Projection Theorems Using Effective Dimension

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Abstract

In this paper we use the theory of computing to study fractal dimensions of projections in Euclidean spaces. A fundamental result in fractal geometry is Marstrand's projection theorem, which shows that for every analytic set E, for almost every line L, the Hausdorff dimension of the orthogonal projection of E onto L is maximal.

We use Kolmogorov complexity to give two new results on the Hausdorff and packing dimensions of orthogonal projections onto lines. The first shows that the conclusion of Marstrand's theorem holds whenever the Hausdorff and packing dimensions agree on the set E, even if E is not analytic. Our second result gives a lower bound on the packing dimension of projections of arbitrary sets. Finally, we give a new proof of Marstrand's theorem using the theory of computing.

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1 Introduction

The field of fractal geometry studies the fine-grained structure of irregular sets. Of particular importance are fractal dimensions, especially the Hausdorff dimension, $\dim_H(E)$, and packing dimension, $\dim_P(E)$, of sets $E \subseteq \mathbb{R}^n$. Intuitively, these dimensions are alternative notions of size that allow us to quantitatively classify sets of measure zero. The books of Falconer [8] and Mattila [23] provide an excellent introduction to this field.

A fundamental problem in fractal geometry is determining how projection mappings affect dimension [9, 24]. Here we study orthogonal projections of sets onto lines. Let e be a point on the unit $(n-1)$-sphere S^{n-1}, and let L_e be the line through the origin and e. The *projection of E onto L_e* is the set

$$\operatorname{proj}_e E = \{e \cdot x : x \in E\},$$

where $e \cdot x$ is the usual dot product, $\sum_{i=1}^n e_i x_i$, for $e = (e_1, \dots, e_n)$ and $x = (x_1, \dots, x_n)$. We restrict our attention to lines through the origin because translating the line L_e will not affect the Hausdorff or packing dimension of the projection.

Notice that $\operatorname{proj}_e E \subseteq \mathbb{R}$, so the Hausdorff dimension of $\operatorname{proj}_e E$ is at most 1. It is also simple to show that $\dim_H(\operatorname{proj}_e E)$ cannot exceed $\dim_H(E)$ [8]. Given these bounds, it is natural to ask whether $\dim_H(\operatorname{proj}_e E) = \min\{\dim_H(E), 1\}$. Choosing E to be a line

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orthogonal to L_e shows that this equality does not hold in general. However, a fundamental theorem due to Marstrand [21] states that, if $E \subseteq \mathbb{R}^2$ is analytic, then for *almost all* $e \in S^1$, the Hausdorff dimension of $\text{proj}_e E$ is maximal. Subsequently, Mattila [22] showed that the conclusion of Marstrand's theorem also holds in higher-dimensional Euclidean spaces.

▶ **Theorem 1** ([21, 22]). *Let $E \subseteq \mathbb{R}^n$ be an analytic set with $\dim_H(E) = s$. Then for almost every $e \in S^{n-1}$,*

$$\dim_H(\text{proj}_e E) = \min\{s, 1\}.$$

In recent decades, the study of projections has become increasingly central to fractal geometry [9]. The most prominent technique has been the potential theoretic approach of Kaufman [14]. While this is a very powerful tool in studying the dimension of a set, it requires that the set be analytic. We will show that techniques from theoretical computer science can circumvent this requirement in some cases.

Our approach to this problem is rooted in the effectivizations of Hausdorff dimension [16] by J. Lutz and of packing dimension by Athreya et al. [1]. The original purpose of these effective dimension concepts was to quantify the size of complexity classes, but they also yield geometrically meaningful definitions of dimension for *individual points* in $\mathbb{R}^n$ [18]. More recently, J. Lutz and N. Lutz established a bridge from effective dimensions back to classical fractal geometry by showing that the Hausdorff and packing dimensions of a set $E \subseteq \mathbb{R}^n$ are characterized by the corresponding effective dimensions of the individual points in E, taken relative to an appropriate oracle [17].

This result, a *point-to-set principle* (Theorem 7 below), allows researchers to use tools from algorithmic information theory to study problems in classical fractal geometry. Although this connection has only recently been established, there have been several results demonstrating the usefulness of the point-to-set principle: J. Lutz and N. Lutz [17] applied it to give a new proof of Davies' theorem [4] on the Hausdorff dimension of Kakeya sets in the plane; N. Lutz and Stull [20] applied it to the dimensions of points on lines in $\mathbb{R}^2$ to give improved bounds on generalized Furstenberg sets; and N. Lutz [19] used it to show that a fundamental bound on the Hausdorff dimension of intersecting fractals holds for arbitrary sets.

In this paper, we use algorithmic information theory, via the point-to-set principle, to study the Hausdorff and packing dimensions of orthogonal projections onto lines. Given the statement of Theorem 1, it is natural to ask whether the requirement that E is analytic can be removed. Without further conditions, it cannot; Davies [5] showed that, assuming the continuum hypothesis, there are non-analytic sets for which Theorem 1 fails. Indeed, Davies constructed a set $E^* \subseteq \mathbb{R}^2$ such that $\dim_H(E^*) = 1$ but $\dim_H(\text{proj}_e E^*) = 0$ for every $e \in S^1$.

