Maximum Area Axis-Aligned Square Packings

Hugo A. Akitaya
Tufts University, Medford, MA, USA
hugo.alves_akitaya@tufts.edu
 https://orcid.org/0000-0002-6827-2200

Matthew D. Jones
Tufts University, Medford, MA, USA
matthew.jones@tufts.edu

David Stalfa
Northeastern University, Boston, MA, USA
stalfa@ccis.neu.edu
 https://orcid.org/0000-0003-2101-8675

Csaba D. Tóth
California State University Northridge, Los Angeles, CA, USA
csaba.toth@csun.edu
 https://orcid.org/0000-0002-8769-3190

──── **Abstract** ────

Given a point set $S = \{s_1, \ldots, s_n\}$ in the unit square $U = [0,1]^2$, an *anchored square packing* is a set of n interior-disjoint empty squares in U such that s_i is a corner of the ith square. The *reach* $R(S)$ of S is the set of points that may be covered by such a packing, that is, the union of all empty squares anchored at points in S.

It is shown that $\text{area}(R(S)) \geq \frac{1}{2}$ for every finite set $S \subset U$, and this bound is the best possible. The region $R(S)$ can be computed in $O(n \log n)$ time. Finally, we prove that finding a maximum area anchored square packing is NP-complete. This is the first hardness proof for a geometric packing problem where the size of geometric objects in the packing is unrestricted.

1 Introduction

Let $S = \{s_1, \ldots, s_n\}$ be a set of n points in the unit square $U = [0,1]^2$. We say that a square q is *empty* if no point in S lies in the interior of q, and q is *anchored* at a point s if one of its four corners is s. An *anchored square packing for S* is a set $Q = \{q_1, \ldots, q_n\}$ of interior-disjoint axis-aligned empty squares that lie in U such that q_i is anchored at s_i for $i = 1, \ldots, n$. A *lower-left* anchored square packing is an anchored square packing in which s_i is the lower-left corner of q_i, for $i = 1, \ldots, n$ [2]. No polynomial-time algorithm is known for
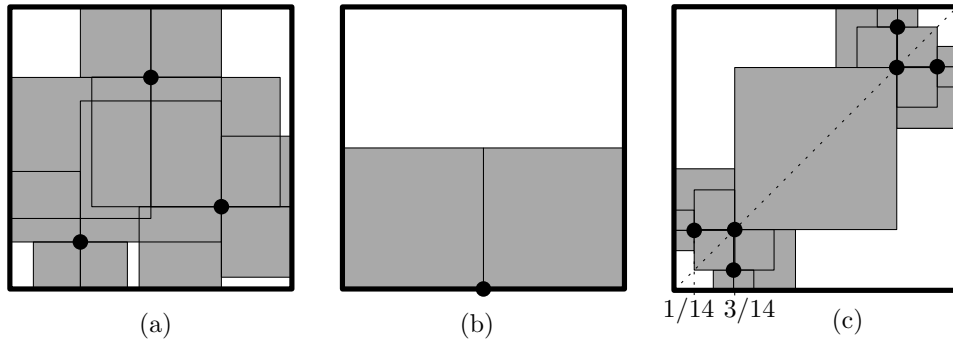
Figure 1 (a) The reach $R(S)$ for a set S of three points. (b) The area of $R(S)$ is $\frac{1}{2}$ for $S = \{(\frac{1}{2}, 0)\}$. (c) The reach $R(S)$ touches all four sides of U, and its area is $\frac{4}{7}$.

computing the maximum area of an anchored square packing for a given point set S; the problem admits a PTAS using a reduction to the maximum weight independent set problem (MWIS) [1]. The empty squares anchored at S do not always cover U entirely (Fig. 1(a)). For finding a maximum anchored square packing for S, it suffices to consider the subset of U that can be reached by anchored empty squares. Specifically, we define the *reach* of S, denoted $R(S)$, as the union of all axis-aligned empty squares contained in U and anchored at some point in S.

For computing the reach $R(S)$, we can take the union of all maximal empty squares anchored at the points in S, as follows. For $i = 1, \ldots, n$, let q_i^1 be the maximal axis-aligned empty square in U whose lower-left corner is s_i, and similarly define q_i^2, q_i^3, and q_i^4 where s_i is the upper-left, upper-right, and lower-right corner, respectively. We say that a point $s \in S$ *blocks* a square q_i^1 if s is incident to the top or right edge of q_i^1. Similarly, s blocks q_i^j if $j = 2$ (resp., 3, 4) and s is incident to the bottom or right edges of q_i^j (resp., bottom or left edges, or top or left edges of q_i^j). It is now clear that $R(S) = \bigcup_{i=1}^n \bigcup_{j=1}^4 q_i^j$.

Summary of Results.   We prove that for every finite set $S \subset U$, the area of $R(S)$ is at least $\frac{1}{2}$, and this bound is the best possible (Section 2). This settles in the affirmative a conjecture by Balas et al. [1]. We show how to compute $R(S)$ in $O(n \log n)$ time where $n = |S|$ (Section 3). We also show that finding the maximum area anchored square packing for a given point set S is NP-complete (Section 4). This is the first NP-hardness result for a geometric packing problem, where the size of the geometric objects in the packing is unrestricted. We conclude with related open problems (Section 5).

Motivation and Related Previous Work.   Geometric packing and covering problems have a long and revered history, going back to Kepler's problem about the densest packing of congruent balls in Euclidean space. In a classical packing problem, we are given a container region C, and a set O of geometric objects, and we wish to find a maximum subset $O' \subseteq O$ such that congruent copies (or translates) of the objects in O' fit in C without overlap.

Anchored variants, where each geometric object needs to contain a given point (*anchor*) initially emerged in VLSI design, where the anchors represent the endpoints of wires. Allen Freedman [15] conjectured that for every finite set $S \subset [0,1]^2$, which contains the origin (i.e., $\mathbf{0} \in S$), there is a lower-left anchored rectangle packing of area at least $\frac{1}{2}$. This lower bound would match an easy upper bound construction, where n points are equally distributed on the diagonal. The current best lower bound is 0.091 [6].

More recently, a broad family of anchored packing problems were proposed in the context of *map labelling*, where the anchors represent cities in a map, and axis-aligned rectangles represent labels [7, 8, 9, 10, 12, 13, 16]. Variants of the problem require the anchor to be at a corner, at a side, or anywhere in the rectangle, and the objective is to maximize the number of labels that can be packed in the map. Many of these problems are known to be NP-complete. However, in all previous reductions, the label boxes have a finite number of possible sizes [7, 12, 16] or bounded size [8].

In this paper, we consider the variant of Freedman's problem: We need to place an axis-aligned square at each anchor, and the sizes of the squares are not given in advance. Our objective is to maximize the total area of an anchored square packing. Balas et al. [1] showed that a greedy strategy finds an $\frac{5}{32}$-approximation, and a reduction to MWIS yields a PTAS that achieves an $(1 - \varepsilon)$-approximation in time $n^{O(1/\varepsilon)}$. It is known that the number of maximum-area square anchored packings may be exponential in n [2].

