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Families of Strictly Pseudoconvex Domains and Peak Functions

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Abstract We prove that given a family (G_t) of strictly pseudoconvex domains varying in $\mathcal{C}^2$ topology on domains, there exists a continuously varying family of peak functions $h_{t,\zeta}$ for all G_t at every $\zeta \in \partial G_t$.

Keywords Strictly pseudoconvex domains · Peak functions · Peak points

Mathematics Subject Classification Primary 32T40 · Secondary 32T15

1 Introduction

Let $D \subset \mathbb{C}^n$ be a bounded domain and let ζ be a boundary point of D. It is called a *peak point* with respect to $\mathcal{O}(\overline{D})$, the family of functions which are holomorphic in a neighborhood of $\overline{D}$, if there exists a function $f \in \mathcal{O}(\overline{D})$ such that $f(\zeta) = 1$ and $f(\overline{D}\backslash\{\zeta\}) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Such a function is a *peak function for* D *at* ζ. The concept of peak functions appears to be a powerful tool in complex analysis with many applications. It has been used to show the existence of (complete) proper holomorphic embeddings of strictly pseudoconvex domains into the unit ball $\mathbb{B}^N$ with large N [3,5], to estimate the boundary behavior of Carathéodory and Kobayashi metrics [1,7], or to construct the solution operators for $\overline{\partial}$ problem with L^∞ or Hölder estimates [4,10], just to name a few of those applications.

It is well known that every boundary point of strictly pseudoconvex domain is a peak point. Even more is true, in [7] it is showed that, given a strictly pseudoconvex

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domain G, there exists an open neighborhood $\widehat{G}$ of G, and a continuous function $h : \widehat{G} \times \partial G \to \mathbb{C}$ such that for $\zeta \in \partial G$, the function $h(\cdot;\zeta)$ is a peak function for G at ζ.

In a recent paper [2], the following question has been posed:

Problem 1.1 Let $\rho : \mathbb{D} \times \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function of class $\mathcal{C}^{2+k}$, $k \in \mathbb{N} \cup \{0\}$, such that for any $z \in \mathbb{D}$ the truncated function $\rho|_{\{z\}\times\mathbb{C}^n}$ is strictly plurisubharmonic. Define $G_z := \{w \in \mathbb{C}^n : \rho(z,w) < 0\}$, $z \in \mathbb{D}$. This can be understood as a family of strictly pseudoconvex domains over $\mathbb{D}$. Does there exist a $\mathcal{C}^k$-continuously varying family $(h_{z,\zeta})_{z\in\mathbb{D},\zeta\in\partial G_z}$ of peak functions for G_z at ζ?

We answer this question affirmatively in the case $k = 0$ and under additional assumption that, roughly speaking, the function ρ keeps its regularity up to the set $\Omega \times \mathbb{C}^n$, where Ω is some open neighborhood of $\overline{\mathbb{D}}$ (however, see Remark 1.5 below). Namely, let us consider the following:

Situation 1.2 Let $(G_t)_{t\in T}$ be a family of bounded strictly pseudoconvex domains, where T is a compact metric space with associated metric d. Suppose we have a domain $U \subset\subset \mathbb{C}^n$ such that

(1) $\bigcup_{t\in T} \partial G_t \subset\subset U$,
(2) for each $t \in T$ there exists a defining function r_t for G_t satisfying with neighborhood $\partial G_t \subset U$ all the conditions (A)–(D) below (see Sect. 2),
(3) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $s, t \in T$ with $d(s,t) \le \delta$ there is $\|r_t - r_s\|_{\mathcal{C}^2(U)} < \varepsilon$.

Observe that the above setting is completely in the spirit of the formulation of Problem 1.1:

(i) The assumption that all the functions r_t satisfy (A)–(D) with common neighborhood $\partial G_t \subset U$ stays in relation with the fact that in Problem 1.1 all the defining functions for domains D_z have the same domain of definition ($\mathbb{C}^n$).
(ii) The assumption (3) comes from the fact that the function ρ in Problem 1.1 is of class at least $\mathcal{C}^2$.
(iii) The compactness of the set of parameters (T) reflects the above-mentioned assumption that ρ continues to be of class $\mathcal{C}^2$ up to $\Omega \times \mathbb{C}^n$, with Ω being some neighborhood of $\overline{\mathbb{D}}$.