Our first main theorem shows that if the Hausdorff and packing dimensions of E agree, then we can remove the requirement that E is analytic.

▶ **Theorem 2.** *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = \dim_P(E) = s$. Then for almost every $e \in S^{n-1}$,*

$$\dim_H(\text{proj}_e E) = \min\{s, 1\}.$$

Our second main theorem applies to projections of *arbitrary* sets. Davies' construction precludes any non-trivial lower bound on the Hausdorff dimension of projections of arbitrary sets, but we are able to give a lower bound on the *packing* dimension.

▶ **Theorem 3.** *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = s$. Then for almost every $e \in S^{n-1}$,*

$$\dim_P(\text{proj}_e E) \geq \min\{s, 1\}.$$

Lower bounds on the packing dimension of projections have been extensively studied for restricted classes sets such as Borel and analytic [6, 7, 10, 12, 26]. To the best of our knowledge, our result is the first non-trivial lower bound of this type for arbitrary sets. It is known that the analogue of Marstrand's theorem for packing dimension does *not* hold [13].

Our other contribution is a new proof of Marstrand's projection theorem (Theorem 1). In addition to showing the power of theoretical computer science in geometric measure theory, this proof introduces a new technique for further research in this area. We show that the assumption that E is analytic allows us to use an earlier, restricted point-to-set principle due to J. Lutz [16] and Hitchcock [11]. While less general than that of J. Lutz and N. Lutz, it is sufficient for this application and involves a simpler oracle. Informally, this allows us to reverse the order of quantifiers in the statement of Theorem 1. This will be both beneficial for further research, as well as clarifying the role of the analytic assumption of E.

2 Preliminaries

We begin with a brief description of algorithmic information quantities and their relationships to Hausdorff and packing dimensions.

2.1 Kolmogorov Complexity in Discrete and Continuous Domains

The *conditional Kolmogorov complexity* of a binary string $\sigma \in \{0,1\}^*$ given a binary string $\tau \in \{0,1\}^*$ is the length of the shortest program π that will output σ given τ as input. Formally, the conditional Kolmogorov complexity of σ given τ is

$$K(\sigma \mid \tau) = \min_{\pi \in \{0,1\}^*} \{\ell(\pi) : U(\pi, \tau) = \sigma\} ,$$

where U is a fixed universal prefix-free Turing machine and $\ell(\pi)$ is the length of π. Any π that achieves this minimum is said to *testify* to, or be a *witness* to, the value $K(\sigma \mid \tau)$. The *Kolmogorov complexity* of a binary string σ is $K(\sigma) = K(\sigma \mid \lambda)$, where λ is the empty string. These definitions extend naturally to other finite data objects, e.g., vectors in $\mathbb{Q}^n$, via standard binary encodings; see [15] for details.

One of the most useful properties of Kolmogorov complexity is that it obeys the *symmetry of information*. That is, for every $\sigma, \tau \in \{0,1\}^*$,

$$K(\sigma, \tau) = K(\sigma) + K(\tau \mid \sigma, K(\sigma)) + O(1) .$$

Kolmogorov complexity can be naturally extended to points in Euclidean space, as we now describe. The *Kolmogorov complexity* of a point $x \in \mathbb{R}^m$ at *precision* $r \in \mathbb{N}$ is the length of the shortest program π that outputs a *precision-r* rational estimate for x. Formally, this is

$$K_r(x) = \min \{K(p) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} ,$$

where $B_\varepsilon(x)$ denotes the open ball of radius ε centered on x. The *conditional Kolmogorov complexity of x at precision r given $y \in \mathbb{R}^n$ at precision $s \in \mathbb{R}^n$* is

$$K_{r,s}(x \mid y) = \max \big\{ \min\{K_r(p \mid q) \,:\, p \in B_{2^{-r}}(x) \cap \mathbb{Q}^m\} \,:\, q \in B_{2^{-s}}(y) \cap \mathbb{Q}^n \big\} .$$

When the precisions r and s are equal, we abbreviate $K_{r,r}(x \mid y)$ by $K_r(x \mid y)$. Given any positive real as a precision parameter, we round up to the next integer; for example, $K_r(x)$ denotes $K_{\lceil r \rceil}(x)$ whenever $r \in (0, \infty)$.

We will need the following technical lemmas which show that versions of the symmetry of information hold for Kolmogorov complexity in $\mathbb{R}^n$. The first Lemma 4 was proved in our previous work [20].

▶ **Lemma 4** ([20]). *For every $m, n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \mathbb{N}$ with $r \geq s$,*

i. $\left|K_r(x \mid y) + K_r(y) - K_r(x, y)\right| \leq O(\log r) + O(\log\log \|y\|)$.

ii. $|K_{r,s}(x \mid x) + K_s(x) - K_r(x)| \leq O(\log r) + O(\log\log \|x\|)$.

