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The significance of the Schott energy in the electrodynamics of charged particles and their fields

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Abstract The significance of the Schott four-momentum in the energy-momentum conservation account of a charged particle and its electromagnetic field is analyzed Periods with preacceleration and run away motion are discussed The existence of pre-radiation is demonstrated The Schott energy is identified as field energy localized just outside the particle Giving first a description with reference to an inertial frame, the analysis is then extended to a more general description valid also with reference to a uniformly accelerated frame It is shown that the Schott energy accounts for the energy radiated by a freely falling particle The non-invariance of electromagnetic radiation from a charged particle under transformations involving accelerated motion is discussed

Keywords Electrodynamics, accelerated charge, electromagnetic radiation Schott energy, equation of motion, energy-momentum conservation

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1. Introduction

The Schott energy-momentum is often neglected in treatises on electrodynamics However, it must be included in order to obtain a proper energy-momentum account for an accelerated charge and its electromagnetic field [1]

Consider for example a freely falling charge moving vertically along a geodesic world line In this case there is no radiation reaction Hence a neutron and a proton falling vertically besides each other will proceed to move together Yet the proton radiates, but not the neutron. Where does the radiated energy come from ? What is the difference between the energy-momentum accounts of these particles ?

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We shall here show how the Schott energy-momentum provides an answer to this question. It is also made clear that the Schott energy can be localised as a field energy which is found in the immediate neighbourhood of the particle.

2. Electrodynamics of an accelerated charge

The relativistic equation of motion of a particle with rest mass m_0 and charge Q is the Lorentz-Abraham-Dirac equation [2–4] (the LAD-equation) and may be written

$$F^{\mu}_{ext} + \Gamma^{\mu} = m_0 \dot{U}^{\mu}, \tag{2.1}$$

where

$$\Gamma^{\mu} \equiv \frac{2}{3} Q^2 \left(\dot{A}^{\mu} - A^{\alpha} A_{\alpha} U^{\mu} \right), \tag{2.2}$$

and the dot denotes differentiation with respect to the proper time of the charge. The vector Γ^{μ} is called the *Abraham four vector* and may be written

$$\Gamma^{\mu} = \gamma \left(\mathbf{v} . \boldsymbol{\Gamma}, \boldsymbol{\Gamma} \right), \tag{2.3}$$

where $\boldsymbol{\Gamma}$ is the three-dimensional force called the *field reaction force* [5], and v is the ordinary velocity of the particle. The Abraham four vector may be written

$$\Gamma^{\mu} = \frac{2}{3} Q^2 \gamma \left(\mathbf{v} . \dot{\mathbf{g}}, \dot{\mathbf{g}} \right). \tag{2.4}$$

Hence

$$\boldsymbol{\Gamma} = \frac{2}{3} Q^2 \dot{\mathbf{g}} . \tag{2.5}$$

According to the relativistic Larmor formula, valid with reference to inertial systems, the energy radiated by the particle per unit time is

$$\mathcal{R} = \frac{2}{3} Q^2 A^{\alpha} A_{\alpha} = \frac{2}{3} Q^2 g^2 \tag{2.6}$$

where $g = \left(A^{\alpha} A_{\alpha} \right)^{1/2}$ is the proper acceleration of the charge. The radiated momentum per unit proper time is

$$P^{\mu}_{R} = \mathcal{R} U^{\mu} . \tag{2.7}$$

From the equation of motion (2.1) we get the energy equation

$$\gamma \mathbf{v} \cdot \mathbf{F}_{ext} = m_0 \dot{U}^0 - \Gamma^0 = m_0 \gamma^4 \mathbf{v} \cdot \mathbf{a} - \gamma \mathbf{v} \cdot \boldsymbol{\Gamma} = \frac{dE_K}{dT} - \gamma \mathbf{v} \cdot \boldsymbol{\Gamma} , \tag{2.8}$$

where $E_k = (\gamma - 1)m_0$ is the kinetic energy of the particle. Note that the energy supplied by the external force is equal to the change of the kinetic energy of the charge when the Abraham four force vanishes [6]. Hence, it is tempting to conclude from the Abraham Lorentz theory, *i.e.* from eqs. (2.5) and (2.8), that a charge having constant acceleration does not radiate. This is, however, not the case. The power due to the field reaction force is

$$\cdot \Gamma = \frac{d}{dT}\left(\frac{2}{3}Q^2\gamma^4\mathbf{v}\cdot\mathbf{a}\right) - \mathcal{R} = -\frac{dE_S}{dT} \quad \frac{dE_R}{dT}, \tag{2.9}$$

where E_R is the energy of the radiation field, and we have defined the Schott energy,

$$-\frac{2}{3}Q^2\gamma^4\mathbf{v}\cdot\mathbf{a} = -\frac{2}{3}Q^2A^0. \tag{2.10}$$

Hence, in the case of constant acceleration, when the Abraham four-force vanishes, the charge radiates in accordance with Larmor's formula, eq. (2.6), and the rate of radiated energy is equal to minus the rate of change of the Schott energy. The energy equation may now be written

$$\frac{dW_{ext}}{dT} = \mathbf{v}\cdot\mathbf{F}_{ext} = \frac{d}{dT}(E_K + E_S + E_R), \tag{2.11}$$

where W_{ext} is the work on the particle due to the external force.

Let $\mathbf{P}_{ext}^{*}$ be the momentum delivered to the particle from the external force. Then $d\mathbf{P}_{ext}/dT = \mathbf{F}_{ext}$, and by means of eqs. (2.1), (2.2) and (2.7) we get,

$$\frac{d\mathbf{P}_{ext}}{dT} = \mathbf{F}_{ext} = m_0\frac{d\mathbf{U}}{dT} - \frac{2}{3}Q^2\left(\frac{d\mathbf{A}}{dT} - g^2\mathbf{v}\right) = \frac{d\mathbf{P}_M}{dT} \quad \frac{d\mathbf{P}_S}{dT} \quad \frac{d\mathbf{P}_R}{dT} \tag{2.12}$$

Thus, according to eqs. (2.11) and (2.12) the momentum of the particle takes the form

$$P^\mu = P_M^\mu + P_S^\mu, \tag{2.13}$$

where

$$P_M^\mu = m_0U^\mu, \quad P_S^\mu = -\frac{2}{3}Q^2A^\mu \tag{2.14}$$

are the mechanical momentum of the particle and the Schott-momentum, respectively. In addition we have the momentum of the radiation field, which is not a state function of the particle.

3. Localization of the Schott energy

The Schott energy was introduced in the electrodynamics of moving charges in 1915 as an "acceleration energy," $E_A = - E_S$, because, as stated by Schott, "it must be regarded as work stored in the electron in virtue of its acceleration" [7] This sounds somewhat mysterious, and does not give us any idea of what sort of energy the Schott energy is It vanishes for a charge instantaneously at rest, so one might wonder if it is some sort of acceleration dependent kinetic energy However since it vanishes for electrically neutral particles, this does not seem to be a natural interpretation

In order to search for a more natural identification of the Schott energy we have investigated the energy of the electromagnetic field of accelerated charges [8]

The field at a point of time T at an arbitrary point P in space originates from the particle at an earlier, retarded point T_Q The corresponding point P_Q on the world line of the particle is given by $R = c\left(T - T_Q\right)$ where R is the distance from P_Q to P Thus, the field which a charged particle creates at a certain point of time is later found on a spherical surface with center at the source point which is expanding with the velocity of light Such a spherical surface is called an eikonal

The field of a particle may be written in terms of the field tensor as $F^{\mu\nu} = F_I^{\mu\nu} + F_{II}$ where $F_I^{\mu\nu}$ is the generalized Coulomb field, and $F_{II}^{\mu\nu}$ is the radiation field The field component $F_I^{\mu\nu}$ decreases as $1/R^2$ and is independent of the particle's acceleration while the component $F_{II}^{\mu\nu}$ decreases as $1/R$ and is of first degree in the acceleration According to Teitelboim [9] the energy-momentum tensor may be written as the sum of three symmetrical terms, $T^{\mu\nu} = T_{I\,I}^{\mu\nu} + T_{I\,II}^{\mu\nu} + T_{II\,II}^{\mu\nu}$ Here $T_{I\,I}^{\mu\nu}$ is the energy-momentum tensor of the generalized Coulomb field, $T_{I\,II}^{\mu\nu}$ the energy-momentum tensor due to the interaction between $F_I^{\mu\nu}$ and $F_{II}^{\mu\nu}$, and $T_{II\,II}^{\mu\nu}$ is the energy-momentum tensor of the radiation field The tensors $T_I^{\mu\nu} = T_{I\,I}^{\mu\nu} + T_{I\,II}^{\mu\nu}$ and $T_{II}^{\mu\nu} = T_{II\,II}^{\mu\nu}$ are covariant divergence free outside the world line of the particle