2   The Minimum Area of the Reach

In this section, we prove $\text{area}(R(S)) \geq \frac{1}{2}$ for every set S of n points in $U = [0, 1]^2$ (Theorem 12). Note that this bound is the best possible for all $n \in \mathbb{N}$. Indeed, if S is the one-element set $S = \{(\frac{1}{2}, 0)\}$, then $\text{area}(R(S)) = \frac{1}{2}$; see Fig. 1(b). By placing n points in an ε-neighborhood of $(\frac{1}{2}, 0)$ in U, we see that for every $\varepsilon > 0$ and every $n \in \mathbb{N}$, there exists a set S of n points in U such that $\text{area}(R(S)) < \frac{1}{2} + \varepsilon$. Note that in this construction all maximal anchored squares are disjoint from the top side of U. Under this constraint, the upper bound $\frac{1}{2}$ is always attained.

We call a point set S *trivial* if $R(S)$ is disjoint from one of the sides of U. The following lemma shows that $\text{area}(R(S)) \geq \frac{1}{2}$ for trivial instances.

▶ **Lemma 1.** *If $R(S)$ does not touch one of the sides of U, then $\text{area}(R(S)) \geq \frac{1}{2}$.*

Proof. Without loss of generality, $R(S)$ does not touch the top side of U. Let $s = (x, y)$ be a point in S with maximum y-coordinate. Consider the maximal empty squares whose lower-left and lower-right corners are at s. Since these squares do not touch the top side of U, and s has maximum y-coordinate, they touch the left and right side of U, respectively. Consequently, their combined area is $x^2 + (1-x)^2 \geq (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}$. Hence $\text{area}(R(S)) \geq \frac{1}{2}$, as claimed. ◀

▶ **Remark.** We do not know of any nontrivial point set S that attains the lower bound $\text{area}(R(S)) \geq \frac{1}{2}$. Our best lower bound construction for nontrivial instances yields $\frac{4}{7}$; see Fig. 1(c).

Outline.   In the remainder of Section 2, we consider nontrivial instances $S \subset U$. A *gap* is a connected component of $U \setminus R(S)$, i.e., of the complement of the reach. Section 2.1 presents basic properties of $R(S)$ and its gaps, Section 2.2 classifies the possible gaps into five types, and Section 2.3 presents a charging scheme in which we define for every gap C a region $R_C \subset R(S)$ such that $\text{area}(C) \leq \text{area}(R_C)$, and the regions R_C are pairwise interior-disjoint. Summation over all gaps yields $\text{area}(U \setminus R(S)) \leq \sum_C \text{area}(R_C) \leq \text{area}(R(S))$, consequently $\text{area}(R(S)) \geq \frac{1}{2}\text{area}(U) = \frac{1}{2}$.
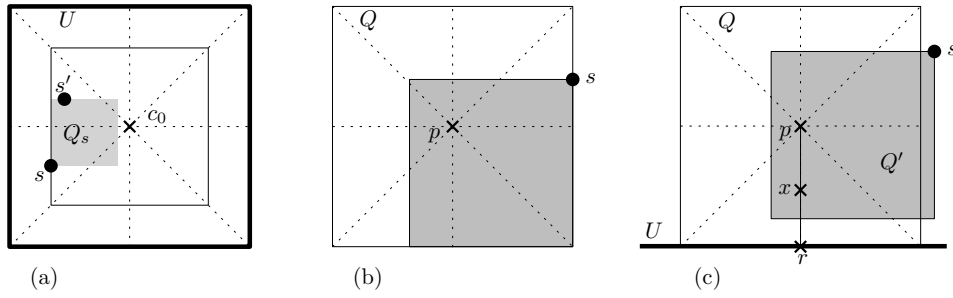
Figure 2 (a) A point $s \in S$ where the anchored square Q_s does not contain c_0. (b) If $s \in \partial Q$, $x(p) \leq x(s)$, and $y(p) \leq y(s)$, then the lower-left square anchored at s contains p. (c) ∂Q intersects the bottom side of U, and r lies below p.

2.1   Properties of the Reach and its Gaps

▶ **Lemma 2.** *For every finite set $S \subset U$, the reach $R(S)$ is connected.*

Proof. Let $S \subset U$ be a finite set, and let $c_0 = (\frac{1}{2}, \frac{1}{2})$ denote the center of U. We show that for each $s \in S$, there is an empty square Q_s anchored at s that contains c_0 **or** whose boundary contains an anchor $s' \in S$ such that $\|s' - c_0\|_\infty < \|s - c_0\|_\infty$ (i.e., s' is closer to c_0 in L_∞ norm than s). This implies that Q_s (hence $R(S)$) contains a line segment from s to c_0 or to s'. Consequently, $R(S)$ contains a polyline from every $s \in S$ to c_0. By the definition of $R(S)$, this further implies that $R(S)$ contains a polyline between any two points in $R(S)$.

It remains to prove the claim. Let $s \in S$. We may assume without loss of generality that $x(s) \leq y(s) \leq \frac{1}{2}$, hence $\|s - c_0\|_\infty = \frac{1}{2} - x(s)$. Let Q_s be the maximal empty square whose lower-left corner is s. Refer to Fig. 2(a). If $c_0 \in Q_s$, then our proof is complete. Otherwise, the side length of Q_s is $a_s < \frac{1}{2} - x(s)$, and there is a point s' in the right or the top side of Q_s. The anchor s' lies in the interior of the L_∞-ball of radius $\frac{1}{2} - x(s)$ centered at c_0, hence $\|s' - c_0\|_\infty < \|s - c_0\|_\infty$, as claimed.   ◀

▶ **Lemma 3.** *For every point $p \in U \setminus R(S)$, there exists a point $r \in \partial U$ such that the line segment pr is horizontal or vertical; and $pr \subset U \setminus R(S)$.*

Proof. Let $p \in U \setminus R(S)$, and let Q be the maximal empty axis-aligned square centered at p. Refer to Fig. 2(b). The boundary of this square, ∂Q, intersects S or ∂U, otherwise Q would not be maximal.

First assume that ∂Q contains a point $s \in S$. Without loss of generality, we may assume that $x(p) \leq x(s)$ and $y(p) \leq y(s)$. Since Q is empty, the maximal anchored square with upper-right corner at s contains p, hence $p \in R(S)$, contradicting our assumption that $p \notin R(S)$.

We can now assume that ∂Q intersects ∂U. Without loss of generality, $\partial Q \cap \partial U$ lies in the bottom side of both Q and U. Let $r \in \partial Q \cap \partial U$ be a point vertically below p (see Fig. 2(c) for an example). Suppose that segment pr intersects $R(S)$. Then some point $x \in pr$ lies in a square Q' anchored at a point $s \in S$. Since Q is empty, the anchor s lies outside of Q, and so the side length of Q' is at least half of that of Q, i.e., the side length of Q' is at least $|pr|$. However, then $y(s) \geq |pr|$, and the square Q' contains the segment px, contradicting our assumption that $p \notin R(S)$. Therefore there is no such point $x \in pr$, and $pr \subset U \setminus R(S)$, as claimed.   ◀
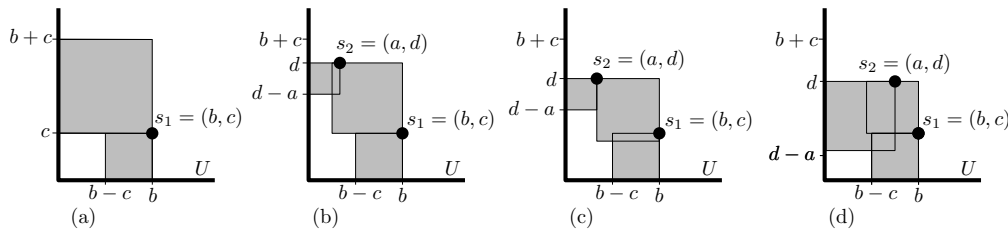
Figure 3 (a) A corner gap of type 1. (b–d) Corner gaps of type 2.