We shall prove the following:

Theorem 1.3 *Let $(G_t)_{t\in T}$ be a family of strictly pseudoconvex domains as in Situation 1.2. Then there exists an $\varepsilon > 0$ such that for any $\eta_1 < \varepsilon$ there exist an $\eta_2 > 0$ and positive constants d_1, d_2 such that for any $t \in T$ there exist a domain $\widehat{G_t}$ containing $\overline{G_t}$, and functions $h_t(\cdot;\zeta) \in \mathcal{O}(\widehat{G_t})$, $\zeta \in \partial G_t$ fulfilling the following conditions:*

(a) $h_t(\zeta;\zeta) = 1$, $|h_t(\cdot;\zeta)| < 1$ *on* $\overline{G_t}\setminus\{\zeta\}$ *(in particular, $h_t(\cdot;\zeta)$ is a peak function for G_t at ζ),*
(b) $|1 - h_t(z;\zeta)| \le d_1\|z-\zeta\|$, $z \in \widehat{G_t} \cap \mathbb{B}(\zeta,\eta_2)$,
(c) $|h_t(z;\zeta)| \le d_2 < 1$, $z \in \overline{G_t}$, $\|z-\zeta\| \ge \eta_1$.

Moreover, the constants $\varepsilon, \eta_2, d_1, d_2$, *domains* $\widehat{G_t}$, *and functions* $h_t(\cdot;\zeta)$ *may be chosen in such a way that for any* $\alpha > 0$ *and any fixed triple* (t_0, ζ_0, z_0), *where* $t_0 \in T$, $\zeta_0 \in \partial G_{t_0}$, *and* $z_0 \in \widehat{G_{t_0}}$, *there exists a* $\delta > 0$ *such that whenever the triple* (s, ξ, w) *satisfies* $s \in T, \xi \in \partial G_s, w \in \widehat{G_s}$, *and* $\max\{d(s,t_0), \|\xi - \zeta_0\|, \|w - z_0\|\} < \delta$, *then* $|h_{t_0}(z_0;\zeta_0) - h_s(w;\xi)| < \alpha$.

The latter property will be referred to as *continuity*.

Remark 1.4 It is known that for each $t \in T$ there exists an $\varepsilon = \varepsilon(t) > 0$ such that for any $\eta_1 < \varepsilon$ there exist a positive $\eta_2 = \eta_2(t) < \eta_1$, constants $d_1 = d_1(t), d_2 = d_2(t) \in \mathbb{R}$, domain $\widetilde{G_t}$ containing $\overline{G_t}$, and functions $h_t(\cdot;\zeta) \in \mathcal{O}(\widehat{G_t})$, $\zeta \in \partial G_t$ satisfying (a)–(c). This is a subject of Theorem 19.1.2 from [8]. The strength of our result dwells in the fact that all the constants $\varepsilon, \eta_2, d_1, d_2$ are chosen independently of t and in the continuity property.

Remark 1.5 As noticed by the referee, our result can be strengthened in the spirit of Theorem 5.1 from [6]. It gives the construction of Henkin–Ramírez functions for variable strictly pseudoconvex open sets (with boundaries of class $\mathcal{C}^{2+a,j}$; see Definition 2.5 therein) depending $\mathcal{C}^{1+a,j}$-smoothly on a parameter. Under similar assumptions as in [6], and by merging the method of proof of our Theorem 1.3 with the method of proof of Theorem 5.1 from [6], we can get similar regularity for the dependence of our peak functions on the parameter.

In Sect. 2, we recall some preliminaries concerning the strictly pseudoconvex domains. The proof of Theorem 1.3 is presented in Sect. 3.

2 Strictly Pseudoconvex Domains

Let $D \subset\subset \mathbb{C}^n$ be a domain. It is called a *strictly pseudoconvex* if there exist a neighborhood U of ∂D and a *defining function* $r : U \to \mathbb{R}$ of class $\mathcal{C}^2$ and such that

(A) $D \cap U = \{z \in U : r(z) < 0\}$,
(B) $(\mathbb{C}^n \setminus \overline{D}) \cap U = \{z \in U : r(z) > 0\}$,
(C) $\nabla r(z) \neq 0$ for $z \in \partial D$, where $\nabla r(z) := \left(\frac{\partial r}{\partial \bar{z}_1}(z), \ldots, \frac{\partial r}{\partial \bar{z}_n}(z)\right)$,

together with

$$\mathcal{L}_r(z; X) > 0 \text{ for } z \in \partial D \text{ and nonzero } X \in T_z^{\mathbb{C}}(\partial D),$$

where $\mathcal{L}_r$ denotes the Levi form of r and $T_z^{\mathbb{C}}(\partial D)$ is the complex tangent space to ∂D at z.