As an illustration the energy density may be written

$$u = \frac{1}{8\pi}\left[\left(\boldsymbol{E}_I + E_{II}\right)^2 + \left(B_I + B_{II}\right)^2\right] = u_{I\,I} + u_{I\,II} + u_{II\,II} = u_I + u_{II} \tag{3 1}$$

where $u_I = u_{I\,I} + u_{I\,II}$ is the "bound energy density" and $u_{II} = u_{II\,II}$ is the radiation energy density,

$$u_{I\,I} = \frac{1}{8\pi}\left(\boldsymbol{E}_I^2 + \boldsymbol{B}_I^2\right),\ u_{I\,II} = \frac{1}{4\pi}\left(\boldsymbol{E}_I \cdot \boldsymbol{E}_{II} + \boldsymbol{B}_I \cdot \boldsymbol{B}_{II}\right),\ u_{II\,II} = \frac{1}{8\pi}\left(\boldsymbol{E}_{II}^2 + \boldsymbol{B}_{II}^2\right) \tag{3 2}$$

We have evaluated the bound four-momentum, $P_I^{\mu} = P_{I\,I}^{\mu} + P_{I\,II}^{\mu}$, in the total space outside the particle (which we consider to be a sphere of radius ε in its rest system, *i e* a

Lorentz contracted sphere formed like an ellipsoid in the laboratory system) and found a remarkable property of the four-momentum, namely that it is a state function of the particle. With the charge concentrated at the midpoint of the ellipsoid, the result of the integration is

$$P_I^{\mu} = m_0 U^{\mu} - \frac{2}{3} Q^2 A^{\mu} = P_M^{\mu} + P_S^{\mu} \tag{3.3}$$

where the rest mass m_0 is introduced by the usual procedure of renormalization. This is just eq. (2.13). Hence, we have obtained an interpretation of the Schott energy-momentum as contained in the electromagnetic field of the particle.

Rowe [10] has made a different partition of the bound energy-momentum tensor $T_I^{\mu\nu}$ into two tensors that are divergence free outside the world line of the particle, $T_I^{\mu\nu}$ and $T_2^{\mu\nu}$. Here $T_I^{\mu\nu}$ is the Coulomb energy-momentum tensor $T_{I,I}^{\mu\nu}$ amplified with an interaction term, and $T_2^{\mu\nu} = T_I^{\mu\nu} - T_1^{\mu\nu}$. Considering the field from a point particle Rowe notes that in spite of the fact that $T_1^{\mu\nu}$ and $T_2^{\mu\nu}$ are symmetical outside the particle, this is not the case at the particle. He then constructs corresponding tensors that are symmetrical also at the position of the particle by adding certain delta-function terms. They are called $T_{1new}^{\mu\nu}$, $T_{2new}^{\mu\nu}$.

We have shown by direct calculation that the integral of the energy over all of space of type 1-new is equal to the Coulomb energy, and of type 2-new the Schott energy. Furthermore, the energy-momentum of type 2-new (and type 2) in the space outside K vanishes [8]. The energy of type 2-new inside K was found to be equal to E_S with a contribution $2E_S$ localized at the particle represented by the delta function, and a contribution $-E_S$ between the charge and an eikonal of arbitrary size. The energy of type 1-new contains an energy $-2E_S$ at the particle. Thus, the bound energy, which is the sum of energies of type 1-new and type 2-new, contains no Schott energy at the particle, only in the space outside the particle and inside the eikonal K.

The Schott energy can now be localized in space. The surface K (which is spherical in the laboratory frame) may be chosen so small that it touches the charged particle (having a surface which is a Lorentz contracted sphere in the laboratory frame). This means that *the Schott energy is field energy localized close to the charge.* Assuming that the charged particle is spherical with a radius ε in its rest frame it is Lorentz contracted in the laboratory frame. The radius of the smallest eikonal K is then $T - T_{Q2} \approx \gamma(1+v)\varepsilon$. In the case of an accelerated charge increasing its velocity the Schott energy is a sort of energy reservoir. The Schott energy gets increasingly negative for an accelerated charge increasing its velocity. Together with the work performed by the external force, it accounts for the increase of kinetic energy of the charge and the energy it radiates.

4. Preradiation and Schott energy

The LAD-equation has two strange consequences [11–19]; pre-acceleration which is accelerated motion before a force acts, and run-away motion which is accelerated motion of a charge after a force which acted upon it, has ceased to act.

It has been claimed that during a period of pre-acceleration, before a charge is acted upon by an external force, the charge will not emit radiation [20]. In this section we will review a recent demonstration we have given where it was shown that a charge emits radiation during a period of pre-acceleration and that the radiation energy then comes from the Schott energy which decreases during this period [21].

We shall consider a particle with charge Q and rest mass m_0 moving in an inertial frame and acted upon by an external force of finite duration. The energy-momentum P^ν of the particle and its field is

$$P^\nu = P_M^\nu + P_S^\nu + P_R^\nu \tag{4.1}$$

where

$$P_R^\nu = \int^{\tau} \mathcal{R} U^\nu d\tau \tag{4.2}$$

is the 4-momentum of the radiation field.

The LAD-equation can now be written

$$F^\nu = \dot{P}^\nu = m_0 \dot{U}^\nu + \dot{P}_S^\nu + \dot{P}_R^\nu . \tag{4.3}$$

In the following we restrict ourselves to linear motion (along the x-axis). The equations will be simplified by introducing the rapidity of the particle,

$$\alpha = \operatorname{artanh} v \tag{4.4}$$

Hence,

$$v = \tanh\alpha, \quad \gamma = \cosh\alpha , \quad \gamma v = \sinh\alpha ,$$

$$a = \frac{dv}{dt} = \frac{d\tau}{dt}\frac{dv}{d\tau} = \frac{\dot{v}}{\cosh\alpha} = \frac{\dot{\alpha}}{\cosh^3\alpha} . \tag{4.5}$$

$$U^\nu = (\cosh\alpha, \sinh\alpha, 0, 0), \qquad A^\nu = \dot{U}^\nu = \left(\dot{\alpha}\sinh\alpha,\ \dot{\alpha}\cosh\alpha,\ 0, 0\right) \tag{4.6}$$

where $\dot{\alpha} = a_0$, *i.e.* $\dot{\alpha}$ is the acceleration in the inertial rest frame. The components of the mechanical-, Schott- and radiation four-momenta may then be expressed as

$$m_0 U^\nu = m_0\left(\cosh\alpha,\ \sinh\alpha\right) \tag{4.7a}$$

$$P_S^\nu = -\frac{2}{3}Q^2\dot{\alpha}\,(\sinh\alpha,\ \cosh\alpha) \tag{4.7b}$$

$$P_R^\nu = \frac{2}{3}Q^2 \int_{-\infty}^{\tau} \dot{\alpha}^2\,(\cosh\alpha,\ \sinh\alpha)\,d\tau\,. \tag{4.7c}$$

Differentiation gives

$$m_0\dot{U}^\nu = m_0\dot{\alpha}(\sinh\alpha,\ \cosh\alpha) \tag{4.8a}$$

$$\dot{P}_S^\nu = -\frac{2}{3}Q^2\ddot{\alpha}(\sinh\alpha,\ \cosh\alpha) - \frac{2}{3}Q^2\dot{\alpha}^2(\cosh\alpha,\ \sinh\alpha) \tag{4.8b}$$

$$\dot{P}_R^\nu = \frac{2}{3}Q^2\dot{\alpha}^2(\cosh\alpha,\ \sinh\alpha) \tag{4.8c}$$

with the sum

$$\dot{P}^\nu = m_0\dot{U}^\nu + \dot{P}_S^\nu + \dot{P}_R^\nu = \left(m_0\dot{\alpha} - \frac{2}{3}Q^2\ddot{\alpha}\right)(\sinh\alpha,\ \cosh\alpha)\,. \tag{4.9}$$