▶ **Corollary 4.** *The reach is simply connected.*

Proof. By Lemma 2, $R(S)$ is connected. Suppose that $R(S)$ is not simply connected. Then there is a gap $C \subset U \setminus R(S)$ such that $\partial C \subset R(S)$. Let $p \in \text{int}(C)$ be an arbitrary point in the interior of C. By Lemma 3, there is a point $r \in \partial U$ such that $pr \subset U \setminus R(S)$, which implies $r \in \partial C$, contradicting our assumption $\partial C \subset R(S)$. Therefore $R(S)$ is simply connected, as required. ◀

2.2 Classification of Gaps

In this section we classify the possible shapes of the gaps in $U \setminus R(S)$ for nontrivial instances. To simplify our analysis, we assume that $S \subset \text{int}(U)$ and no two points in S have the same x- or y-coordinates. This assumption is justified by the following lemma.

▶ **Lemma 5.** *If* $\text{area}(R(S)) \geq \frac{1}{2}$ *for every finite point set* $S \subset U$ *such that* $S \subset \text{int}(U)$ *and no two points in* S *have the same x- or y-coordinates, then* $\text{area}(R(S)) \geq \frac{1}{2}$ *for every finite point set* $S \subset U$.

Proof. Let $S \subset U$ be a finite point set that contains a point in ∂U or two points with the same x- or y-coordinate. Let ε_0 be minimum positive difference between x- and y-coordinates of points in S. For every $\varepsilon \in (0, \varepsilon_0/2)$, translate each point in S by a random vector of length at most ε into $\text{int}(U)$. The resulting point set S_ε lies in $\text{int}(U)$ and have distinct x- and y-coordinates with probability 1; the side length of each maximal anchored square may increase by at most 2ε, but could decrease substantially. Consequently, $\text{area}(R(S_\varepsilon)) \leq \text{area}(R(S)) + 4n\varepsilon$, hence $\lim_{\varepsilon \to 0} \text{area}(R(S_\varepsilon)) \leq \text{area}(R(S))$. ◀

We distinguish a *corner gap*, which is incident to a corner of U; and a *side gap*, which is adjacent to exactly one side of U. We show that every gap is bounded by ∂U and by squares anchored at up to three points in S. We define five types of gaps (two types of corner gaps and three types of side gaps). Each type is defined together with an empty rectangle $B \subset U$ and 1–3 anchors on the boundary of B. In each case, the gap is determined by the maximal empty squares that lie entirely in B and are anchored at points in $S \cap B$.

We describe each type modulo the symmetry group of U (i.e., the dyhedral group D_4). Specifically, we restrict ourselves to corner gaps incident to the lower-left corner of U, and side gaps adjacent to the bottom side of U. Reflection in the line $x = y$ (resp., $x = \frac{1}{2}$) maintains corner gaps incident to the origin (resp., side gaps along the bottom side of U); and we describe only one variant modulo reflection.

1. Let $0 < c < b < 1$. If $B = [0, b] \times [0, b+c]$ is empty and $s_1 = (b, c) \in S$, then the squares anchored at s_1 form a corner gap $[0, b-c] \times [0, c]$. See Fig. 3(a).
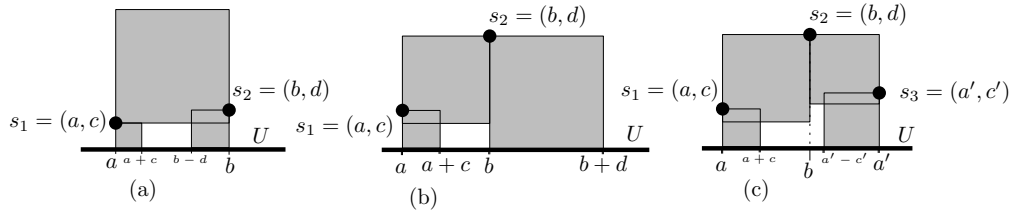
Figure 4 (a–c) Side gaps of type 3, 4, and 5, respectively.

2. Let $0 < a < b < 1$ and $0 < c < d < 1$ such that $c < b$ and $d < b + c$. If $B = [0, b] \times [0, d]$ is empty and $s_1 = (b, c), s_2 = (a, d) \in S$, then the squares anchored at s_1 and s_2 form a corner gap $[0, b - c] \times [0, \min(c, d - a, d - b + a)] \cup [0, \min(a, b - c, b - d + c)] \times [0, d - a]$. See Fig. 3(b–d).

3. Let $0 < a < b < 1$ and $0 < c, d < 1$ with $\max(c, d) < b - a$. If $B = [a, b] \times [0, \min(c, d) + (b - a)]$ is empty and $s_1 = (a, c), s_2 = (b, d) \in S$, then the squares anchored at s_1 and s_2 form a side gap $[a + c, b - d] \times [0, \min(c, d)]$. See Fig. 4(a).

4. Let $0 < a < b < 1$ and $0 < c < d < 1$ with $b - a < d$. If $B = [a, b + d] \times [0, d]$ is empty and $s_1 = (a, c), s_2 = (b, d) \in S$, then the squares anchored at s_1 and s_2 form a side gap $[a + c, b] \times [0, \min(c, d - b + a)]$. See Fig. 4(b).

5. Let $0 < a < b < a' < 1$ and $0 < c < c' < d < 1$ with $b - a < d$ and $a' - b < d$. If $B = [a, a'] \times [0, d]$ is empty and $s_1 = (a, c), s_2 = (b, d), s_3 = (a', c') \in S$, then the squares anchored at $s_1, s_2,$ and s_3 form a side gap $[a + c, \min(b, a' - d + c')] \times [0, \min(c, d - b + a)] \cup [\min(b, a' - d + c'), a' - c'] \times [0, \min(c', d - a' - b)]$. See Fig. 4(c) for an example.

▶ **Lemma 6.** *Every gap C of type 1–5 is disjoint from all empty squares that are anchored at points in S and lie in the exterior of the defining box B of C. Consequently, C is bounded by ∂U and some empty squares anchored at points in $S \cap B$.*

Proof. In each of the five cases, $\partial B \cap \mathrm{int}(U)$ is covered by empty squares anchored at the points in S that define C. More precisely, each point in $\partial B \cap \mathrm{int}(U)$ lies in an empty square anchored at a point in $S \cap \partial B$ blocked by some point in $S \cap \partial B$ or $\partial U \cap \partial B$. For $s_i \in S$ lying in the exterior of B, let Q_i be a square anchored at s_i. If Q_i intersects B, then its interior intersects $\partial B \cap \mathrm{int}(U)$, hence it intersects a square Q_j anchored at some $s_j \in \partial B$ and blocked by some point $p_j \in S \cap \partial B$ or $\partial U \cap \partial B$. Since $\mathrm{int}(Q_i)$ contains neither s_j nor p_j, we have $Q_i \cap B \subset Q_j$, and so Q_i is disjoint from the gap C, as claimed. ◀

We prove the following classification result for the gaps in the full paper.