It is known that U and r can be chosen to satisfy (A)-(C) and, additionally,

(D) $\mathcal{L}_r(z; X) > 0$ for $z \in U$ and all nonzero $X \in \mathbb{C}^n$,

cf. [9].

Note that for a function r as above and a point $\zeta \in \partial G$, Taylor expansion of r at ζ has the following form:

$$r(z) = r(\zeta) - 2\mathrm{Re}P(z;\zeta) + \mathcal{L}_r(\zeta; z - \zeta) + o(\|z - \zeta\|^2), \tag{2.1}$$

where

$$P(z;\zeta) := -\sum_{j=1}^{n}\frac{\partial r}{\partial z_j}(\zeta)(z_j-\zeta_j) - \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 r}{\partial z_i \partial z_j}(\zeta)(z_i-\zeta_i)(z_j-\zeta_j)$$

is the *Levi polynomial of r at ζ*.

3 Proof of Theorem 1.3

We divide the proof into two parts. First we give the construction of $\widehat{G_t}$ and $h_t(\cdot;\zeta)$, $t \in T$, and define the constants ε, η_2, d_1, and d_2, all independent of t. This is refinement of the construction from the proof of Theorem 19.1.2 from [8]. Note that in order to get the independence of all the constants from t, we must be more careful here. In the second part, we prove the continuity property.

Construction of $\widehat{G_t}$ and $h_t(\cdot;\zeta)$ and the choice of ε, η_2, d_1, and d_2. For $t \in T$ and $\zeta \in \partial G_t$ let $P_t(z;\zeta)$ be the Levi polynomial of r_t at ζ. □

Fix an $\varepsilon_1 > 0$ such that $U' := \bigcup_{t\in T,\zeta\in\partial G_t}\mathbb{B}(\zeta,\varepsilon_1) \subset\subset U$.

There exists a constant $C_1 = C_1(t) < 1$ such that

$$\mathcal{L}_{r_t}(z;X) \geq C_1\|X\|^2, \quad z \in U', X \in \mathbb{C}^n.$$

Indeed, $\mathcal{L}_{r_t}$ is continuous and positive on $U \times (\mathbb{C}^n\setminus\{0\})$, so it attains its minimum $C_1(t) > 0$ on $\overline{U'} \times \mathbb{S}^{n-1}$. Since for any nonzero $X \in \mathbb{C}^n$ we have $\frac{X}{\|X\|} \in \mathbb{S}^{n-1}$, we get the required inequality. Moreover, from the assumption (3) it follows that for s from some neighborhood of t, we have

$$\mathcal{L}_{r_s}(z;X) \geq \frac{C_1(t)}{2}\|X\|^2, \quad z \in U', X \in \mathbb{C}^n.$$

The compactness argument then gives that C_1 may be chosen independently of t.

Taylor formula (2.1) yields that with some $0 < C_2 < C_1$ there is

$$r_t(z) \geq -2\mathrm{Re}P_t(z;\zeta) + C_2\|z-\zeta\|^2 \tag{3.1}$$

for $\|z-\zeta\| < \varepsilon_2(t) < \varepsilon_1$, $\zeta \in \partial G_t$, where $\varepsilon_2(t)$ is independent of $\zeta \in \partial G_t$ (and even of $\zeta \in W \subset\subset U$, some neighborhood of ∂G_t—see [11], Proposition II.2.16). Moreover, from the proof of Theorem V.3.6 from [11], it follows that for s close enough to t we have

$$r_s(z) \geq r_s(\zeta) - 2\mathrm{Re}P_s(z;\zeta) + \frac{C_2}{2}\|z-\zeta\|^2, \quad \zeta \in W, \|z-\zeta\| < \varepsilon_2(t).$$

Therefore, for s near to t, and for $\xi \in \partial G_s$, the following estimate holds true:

$$r_s(z) \geq -2\mathrm{Re}P_s(z;\xi) + \frac{C_2}{2}\|z-\xi\|^2, \quad \|z-\xi\| < \varepsilon_2(t).$$

The compactness argument then implies that C_2 and ε_2 in (3.1) may be chosen independently of t.

Let $0 < \eta_1 < \varepsilon_2$ and $\widehat{\chi} \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ be such that $\widehat{\chi}(t) = 1$ for $t \le \frac{\eta_1}{2}$ and $\widehat{\chi}(t) = 0$ for $t \ge \eta_1$. Put $\chi(z;\zeta) := \widehat{\chi}(\|z-\zeta\|)$. This is a smooth function on $\mathbb{C}^n \times \mathbb{C}^n$, taking its values in $[0, 1]$.

