsquigarrow T^*$ given by*

$$A^{<1>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = \mathcal{L}_{X^2}\omega^1$$
$$A^{<2>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = \mathcal{L}_{X^1}\omega^2$$
$$A^{<3>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = d < X^1 \oplus \omega^1, X^2 \oplus \omega^2 >_+$$
$$A^{<4>}(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = d < X^1 \oplus \omega^1, X^2 \oplus \omega^2 >_- .$$

Proof Clearly, A is determined by the values

$$< A(X^1 \oplus \omega^1, X^2 \oplus \omega^2)_{|_0}, v >\in \mathbb{R}$$

for all $X^1, X^2 \in \mathcal{X}(\mathbb{R}^m)$, $\omega^1, \omega^2 \in \Omega^1(\mathbb{R}^m)$, $v \in T_0\mathbb{R}^m$. Consequently, using the bilinearity of A, A is determined by the values

$$< A(0 \oplus \omega^1, 0 \oplus \omega^2)_{|_0}, v > , \ < A(0 \oplus \omega^1, X^2 \oplus 0)_{|_0}, v > ,$$

$$< A(X^1 \oplus 0, 0 \oplus \omega^2)_{|_0}, v > , \ < A(X^1 \oplus 0, X^2 \oplus 0)_{|_0}, v >$$

for all $X^1, X^2 \in \mathcal{X}(\mathbb{R}^m)$, $\omega^1, \omega^2 \in \Omega^1(\mathbb{R}^m)$, $v \in T_0\mathbb{R}^m$. Using the invariance of A with respect to the homotheties and the bilinearity of A and then applying the homogeneous function theorem, we easily deduce that

$$< A(0 \oplus \omega^1, 0 \oplus \omega^2)_{|_0}, v >= 0.$$

By the same argument, $< A(0 \oplus \omega^1, X^2 \oplus 0)_{|_0}, v >$ depends on $j^1_0\omega^1$ and $j^1_0X^2$ only, and (symmetrically) $< A(X^1 \oplus 0, 0 \oplus \omega^2)_{|_0}, v >$ depends on $j^1_0\omega^2$ and $j^1_0X^1$ only, and (similarly) $< A(X^1 \oplus 0, X^2 \oplus 0)_{|_0}, v >$ depends on $j^3_0X^1$ and $j^3_0X^2$ only. Next, by the regularity of A we may assume $X^1_{|_0}$ and v are linearly independent, and then by the Frobenius theorem we may assume that $X^1 = \partial_1$ and $v = \partial_{2|_0}$. Then, using the invariance of A with respect to $(t_1x^1, ..., t_mx^m)$ for $t_l \in \mathbb{R}_+$, $l = 1, ..., m$, we may assume $< A(X^1 \oplus 0, 0 \oplus \omega^2)_{|_0}, v >$ is determined by

$$c_1 :=< A(\partial_1 \oplus 0, 0 \oplus x^1dx^2)_{|_0}, \partial_{2|_0} > \ \text{and} \ c_3 :=< A(\partial_1 \oplus 0, 0 \oplus x^2dx^1)_{|_0}, \partial_{2|_0} > .$$

By similar arguments, we may assume $< A(0 \oplus \omega^1, X^2 \oplus 0)_{|_0}, v >$ is determined by

$$c_2 :=< A(0 \oplus x^1dx^2, \partial_1 \oplus 0)_{|_0}, \partial_{2|_0} > \ \text{and} \ c_4 :=< A(0 \oplus x^2dx^1, \partial_1 \oplus 0)_{|_0}, \partial_{2|_0} > ,$$

and (similarly) we may assume $< A(X^1 \oplus 0, X^2 \oplus 0)_{|_0}, v >$ is determined by

$$d_k :=< A(\partial_1 \oplus 0, x^1x^2x^k\partial_k \oplus 0)_{|_0}, \partial_{2|_0} > , \ k = 1, ..., m .$$

The above facts imply that A is determined by the real numbers d_k and

$$b_1 := c_2,\ b_2 := c_1,\ b_3 := c_3 + c_4,\ b_4 := c_4 - c_3\ .$$

We prove that $A = \sum_{j=1}^4 b_j A^{<j>}$.