▶ **Lemma 7.** *Every gap of a nontrivial instance is of one of the five types defined above.*

It is now easy to check that the following properties hold for all five types of gaps.

▶ **Corollary 8.**
  (i) *Each gap is either a rectangle incident to a side of U, or the union of two rectangles incident to the same side of U (which we call an L-shaped gap).*
  (ii) *Every edge uv of a gap is contained in either ∂U or a maximal anchored rectangle. Consequently, the square built on the side uv outside of the gap lies either outside of U or in $R(S)$.*

We say that a point $p \in \partial U$ is a *lead* if it is a vertex of a maximal anchored square q_i^j, and ps_i is a diagonal of q_i^j. We observe that one or two vertices of a gap along ∂U is a lead.

Figure 5 An L-shaped side gap C is subdivided into two rectangles $C_1, C_2 \in \mathcal{C}^*$.

▶ **Corollary 9.**

▬ *If C is a rectangular gap, then at least one endpoint of $C \cap \partial U$ is a lead,*

▬ *otherwise both endpoints of $C \cap \partial U$ are leads.*

2.3 Charging Scheme

For every gap C, we define a region $R_C \subset R(S)$; and then we show that area$(R_C) \geq$ area(C) and the regions R_C are pairwise interior-disjoint.

For ease of exposition, we subdivide every L-shaped side gap C into two rectangles $C = C_1 \cup C_2$, then define interior-disjoint regions R_{C_1} and R_{C_2}, and let $R_C := R_{C_1} \cup R_{C_2}$. Specifically, let $\mathcal{C}^*$ be a set of regions that contains: (1) all corner gaps, (2) all rectangular side gaps, and (3) for each L-shaped side gap C, the two interior-disjoint rectangles C_1 and C_2, such that $C = C_1 \cup C_2$ and both C_1 and C_2 have a common side with ∂U (see Fig. 5 for an example). By Corollary 9, at least one vertex of every rectangle in $\mathcal{C}^*$ is a lead, and two vertices of every L-shaped corner gap in $\mathcal{C}^*$ are leads.

We are now ready to define a region R_C for each region $C \in \mathcal{C}^*$.

▬ Let $C = (a, b, c, d)$ be a rectangle in $\mathcal{C}^*$. Assume w.l.o.g. that bc is contained in the bottom side of U, and c is a lead (a symmetric construction applies if bc is contained in another side of U or b is the only lead). Refer to Fig. 6(a). Let ℓ_1 and ℓ_2 be lines of slope 1 passing through a and c, respectively. Let p_1 be the intersection of ℓ_1 with the vertical line through cd, and let p_2 be the intersection of ℓ_2 with the horizontal line through da. Let z_1 (resp., z_2) be the intersection point of ℓ_2 (resp., ℓ_1) with the line of slope -1 passing through p_1 (resp., p_2). Then R_C is the smaller pentagon out of (a, d, c, z_1, p_1) and (a, d, c, p_2, z_2).

▬ Let $C = (a, b, c, d, e, f)$ be a L-shaped corner gap in $\mathcal{C}^*$. Assume w.l.o.g. that b is the lower-left corner of U. By Corollary 9, both a and c are leads. Refer to Fig. 6(b). Let ℓ_1 and ℓ_2 be lines of slope 1 passing through a and c, respectively. Let $p_1, p_2 \in S$ be the anchors on ℓ_1 and ℓ_2, respectively (which exist since both a and c are leads). Let z_1 (resp., z_2) be the intersection point of line ℓ_2 (resp., ℓ_1) with the line of slope -1 passing through p_1 (resp., p_2). Then R_C is the smaller heptagon out of $(a, f, e, d, c, z_1, p_1)$ and $(a, f, e, d, c, p_2, z_2)$.

▶ **Lemma 10.** *For every $C \in \mathcal{C}^*$, we have:*

(P1) $R_C \subseteq R(S)$,

(P2) int(R_C) *does not contain any anchors, and*

(P3) area$(C) \leq$ area(R_C).

Proof. The region $C \in \mathcal{C}^*$ is either a gap or a rectangle within an L-shaped side gap; Fig. 6(b). Let C^* be the gap that contains C, and B the box defining the gap C^*. For all five types of gaps, int(B) does not contain any anchor, and $B \setminus C^* \subset R(S)$. By Corollaries 8(ii) and 9, the points p_1 and p_2 lie in B. Consequently, $R_C \subset B$, hence $R_C \subset B \setminus C^*$. This confirms (P1) and (P2).

To prove (P3), we distinguish two cases. First assume that C is an $x \times y$ rectangle. Let T be an isosceles right triangle whose hypotenuse has length $x + y$. It is easy to check that area$(C) \leq$ area(T). Indeed, area$(T) = \left(\frac{1}{2}(x + y)\right)^2 = \frac{1}{2}\left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right) \geq xy =$ area(C).
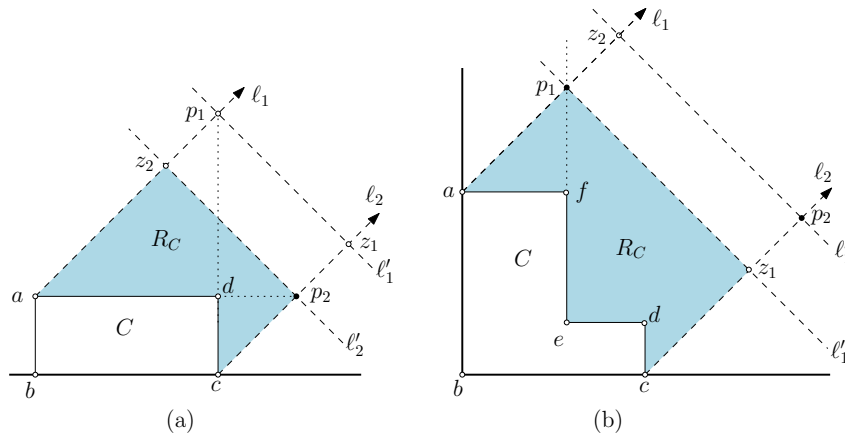
Figure 6 Region R_C. (a) $C \in \mathcal{C}^*$ is a rectangle. (b) $C \in \mathcal{C}^*$ is an L-shaped corner gap.

By definition, R_C contains a triangle congruent to T, consequently $\mathrm{area}(C) \leq \mathrm{area}(T) \leq \mathrm{area}(R_C)$, as claimed.