Define

$$\varphi_t(z;\zeta) := \chi(z;\zeta)P_t(z;\zeta) + (1-\chi(z;\zeta))\|z-\zeta\|^2, \quad z \in \mathbb{C}^n.$$

Observe that if $\|z-\zeta\| \le \frac{\eta_1}{2}$, then $\varphi_t(z;\zeta) = P_t(z;\zeta)$. In particular $\varphi_t(\cdot;\zeta) \in \mathcal{O}(\mathbb{B}(\zeta, \frac{\eta_1}{2}))$. Furthermore, for z satisfying $\|z-\zeta\| \ge \frac{\eta_1}{2}$ and $r_t(z) < C_2\frac{\eta_1^2}{8}$ the following estimate holds true:

$$2\mathrm{Re}\varphi_t(z;\zeta) \ge C_2\frac{\eta_1^2}{8} > 0. \tag{3.2}$$

Take $0 < \eta_t < C_2\frac{\eta_1^2}{8}$ such that the connected component $\widetilde{G}_t$ containing $\overline{G_t}$ of the open set

$$G_t \cup \{z \in U' : r_t(z) < \eta_t\}$$

is a strictly pseudoconvex domain, relatively compact in $G_t \cup U'$. Because of the assumption (3), there exists a positive number β such that for s close to t the connected component $\widetilde{G}_s$ containing $\overline{G_s}$ of the set

$$G_s \cup \{z \in U' : r_s(z) < \eta_t - \beta\}$$

is a strictly pseudoconvex domain, relatively compact in $G_s \cup U'$. Making again use of the compactness of T, we conclude that in fact $\eta = \eta_t$ may be taken independently of t. Note that, for the family $(\widetilde{G}_t)_{t\in T}$, the assumption (3) remains true.

The function $\varphi_t(\cdot;\zeta) \in \mathcal{C}^\infty(\mathbb{C}^n)$ does not vanish on $\widetilde{G}_t \setminus \mathbb{B}(\zeta, \frac{\eta_1}{2})$ and is in $\mathcal{O}(\mathbb{B}(\zeta, \frac{\eta_1}{2}))$. Therefore $\bar\partial \frac{1}{\varphi_t(\cdot;\zeta)}$ defines a $\bar\partial$-closed $\mathcal{C}^\infty$ form

$$\alpha_t(\cdot;\zeta) = \sum_{j=1}^n \alpha_{t,j}(\cdot;\zeta)d\bar{z}_j$$

on $\widetilde{G}_t$, where

$$\alpha_{t,j} = \begin{cases} 0, & z \in \widetilde{G}_t \cap \mathbb{B}(\zeta; \frac{\eta_1}{2}), \\ -\frac{\partial\varphi_t}{\partial\bar{z}_j}(z;\zeta)\cdot\frac{1}{\varphi_t^2(z;\zeta)}, & z \in \widetilde{G}_t \setminus \mathbb{B}(\zeta; \frac{\eta_1}{2}). \end{cases}$$

Thanks to (3.2) we have $\|\alpha_{t,j}(\cdot;\zeta)\|_{\widetilde{G}_t} \le C_3$, where, utilizing the compactness of T together with the assumption (3), we deliver that C_3 is independent of t and $\zeta \in \partial G_t$. [11, Theorem V.2.7] gives then the functions $v_t(\cdot;\zeta) \in \mathcal{C}^\infty(\widetilde{G}_t)$ with $\bar\partial v_t(\cdot;\zeta) = \alpha_t(\cdot;\zeta)$ and

$$\|v_t(\cdot;\zeta)\|_{\widetilde{G_t}} \le C_4,$$

where C_4 does not depend on $\zeta \in \partial G_t$. Moreover, by [11, Theorem V.3.6] and the compactness of T, C_4 may be chosen to be independent of t.

Define

$$f_t(\cdot;\zeta) := \frac{1}{\varphi_t(\cdot;\zeta)} + C_4 - v_t(\cdot;\zeta), \quad z \in \widetilde{G_t}\backslash Z_t(\zeta),$$

where

$$Z_t(\zeta) := \{z \in \widetilde{G_t} : \varphi_t(z;\zeta) = 0\}.$$

Then $f_t(\cdot;\zeta) \in \mathcal{O}(\widetilde{G_t}\backslash Z_t(\zeta))$ as well as