In terms of the rapidity the Minkowski force eq. (4.3) reads

$$F^\nu = (\gamma v F,\ \gamma F) = F(\sinh\alpha,\ \cosh\alpha)\,. \tag{4.10}$$

The LAD-equation, $F^\nu = \dot{P}^\nu$, for linear motion then takes the form

$$F = m_0\dot{\alpha} - \frac{2}{3}Q^2\ddot{\alpha} \tag{4.11}$$

or

$$F = m_0\,(\dot{\alpha} - \tau_0\ddot{\alpha}) \tag{4.12}$$

where

$$\tau_0 = \frac{2}{3}\frac{Q^2}{m_0} \tag{4.13}$$

is the time taken by a light signal to travel a distance equal to two thirds of the charged particle's classical radius. In the case of an electron $\tau_0 = 6,2.10^{-24}\,\mathrm{s}$. Note that eq. (4.12) transforms into the non-relativistic equation of motion when α is replaced by v and proper time by laboratory time [22]. Eq. (4.12) may be written

$$\frac{d}{d\tau}\left(e^{-\tau/\tau_0}\dot{\alpha}\right) = -\frac{F}{m_0\tau_0}e^{-\tau/\tau_0} \tag{4.14}$$

Let τ_1 and τ_2 be two points of proper time with $\tau_1 < \tau_2$ and F a function of τ such that $F(\tau) = 0$ for $\tau < \tau_2$ and $\tau > \tau_2$. Then the general solution of eq. (4.14) may be written

$$e^{-\tau/\tau_0}\dot{\alpha}(\tau) = \int_{\tau}^{\tau_2} \frac{F(\tau')}{m_0\tau_0} e^{-\tau/\tau_0} d\tau' + C_0 \tag{4.15}$$

where C_0 is a constant. For $\tau > \tau_2$ the integral is zero, and

$$\dot{\alpha}(\tau) = C_0 e^{\tau/\tau_0}, \quad i.e. \quad \alpha(\tau) = C_0\tau_0 e^{\tau/\tau_0} + const\,. \tag{4.16}$$

When $C_0 \neq 0$ this is a "run away" solution. The rapidity increases without any boundary, and the velocity approaches the velocity of light when $\tau \to \infty$. From a physical point of view the "run away" solutions are not acceptable, so we put $C_0 = 0$ which gives the following solution of eq. (4.14),

$$\dot{\alpha}(\tau) = e^{-\tau/\tau_0} \int_{\tau}^{\tau_2} \frac{F(\tau')}{m_0\tau_0} e^{-\tau'/\tau_0} d\tau'\,. \tag{4.17}$$

The integral has the same value for all $\tau < \tau_1$ and is equal to zero for $\tau > \tau_2$. For convenience we introduce the notation

$$f(\tau) \equiv \int_{\tau}^{\tau_2} \frac{F(\tau')}{m_0\tau_0} e^{-\tau'/\tau_0} d\tau'\,. \tag{4.18}$$

We put $\alpha(-\infty) = 0$ and get from eqs. (4.17) and (4.18),

$$\text{for} \quad \tau < \tau_1, \ \dot{\alpha} = e^{\tau/\tau_0} f(\tau_1), \quad \alpha = \tau_0 e^{\tau/\tau_0} f(\tau_1) \tag{4.19a}$$

$$\text{for } \tau_1 < \tau < \tau_2, \ \dot{\alpha} = e^{\tau/\tau_0} f(\tau), \quad \alpha = \tau_0 e^{\tau/\tau_0} f(\tau) + \frac{1}{m_0}\int_{\tau_1}^{\tau} F(\tau')\,d\tau' \tag{4.19b}$$

$$\text{for } \tau < \tau_2, \ \dot{\alpha} = 0, \quad \alpha = \alpha(\tau_2) = \frac{1}{m_0}\int_{\tau_1}^{\tau_2} F(\tau')\,d\tau'. \tag{4.19c}$$

Note that $\dot{\alpha} = 0$ for $\tau < \tau_1$ if $f(\tau_1) = 0$. That is : there is no pre-acceleration if $\int^{\tau_2} F(\tau') e^{-\tau'/\tau_0}\, d\tau' = 0$.

In order to discuss the energy and momentum of the particle and its field we consider the formulation (4.3) of the LAD-equation which is a conservation equation of energy and momentum in differential form. Let τ_a and τ be two points of proper time with $\tau < \tau_a$. Then according to eq. (4.3)

$$\int F^{\nu} d\tau = \Delta\left(m_0 U^{\nu}\right) + \Delta P_S^{\nu} + \Delta P_R^{\nu} \,. \tag{4.20}$$

For $\nu = 0$ the left hand side is the work done by the external force, and for $\nu = 1$ it is the delivered momentum. The Δ-symbols refer to the increments from τ_a to τ.

As seen from eq. (4.19a) the energies and momenta in the pre-acceleration period are given by eqs. (4.7) when we put $\alpha = \alpha/\tau_0$. The integrals in eq. (4.17c) are then solved by introducing $d\tau = \tau_0 d\alpha/\alpha$. From eq. (4.13) we have $(2/3)Q^2 = \tau_0 m_0$. We put $\tau_a = -\infty$ and $\tau < \tau_1$. Due to the initial condition $\alpha(-\infty) = 0$ we get

$$\Delta\left(m_0 U^{\nu}\right) = \left(E_{kin}(\tau), P(\tau)\right) = m_0\left(\cosh\alpha - 1,\ \sinh\alpha\right) \tag{4.21a}$$

$$\Delta P_S^{\nu} = P_S^{\nu}(\tau) = m_0\left(\alpha \sinh \alpha,\ -\alpha \cosh \alpha\right) \tag{4.21b}$$

$$\Delta P_R^{\nu} = P_R^{\nu}(\tau) = m_0\left(-\alpha \sinh \alpha,\ -\cosh \alpha + 1,\ \alpha\cosh\alpha - \sinh\alpha\right) \tag{4.21c}$$

where

$$\alpha = \tau_0 e^{\tau/\tau_0} f(\tau_1) \tag{4.22}$$

This leads to

$$\Delta\left(m_0 U^{\nu}\right) + \Delta P_S^{\nu} + \Delta P_R^{\nu} = 0 \,. \tag{4.23}$$

This equation says that the total increment of the energy and momentum of the system is zero, as it must be since the external force in the interval is zero.

A simple illustration of the above results is obtained by considering the special case where F is constant. We then put $g = F/m_0$, and the solution (4.5) takes the form

$$\tau < \tau_1, \quad \alpha = g\tau_0 e^{(\tau-\tau_1)/\tau_0}\left(1 - e^{-(\tau_2-\tau_1)/\tau_0}\right) \tag{4.24a}$$

$$\tau_1 < \tau < \tau_2, \quad \alpha = g\tau_0\left(1 - e^{(\tau-\tau_2)/\tau_0}\right) + g\left(\tau - \tau_1\right) \tag{4.24b}$$

$$\tau > \tau_2, \quad \alpha = g\left(\tau_2 - \tau_1\right) \tag{4.24c}$$

The rapidity α and its rate of change times τ_0 are shown graphically in Figure 1 The corresponding curves for the work performed by the external force, $W = \int_{\tau_1}^{\tau} F^0 d\tau$, the kinetic energy of the particle, the radiation energy and the Schott energy, as given in eqs. (26), are shown in Figure 2.