Next assume that $C \in \mathcal{C}^*$ is an L-shaped corner gap; Fig. 6(b). Assume that C is formed by three interior-disjoint axis-aligned rectangles defined by diagonals ae, be, and ce. Let their dimensions respectively be $x \times y$, $x \times z$, and $w \times z$. Let T_1 and T_2 be isosceles right triangles whose hypotenuses are of length $x + y$ and $w + z$, respectively. Let T_3 and T_4 be isosceles right triangles whose legs are of length x and z, respectively. By definition, R_C contains interior-disjoint triangles congruent to T_1, T_2, T_3, and T_4: the hypotenuses of the respective triangles are in the same supporting lines as ef, ed, ap_1, and cp_2 respectively. Using the same argument as in the previous case, we can show that $\mathrm{area}(T_1)$ and $\mathrm{area}(T_2)$ are, respectively, greater or equal than the areas of the $x \times y$ and $w \times z$ rectangles. It remains to show that $\mathrm{area}(T_3) + \mathrm{area}(T_4)$ is greater or equal than the area of the $x \times z$ rectangle. By definition, we have $\mathrm{area}(T_3) + \mathrm{area}(T_4) = \left(\frac{x^2}{2} + \frac{z^2}{2} \right) \geq xz$ for all $x, z > 0$. ◀

We prove that the regions R_C, $C \in \mathcal{C}^*$, are pairwise interior-disjoint in the full paper.

▶ **Lemma 11.** *For every two regions* $C, C' \in \mathcal{C}^*$, $C \neq C'$, *we have* $\mathrm{int}(R_C) \cap \mathrm{int}(R_{C'}) = \emptyset$.

▶ **Theorem 12.** *For every finite set* $S \subset U$, *we have* $\mathrm{area}(R(S)) \geq \frac{1}{2}$, *and this bound is the best possible.*

Proof. By Lemma 5, it suffices to prove the lower bound when $S \subset \mathrm{int}(U)$ and no two points in S have the same x- or y-coordinate. For all gaps $C \subset U \setminus R(S))$, we have defined interior-disjoint regions $R_C \subset R(S)$ such that $\mathrm{area}(C) \leq \mathrm{area}(R_C)$. Consequently, $\sum_C \mathrm{area}(C) \leq \sum_C \mathrm{area}(R_C) \leq \mathrm{area}(R(S))$, which immediately yields $\mathrm{area}(R(S)) = 1 - \sum_C \mathrm{area}(C) \geq 1 - \mathrm{area}(R(S))$, and $\mathrm{area}(R(S)) \geq 1/2$, as claimed. This bound is the best possible: the point set $S = \{(\frac{1}{2}, 0)\}$ attains $\mathrm{area}(R(S)) = \frac{1}{2}$. ◀

3 Algorithm for Computing the Reach

In this section, we show how to compute efficiently the reach of a given point set.

▶ **Theorem 13.** *For a set* $S \subset U$ *of* n *points,* $R(S)$ *can be computed in* $O(n \log n)$ *time.*
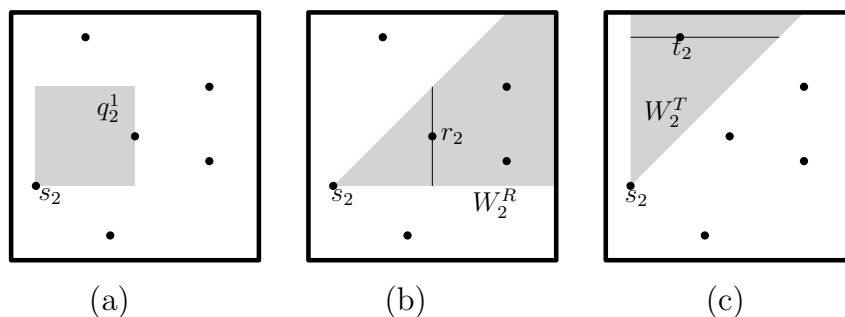
Figure 7 A set S of 6 anchors in $U = [0,1]^2$. (a) The maximal empty anchored square q_2^1; (b) wedge W_2^R; and (c) wedge W_2^T.

Recall that, for a set $S = \{s_1, \ldots, s_n\}$, the reach is defined as a union of $4n$ squares, $R(S) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{4} q_i^j$, where q_i^1 is the maximal axis-aligned empty square in U whose lower-left corner is s_i, and q_i^2, q_i^3, and q_i^4 are defined similarly where s_i is the upper-left, upper-right, and lower-right corner, respectively. Since any two squares cross in at most two points, the $4n$ squares q_i^j ($i = 1, \ldots, n$ and $j = 1, \ldots, 4$) form a pseudo-circle arrangement. It is well known that the union of $O(n)$ pseudo-circles has $O(n)$ vertices [11]. The union of $4n$ axis-aligned *squares* can be computed by a sweep-line algorithm in $O(n \log n)$ time [3].

We note that Bentley's sweep-line algorithm can compute the *area* of the union of n axis-aligned *rectangles* in $O(n \log n)$ time (without computing the union itself, which may have $\Theta(n^2)$ complexity). Computing the volume of the union of axis-aligned hyper-rectangles in $\mathbb{R}^d$ is known as *Klee's measure problem*, and the current best algorithms [4] for $d \geq 3$ run in $O(n^{d/2})$ time in general, and in $O(n^{(d+1)/3} \, \mathrm{polylog}(n))$ time for hypercubes (see also [17]).

It remains to compute the $4n$ anchored maximal empty squares q_i^j. We focus on the n *lower-left* anchored squares q_i^1 ($i = 1, \ldots, n$), the other three types can be computed analogously. For every $i = 1, \ldots, n$, the lower-left corner of q_i^1 is s_i, and its left *or* top side contains another anchor or a point in ∂U; we say that this point is the *blocker* of q_i^1. For each i, we find a first point that may block the square q_i^1 on the left and on the top side, independently. The blocker of q_i^1 is the points closest to s_i in L_∞ norm. We continue with the details. We define two wedges with apex at the origin: Let $W^L = \{(x,y) \in \mathbb{R}^2 : 0 < y < x\}$ and $W^T = \{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$; see Fig. 7(b–c). The Minkowski sums $W_i^L := s_i + W_L$ and $W_i^T := s_i + W^T$ are the translates of these wedges with apex at s_i. Let r_i be a point of minimum x-coordinate in $W_i^L \cap (S \cup \partial U)$; and let t_i be a point of minimum y-coordinate in $W_i^T \cap (S \cup \partial U)$. Then the blocker of q_i^1 is either r_i or t_i, whichever is closer to s_i in L_∞ norm.

For every $i = 1, \ldots, n$, we find points r_i and t_i, independently. Consider the points $r_i \in W_i^R$, for $i = 1, \ldots, n$ (the case of the points $t_i \in W_t^T$ is analogous). We use a data structure originally developed for computing Θ-graphs in the context of geometric spanners by Narasimhan and Smid [14, Section 4.1.2]. They developed the following dynamic data structure for n points in the plane:

▶ **Lemma 14** ([14], Lemma 4.1.9). *Let H be a nonvertical line through the origin. There is a data structure that maintains a set P of n points in the plane and supports the following queries: (i)* MinBelow(p): *Given a query point $p \in P$, compute a point with the minimum x-coordinate among all points in P that are below $p + H$; (ii) insert a point into P; (iii) delete a point from P. The data structure has $O(n)$ space, $O(n \log n)$ preprocessing time, and $O(\log n)$ query time.*

▶ **Corollary 15.** *Given a point set $S = \{s_i : i = 1 \ldots, n\} \subset U$, the points r_i and t_i ($i = 1, \ldots, n$) can be computed in $O(n \log n)$ time. Consequently, the squares q_i^1 can also be computed in $O(n \log n)$ time.*

Proof. Assume that S is sorted in decreasing order by their y-coordinates. We use the data structure in Lemma 14 with the line $H : y = x$ as follows. Initially $P = \emptyset$. For $i = 1, \ldots, n$, we insert s_i into P. If MinBelow(s_i) returns a point in P, then let this be r_i, otherwise let r_i be the point in the right side of U that has the same y-coordinate as s_i. Since P contains all points in S whose y-coordinates are greater or equal to that of s_i, if wedge W_i^R contains any anchor, then MinBelow(s_i) returns one with the minimum x-coordinate. This shows that r_i is computed correctly for all $i = 1, \ldots, n$.