$$\operatorname{Re} f_t(\cdot;\zeta) > 0$$

on the set $(\widetilde{G_t}\backslash\mathbb{B}(\zeta, \frac{\eta_1}{2})) \cup (\overline{G_t}\backslash\{\zeta\})$, in virtue of (3.1) and (3.2). Since for any $\zeta \neq z_0 \in \partial G_t \cap \overline{\mathbb{B}}(\zeta, \frac{\eta_1}{2})$ there exists a neighborhood U_{z_0} of z_0 such that $\operatorname{Re} f_t(\cdot;\zeta) > 0$ on U_{z_0}, we conclude that there exists a neighborhood $U_{t,\zeta}$ of $\overline{G_t}\backslash\{\zeta\}$ such that the function

$$h_t(\cdot;\zeta) := \exp(-g_t(\cdot;\zeta)),$$

where $g_t(\cdot;\zeta) := \frac{1}{f_t(\cdot;\zeta)}$, is holomorphic on $H_{t,\zeta} := (\widetilde{G_t}\backslash\mathbb{B}(\zeta, \frac{\eta_1}{2})) \cup U_{t,\zeta}$. Note that h_t takes its values in $\mathbb{D}$.

There exists a $C_5 > 0$, independent of t, such that

$$|P_t(z;\zeta)| \le C_5\|z - \zeta\|, \quad \zeta, z \in U'.$$

Therefore, since for $0 < \eta_2 < \min\left\{\frac{\eta_1}{2}, \frac{1}{4C_4C_5}\right\}$, which now is independent of t, and for $z \in (\widetilde{G_t} \cap \mathbb{B}(\zeta, \eta_2))\backslash Z_t(\zeta)$ the following equality holds true:

$$g_t(z;\zeta) = \frac{P_t(z;\zeta)}{1 - P_t(z;\zeta)(v_t(z;\zeta) - C_4)},$$

we conclude that $g_t(\cdot;\zeta)$ is bounded near $Z_t(\zeta)$, which yields it extends to be holomorphic on $\widetilde{H_{t,\zeta}} := H_{t,\zeta} \cup (\mathbb{B}(\zeta, \eta_2) \cap \widetilde{G_t})$.

Now $\widetilde{H_{t,\zeta}}$ depends on ζ, but using the inclusion $\overline{G_t} \subset \widetilde{H_{t,\zeta}}$, we may find some $\widehat{G_t}$, strictly pseudoconvex domain which is independent on $\zeta \in \partial G_t$, such that $\overline{G_t} \subset \widehat{G_t} \subset \widetilde{H_{t,\zeta}}$ for each $\zeta \in \partial G_t$, and with the property that $h_t(\cdot;\zeta) \in \mathcal{O}(\widehat{G_t})$, $\zeta \in \partial G_t$ (use the joint continuity of φ_t with respect to z and ζ to shrink $\widetilde{H_{t,\zeta}}$ little bit to get some domain with desired properties, independent on ξ close to ζ, and finally apply the compactness of ∂G_t).

Let C_6, independent on t and $\zeta \in \partial G_t$, such that for $z \in \widehat{G_t}$ with $\|z - \zeta\| < \eta_2$ we have

$$|g_t(z;\zeta)| \le \frac{C_5\|z-\zeta\|}{1-2C_4C_5\|z-\zeta\|} \le C_6\|z-\zeta\|.$$

This implies

$$|1-h_t(z;\zeta)| \le C_7|g_t(z;\zeta)| \le C_6C_7\|z-\zeta\| =: d_1\|z-\zeta\| \tag{3.3}$$

for $z \in \widehat{G_t}$, $\|z-\zeta\| < \eta_2$, $\zeta \in \partial G_t$, if only C_7 is chosen so that

$$|e^\lambda - 1| \le C_7|\lambda|, \quad |\lambda| \le C_6\eta_2.$$

In particular, d_1 does not depend on t and we have $h_t(\zeta;\zeta) = 1$.

Furthermore, for $z \in \overline{G_t}$, $\|z-\zeta\| \ge \eta_1$ there is

$$\begin{aligned}\mathrm{Re} g_t(z;\zeta) &= \|z-\zeta\|^2 \frac{1+\|z-\zeta\|^2(C_4 - \mathrm{Re} v_t(z;\zeta))}{|1-\|z-\zeta\|^2(v_t(z;\zeta)-C_4)|^2} \\ &\ge \frac{\eta_1^2}{(1+2(\mathrm{diam} U)^2C_4)^2} =: C_8,\end{aligned}$$

which gives

$$|h_t(z;\zeta)| \le e^{-C_8} =: d_2 < 1.$$

Observe that d_2 is independent on t. □

Proof of continuity Fix $\alpha > 0$, $t_0 \in T$, $\zeta_0 \in \partial G_{t_0}$, and $z_0 \in \widehat{G_{t_0}}$. Let K_0 be a compact subset of $\widetilde{G_{t_0}}$, containing in its interior the set $\overline{G_{t_0}} \cup \{z_0\}$. In the sequel, we shall use the following convention: whenever we say that the triple (s, ξ, w) is near to (t_0, ζ_0, z_0), it will carry the additional information that $\xi \in \partial G_s$, $w \in \widehat{G_s}$, unless explicitly stated otherwise.