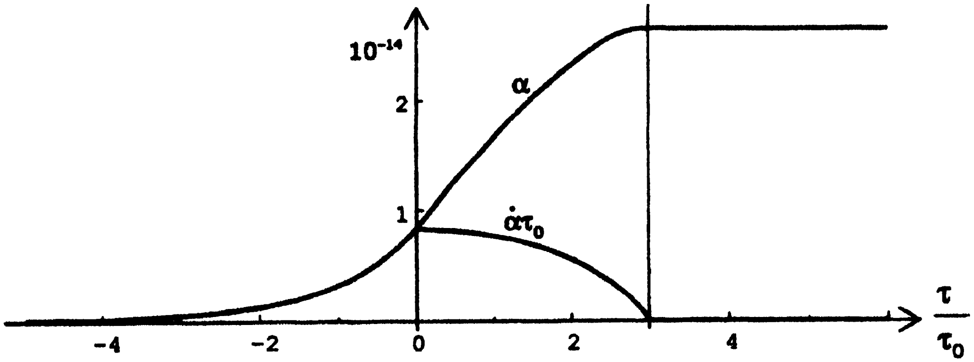

Figure 1. α is the rapidity of an electron and τ_0 is the time taken by a light signal to travel a distance equal to two thirds of the classical electron radius A constant force acts from the proper time $\tau_1 = 0$ to proper time $\tau_2 = 3\tau_0$ The electron, originally at rest, gets a motion (pre-acceleration) before the force acts

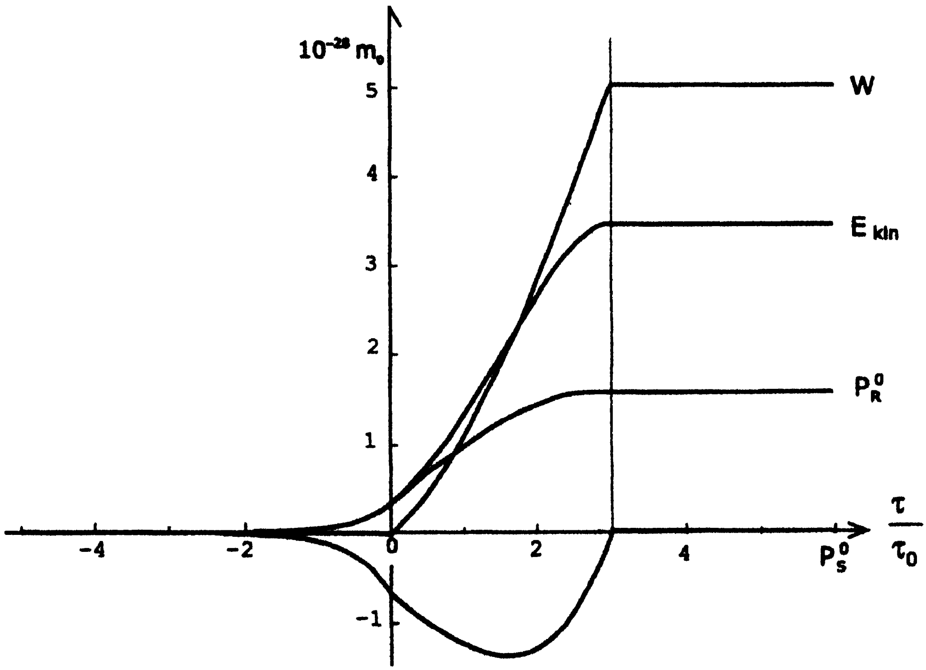

Figure 2. The situation is the same as in Figure 1 The graphs show the kinetic energy E_{kin}, the radiated energy P_R^0, the Schott energy P_S^0 and the external work W as functions of τ/τ_0 Note that $W = E_{kin} + P_R^0 + P_S^0$ In the pre-acceleration period $P_R^0 = E_{kin}$

In order to obtain some intuition about the quantities involved, we may refer to the figures, where we have put $\tau_2 - \tau_1 = 3\tau_0$, and the external force is due to the critical electrical field in air, E = 2, 4.10^6 V/m. Then for an electron, g = 4, 2.10^{17} m/s^2 and $g\tau_0 = 2,6.10^{-6}$ m/s.. In ordinary units, where c is not taken to be 1, the factor $g\tau_0$ in eq. (4.24a), say, should be replaced by $g\tau_0/c = 0,88.10^{-14}$. Hence, according to eq. (4.24a) the rapidity in the pre-acceleration period is of the order 10^{-14}, $\alpha(\tau_1) = 0,84.10^{-14}$, $v(\tau_1) = c \tanh \alpha(\tau_1) = 2,5.10^{-6}\mathrm{m/s}$. To lowest order in α (the next order is of the magnitude 10^{-42}) the expressions (36) for the changes of the kinetic energy and the Schott energy, and the emitted radiation energy in the pre-acceleration period reduce to

$$E_{kin} = m_0(\cosh\alpha - 1) \approx \frac{1}{2} m_0 \alpha^2 \tag{4.25a}$$

$$P_S^0 = -m_0 \alpha \sinh\alpha \approx -m_0 \alpha^2 \tag{4.25b}$$

$$P_R^0 = m_0(1 + \alpha \sinh\alpha - \cosh\alpha) \approx \frac{1}{2} m_0 \alpha^2 \tag{4.25c}$$

where α is given by eq. (4.24a). These expressions show that radiated energy is approximately equal to the increase of kinetic energy.

5. Run away motion and Schott momentum

Run-away acceleration seems to be in conflict with the conservation laws of energy and momentum. The momentum and the kinetic energy of the particle increase even when no force acts upon the article. The charge even puts out energy in the form of radiation. Where do the energy and the momentum come from ?

We shall here show that the source of energy and momentum in run-away motion is the so-called Schott energy and momentum. During motion of a charge in which the velocity increases, the Schott energy has an increasingly negative value and there is an increasing Schott momentum directed oppositely to the direction of the motion of the charge.

We shall consider a charged particle performing run away motion along the x-axis. Introducing the rapidity α of he particle its velocity and acceleration may be expressed as

$$v = \frac{dx}{dt} = \tanh\alpha,\ \gamma = \left(1 - v^2\right)^{-1/2} = \cosh\alpha,\ \gamma v = \sinh\alpha\ , \tag{5.1}$$

$$a = \frac{dv}{dt} = \frac{1}{\gamma}\frac{dv}{dt} = \frac{\dot{\alpha}}{\cosh^3\alpha},\quad \bullet = \frac{d}{d\tau}.$$

The Lorentz-Abraham-Dirac (LAD) equation then takes the form [23]

$$\alpha - \tau_0 \alpha = F/m_0 \,, \tag{5 2a}$$

where F is the external force, and $\tau_0 = 2Q^2/3m_0$ is the time taken by a light signal to travel a distance equal to two thirds of the charged particle's classical radius For an electron $\tau_0 = 6,\ 2\ 10^{-24} s$ Eq (5 2a) may be written

$$\frac{d}{d\tau}\left(e^{-\tau/\tau_0}\alpha\right) = -\frac{F}{m_0 \tau_0} e^{-\tau/\tau_0} \tag{5 2b}$$

For $F = 0$, *i e* for a free particle, the solutions of the LAD-equation are

$$1) \quad \alpha = 0, \ \ i\,e \quad \alpha = const\,,\ v = const \tag{5 3a}$$

which is consistent with Newton's 1 law

$$2) \quad \alpha = k e^{\tau/\tau_0}, \ k \neq 0, \ \ i\,e \ \ a \neq 0 \tag{5 3b}$$

This is the run away solution

As pointed out by Dirac [4] a particle in state 1) or 2) will remain in that state as long as no external force is acting We shall here consider a particle which is at rest, *i e* in state 1), until it is acted upon by a force $F(\tau)$ pointing in the positive x-direction, *i e* we consider a solution of the LAD-equation without pre-acceleration The force is acting from τ_1 to τ_2 For $\tau > \tau_2$ the particle is again free

According to the LAD-equation (5 2b) α is in the present case given by

$$\alpha(\tau) = -\frac{e^{\tau/\tau_0}}{m_0 \tau_0} \int_{-\iota}^{\tau} F(\tau')\, e^{-\tau'/\tau_0} d\tau' \tag{5 4}$$

The integral vanishes for $\tau > \tau_1$, which gives $\alpha = 0$ (and $\alpha = 0$) For $\tau > \tau_2$ the integral is independent of τ and we get the run away motion eq (5 3b) If the integral limit $-\infty$ in eq (44) is replaced by ∞, pre-acceleration is introduced, and run away motion disappears