Replacing A by $A - \sum_{j=1}^4 b_j A^{<j>}$, we may assume $b_1 = b_2 = b_3 = b_4 = 0$, i.e. we may assume that A is determined by the values d_k, i.e. we may assume that A is determined by the value

$$< A(\partial_1 \oplus 0, (x^1)^2 x^2 \partial_1 \oplus 0)_{|0}, \partial_{2|0} > \in \mathbb{R}$$

together with the values

$$A(\partial_1 \oplus 0, x^1 Y \oplus 0)_{|0} \in T_0^* \mathbb{R}^m$$

for all vector fields $Y \in \mathcal{X}(\mathbb{R}^{m-1})$ (depending on $x^2, , ..., x^m$). Next, by the regularity of A, we may assume $Y_{|0} \neq 0$, and then, by the invariance of A with respect to local diffeomorphisms of the form $id_{\mathbb{R}} \times \psi(x^2, ..., x^m)$ and the Frobenius theorem, we may assume $Y = \partial_2$. However,

$$A(\partial_1 \oplus 0, x^1 \partial_2 \oplus 0)_{|0} = 0$$

because of the invariance of A with respect to the homotheties. Consequently, A is determined by the $\mathcal{M}f_m$-natural operator $B : T \oplus T^{(0,0)} \rightsquigarrow T^*$ given by

$$B(X \oplus f) := A(X \oplus 0, fX \oplus 0)\ ,$$

$M \in obj(\mathcal{M}f_m)$, $X \in \mathcal{X}(M)$, $f \in \mathcal{C}^\infty(M)$. Clearly, B satisfies the assumptions of Lemma 5.1. Then we have $\lambda \in \mathbb{R}$ such that

$$B(X \oplus f) = \lambda d(XXf)$$

for any $M \in obj(\mathcal{M}f_m)$, $X \in \mathcal{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. In particular,

$$A(x^1\partial_1 \oplus 0, \partial_1 \oplus 0) = B(x^1 \partial_1 \oplus \frac{1}{x^1}) = \lambda d[(x^1\partial_1) \circ (x^1 \partial_1)(\frac{1}{x^1})] = \lambda \frac{1}{x^1} dx^1$$

over $\mathbb{R}^m \setminus \{0\}$. Then $\lambda = 0$ because $A(x^1\partial_1 \oplus 0, \partial_1 \oplus 0)$ is a smooth form on all $\mathbb{R}^m$ and $\frac{1}{x^1}dx^1$ is not extendable to a smooth form on $\mathbb{R}^m$. Then $B = 0$, and then $A = 0$ (under the additional assumption). It means $A = \sum_{j=1}^4 b_j A^{<j>}$, where the numbers b_j are defined above. The proof of Proposition 6.1 is complete. □

7. Proof of Theorem 3.1

Proof Theorem 3.1 is an immediate consequence of Propositions 4.1 and 6.1. □

8. Proof of Theorem 3.2

In this section we prove Theorem 3.2 as follows:

Proof Let $A : (T \oplus T^*) \times (T \oplus T^*) \rightsquigarrow T \oplus T^*$ be a bilinear $\mathcal{M}f_m$-natural operator satisfying the Leibniz rule. By Theorem 3.1, A is of the form

$$A(X^1 \oplus \omega^1, X^2 \oplus \omega^2) = a[X^1, X^2] \oplus (b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d < X^2, \omega^1 > + c_2 d < X^1, \omega^2 >)$$

for (uniquely determined by A) real numbers a, b_1, b_2, c_1, c_2, where $< -, - >$ is as in Example 2.7. Then for any $X^1, X^2, X^3 \in \mathcal{X}(M)$ and $\omega^1, \omega^2, \omega^3 \in \Omega^1(M)$ we have

$$A(X^1 \oplus \omega^1, A(X^2 \oplus \omega^2, X^3 \oplus \omega^3)) = a^2[X^1, [X^2, X^3]] \oplus \Omega \ ,$$

$$A(A(X^1 \oplus \omega^1, X^2 \oplus \omega^2), X^3 \oplus \omega^3) = a^2[[X^1, X^2], X^3] \oplus \Theta \ ,$$

$$A(X^2 \oplus \omega^2, A(X^1 \oplus \omega^1, X^3 \oplus \omega^3)) = a^2[X^2, [X^1, X^3]] \oplus \mathcal{T} \ ,$$