Observe that for (s, ξ) close to (t_0, ζ_0) (even without requiring that $\xi \in \partial G_s$), and any $z \in U'$ we have

$$|P_{t_0}(z;\zeta_0) - P_s(z;\xi)| < M_0\alpha$$

with some positive M_0. In particular, for w close to z_0 the following estimate is true

$$|P_{t_0}(z_0;\zeta_0) - P_s(w;\xi)| < M_1\alpha,$$

where $M_1 := M_0 + 1$.

Further, using the fact that all the functions φ_t are continuous as functions of both variables, we conclude that for (s, ξ) close to (t_0, ζ_0) we have

$$\|\varphi_{t_0}(\cdot;\zeta_0) - \varphi_s(\cdot;\xi)\|_{U'} < M_2\alpha$$

with some positive M_2.

For (s,ξ) near (t_0,ζ_0) we have

$$\left\| \frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(\cdot;\zeta_0) - \frac{\partial \varphi_s}{\partial \bar{z}_j}(\cdot;\xi) \right\|_{U'} < M_3\alpha$$

with some positive M_3. Furthermore, for (s,ξ) close to (t_0,ζ_0) and $z \in \widetilde{G_s} \cap \widetilde{G_{t_0}}$, the following estimates hold true:

(I) If $z \notin \mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cup \mathbb{B}(\xi, \frac{\eta_1}{2})$, then

$$|\alpha_{t_0,j}(z;\zeta_0) - \alpha_{s,j}(z;\xi)| < L\alpha,$$

where positive constant L does not depend on z as above. Indeed,

$$\begin{aligned}
&\left| \frac{\frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(z;\zeta_0)}{\varphi_{t_0}^2(z;\zeta_0)} - \frac{\frac{\partial \varphi_s}{\partial \bar{z}_j}(z;\xi)}{\varphi_s^2(z;\xi)} \right| = \left| \frac{\varphi_s^2(z;\xi)\frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(z;\zeta_0) - \varphi_{t_0}^2(z;\zeta_0)\frac{\partial \varphi_s}{\partial \bar{z}_j}(z;\xi)}{\varphi_{t_0}^2(z;\zeta_0)\varphi_s^2(z;\xi)} \right| \\
&\leq \frac{64}{C_2^2\eta_1^4} \left| \varphi_s^2(z;\xi)\frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(z;\zeta_0) - \varphi_{t_0}^2(z;\zeta_0)\frac{\partial \varphi_s}{\partial \bar{z}_j}(z;\xi) \right| \\
&\leq \frac{64}{C_2^2\eta_1^4} \left(\|\varphi_s^2\|_{U'} \left\| \frac{\partial \varphi_{t_0}}{\partial \bar{z}_j}(\cdot;\zeta_0) - \frac{\partial \varphi_s}{\partial \bar{z}_j}(\cdot;\xi) \right\|_{U'} + \left\| \frac{\partial \varphi_s}{\partial \bar{z}_j}(\cdot;\xi) \right\|_{U'} \left\| \varphi_s^2 - \varphi_{t_0}^2 \right\|_{U'} \right) \\
&\leq \frac{64}{C_2^2\eta_1^4}(L_1M_3\alpha + L_2M_2\alpha) =: L\alpha,
\end{aligned}$$

where the first inequality is the consequence of (3.2).

(II) If $z \in \mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cup \mathbb{B}(\xi, \frac{\eta_1}{2})$:
Observe that letting ξ close to ζ_0, we may make the balls arbitrarily close to each other. Using then the assumption (3), the fact that η were chosen to be strictly smaller than $C_2\frac{\eta_1^2}{8}$, and the strictness of uniform estimate (3.2), we see that for (s,ξ) close enough to (t_0,ζ_0) the estimate similar to the previous one holds true for $z \in S := \bigcup_{w:\|w-\zeta_0\|=\frac{\eta_1}{2}} \mathbb{B}(w,\gamma)$ with some sufficiently small $\gamma > 0$ (and is independent on such z). Additionally, (s,ξ) may be chosen so that $S' := (\mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cup \mathbb{B}(\xi, \frac{\eta_1}{2})) \setminus S \subset \mathbb{B}(\zeta_0, \frac{\eta_1}{2}) \cap \mathbb{B}(\xi, \frac{\eta_1}{2})$.