In the following we examine eq (5 4) when the force F has constant value F_0 between τ_1 and τ_2, and is equal to zero outside this interval The solution of the equation of motion is then

$$\tau < \tau_1, \quad \alpha = 0, \ \ \alpha = 0\,, \tag{5 5a}$$

$$\tau_1 < \tau < \tau_2, \ \ \alpha = \frac{F_0}{m_0} - \frac{F_0}{m_0} e \qquad \alpha = \frac{F_0}{m_0}(\tau - \tau_1) - \frac{F_0 \tau_0}{m_0}\left(e^{\frac{\tau - \tau_1}{\tau_0}} - 1\right), \tag{5 5b}$$

$$\tau_2 < \tau, \ \dot{\alpha} = -\frac{F_0}{m_0}\left(e^{-\frac{\tau_1}{\tau_0}} - e^{-\frac{\tau_2}{\tau_0}} \right) e^{\frac{\tau}{\tau_0}}, \ \alpha = \frac{F_0}{m_0}(\tau_2 - \tau_1) - \frac{F_0\tau_0}{m_0}\left(e^{-\frac{\tau_1}{\tau_0}} - e^{-\frac{\tau_2}{\tau_0}} \right) e^{\frac{\tau}{\tau_0}} \qquad (5.5c)$$

Eq. (5.5b) shows a strange aspect of the motion. The quantity $\dot{\alpha}$ contains two terms. The first expresses the relativistic version of Newton's 2 law, *i.e.* $F_0 = d(\gamma m_0 v)/dt$. However, the second term represents a run away motion *oppositely directed* relative to the external force F_0, a highly unexpected mathematical result. According to eq. (5.5b) $\dot{\alpha}$ and α are oppositely directed relative to F_0 during the whole of the time interval $\tau_1 < \tau < \tau_2$.

At the point of time $\tau = \tau_2$,

$$\dot{\alpha}(\tau_2) = \frac{F_0}{m_0}\left(1 - e^{\frac{\tau_2 - \tau_1}{\tau_0}} \right), \qquad (5.6a)$$

$$\alpha(\tau_2) = \frac{F_0}{m_0}\left(\tau_2 - \tau_1 + \tau_0 - \tau_0 e^{\frac{\tau_2 - \tau_1}{\tau_0}} \right). \qquad (5.6b)$$

In order to simplify the expressions we let $\tau_2 - \tau_1 \to 0$ and $F_0 \to \infty$ keeping the product $(\tau_2 - \tau_1)/F_0 \equiv P$ constant. We then find the limits

$$\dot{\alpha}(\tau_2) = -P/m_0\tau_0, \quad i.e. \quad a = -P/m_0\tau_0, \qquad (5.7a)$$

$$\alpha(\tau_2) = 0, \quad i.e. \quad v = 0. \qquad (5.7b)$$

In this limit the external force is expressed by a δ-function

$$F(\tau) = \delta(\tau - \tau_1)\, P \qquad (5\ 8)$$

Putting $\tau_1 = 0$ we have the situation : For $\tau < 0$ the particle stays at rest. At $\tau = 0$ it is acted upon by the force

$$F = \delta(\tau)P \qquad (5.9)$$

giving the particle an acceleration oppositely directed relatively to the force and a vanishing initial velocity,

$$a(0) = a_0 = -P/m_0\tau_0, \quad v(0) = 0. \qquad (5.10)$$

According to eqs. (5.3) and (5.10) the motion is as follows

$$\tau < 0,\ \dot{\alpha} = 0,\ \alpha = 0, \tag{5.11a}$$

$$\tau > 0,\ \dot{\alpha} = a_0 e^{\frac{\tau}{\tau_0}},\ \ \alpha = \tau_0 a_0 e^{\frac{\tau}{\tau_0}} - \tau_0 a_0. \tag{5.11b}$$

The run away motion for $\tau > 0$ is accelerated, and the velocity $v = \tanh\alpha$ approaches the velocity of light as an unobtainable limit.

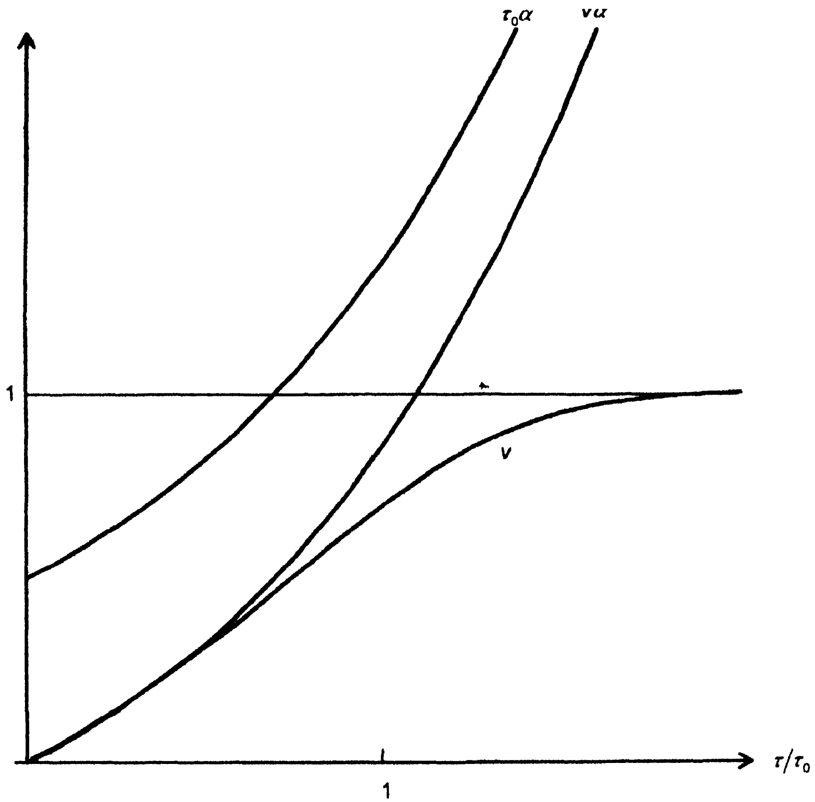

Figure 3. The proper acceleration, $\dot{\alpha}$, the velocity parameter, α, and the velocity, $v = \tanh\alpha$, as functions of the proper time for a particle performing run away motion, starting from rest with positive acceleration The quantity τ_0 is the time taken by a light signal to travel a distance equal to two thirds of the particle's classical radius.

The problem is to explain how this is possible for a particle not acted upon by any external force. It must be possible to demonstrate that the energy and momentum of the particle and its electromagnetic field is conserved, and find the force causing the acceleration. Of essential importance in this connection is the Schott energy and the Schott momentum.

Noting that $\dot{\alpha}$ is the acceleration in the instantaneous inertial rest frame of the particle and that $(2/3)Q^2 = m_0\tau_0$, we find the energies expressed by the rapidity utilizing, from eq (5.11b), that $\dot{\alpha} = a_0 + \alpha/\tau_0$. The kinetic energy of the particle is

$$E_{kin} = m_0(\gamma - 1) = m_0(\cosh\alpha - 1). \qquad (5.12)$$

The radiation energy is

$$E_R = \frac{2}{3}Q^2\int^{\tau}\dot{\alpha}^2\cosh\alpha\, d\tau = m_0(\alpha\sinh\alpha + a_0\tau_0\sinh\alpha - \cosh\alpha + 1). \qquad (5.13)$$

The Schott energy (also called acceleration energy) is

$$E_S = -\frac{2}{3}Q^2\gamma^4 va = -m_0\tau_0\dot{\alpha}\sinh\alpha = -m_0(\alpha + a_0\tau_0)\sinh\alpha. \qquad (5.14)$$

The sum of the energies is constant and equal to the initial value zero.