where

$$\begin{aligned} \Omega = \quad & b_1 \mathcal{L}_{a[X^2,X^3]}\omega^1 + c_1 d < a[X^2, X^3], \omega^1 > \\ & + b_2 \mathcal{L}_{X^1}(b_1 \mathcal{L}_{X^3}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^3 + c_1 d < X^3, \omega^2 > + c_2 d < X^2, \omega^3 >) \\ & + c_2 d < X^1, b_1 \mathcal{L}_{X^3}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^3 + c_1 d < X^3, \omega^2 > + c_2 d < X^2, \omega^3 >> \ , \end{aligned}$$

$$\begin{aligned} \Theta = \quad & b_2 \mathcal{L}_{a[X^1,X^2]}\omega^3 + c_2 d < a[X^1, X^2], \omega^3 > \\ & + b_1 \mathcal{L}_{X^3}(b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d < X^2, \omega^1 > + c_2 d < X^1, \omega^2 >) \\ & + c_1 d < X^3, b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d < X^2, \omega^1 > + c_2 d < X^1, \omega^2 >> \ , \end{aligned}$$

$$\begin{aligned} \mathcal{T} = \quad & b_1 \mathcal{L}_{a[X^1,X^3]}\omega^2 + c_1 d < a[X^1, X^3], \omega^2 > \\ & + b_2 \mathcal{L}_{X^2}(b_1 \mathcal{L}_{X^3}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^3 + c_1 d < X^3, \omega^1 > + c_2 d < X^1, \omega^3 >) \\ & + c_2 d < X^2, b_1 \mathcal{L}_{X^3}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^3 + c_1 d < X^3, \omega^1 > + c_2 d < X^1, \omega^3 >> \ . \end{aligned}$$

The Leibniz rule of A is equivalent to

$$\Omega = \Theta + \mathcal{T}.$$

Applying the differentiation d to both sides of the last equality and using the well-known formula $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$ we get

$$\begin{aligned} & b_1 a \mathcal{L}_{[X^2,X^3]} d\omega^1 + b_2 b_1 \mathcal{L}_{X^1} \mathcal{L}_{X^3} d\omega^2 + b_2^2 \mathcal{L}_{X^1} \mathcal{L}_{X^2} d\omega^3 \\ & = (b_2 a \mathcal{L}_{[X^1,X^2]} d\omega^3 + b_1^2 \mathcal{L}_{X^3} \mathcal{L}_{X^2} d\omega^1 + b_1 b_2 \mathcal{L}_{X^3} \mathcal{L}_{X^1} d\omega^2) \\ & + (b_1 a \mathcal{L}_{[X^1,X^3]} d\omega^2 + b_2 b_1 \mathcal{L}_{X^2} \mathcal{L}_{X^3} d\omega^1 + b_2^2 \mathcal{L}_{X^2} \mathcal{L}_{X^1} d\omega^3) \ . \end{aligned}$$

If we put $X^1 = \partial_1$, $X^2 = x^1 \partial_1$, $X^3 = 0$ and $\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = (x^1)^2 dx^2$, we get

$$4b_2^2 dx^1 \wedge dx^2 = 2b_2 a dx^1 \wedge dx^2 + 2b_2^2 dx^1 \wedge dx^2 \ .$$

If we put $X^1 = 0$, $X^2 = \partial_1$, $X^3 = x^1\partial_1$ and $\omega^1 = (x^1)^2 dx^2$,$\omega^2 = 0$, $\omega^3 = 0$, we get

$$2b_1 a dx^1 \wedge dx^2 = 2b_1^2 dx^1 \wedge dx^2 + 4b_2 b_1 dx^1 \wedge dx^2 \ .$$

If we put $X^1 = \partial_1$,$X^2 = 0$, $X^3 = x^1\partial_1$ and $\omega^1 = 0$, $\omega^2 = (x^1)^2 dx^2$, $\omega^3 = 0$, we get

$$4b_2 b_1 dx^1 \wedge dx^2 = 2b_1 b_2 dx^1 \wedge dx^2 + 2b_1 a dx^1 \wedge dx^2 \ .$$

Thus,

$$b_2 a = b_2^2 \ , \quad b_1 a = b_1^2 + 2b_1 b_2 \ , \quad b_1 b_2 = b_1 a \ .$$