▶ **Lemma 5.** *Let $m, n \in \mathbb{N}$, $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$, $\varepsilon > 0$ and $r \in \mathbb{N}$. If $K^x_r(z) \geq K_r(z) - \varepsilon r$, then the following hold for all $s \leq r$.*

i. $|K^x_s(z) - K_s(z)| \leq \varepsilon r - O(\log r)$.

ii. $|K_{s,r}(x \mid z) - K_s(x)| \leq \varepsilon r - O(\log r)$.

Proof. We first prove item (i). By Lemma 4(ii),

$$\begin{aligned}
\varepsilon r &\geq K_r(z) - K^x_r(z) \\
&\geq K_s(z) + K_{r,s}(z \mid z) - (K^x_s(z) + K^x_{r,s}(z \mid z)) - O(\log r) \\
&\geq K_s(z) - K^x_s(z) + K_{r,s}(z \mid z) - K^x_{r,s}(z \mid z) - O(\log r).
\end{aligned}$$

Rearranging, this implies that

$$\begin{aligned}
K_s(z) - K^x_s(z) &\leq \varepsilon r + K^x_{r,s}(z \mid z) - K_{r,s}(z \mid z) + O(\log r) \\
&\leq \varepsilon r + O(\log r),
\end{aligned}$$

and the proof of item (i) is complete.

To prove item (ii), by Lemma 4(i) we have

$$\begin{aligned}
\varepsilon r &\geq K_r(z) - K_r(z \mid x) \\
&\geq K_r(z) - (K_r(z, x) - K_r(x)) - O(\log r) \\
&\geq K_r(z) - (K_r(z) + K_r(x \mid z) - K_r(x)) - O(\log r) \\
&= K_r(x) - K_r(x \mid z) - O(\log r).
\end{aligned}$$

Therefore, by Lemma 4(ii),

$$\begin{aligned}
K_s(x) - K_{s,r}(x \mid z) &= K_r(x) - K_{r,s}(x \mid x) - (K_r(x \mid z) - K_{r,s,r}(x \mid x, z)) \\
&\leq \varepsilon r + O(\log r) + K_{r,s,r}(x \mid x, z) - K_{r,s}(x \mid x) \\
&\leq \varepsilon r + O(\log r),
\end{aligned}$$

and the proof is complete. ◀

2.2 Effective Hausdorff and Packing Dimensions

J. Lutz [16] initiated the study of effective dimensions by effectivizing Hausdorff dimension using betting strategies called *gales*, which generalize martingales. Subsequently, Athreya et al. defined effective packing dimension, also using gales [1]. Mayordomo showed that effective Hausdorff dimension can be characterized using Kolmogorov complexity [25], and Mayordomo and J. Lutz [18] showed that effective packing dimension can also be characterized in this way. In this paper, we use these characterizations as definitions. The *effective Hausdorff dimension* and *effective packing dimension* of a point $x \in \mathbb{R}^n$ are

$$\dim(x) = \liminf_{r\to\infty} \frac{K_r(x)}{r} \quad \text{and} \quad \mathrm{Dim}(x) = \limsup_{r\to\infty} \frac{K_r(x)}{r}.$$

Intuitively, these dimensions measure the density of algorithmic information in the point x. J. Lutz and N. Lutz [17] generalized these definitions by defining the *lower* and *upper conditional dimension* of $x \in \mathbb{R}^m$ given $y \in \mathbb{R}^n$ as

$$\dim(x \mid y) = \liminf_{r\to\infty} \frac{K_r(x \mid y)}{r} \quad \text{and} \quad \mathrm{Dim}(x \mid y) = \limsup_{r\to\infty} \frac{K_r(x \mid y)}{r}.$$

2.3 The Point-to-Set Principle

By letting the underlying fixed prefix-free Turing machine U be a universal *oracle* machine, we may *relativize* the definitions in this section to an arbitrary oracle set $A \subseteq \mathbb{N}$. The definitions of $K^A(\sigma|\tau)$, $K^A(\sigma)$, $K^A_r(x)$, $K^A_r(x \mid y)$, $\dim^A(x)$, $\mathrm{Dim}^A(x)$ $\dim^A(x \mid y)$, and $\mathrm{Dim}^A(x \mid y)$ are then all identical to their unrelativized versions, except that U is given oracle access to A. We will frequently consider the complexity of a point $x \in \mathbb{R}^n$ *relative to a point* $y \in \mathbb{R}^m$, i.e., relative to an oracle set A_y that encodes the binary expansion of y is a standard way. We then write $K^y_r(x)$ for $K^{A_y}_r(x)$.

The following *point-to-set principles* show that the classical notions of Hausdorff and packing dimension of a set can be characterized by the effective dimension of its individual points. The first point-to-set principle we use here, which applies to a restricted class of sets, was implicitly proven by J. Lutz [16] and Hitchcock [11].