Noting that for $z \in S'$ and (s,ξ) as above $\alpha_{t_0,j}(z;\zeta_0) = \alpha_{s,j}(z;\xi) = 0$, we conclude that

$$\|\alpha_{t_0}(\cdot;\zeta_0) - \alpha_s(\cdot;\xi)\|_{\widetilde{G_{t_0}} \cap \widetilde{G_s}} \leq M_4\alpha$$

with some positive M_4.

Ofcourse $\overline{G_{t_0}} \subset \widetilde{G_{t_0}}$. This yields that for s close to t_0 we have $\overline{G_{t_0}} \subset \widetilde{G_s}$ as well as $\overline{G_s} \subset \widetilde{G_{t_0}}$ (the assumption (3) remains true for the family $(\widetilde{G_t})_{t\in T}$). For s close to t_0 we may now pick some $G_{t_0,s}$, a strictly pseudoconvex domain with smooth boundary and such that

$$\overline{G_s} \cup \overline{G_{t_0}} \subset K_0 \subset\subset G_{t_0,s} \subset\subset \widetilde{G_s} \cap \widetilde{G_{t_0}}.$$

Again thanks to the property (3), $G_{t_0,s}$ may be chosen independently of s if s is close enough to t_0. For such s, denote it by G^{t_0}. Then, using Lemma 2 from [7], we find some positive constant Γ such that

$$\|v_{t_0}(\cdot;\zeta_0) - v_s(\cdot;\xi)\|_{K_0} \le \Gamma \|\alpha_{t_0}(\cdot;\zeta_0) - \alpha_s(\cdot;\xi)\|_{G^{t_0}} \le \Gamma M_4 \alpha =: M_5\alpha.$$

Notice that Γ may be chosen independently of s. Consequently, for (s, ξ, w) close to (t_0, ζ_0, z_0) there is

$$|v_{t_0}(z_0;\zeta_0) - v_s(w;\xi)| \le |v_{t_0}(z_0;\zeta_0) - v_{t_0}(w;\zeta_0)| + |v_{t_0}(w;\zeta_0) - v_s(w;\xi)| \le M_6\alpha$$

for some positive M_6 (use the smoothness of $v_{t_0}(\cdot;\zeta_0)$).

There are two cases to be considered:

Case 1. $z_0 \in H_{t_0,\zeta_0} \cap \operatorname{int} K_0$. Then $\varphi_{t_0}(z_0;\zeta_0) \neq 0$ and for (s, ξ, w) near (t_0, ζ_0, z_0) we have $\varphi_s(w;\xi) \neq 0$. For such (s, ξ, w) we have

$$\begin{aligned} |f_{t_0}(z_0;\zeta_0) - f_s(w;\xi)| &\le \left| \frac{1}{\varphi_{t_0}(z_0;\zeta_0)} - \frac{1}{\varphi_s(w;\xi)} \right| + |v_{t_0}(z_0;\zeta_0) - v_s(w;\xi)| \\ &\le \left| \frac{\varphi_s(w;\xi) - \varphi_{t_0}(z_0;\zeta_0)}{\varphi_{t_0}(z_0;\zeta_0)\varphi_s(w;\xi)} \right| + M_6\alpha. \end{aligned}$$

Considering the last but one term, its denominator is bounded below by some positive constant for (s, ξ, w) close to (t_0, ζ_0, z_0), and the numerator is estimated from above by $(M_2 + 1)\alpha$. Thus for (s, ξ, w) close to (t_0, ζ_0, z_0)

$$|f_{t_0}(z_0;\zeta_0) - f_s(w;\xi)| \le M_7\alpha$$

for some positive M_7.

In our situation, the function $g_{t_0}(\cdot;\zeta_0)$ is holomorphic in a neighborhood of z_0 and so is $g_s(\cdot;\xi)$ for (s, ξ) close to (t_0, ζ_0). We conclude that for (s, ξ, w) close to (t_0, ζ_0, z_0) there is

$$|g_{t_0}(z_0;\zeta_0) - g_s(w;\xi)| \le M_8\alpha$$

for some positive M_8, and

$$|h_{t_0}(z_0;\zeta_0) - h_s(w;\xi)| = \left|\exp(-g_{t_0}(z_0;\zeta_0)) - \exp(-g_s(w;\xi))\right| \le M_9\alpha$$

for some positive M_9.

Case 2. $z_0 \in (\widetilde{G_{t_0}} \cap \mathbb{B}(\zeta_0, \eta_2)) \cap \operatorname{int} K_{t_0}$.