The points t_i ($i = 1, \ldots, n$) can be computed analogously in $O(n \log n)$ time. In $O(1)$ additional time for each $i = 1, \ldots, n$, we can compare r_i and t_i, find the blocker of q_i^1, and determine the maximal anchored square q_i^1.   ◀

Proof of Theorem 13. By a repeated application of Corollary 15, we can compute all $4n$ anchored squares q_i^j ($i = 1, \ldots, n; j = 1, \ldots, 4$). As noted above, a sweep-line algorithm can compute the union $R(S) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{4} q_i^j$ in $O(n \log n)$ time. This completes the proof.   ◀

## 4   NP-Hardness of Maximum-Area Anchored Square Packings

We now prove that the maximum-area anchored square packing problem is NP-complete. We define the decision version of the problem as follows. Instead of the unit square $[0,1]^2$, we use the square $U = [0,W]^2$, for some integer $W > 0$. For a finite set $S \subset [0,W]^2$ of anchors with integer coordinates, we ask whether there is an anchored square packing of area W^2.

We prove NP-hardness by a reduction from PLANAR-MONOTONE-3SAT (described below). For every instance of PLANAR-MONOTONE-3SAT, we construct an instance $S \subset [0,W]^2$. We say that an anchored empty square is *forced* if every packing of area W^2 contains it. An anchor in S is *forced* if it is the anchor of a forced square; otherwise it is *free*. A forced square A and its anchor $s \in S$ form a *forced pair* (A, s). We construct an instance in which most of the anchors are forced, and a small number of anchors encode the truth value of the variables in a 3SAT instance.

To prove that the two instances are equivalent, we shall argue that a set of squares and anchors are forced. In an intermediate step, we assume that $\mathcal{F} = \{(A_i, s_i) : i = 1 \ldots, f\}$ is a set of forced square-anchor pairs, and we would like to show that another square-anchor pair (A, s) is also forced. Let $P = U \setminus \bigcup_{i=1}^{f} A_i$ be the complement of the forced squares in $\mathcal{F}$. By construction, P is an orthogonal polygon (possibly with holes). An anchor s is *undecided* if there is no forced pair (A, s) anchored at s in $\mathcal{F}$ (i.e., s is either free or its forced pair is not in $\mathcal{F}$).

We show (Lemma 16) that the following two properties each imply that the pair (A, s) is forced (given that all pairs in $\mathcal{F}$ are forced). We define both properties for the orientations shown in Fig. 8, but they generalize to all other orientations obtained through the symmetry group of U. Let $s \in S$, and let A be a maximal empty square anchored at s such that $\text{int}(A) \subset P$. Without loss of generality, assume that s is the upper-right corner of A.

1. The lower-left corner of A is a convex vertex of P and there is no undecided anchor in the closure of the bottom and left edges of A.
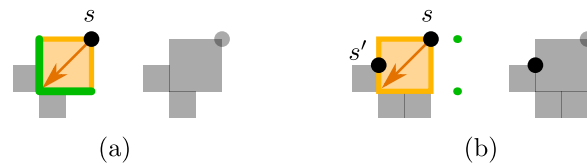
Figure 8 Identifying forced points. (a) Property 1. (b) Property 2.

2. The lower-left corner of A is a convex vertex of P, the side length of A is greater than 1, and the bottom edge of A is contained in ∂P. There is a unique undecided anchor $s' \in \partial A$ located one unit above the lower-left corner of A. There is no undecided anchor one unit to the right of s or to the right of the lower-right corner of A.

▶ **Lemma 16.** *Given a set of forced pairs $\mathcal{F} = \{(A_i, s_i) : i = 1 \dots, f\}$, if a pair (A, s) has properties 1 or 2, then (A, s) is a forced pair.*

Proof. Suppose, to the contrary, that there is a anchored square packing Q of area W^2 that does not use the square A anchored at s. Let $B \subseteq A$ be the unit square incident to the lower-left corner of A. Since all anchors have integer coordinates, every empty square containing B is also contained in A. If a pair (A, s) has property 1, apart from A, no such empty square has a corner at an undecided anchor and, hence, B cannot be covered. If a pair (A, s) has property 2, B must be covered by a square anchored at its upper-left corner, which is undecided by hypothesis. Hence, $B \in Q$. Let B' be a unit square with integer coordinates to the right of B. Then the maximal empty square in $P \setminus B$ containing B' satisfies property 1, but that there is no point at its upper-right corner. In this case, B' is not covered.   ◀

▶ **Theorem 17.** *It is NP-hard to compute the maximum area anchored square packing of a given set S of n anchors with integer coordinates in a square $U = [0, W]^2$.*

Proof. We reduce from PLANAR-MONOTONE-3SAT which is NP-complete [5]. An instance of such problem consists of a boolean formula Φ in 3CNF with n variables $\{x_1, \dots, x_n\}$ and m clauses, and a planar rectilinear drawing of the a bipartite graph of Φ. The drawing given by an PLANAR-MONOTONE-3SAT instance represents variables and clauses by rectangles, and edges by vertical line segments. It has the additional property that the rectangles of variables (and only variables) intersect the line $y = 0$ and the rectangles of clauses lies in the upper (resp., lower) half-plane contain only positive (resp., negative) literals. A literal is called *negative* if it is the negation of a variable, and *positive* otherwise. We need to decide whether we can satisfy all m clauses, each of which is a disjunction of three literals.