A set $E \subseteq \mathbb{R}^n$ is a $\mathbf{\Sigma}^0_2$ set if it is a countable union of closed sets. The computable analogue of $\mathbf{\Sigma}^0_2$ is the class Σ^0_2, consisting of sets $E \subseteq \mathbb{R}^n$ such that there is a uniformly computable sequence $\{C_i\}_{i\in\mathbb{N}}$ satisfying

$$E = \bigcup_{i=0}^{\infty} C_i \,,$$

and each set C_i is *computably closed*, meaning that its complement is the union of a computably enumerable set of open balls with rational radii and centers. We will use the fact that every $\mathbf{\Sigma}^0_2$ set is Σ^0_2 relative to some oracle.

▶ **Theorem 6** ([16, 11]). *Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$ be such that E is a Σ^0_2 set relative to A. Then*

$$\dim_H(E) = \sup_{x\in E} \dim^A(x) \,.$$

J. Lutz and N. Lutz [17] showed that the Hausdorff and packing dimension of *any* set $E \subseteq \mathbb{R}^n$ is characterized by the corresponding effective dimensions of individual points, relativized to an oracle that is optimal for the set E.

▶ **Theorem 7** (Point-to-set principle [17]). *Let $n \in \mathbb{N}$ and $E \subseteq \mathbb{R}^n$. Then*

$$\dim_H(E) = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \dim^A(x), \quad \textit{and}$$
$$\dim_P(E) = \min_{A\subseteq\mathbb{N}} \sup_{x\in E} \mathrm{Dim}^A(x) \,.$$

It is worth noting that the point-to-set principle is taking the *minimum* over all oracles, not simply the *infimum*.

3 Bounding the Complexity of Projections

In this section, we will focus on bounding the Kolmogorov complexity of a projected point *at a given precision*. In Section 4, we will use these results in conjunction with the point-to-set principle to prove our main theorems.

We begin by giving intuition of the main idea behind this lower bound. We will show that under certain conditions, given an approximation of $e \cdot z$ and e, we can compute an approximation of the original point z. Informally, these conditions are the following.

1. The complexity $K_r(z)$ of the original point is small.
2. If $e \cdot w = e \cdot z$, then either $K_r(w)$ is large, or w is close to z.

Assuming that both conditions are satisfied, we can recover z from $e \cdot z$ by enumerating over all points u of low complexity such that $e \cdot u = e \cdot z$. By our assumption, any such point u must be a good approximation of z. We now formally state and prove this lemma.

▶ **Lemma 8.** *Suppose that $z \in \mathbb{R}^n$ $e \in S^{n-1}$, $r \in \mathbb{N}$, $\delta \in \mathbb{R}_+$, and $\varepsilon, \eta \in \mathbb{Q}_+$ satisfy $r \geq \log(2\|z\| + 5) + 1$ and the following conditions.*

i. $K_r(z) \leq (\eta + \varepsilon)\, r$.

ii. *For every $w \in B_1(z)$ such that $e \cdot w = e \cdot z$,*

$$K_r(w) \geq (\eta - \varepsilon)\, r + (r - t)\delta\,,$$

whenever $t = -\log \|z - w\| \in (0, r]$.

Then for every oracle set $A \subseteq \mathbb{N}$,

$$K_r^{A,e}(e \cdot z) \geq K_r^{A,e}(z) - \frac{n\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r)\,.$$

Proof. Suppose z, e, r, δ, ε, η, and A satisfy the hypothesis.

Define an oracle Turing machine M that does the following given oracle (A, e) and input $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$ such that $U^A(\pi_1) = q \in \mathbb{Q}$, $U(\pi_2) = h \in \mathbb{Q}^n$, $U(\pi_3) = s \in \mathbb{N}$, $U(\pi_4) = \zeta \in \mathbb{Q}$, and $U(\pi_5) = \iota \in \mathbb{Q}$.

For every program $\sigma \in \{0,1\}^*$ with $\ell(\sigma) \leq (\iota + \zeta)s$, in parallel, M simulates $U(\sigma)$. If one of the simulations halts with some output $p = (p_1, \ldots, p_n) \in \mathbb{Q}^n \cap B_{2^{-1}}(h)$ such that $|e \cdot p - q| < 2^{-s}$, then $M^{A,e}$ halts with output p. Let c_M be a constant for the description of M.

Let π_1, π_2, π_3, π_4, and π_5 testify to $K_r^{A,e}(e \cdot z)$, $K_1(z)$, $K(r)$, $K(\varepsilon)$, and $K(\eta)$, respectively, and let $\pi = \pi_1\pi_2\pi_3\pi_4\pi_5$. Let σ be a program of length at most $(\eta+\varepsilon)r$ such that $\|p - z\| \leq 2^{-r}$, where $U(\sigma) = p$. Note that such a program must exist by condition (i) of our hypothesis. Then it is easily verified that

$$|e \cdot z - e \cdot p| \leq 2^{-r}\,.$$

Therefore $M^{A,e}$ is guaranteed to halt on π.