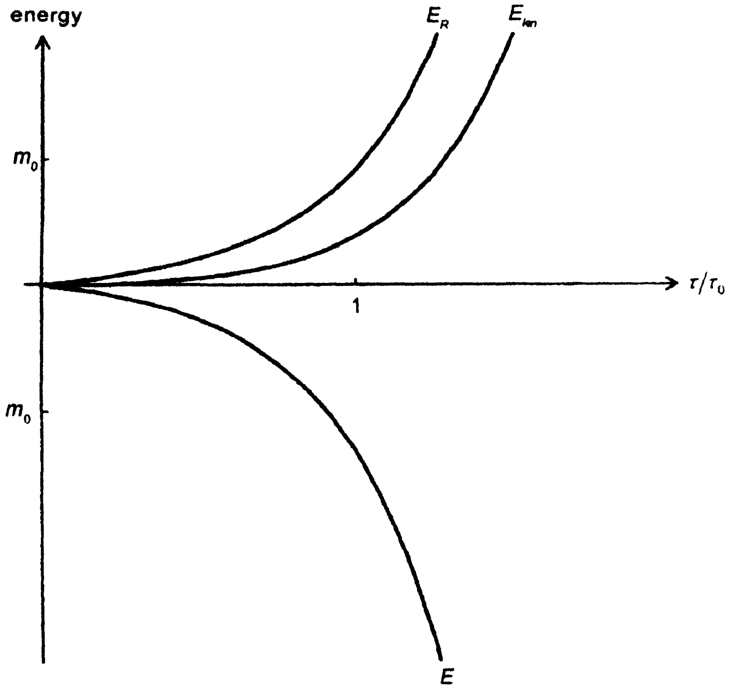

Figure 4. The energies of a particle and its electromagnetic field while the particle performs run away motion, as functions of τ/τ_0. Here E_{kin} is kinetic energy, E_R is radiated energy, and E_S is Schott (or acceleration) energy

Next we consider the momenta. The momentum of the particle is

$$P_{kin} = m_0\gamma v = m_0\sinh\alpha. \qquad (5.15)$$

The momentum of the radiation is

$$P_R = \frac{2}{3}Q^2\int^{\tau}\dot{\alpha}^2 \sinh\alpha d\tau = m_0\left(\alpha\cosh\alpha + a_0\tau_0\cosh\alpha - \sinh\alpha - a_0\tau_0\right). \quad (5.16)$$

The Schott momentum (acceleration momentum) is

$$P_S = -\frac{2}{3}Q^2\gamma^4 a = -\frac{2}{3}Q^2\dot{\alpha}\cosh\alpha = -m_0\left(\alpha + a_0\tau_0\right)\cosh\alpha. \quad (5.17)$$

The sum of the momenta is constant and is equal to $-m_0 a_0 \tau_0$, which is the initial Schott momentum.

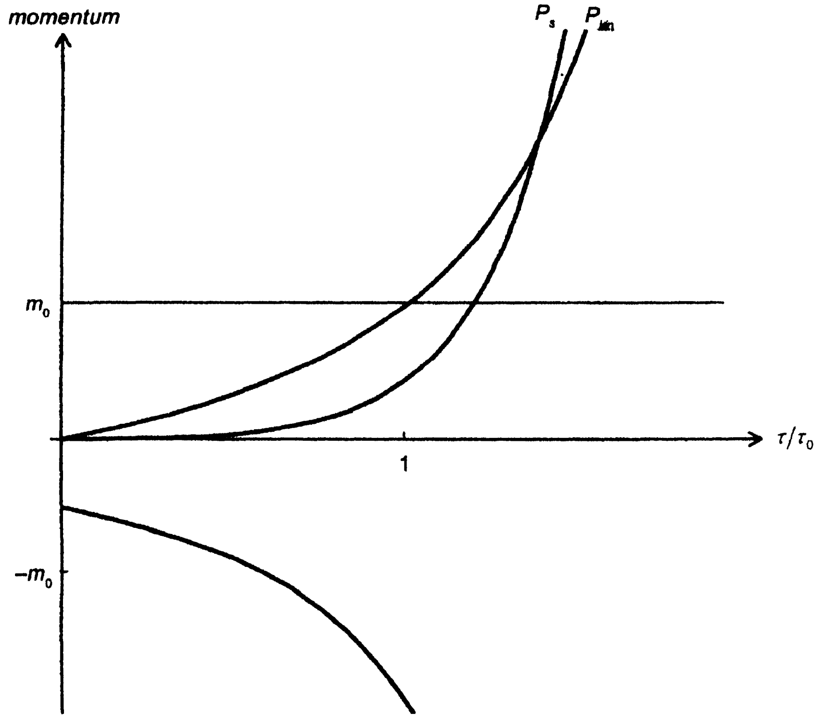

Figure 5. The momenta of a particle and its electromagnetic field while the particle performs run away motion, as functions of τ/τ_0. Here P_{kin} is kinetic momentum, P_R is radiated momentum, and P_S is Schott (or acceleration) momentum

The forces which are responsible for the increase in the momentum of the particle (internal forces) are the following (for rectilinear motion in general) : The radiation reaction force,

$$\Gamma_R = -\frac{dP_R}{dt} = -\frac{2}{3}Q^2\dot{\alpha}^2\tanh\alpha, \quad (5.18)$$

and the acceleration reaction force,

$$\Gamma_A = -\frac{dP_S}{dt} = -\frac{1}{\cosh\alpha}\dot{P}_S = \frac{2}{3}Q^2\left(\ddot{\alpha} + \dot{\alpha}^2 \tanh\alpha\right). \tag{5.19}$$

The total field reaction force (also called the self force) is

$$\Gamma = \Gamma_R + \Gamma_A = \frac{2}{3}Q^2\ddot{\alpha} \tag{5.20}$$

By means of eq. (5.11b) the forces are shown as functions of τ/τ_0 in Figure 6.

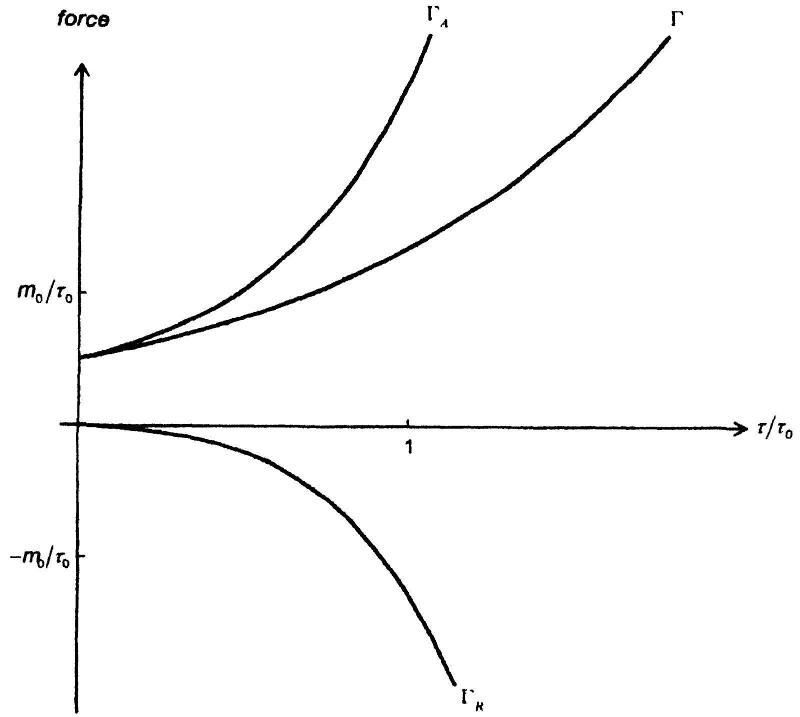

Figure 6. The forces due to the electromagnetic field of a particle acting on the particle while it performs run away motion, as functions of τ/τ_0. Here Γ_R is the radiation reaction force, Γ_A is the Schott (or acceleration) reaction force. Their sum is the field reaction force, $\Gamma = \Gamma_R + \Gamma_A$.

Eq. (5.18) shows that the radiation reaction force Γ_R is a force that retards the motion, acting like friction in a fluid. The "push" in the direction of the motion is provided by the acceleration reaction force, which is opposite to the change of Schott momentum per unit time. This force is opposite to the direction of the external force, *i.e.* it has the same direction as the run away motion.

There is a rather strange point here. We have earlier identified the Schott energy as a field energy localized close to the charge [8]. Yet, in the present case the Schott momentum is oppositely directed to the motion of the charge. This is due to the fact that the Schott energy is negative. Hence even if the Schott momentum has a direction opposite to that of the velocity of the charge, it represents a motion of negative energy in the same direction as that of the charge.