For a given instance of PLANAR-MONOTONE-3SAT, we construct an instance $S \subset [0, W]^2$ of the maximal area anchored square packing problem, and then show that the two instances are equivalent. We first modify the rectilinear graph of the PLANAR-MONOTONE-3SAT instance in the following way. Replace each rectangle by a cycle along its boundary and denote by G the resulting geometric graph. Delete the left, right, and the top (resp., bottom) edges of the rectangles representing positive (resp., negative) clauses. Each clause is now represented by a horizontal segment (a path of length 2 in G). We designate the middle vertex of this path, which has degree 3, as a *clause vertex*. For each cycle in G that represents a variable, delete the right vertical edges, and designate the left vertical edge as a *variable edge*. All remaining edges in G called *wires* and all remaining vertices of degree 3 are called *split vertices*. We orient the wires such that they form directed paths from the variable edges to clause vertices. Assume that the feature size of the resulting rectilinear graph is 1 and the
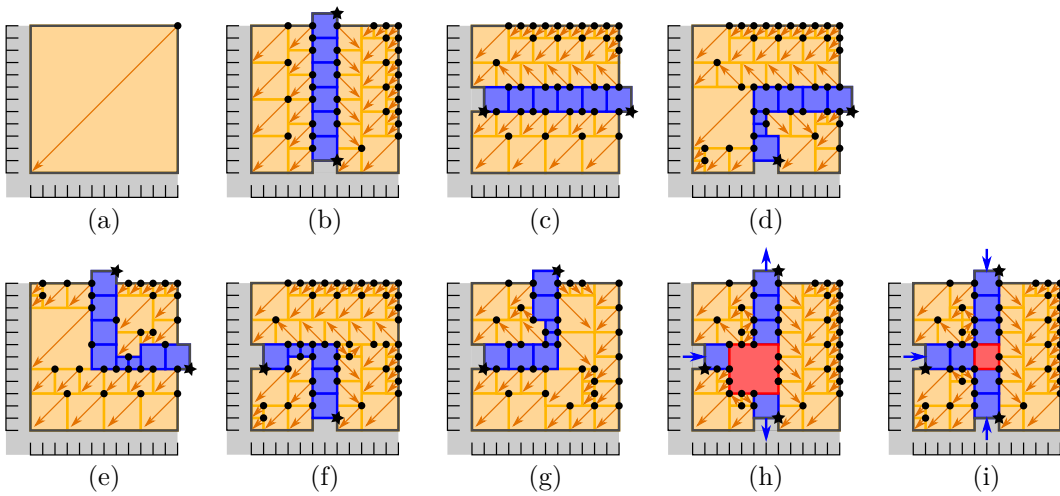
Figure 9 Gadgets. (a) is a *filler* gadget, (b–c) are *wire* gadgets, (d–g) are *turn* gadgets, (h) is a split gadget, and (i) is a *clause* gadget. The rhombus in (h) represents 2 anchors placed at the same position.
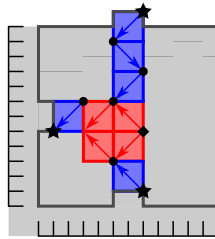

Figure 10 Square packing for the split gadget connected to negative wires.

side length of a minimum enclosing axis-aligned square is k. We set $W = 48k + 48$ and let $U = [0, W]^2$. Scale up the drawing by a factor of 48 and place it in U so that every vertex is at distance at least 24 from ∂U.

We tile U with orthogonal polygons. Every tile is congruent to one of the tiles shown in Fig. 9. We call these tiles *gadgets*: (a) is a *filler* gadget, (b–c) are *wire* gadgets, (d–g) are *turn* gadgets, (h) is a split gadget, and (i) is a *clause* gadget. The filer gadget is a 12×12 square, all other tiles are constructed from a 12×12 square by possibly adding or deleting 1×2 rectangular features in two or three side of the squares. In a tiling of $[0, W]^2$, each such feature matches a feature of an adjacent tile. Choose a tile for each variable that contains part of the variable edge and add the anchors shown in Fig. 9(b) (only the star contained in the 12×12 square is added). Do the same for split and clauses using the tiles shown in Fig. 9(h) and (i), respectively. Connect the gadgets as they are connected in the original drawing using wires and turns. The directions of the wires attached to split and clause gadgets are indicated by blue arrows in Fig. 9. For all remaining tiles, we use filler gadgets with one anchor. This completes the description of the instance $S \subset [0, W]^2$.

We now prove that a PLANAR-MONOTONE-3SAT instance admits a positive solution if and only if the corresponding point set $S \subset [0, W]^2$ admits an anchored square packing of area W^2.
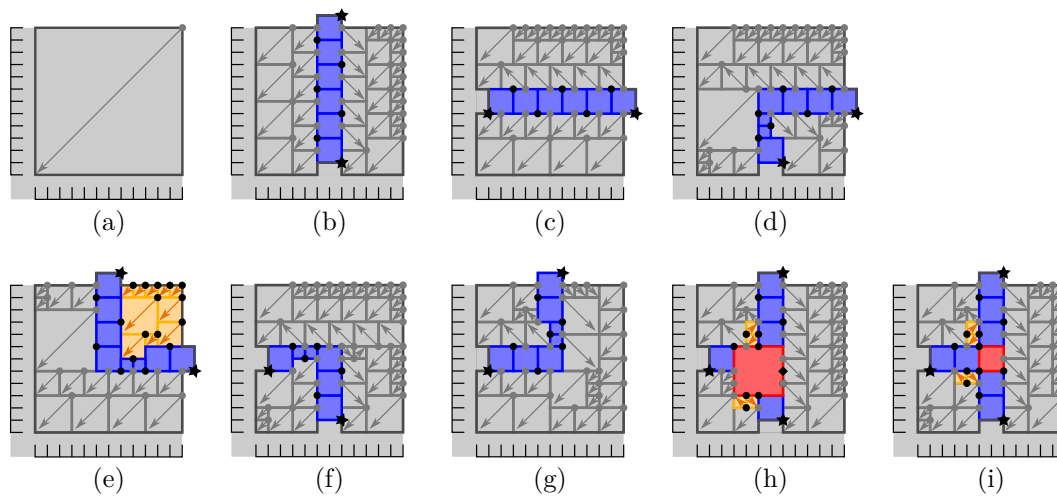
Figure 11 Forced squares and anchors.

Assume that the PLANAR-MONOTONE-3SAT instance admits a positive solution. We show that the corresponding instance $S \subset [0, W]^2$ admits an anchored square packing of area W^2. Choose every orange square in each gadget assigning its anchor as the origin of the arrow contained in it (as shown in Fig. 9). For each of the n wire gadgets placed on variable edges, if the corresponding variable is assigned `true` (resp., `false`) add all blue squares assigning its anchor as the only point on its bottom (resp., top) edge. Every connected component formed by blue squares represents a path of wires. We say that a wire is *positive* if it is in the upper half of U and its corresponding variable is assigned `true`, or if it is in the lower half of U and its corresponding variable is assigned `false`. A wire is *negative* otherwise. If a wire is positive (resp., negative), assign the corner that is behind (resp., ahead of) the blue square as its anchor, considering the direction of the wire. For split gadgets connected to positive wires assign the top-left corner of the red square as its anchor. For negative wires, the red square is reached by four equal squares as shown in Fig. 10. Since there exists at least one positive wire connected to a clause vertex, there will be at least one point in a corner of the red square in the clause gadget that has not yet been assigned a square. We complete the square packing by adding such a square with a corresponding anchor. Since the anchored squares cover all gadgets, the overall area of the square packing is W^2.

Assume that the anchored square packing instance $S \subset [0, W]^2$ admits a positive solution (of area W^2). Recall that $[0, W]^2$ is tiled with gadgets. Sort them in lexicographic order by the coordinates of their lower-left corners (i.e., the first gadget is incident to the origin). We use Lemma 16 to prove the following property for each gadget:

> **Property (i).** If the left and bottom boundaries are part of the perimeter of a forced polygon P and contain no free anchor relative to P except for the points shown by a star, then (i.a) every orange and blue square shown in Fig. 9 in the corresponding gadget is forced; and (i.b) if P' is the union of P and the orange and blue squares inside the gadget, then there is no free anchor relative to P' on the boundary of the gadget except for points shown by a star.