Let $M^{A,e}(\pi) = p = (p_1, \ldots, p_n) \in \mathbb{Q}^n$. Another routine calculation shows that there is some

$$w \in B_{2^{\gamma - r}}(p) \subseteq B_{2^{-1}}(p) \subseteq B_{2^0}(z)$$

such that $e \cdot w = e \cdot z$, where γ is a constant depending only on z and e. Then,

$$\begin{aligned} K_r^{A_e}(w) &\leq |\pi| + c_M \\ &\leq K_r^{A,e}(e \cdot z) + K_1(z) + K(r) + K(\varepsilon) + K(\eta) + c_M \\ &= K_r^{A,e}(e \cdot z) + K(\varepsilon) + K(\eta) + O(\log r)\,. \end{aligned}$$

Rearranging this yields

$$K_r^{A,e}(e \cdot z) \geq K_r^{A,e}(w) - K(\varepsilon) - K(\eta) - O(\log r)\,. \tag{1}$$

Let $t = -\log \|z - w\|$. If $t \geq r$, then the proof is complete. If $t < r$, then $B_{2^{-r}}(p) \subseteq B_{2^{1-t}}(z)$, which implies that $K_r^{A,e}(w) \geq K_{t-1}^{A,e}(z)$. Therefore,

$$K_r^{A,e}(w) \geq K_r^{A,e}(z) - n(r - t) - O(\log r)\,. \tag{2}$$

We now bound $r - t$. By our construction of M,

$$\begin{aligned}(\eta+\varepsilon)r &\geq K(p)\\ &\geq K_r(w) - O(\log r)\,.\end{aligned}$$

By condition (ii) of our hypothesis, then,

$$(\eta+\varepsilon)r \geq (\eta-\varepsilon)r + \delta(r-t)\,,$$

which implies that

$$r - t \leq \frac{2n\varepsilon}{\delta} r + O(\log r)\,.$$

Combining this with inequalities (1) and (2) concludes the proof. ◀

With the above lemma in mind, we wish to give a lower bound on the complexity of points w such that $e \cdot w = e \cdot z$. Our next lemma gives a bound based on the complexity, relative to z, of the direction $e \in S^{n-1}$. This is based on the observation that we can solve for $e = (e_1, \ldots, e_n)$ given w, z and $e_3, \ldots, e_n$. This follows from solving the system of two equations

$$\begin{aligned}e \cdot (z - w) &= 0\\ e_1^2 + \ldots + e_n^2 &= 1\,.\end{aligned}$$

This suggests that

$$K_r^{z,e_3,\ldots,e_n}(e) \leq K_r^{z,e_3,\ldots,e_n}(w)\,.$$

However, for our purposes, we must be able to recover (an approximation of) e given *approximations* of w and z. Intuitively, the following lemma shows that we can algorithmically compute an approximation of e whose error is linearly correlated with the distance between w and z. We can then bound the complexity of w using a symmetry of information argument.

▶ **Lemma 9.** *Let $z \in \mathbb{R}^n$, $e \in S^{n-1}$, and $r \in \mathbb{N}$. Let $w \in \mathbb{R}^n$ such that $e \cdot z = e \cdot w$. Then there are numbers $i, j \in \{1, \ldots, n\}$ such that*

$$K_r(w) \geq K_t(z) + K_{r-t,r}^{e-\{e_i,e_j\}}(e \mid z) + O(\log r)\,,$$

where $t = -\log \|z - w\|$.

Proof. Let z, w, e, and r be as in the statement of the lemma. We first choose i so that $|z_i - w_i|$ is maximal. We then choose j so that

$$\begin{aligned}&\operatorname{sgn}((z_i - w_i)e_i) \neq \operatorname{sgn}((z_j - w_j)e_j), \text{ and}\\ &|z_j - w_j| > 0\,,\end{aligned}$$

where sgn denotes the sign. Note that such a j must exist since $(z - w) \cdot e = 0$. For the sake of removing notational clutter, we will assume, without loss of generality, that $i = 1$ and $j = 2$.

We first show that

$$K_{r-t,r}^{e_3,\ldots,e_n}(e_2 \mid z) \leq K_r(w \mid z) + O(1)\,. \tag{3}$$

As mentioned in the informal discussion preceding this lemma, note that

$$e_2 = \frac{-b + (-1)^h\sqrt{b^2 - 4ac}}{2a}\,, \tag{4}$$

where

- $h \in \{0,1\}$,
- $a = (z_1 - w_1)^2 + (w_2 - z_2)^2$,
- $b = 2(w_2 - z_2)\sum_{i=3}^{n}(w_i - z_i)e_i$, and
- $c = (\sum_{i=3}^{n}(w_i - z_i)e_i)^2 + (z_1 - w_1)^2 \sum_{i=3}^{n} e_i^2 - 1$.