In general the Schott energy is

$$E_S = -\frac{2}{3} Q^2 A^0 \tag{5.21}$$

and the Schott momentum is

$$\boldsymbol{P}_S = -\frac{2}{3} Q^2 \boldsymbol{A} \tag{5.22}$$

where $(\mathring{A}, \boldsymbol{A})$ is the four-acceleration of the particle. From the relation $\mathring{A} = \boldsymbol{v}.\boldsymbol{A}$ we get $E_S = \boldsymbol{v} \, . \, \boldsymbol{P}_S$. It follows that for rectilinear motion $\boldsymbol{v}$ and $\boldsymbol{P}_S$ are oppositely directed when E_S is negative.

6. Non invariance of electromagnetic radiation

The nature of electromagnetic radiation is still a mystery. The wave-particle duality is something which seems to transcend our intuitive understanding. The waves of monochromatic light have infinite extension, but a photon is thought of as something having an exceedingly minute extension with a smallness only limited by the Heisenberg uncertainty relations.

Also thinking of electromagnetic radiation as a photon gas, and photons as a sort of objects which you can detect with your apparatus, it seems exceedingly strange to claim that you can make the object vanish just by changing your state of motion. On the other hand that claim does not sound so impossible if you think of electromagnetic radiation as waves. The waves are a state of oscillation of electric and magnetic fields moving through space with the velocity of light. Maybe they can be transformed away ?

That should indeed be possible. Think of a uniformly accelerated charge, radiating out an electromagnetic power. Transforming to the permanent rest frame of the change the magnetic field vanishes. In this frame the charge does not radiate.

Hence, saying that a charge radiates is not a reference independent statement. This conclusion has been arrived at in different ways by some authors. M Kretzschmar and W Fugmann have generalized Larmor's formula to a form which is valid not only in inertial reference frames, but also with respect to accelerated frames [24,25]. A consequence of their formula is that a charge will be observed to emit radiation only if it accelerates relative to the observer. Whether it moves along a geodesic curve is not decisive. A freely falling charge, *i.e.* a charge at rest in an inertial frame may be observed to radiate, and a charge acted upon by non-geodesic forces may be observed not to radiate.

Hirayama [26] has deduced a covariant version of Larmors's formula valid in uniformly accelerated reference frames. The significance of this formula in connection with energy momentum conservation of a charged particle and its electromagnetic field has been thoroughly analysed by Eriksen and Grøn [27].

In the Rindler frame the non vanishing Christoffel symbols are

$$\Gamma^{x}_{tt} = g^2 x, \quad \Gamma^{t}_{tx} = \Gamma^{t}_{xt} = 1/x \tag{6.1}$$

The 4-velocity and the 4-acceleration of a particle moving along the x-axis are

$$v^{\mu} = \frac{dx^{\mu}}{d\tau} = \gamma(1, v, 0, 0), \quad \gamma = \left(g^2 x^2 - v^2\right)^{-1/2} \tag{6.2}$$

$$a^{\mu} = \frac{dv^{\mu}}{d\tau} + \Gamma^{\mu}_{\alpha\beta} v^{\alpha} v^{\beta} = \gamma^4 \left(a + g^2 x - \frac{2v^2}{x} \right) \left(v, g^2 x, 0, 0 \right) \tag{6.3}$$

where $v = dx/dt$ and $a = dv/dt$. Transformation by means of eq. (3.5) gives the following 4-velocity and 4-acceleration in the laboratory system.

$$V^{\mu} = \left(gxv^{t}, v^{x}, 0, 0 \right), \quad A^{\mu} = \left(gxa^{t}, a^{x}, 0, 0 \right) \tag{6.4}$$

Inserting $v = a = 0$ we find the 4-velocity and 4-acceleration of the reference particles in the Rindler frame

$$u^{\mu} = \left(1/gx, 0, 0, 0 \right), \quad g^{\mu} = \left(0, 1/x, 0, 0 \right) \tag{6.5}$$

The proper acceleration $\hat{a}$ (with respect to an instantaneous inertial rest frame) is given by $\hat{a}^2 = a_{\mu} a^{\mu} = A_{\mu} A^{\mu}$, *i.e.*

$$a = \gamma^3 g x \left| \, a + g^2 x - \frac{2v^2}{x} \right. \tag{6.6}$$

For a particle instantaneously at rest at the point $x = x_1$ we get

$$\hat{a} = \frac{a}{g^2 x_1^2} + \frac{1}{x_1} \tag{6.7}$$

where $1/x_1$ is the proper acceleration of the point $x = x_1$. The difference $\hat{a} - 1/x_1$ will be denoted at $\tilde{a}$. We get

$$\tilde{a} = a/g^2 x_1^2 \tag{6.8}$$

which may be interpreted as the acceleration of the particle in the Rindler system, measured by a standard clock carried by the particle.

According to the analysis of Kretzschmar and Fugmann [24,25] the generalized Larmor formula as written out in a uniformly accelerated reference frame takes the form,

$$P = \frac{2}{3} Q^2 (g x_1)^2 \tilde{a}^2 \tag{6.9}$$

We shall now consider a freely falling charge in the Rindler frame. It is permanently at rest in the inertial co-moving frame. Obviously it does not radiate as observed in this frame. But according to eq. (6.9) it radiates as observed in the Rindler frame. In order to understand this from a field theoretic perspective in a similar way as way obtained with reference to an inertial frame in section 3, we may again consider the Teitelboim partition of the field into an a generalized Coulomb field I and a radiation field II. Calculating the flow of field energy of these types out of the Rindler section we arrived in ref. 25 at,

$$P_I = \frac{2}{3} Q^2 g (v - g x_1) \left[\gamma^2 g (v - g x_1) + 2a^x \right] \tag{6.10}$$

$$P_{II} = \frac{2}{3} Q^2 \left(a^x / \gamma \right)^2 \tag{6.11}$$

The emitted energy per emission time is

$$P = P_I + P_{II} = \frac{2}{3} \frac{Q^2}{\gamma^2} \left[a^x + \gamma^2 g (v - g x_1) \right]^2 \tag{6 12}$$

where

$$a^x = \gamma^4 \left(a + g^2 x_1 - 2v^2 / x_1 \right) g^2 x_1^2 \tag{6.13}$$

We now apply these formulae to the special case of a freely falling charge in the Rindler frame. Then the four acceleration vanishes, $a^x = 0$, which gives

$$P_I = \frac{2}{3} Q^2 g^2 \gamma^2 (g x_1 - v)^2, \quad P_{II} = 0 \tag{6.14}$$

In this case there is no emission of type II energy, only of type I.

This example shows the inadequacy of the Teitelboim separation with respect to non-inertial reference frames. In inertial frames radiation is associated with type II energy. However, as is seen from the present results, this is not the case in general. The separation in type I and II energy is based respectively on the vanishing and the non-vanishing of the four-acceleration of the charge, which means whether it is in free fall or not. The emission of radiation, on the other hand depends upon the relative acceleration between the charge and the observer Only in an inertial frame does the vanishing of the four-acceleration mean that the charge is not accelerated relative to the observer It should also be noted that since there is a flux of type-I energy out of the Rindler sector, the type-I energy is not a state function of the charge in the Rindler frame, as it is in an inertial frame

Following Hirayama [26] we shall now present, in a new way, a separation of the electromagnetic field energy in two types, $\tilde{P}_I$ and $\tilde{P}_{II}$, making use of a modified acceleration called α^1 We write $\alpha^x = a^x - \Delta^x$, where Δ^x is a quantity independent of α^x, which is determined from the condition that there shall be no energy of the new type I emitted out of the Rindler system, $\tilde{P}_I = 0$ Inserting $a^x = \alpha^x + \Delta^x$ into eq (6 11) and selecting the term of second order in α^x The result is

$$\tilde{P}_{II} = \frac{2}{3} Q^2 \left(\alpha^x/\gamma\right)^2 \tag{6 15}$$

Since the total transport of energy out of the sector is independent of the partition which is used, we have $\tilde{P}_I = P - \tilde{P}_{II}$ Hence, we get by means of eq (6 13),

$$\tilde{P}_I = \frac{2}{3}\frac{Q^2}{\gamma^2}\left(2\alpha^x + \Delta^x + \gamma^2 g v - \gamma^2 g^2 x_1\right)\left(\Delta^x + \gamma^2 g v - \gamma^2 g^2 x_1\right) \tag{6 16}$$