Initially, in every gadget, we can determine at least one pair of a forced square and a corresponding forced anchor using Lemma 16. Fig. 11 shows the result of recursively adding a forced square into the forced polygon, and applying Lemma 16 to another pair until there

are no more forced pairs in the gadget. We now show that all blue squares are forced. In each case, we can take the lower-left blue square and conclude that if the square packing covers it entirely, then it is covered by a square anchored at one of its corners. After we add this square to the forced polygon P, the same argument holds for every lower-left blue square not in P. Consequently, all blue squares are forced. The remaining orange squares are forced by recursively applying Lemma 16. Then, if property (i) is satisfied, every gadget satisfies (i.a) and (i.b). Property (i) is trivially satisfied for the lower-left gadget and inductively satisfied by assuming that all gadgets to the left and below satisfy (i.a) and (i.b).

We now show how to convert a square packing of area W^2 into a solution of the PLANAR-MONOTONE-3SAT instance. Wire gadgets have two points indicates by a star: one that is ahead and one behind using the direction of the wire (recall that the direction points from the variable edge to the clause). A wire gadget that does not use the star that is ahead in its direction as an anchor for one of the squares contained in it is called *positive*. A wire gadget is called *negative* otherwise. For all wire gadgets satisfying (i.a) and (i.b), if a star is not used as an anchor for a square in the gadget, then the other point marked by a star must be used as anchor in this gadget. This implies that, for a pair of adjacent wire gadgets, if the one ahead in the wire direction is positive, so is the other gadget. Now assume that one of the outputs of the split gadget satisfying (i.a) and (i.b) is connected to a positive gadget. A point in the middle of an edge of the red square in Fig. 11(h) must be used as an anchor of a blue square. Then, the only way to cover all the red area is to use a single square anchored at its upper-left corner. Therefore, the wire connected to the input of the split gadget must also be positive. Finally, assume that the red square in a clause gadget that satisfies (i.a) and (i.b) (see Fig. 11(i)) is covered. Then, it must be anchored at one of its corners. If it is anchored at the upper-left (resp., bottom-left, bottom-right) corner, then the star at the top (resp., left, bottom) of the gadget is used as an anchor of a blue square in this gadget. Therefore, it must be adjacent to a positive wire. Combining all arguments, we set a variable `true` if its first wire gadget (that was placed on the variable edge) is positive and `false` otherwise, and then this assignment will satisfy the boolean formula of the PLANAR-MONOTONE-3SAT instance.   ◄

5   Open Problems

We have shown that at least half of the area of the unit square $U = [0,1]^2$ can be reached by empty squares anchored at S for any finite set $S \subset U$, and this bound is the best possible. We have also given the first NP-hardness proof for a packing problem over geometric objects of arbitrary sizes. Our results raise several intriguing open problems. Does our result generalize to higher dimensions, that is, is there a lower bound for the maximal volume covered by empty hypercubes anchored at a finite set of points in $[0,1]^d$ for $d > 2$? Axis-aligned squares are balls in L_∞-norm: Over all finite sets S of anchors in a unit-diameter ball U in L_p-norm, $p \geq 1$, what is the maximum area of a packing of L_p-balls that each contain an anchor? Is there a polynomial-time algorithm for computing the minimum area *lower-left anchored* square packing for a given set S of n points in the unit square $[0,1]^2$? Is it NP-hard to compute the maximum area anchored *rectangle* packing of a given set $S \subset [0,1]^2$? For the last two problems, simple greedy strategies achieve constant-factor approximations [6], and a QPTAS is available for rectangles and a PTAS for squares [1].

References

1 Kevin Balas, Adrian Dumitrescu, and Csaba D. Tóth. Anchored rectangle and square packings. *Discrete Optimization*, 26:131–162, 2017. `doi:10.1016/j.disopt.2017.08.003`.

2 Kevin Balas and Csaba D. Tóth. On the number of anchored rectangle packings for a planar point set. *Theor. Comput. Sci.*, 654:143–154, 2016. `doi:10.1016/j.tcs.2016.03.007`.

3 Jon L. Bentley. Solutions to Klee's rectangle problems. unpublished manuscript, 1977.

4 Timothy M. Chan. Klee's measure problem made easy. In *Proc. 54th Annual IEEE Symposium on Foundations of Computer Science*, pages 410–419, 2013. `doi:10.1109/FOCS.2013.51`.

5 Mark de Berg and Amirali Khosravi. Optimal binary space partitions for segments in the plane. *Int. J. Comput. Geometry Appl.*, 22(3):187–206, 2012. URL: `http://www.worldscinet.com/doi/abs/10.1142/S0218195912500045`.

6 Adrian Dumitrescu and Csaba D. Tóth. Packing anchored rectangles. *Combinatorica*, 35(1):39–61, 2015. `doi:10.1007/s00493-015-3006-1`.

7 Michael Formann and Frank Wagner. A packing problem with applications to lettering of maps. In Robert L. Scot Drysdale, editor, *Proc. 7th Annual Symposium on Computational Geometry*, pages 281–288. ACM, 1991. `doi:10.1145/109648.109680`.

8 Claudia Iturriaga and Anna Lubiw. Elastic labels around the perimeter of a map. *J. Algorithms*, 47(1):14–39, 2003. `doi:10.1016/S0196-6774(03)00004-X`.

9 Joo-Won Jung and Kyung-Yong Chwa. Labeling points with given rectangles. *Inf. Process. Lett.*, 89(3):115–121, 2004. `doi:10.1016/j.ipl.2003.09.017`.

10 Konstantinos G. Kakoulis and Ioannis G. Tollis. Labeling algorithms. In Roberto Tamassia, editor, *Handbook on Graph Drawing and Visualization.*, pages 489–515. Chapman and Hall/CRC, 2013. URL: `https://www.crcpress.com/Handbook-of-Graph-Drawing-and-Visualization/Tamassia/9781584884125`.

11 Klara Kedem, Ron Livne, János Pach, and Micha Sharir. On the union of Jordan regions and collision-free translational motion amidst polygonal obstacles. *Discrete & Computational Geometry*, 1:59–70, 1986. `doi:10.1007/BF02187683`.

12 Donald E. Knuth and Arvind Raghunathan. The problem of compatible representatives. *SIAM J. Discrete Math.*, 5(3):422–427, 1992. `doi:10.1137/0405033`.

13 Atsushi Koike, Shin-Ichi Nakano, Takao Nishizeki, Takeshi Tokuyama, and Shuhei Watanabe. Labeling points with rectangles of various shapes. *Int. J. Comput. Geometry Appl.*, 12(6):511–528, 2002. `doi:10.1142/S0218195902001018`.

14 Giri Narasimhan and Michiel H. M. Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007.

15 William T. Tutte, editor. *Recent Progress in Combinatorics*, New York, 1969. Academic Press. Proceedings of the 3rd Waterloo Conference on Combinatorics, May, 1968.

16 Marc J. van Kreveld, Tycho Strijk, and Alexander Wolff. Point labeling with sliding labels. *Comput. Geom.*, 13(1):21–47, 1999. `doi:10.1016/S0925-7721(99)00005-X`.

17 Hakan Yildiz and Subhash Suri. Computing Klee's measure of grounded boxes. *Algorithmica*, 71(2):307–329, 2015. `doi:10.1007/s00453-013-9797-9`.