With this in mind, let M be the Turing machine such that, whenever $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ and $U(\pi, q) = p = (p_1, \dots, p_n) \in Q^2$ with $p_1 \neq q_1$,

$$M^{e_3,\dots,e_n}(\pi, q, j) = \frac{-b' + (-1)^h\sqrt{b'^{,2} - 4a'c'}}{2a'} \,,$$

where

- $h \in \{0,1\}$,
- $a' = (q_1 - p_1)^2 + (p_2 - q_2)^2$,
- $b' = 2(p_2 - q_2)\sum_{i=3}^{n}(p_i - q_i)d_i$, and
- $c' = (\sum_{i=3}^{n}(p_i - q_i)d_i)^2 + (q_1 - p_1)^2 \sum_{i=3}^{n} d_i^2 - 1$, and
- $d = (d_3, \dots, d_n) \in \mathbb{Q}^{n-2}$ is an nr-approximation of $(e_3, \dots, e_n)$.

Let $q \in B_{2^{-r}}(z) \cap \mathbb{Q}^n$, and π_q testify to $\hat{K}_r(w \mid q)$. It tedious but straightforward (Lemma 10) to verify that

$$|M^{e_3,\dots,e_n}(\pi_q, q, h) - e_2| \leq 2^{\alpha + t - r} \,,$$

where α is a constant depending only on e. Hence, inequality (3) holds. Since

$$K_s^{e_3,\dots,e_n}(e_2) = K_s^{e_3,\dots,e_n}(e) + O(1)$$

holds for every s, we see that

$$K_{r-t,r}^{e_3,\dots,e_n}(e \mid z) \leq K_r(w \mid z) + O(1) \,. \tag{5}$$

To complete the proof, we note that

$$\begin{aligned} K_r(w \mid z) &\leq K_{r,t}(w \mid z) + O(\log r) \\ &= K_{r,t}(w \mid w) + O(\log r) \\ &= K_r(w) - K_t(w) + O(\log r) \\ &= K_r(w) - K_t(z) + O(\log r) \,. \end{aligned}$$

The lemma follows from rearranging the above inequality, and combining inequality (5). ◀

The previous lemma uses the following technical lemma, whose proof is omitted due to space considerations.

▶ **Lemma 10.** *Let $z, w \in \mathbb{R}^n$, $e \in S^{n-1}$, and $r \in \mathbb{N}$ such that $e \cdot z = e \cdot w$. Let $q = (q_1, \dots, q_n) \in \mathbb{Q}^n$ and $p = (p_1, \dots, p_n) \in \mathbb{Q}^n$ be r-approximations of z and w, respectively. Then*

$$\left| \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b' + \sqrt{b'^{,2} - 4a'c'}}{2a'} \right| \leq 2^{-r+t+\alpha} \,,$$

where a, b, c, a', b' and c' are as defined in Lemma 9, $t = -\log\|z - w\|$ and α is a constant depending only on e.

Finally, to satisfy the condition that $K_r(z)$ is small, we will use an oracle to "artificially" decrease the complexity of z at precision r. We will achieve this by applying the following lemma due to N. Lutz and Stull.

▶ **Lemma 11** ([20]). *Let $n, r \in \mathbb{N}$, $z \in \mathbb{R}^n$, and $\eta \in \mathbb{Q} \cap [0, \dim(z)]$. Then there is an oracle $D = D(n, r, z, \eta)$ and a constant $k \in \mathbb{N}$ depending only on n, z and η satisfying*

i. *For every $t \leq r$,*

$$K_t^D(z) = \min\{\eta r, K_t(z)\} + k \log r \,.$$

ii. *For every $m, t \in \mathbb{N}$ and $y \in \mathbb{R}^m$,*

$$K_{t,r}^D(y \mid z) = K_{t,r}(y \mid z) + k \log r \,,$$

and

$$K_t^{z,D}(y) = K_t^z(y) + k \log r \,.$$

4 Projection Theorems

The main results of the previous section gave us sufficient conditions for strong lower bounds on the complexity of $e \cdot z$ at a given precision, and methods to ensure that the conditions are satisfied. The following theorem encapsulates these results so that we may apply them in the proof of our main theorems.

▶ **Theorem 12.** *Let $z \in \mathbb{R}^n$, $e \in S^{n-1}$, $A \subseteq \mathbb{N}$, $\eta' \in \mathbb{Q} \cap (0,1) \cap (0, \dim(z))$, $\varepsilon' > 0$, and $r \in \mathbb{N}$. Assume the following are satisfied.*

1. *For every $s \leq r$, and $i, j \in \{1, \ldots, n\}$, $K_s^{e-\{e_i,e_j\}}(e) \geq s - \log(s)$.*
2. *$K_r^{A,e}(z) \geq K_r(z) - \varepsilon' r$.*

Then,

$$K_r^{A,e}(e \cdot z) \geq \eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1 - \eta'} r - K(2\varepsilon') - K(\eta') - O_z(\log r) \,.$$

Proof. Assume the hypothesis, and let $\eta = \eta'$, $\varepsilon = 2\varepsilon'$ and $\delta = 1 - \eta'$. Let $D_r = D(n, r, z, \eta')$ be the oracle as defined in Lemma 11.