From the requirement that $\tilde{P}_I = 0$ for all values of α^x follows

$$\Delta^x = \gamma^2 g\left(g x_I - v\right) \tag{6 17}$$

giving

$$\alpha^x = a^x - \gamma^2 g\left(g x_I - v\right) \tag{6 18}$$

and

$$\tilde{P}_I = 0, \quad \tilde{P}_{II} = P \tag{6 19}$$

Here α^x is just the x-component of "the acceleration of the charge relative to the Rindler frame" found by Hirayama using Killing vectors

The covariant expression of the vector is

$$\alpha^\mu = a^\mu - \left(g_\alpha g^\alpha\right)^{1/2} u^\mu - g^\mu - \left(g_\alpha g^\alpha\right)^{1/2} v_\beta u^\beta v^\mu - v_\alpha g^\alpha v^\mu \tag{6 20}$$

Using eqs (6 2) and (6 5) we have in our case

$$\left(g_\alpha g^\alpha\right)^{1/2} = 1/x_I, \quad v_\beta u^\beta = -\gamma g x_I, \quad v_\alpha g^\alpha = \gamma v/x_I \tag{6 21}$$

and Hirayama's vector reads

$$\alpha^{\mu} = a^{\mu} - \frac{\gamma^2 (gx_l - v)}{gx_l^2}\left(v, g^2 x_l^2, 0, 0\right) \tag{6 22}$$

or, by means of eq (6 3),

$$\alpha^{\mu} = \gamma^4 \left| a - \frac{v^2}{} \right| + \frac{\gamma^2 v}{gx_1^2} \left(v, g^2 x_l^2, 0, 0\right) \tag{6 23}$$

It follows that $\left(\alpha^x/\gamma\right)^2 = g^2 x_l^2 \alpha_{\mu}\alpha^{\mu}$ which by means of eq (6 15) gives

$$P = \tilde{P}_{II} = \frac{2}{3} Q^2 g^2 x_l^2 \alpha_{\mu}\alpha^{\mu} \tag{6 24}$$

for the field energy produced per coordinate time which leaves the Rindler sector

It is easily seen that the Hirayama separation is a proper generalization of the Teitelboim separation to accelerated frames, which reduces to the latter in inertial frames To that end we describe the particle by the coordinate $\bar{x} = x_1 - 1/g$ Then the corrdinate time for $\bar{x} = 0$ is equal to the proper time Keeping $\bar{x}$ finite and letting $g \to 0$ we get the limits $x_1 \to \infty$, $gx_1 \to 1$, $ds^2 \to -dt^2 + dx^2$, $\gamma \to \left(1 - v^2\right)^{-1/2}$ From eq (6 20) we then find that $\alpha^{\mu} \to a^{\mu}$ and from eq (6 22) that $P \to (2/3)\, Q^2 a_{\mu} a^{\mu}$

Calculating the bound energy in the Rindler frame we found that the total energy of the charge and its field is [26]

$$\tilde{U} = \tilde{U}_I + \tilde{U}_{II} = -\frac{1}{2} Q^2 g + g^2 x_l^2 \gamma m_0 + \tilde{E}_S + \tilde{E}_R \tag{6 25}$$

The first term at the right hand side has no obvious physical interpretation The second is the mechanical energy of the particle The third term is the acceleration energy or the Schott energy, when the partition of the field is made according to the acceleration α^{μ},

$$\tilde{E}_S = -\frac{2}{3} Q^2 v \alpha^x \tag{6 26}$$

$\tilde{E}_S$ is analogous to the Schott energy

$$E_S = -\frac{2}{3} Q^2 V A^x \tag{6 27}$$

in an inertial system according to the Teitelboim partition. The fourth term is the radiation energy in the α^{μ}-partition,

$$\tilde{E}_R = \frac{2}{3} Q^2 \int g^2 x^2 \alpha_{\mu} \alpha^{\mu} dt \tag{6.28}$$

By differentiation upon the proper time of the particle, *i.e.* $d/d\tau = \gamma (d/dt_1)$, we find the formula

$$\frac{d}{d\tau}\left(\gamma g^2 x_1^2\right) = a^t g^2 x_1^2 = -a_t \tag{6.29}$$

by which the energy eq. (6.25) becomes

$$\frac{d\tilde{U}}{d\tau} = -m_0 a_t + \tilde{\Gamma}_0 \tag{6.30}$$

where

$$\tilde{\Gamma}_0 = \frac{d}{d\tau}\left(\tilde{E}_S + \tilde{E}_R\right) = \frac{2}{3} Q^2 \left(\frac{d\alpha_t}{d\tau} - v_t \alpha_v \alpha \right. \tag{6.31}$$

The quantity $\tilde{\Gamma}_0$ is interpreted as a component of the Abraham vector in the Rindler frame. We will compare it with the time component of the corresponding vector Γ^{μ} given by Hirayama [25]. From his eq. (3.24) we get for the motion in the x-direction

$$\Gamma_0 = \frac{2}{3} Q^2 \left\{ v^{\nu} \nabla_{\nu} \alpha_t - \frac{\alpha_t}{\gamma v x} v_t \alpha_v \alpha^{v} \right. \tag{6.32}$$

Inserting

$$v^{\nu} \nabla_{\nu} \alpha_t = \frac{d\alpha_t}{d\tau} - \Gamma^{\sigma}_{t\beta} v^{\beta} \alpha_{\sigma} = \frac{d\alpha_t}{d\tau} + \frac{\alpha_t}{\gamma v x} \tag{6.33}$$

we get

$$\Gamma_0 = \frac{2}{3} Q^2 \left\{ \frac{d\alpha_t}{d\tau} - v_t \alpha_v \alpha^{v} \right. \tag{6.34}$$

which is equal to $\tilde{\Gamma}_0$ as given by eq. (6.31). Thus for the Abraham vector in the Rindler frame we have

$$\Gamma_0 = \tilde{\Gamma}_0 = \frac{d}{d\tau}\left(\tilde{E}_S + \tilde{E}_R\right) \tag{6.35}$$

Note that when $\Gamma_0 = 0$, the radiation energy is supplied by the Schott energy. This is quite similar to the corresponding case in an inertial frame. From eq. (3.2) in Ref. [8] we then have $\Gamma_T = \frac{d}{d\tau}(E_S + E_R)$, where $E_S = (2/3)Q^2 A_T$ and $E_R = \frac{2}{3}Q^2 \int_{-\infty}^{T} A_\mu A^\mu dT$.

7. Conclusion

Run away motion, or alternatively, pre-acceleration, is a consequence of the Lorentz-Abraham-Dirac equation of motion of a charged, radiating particle. We have earlier considered the significance of the Schott energy-momentum in connection with energy-momentum conservation during a period with pre-acceleration [21]. In the present article we follow up with a corresponding analysis of run away motion.

The run away motion is an accelerated motion that is induced by a force, but proceeds after the forced has ceased to act upon the particle. In the limiting case that the charge is given a blow of infinitely short duration, the run away motion makes the charge accelerate in the opposite direction to that of the blow. During the run away motion it radiates positive energy and a net momentum in the same direction as the motion seemingly in conflict with energy-momentum conservation.

However the electromagnetic field of the charge contains an energy and a momentum called the Schott energy-momentum. The energy is increasingly negative during the run away motion, and the Schott momentum points in the opposite direction of the motion and has an increasing value. It represents a motion in the same direction as that of the charge, of a negative component of its electromagnetic field energy The sum of the kinematic-, radiation- and Schott energy-momentum is conserved during the run away motion.

From the Lorentz invariance of Maxwell's equations follows that the existence of electromagnetic radiation is Lorentz invariant. The quantum mechanical photon picture of radiation suggests that its existence is generally invariant. However, the equations of classical electrodynamics imply that this is not the case. The existence of radiation from a charged particle is not invariant against a transformation involving reference frames that accelerate or rotate relative to each other. Even if a charge accelerates as observed in an inertial frame, it does not radiate as observed from its permanent rest frame.

It has here been shown how this can be understood by making use of Hirayama's generalization of Teitelboim's separation of the electromagnetic field in a generalized Coulomb part and a radiation part. Also we have demonstrated that the energy of for example the vertically falling proton, mentioned in the introduction, comes from a reservoir of Schott energy, which becomes increasingly negative when an accelerated charge increases its velocity.

Acknowledgment

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