(I) Suppose $\varphi_{t_0}(z_0;\zeta_0) \neq 0$.
It is equivalent to $P_{t_0}(z_0;\zeta_0) \neq 0$. This yields that $P_s(w;\xi) \neq 0$ for (s,ξ,w) close to (t_0,ζ_0,z_0). Then

$$\begin{aligned}
&|g_{t_0}(z_0;\zeta_0) - g_s(w;\xi)| \\
&\quad = \left| \frac{P_{t_0}(z_0;\zeta_0)}{1 - P_{t_0}(z_0;\zeta_0)(v_{t_0}(z_0;\zeta_0) - C_4)} - \frac{P_s(w;\xi)}{1 - P_s(w;\xi)(v_s(w;\xi) - C_4)} \right| \\
&\quad \le N|P_{t_0}(z_0;\zeta_0) - P_s(w;\xi)| + N|P_{t_0}(z_0;\zeta_0)P_s(w;\xi)\|v_{t_0}(z_0;\zeta_0) - v_s(w;\xi)| \\
&\quad \le NM_1\alpha + N'M_6\alpha =: M_{10}\alpha,
\end{aligned}$$

and similarly as in the previous case

$$|h_{t_0}(z_0;\zeta_0) - h_s(w;\xi)| \le M_{11}\alpha$$

with some positive N, N', M_{10}, and M_{11}.

(II) Suppose $\varphi_{t_0}(z_0;\zeta_0) = 0$.
This is equivalent to $P_{t_0}(z_0;\zeta_0) = 0$. Then for some positive ρ we have $\mathbb{B}(z_0,\rho) \subset\subset K_0 \cap \mathbb{B}(\zeta_0,\eta_2)$. Similarly, for (s,ξ) close to (t_0,ζ_0) there is $\mathbb{B}(z_0,\rho) \subset\subset K_0 \cap \mathbb{B}(\xi,\eta_2)$. Therefore, because of the choice of d_1 in (3.3), for (s,ξ,w) close to (t_0,ζ_0,z_0), $w \in \mathbb{B}(z_0,\frac{\rho}{2})$ there is

$$\begin{aligned}
|h_{t_0}(w;\zeta_0) - h_s(w;\xi)| &\le |1 - h_{t_0}(w;\zeta_0)| + |1 - h_s(w;\xi)| \\
&\le d_1(\|w - \zeta_0\| + \|w - \xi\|) \le 2d_1\eta_2.
\end{aligned}$$

Consequently, since the functions $h_{t_0}(\cdot;\zeta_0) - h_s(\cdot;\xi)$ are holomorphic in suitable neighborhood of z_0 for (s,ξ) close to (t_0,ζ_0), for some positive $\widetilde{\rho} < \frac{\rho}{2}$, for every $x, y \in \mathbb{B}(z_0,\frac{\widetilde{\rho}}{2})$ we have

$$|h_{t_0}(x;\zeta_0) - h_s(x;\xi) - h_{t_0}(y;\zeta_0) + h_s(y;\xi)| \le \alpha. \tag{3.4}$$

Moreover, $\widetilde{\rho}$ may be chosen so that for $v, w \in \mathbb{B}(z_0,\frac{\widetilde{\rho}}{2})$ there is

$$|h_{t_0}(v;\zeta_0) - h_{t_0}(w;\zeta_0)| \le \alpha, \tag{3.5}$$

by continuity of $h_{t_0}(\cdot;\zeta_0)$. Fix some $w_0 \in \mathbb{B}(z_0,\frac{\widetilde{\rho}}{2})$ such that $P_{t_0}(w_0;\zeta_0) \neq 0$. Then for (s,ξ) near (t_0,ζ_0), by virtue of the subcase (I), we have

$$|h_{t_0}(w_0;\zeta_0) - h_s(w_0;\xi)| \le \alpha.$$

Finally, for $w \in \mathbb{B}(z_0,\frac{\widetilde{\rho}}{2})$ and (s,ξ) close to (t_0,ζ_0) we have

$$\begin{aligned}
|h_{t_0}(z_0;\zeta_0) - h_s(w;\xi)| &\le |h_{t_0}(z_0;\zeta_0) - h_{t_0}(w_0;\zeta_0)| + |h_{t_0}(w_0;\zeta_0) - h_s(w_0;\xi)| \\
&\quad + |h_s(w_0;\xi) - h_s(w;\xi)| \le \alpha + \alpha + 2\alpha = 4\alpha,
\end{aligned}$$

where the last estimate follows from (3.4) and (3.5), which leads us to the conclusion. □

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