First assume that the conditions of Lemma 8, relative to D_r, hold for z, e, r, η, ε and δ. Then we may apply Lemma 8, which, when combined item (2) and Lemma 11, yields

$$\begin{aligned} K_r^{A,D_r,e}(e \cdot z) &\geq K_r^{A,D_r,e}(z) - \frac{n\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r) \\ &\geq K_r^{D_r}(z) - \varepsilon' r - \frac{n\varepsilon}{\delta} r - K(\varepsilon) - K(\eta) - O_z(\log r) \\ &= \eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1 - \eta'} r - K(\varepsilon') - K(\eta') - O_z(\log r) \,. \end{aligned}$$

Therefore, to complete the proof, it suffices to show that the conditions of Lemma 8, relative to D_r, hold.

Item (i) of Lemma 8 holds by our construction of D_r. To see that condition (ii) holds, let $w \in B_1(z)$ such that $e \cdot w = e \cdot z$. By Lemma 9, for some $i, j \in \{1, \ldots, n\}$,

$$K_r^{D_r}(w) \geq K_t^{D_r}(z) + K_{r-t,r}^{D_r, e-\{e_i,e_j\}}(e \mid z) + O(\log r) \,,$$

where $t = -\log \|z - w\|$. Therefore, by condition (2) of the hypothesis and Lemma 5,

$$K_r^{D_r}(w) \geq K_t^{D_r}(z) + K_{r-t,r}^{D_r, e-\{e_i,e_j\}}(e) - \varepsilon' r - O(\log r) \,.$$

By combining this with condition (1) of the present lemma and Lemma 11,

$$\begin{aligned} K_r^{D_r}(w) &\geq K_t^{D_r}(z) + K_{r-t,r}^{D_r, e-\{e_i,e_j\}}(e) - \varepsilon' r - O(\log r) \\ &\geq \eta' t + r - t - \varepsilon' r - O(\log r) \\ &= t(\eta' - 1) + r(1 - \varepsilon') - O(\log r) \\ &\geq (\eta - \varepsilon) r + \delta(r - t)\,. \end{aligned}$$

Hence, the conditions of Lemma 8 are satisfied and the proof is complete. ◀

4.1 Projection Theorems For Non-Analytic Sets

Our first main theorem shows that if the Hausdorff and packing dimensions of E are equal, the conclusion of Marstrand's theorem holds. Essentially this assumption guarantees, for every oracle A and direction e, the existence of a point $z \in E$ such that $\dim^{A,e}(z) \geq \dim_H(E) - \varepsilon$. This allows us to use Theorem 12 at all sufficiently large precisions r.

▶ **Theorem 2.** *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = \dim_P(E) = s$. Then for almost every $e \in S^{n-1}$,*

$$\dim_H(\operatorname{proj}_e E) \geq \min\{s, 1\}\,.$$

Proof. Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = \dim_P(E) = s$. By the point-to-set principle, there is an oracle $B \subseteq \mathbb{N}$ testifying to $\dim_H(E)$ and $\dim_P(E)$. Let $e \in S^{n-1}$ be any point which is random relative to B. That is, let e be any point such that

$$K_r^{B, e-\{e_i,e_j\}}(e) \geq r - \log r\,,$$

for every $i, j \in \{1, \ldots, n\}$. Note that almost every point satisfies this requirement. Let $A \subseteq \mathbb{N}$ be the oracle testifying to $\dim_H(\operatorname{proj}_e E)$. Then, by the point-to-set principle, it suffices to show that for every $\varepsilon > 0$ there is a $z \in E$ such that

$$\dim^A(e \cdot z) \geq \min\{s, 1\} - \varepsilon\,.$$

To that end, let $\eta' \in \mathbb{Q} \cap (0,1) \cap (0,s)$ and $\varepsilon' > 0$. By the point-to-set principle, there is a $z_{\varepsilon'} \in E$ such that

$$\begin{aligned} s - \frac{\varepsilon'}{4} &\leq \dim^{A,B,e}(z_{\varepsilon'}) \\ &\leq \dim^B(z_{\varepsilon'}) \\ &\leq \operatorname{Dim}^B(z_{\varepsilon'}) \\ &\leq s\,. \end{aligned} \tag{6}$$

We now show that the conditions of Theorem 12 are satisfied, relative to B, for all sufficiently large $r \in \mathbb{N}$. We first note that, by inequality (6) and the definition of effective dimension,

$$\begin{aligned} sr - \frac{\varepsilon'}{4} r - \frac{\varepsilon'}{4} r &\leq K_r^{A,B,e}(z_{\varepsilon'}) \\ &\leq K_r^B(z_{\varepsilon'}) + O(1) \\ &\leq sr + \frac{\varepsilon'}{2} r\,, \end{aligned}$$

for all sufficiently large r. Hence, for all such r,

$$K_r^{A,B,e}(z_{\varepsilon'}) \geq K_r^B(z_{\varepsilon'}) - \varepsilon' r\,. \tag{7}$$

Thus the conditions of Theorem 12, relative to B, are satisfied.