From the first equality we get $b_2 = 0$ or $b_2 = a$. From the third one we get $b_1 = 0$ or $b_2 = a$. Adding the first two equalities we get $(b_2 + b_1)a = (b_2 + b_1)^2$, i.e. $b_2 + b_1 = 0$ or $b_2 + b_1 = a$. Consequently,

$$(b_1, b_2) = (0, 0) \text{ or } (b_1, b_2) = (0, a) \text{ or } \ (b_1, b_2) = (-a, a) \ . \tag{1}$$

Then, using the formula $\mathcal{L}_X \mathcal{L}_Y \omega - \mathcal{L}_Y \mathcal{L}_X \omega = \mathcal{L}_{[X,Y]}\omega$ and (1), we get

$$\begin{aligned}
&b_1 a \mathcal{L}_{[X^2,X^3]}\omega^1 + b_2 b_1 \mathcal{L}_{X^1}\mathcal{L}_{X^3}\omega^2 + b_2^2 \mathcal{L}_{X^1}\mathcal{L}_{X^2}\omega^3 \\
&= (b_2 a \mathcal{L}_{[X^1,X^2]}\omega^3 + b_1^2 \mathcal{L}_{X^3}\mathcal{L}_{X^2}\omega^1 + b_1 b_2 \mathcal{L}_{X^3}\mathcal{L}_{X^1}\omega^2) \\
&+(b_1 a \mathcal{L}_{[X^1,X^3]}\omega^2 + b_2 b_1 \mathcal{L}_{X^2}\mathcal{L}_{X^3}\omega^1 + b_2^2 \mathcal{L}_{X^2}\mathcal{L}_{X^1}\omega^3) \ .
\end{aligned}$$

Consequently, the Leibniz rule of A is equivalent to the system of (1) and

$$\begin{aligned}
&c_1 ad < [X^2, X^3], \omega^1 > + b_2 \mathcal{L}_{X^1}(c_1 d < X^3, \omega^2 > + c_2 d < X^2, \omega^3 >) \\
&+ c_2 d < X^1, b_1 \mathcal{L}_{X^3}\omega^2 + b_2 \mathcal{L}_{X^2}\omega^3 + c_1 d < X^3, \omega^2 > + c_2 d < X^2, \omega^3 >> \\
&= c_2 ad < [X^1, X^2], \omega^3 > + b_1 \mathcal{L}_{X^3}(c_1 d < X^2, \omega^1 > + c_2 d < X^1, \omega^2 >) \\
&+ c_1 d < X^3, b_1 \mathcal{L}_{X^2}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^2 + c_1 d < X^2, \omega^1 > + c_2 d < X^1, \omega^2 >> \\
&+ c_1 ad < [X^1, X^3], \omega^2 > + b_2 \mathcal{L}_{X^2}(c_1 d < X^3, \omega^1 > + c_2 d < X^1, \omega^3 >) \\
&+ c_2 d < X^2, b_1 \mathcal{L}_{X^3}\omega^1 + b_2 \mathcal{L}_{X^1}\omega^3 + c_1 d < X^3, \omega^1 > + c_2 d < X^1, \omega^3 >> \ .
\end{aligned} \tag{2}$$

If we put $X^1 = \partial_1$, $X^2 = \partial_2$,$X^3 = 0$ and $\omega^1 = 0$, $\omega^2 = 0$, $\omega^3 = (x^2)^2 dx^1$, we get

$$2c_2 b_2 dx^2 = 2c_2 b_2 dx^2 + 2c_2^2 dx^2 \ .$$

Then $c_2 = 0$.

If we put $X^1 = 0$, $X^2 = \partial_1$, $X^3 = \partial_2$ and $\omega^1 = (x^2)^2 dx^1$, $\omega^2 = 0$, $\omega^3 = 0$ we get

$$0 = 2b_1 c_1 dx^2 + 2c_1^2 dx^2 + 2c_2 b_1 dx^2.$$

Then (as $c_2 = 0$) we get $c_1 = 0$ or $c_1 = -b_1$.

Consequently, we get $(b_1, b_2, c_1, c_2) = (0, 0, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (0, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, 0, 0)$ or $(b_1, b_2, c_1, c_2) = (-a, a, a, 0)$. On the other hand one can directly verify that the operators $A_1, \ldots, A_4$ from Theorem 3.2 satisfy the Leibniz rule. Theorem 3.2 is complete. □

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