We may therefore apply Theorem 12, resulting in

$$K_r^{A,B,e}(e \cdot z_{\varepsilon'}) \geq \eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1-\eta'}r - K(\varepsilon') - K(\eta') - O_{z_{\varepsilon'}}(\log r)\,.$$

Hence,

$$\begin{aligned}
\dim^A(e \cdot z_{\varepsilon'}) &\geq \dim^{A,B,e}(e \cdot z_{\varepsilon'}) \\
&= \liminf_{r\to\infty} \frac{K_r^{A,B,e}(e \cdot z_{\varepsilon'})}{r} \\
&\geq \liminf_{r\to\infty} \frac{\eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1-\eta'}r - K(\varepsilon') - K(\eta') - O_{z_{\varepsilon'}}(\log r)}{r} \\
&= \eta' - \varepsilon' - \frac{2n\varepsilon'}{1-\eta'}\,.
\end{aligned}$$

Since η' was chosen arbitrarily,

$$\dim^A(e \cdot z) \geq \min\{s, 1\} - \frac{\varepsilon'}{4}\,.$$

As ε' was chosen arbitrarily, by the point-to-set principle,

$$\begin{aligned}
\dim_H(\mathrm{proj}_e\, E) &\geq \sup_{z\in E} \dim^A(e \cdot z) \\
&\geq \sup_{\varepsilon>0} \dim^A(e \cdot z_{\varepsilon'}) \\
&= \min\{s, 1\}\,,
\end{aligned}$$

and the proof is complete. ◀

Our second main theorem gives a lower bound for the *packing* dimension of a projection for *general* sets. The proof of this theorem again relies on the ability to choose, for every (A, e), a point z whose complexity is unaffected relative to (A, e). This cannot be assumed to hold for every precision r. However, by the point-to-set principle, we can show that this can be done for infinitely many precision parameters r.

▶ **Theorem 3.** *Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = s$. Then for almost every $e \in S^{n-1}$,*

$$\dim_P(\mathrm{proj}_e\, E) \geq \min\{s, 1\}\,.$$

Proof. Let $E \subseteq \mathbb{R}^n$ be any set with $\dim_H(E) = s$. By the point-to-set principle, there is an oracle $B \subseteq \mathbb{N}$ testifying to $\dim_H(E)$ and $\dim_P(E)$. Let $e \in S^{n-1}$ be any point which is random relative to B. Note that almost every point satisfies this requirement. Let $A \subseteq \mathbb{N}$ be the oracle testifying to $\dim_P(\mathrm{proj}_e\, E)$. Then, by the point-to-set principle, it suffices to show that for every $\varepsilon > 0$ there is a $z \in E$ such that

$$\mathrm{Dim}^A(e \cdot z) \geq \min\{s, 1\} - \varepsilon\,.$$

To that end, let $\eta' \in \mathbb{Q} \cap (0, 1) \cap (0, s)$ and $\varepsilon' > 0$. By the point-to-set principle, there is a $z_{\varepsilon'} \in E$ such that

$$s - \frac{\varepsilon'}{4} \leq \dim^{A,B,e}(z_{\varepsilon'}) \leq \dim^B(z_{\varepsilon'}) \leq s\,. \tag{8}$$

We now show that the conditions of Theorem 12 are satisfied, relative to B, for infinitely many $r \in \mathbb{N}$. We first note that, by equation (8),

$$\begin{aligned} sr - \frac{\varepsilon'}{4}r - \frac{\varepsilon'}{4}r &\leq K_r^{A,B,e}(z_{\varepsilon'}) \\ &\leq K_r^B(z_{\varepsilon'}) + O(1) \\ &\leq sr + \frac{\varepsilon'}{2}r\,, \end{aligned}$$

for infinitely many r. Hence, for all such r,

$$K^{A,B,e}(z_{\varepsilon'}) \geq K^B(z_{\varepsilon'}) - \varepsilon r\,. \tag{9}$$

Thus the conditions of Theorem 12, relative to B, are satisfied for infinitely many $r \in \mathbb{N}$.

We may therefore apply Theorem 12, resulting in

$$K_r^{A,B,e}(e \cdot z_{\varepsilon'}) \geq \eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1-\eta'}r - K(\varepsilon') - K(\eta') - O_{z_{\varepsilon'}}(\log r)\,,$$

for infinitely many $r \in \mathbb{N}$. Hence,

$$\begin{aligned} \mathrm{Dim}^A(e \cdot z_{\varepsilon'}) &\geq \mathrm{Dim}^{A,B,e}(e \cdot z_{\varepsilon'}) \\ &= \limsup_{r\to\infty} \frac{K_r^{A,B,e}(e \cdot z_{\varepsilon'})}{r} \\ &\geq \limsup_{r\to\infty} \frac{\eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1-\eta'}r - K(\varepsilon') - K(\eta') - O_{z_{\varepsilon'}}(\log r)}{r} \\ &= \eta' - \varepsilon' - \frac{2n\varepsilon'}{1-\eta'}\,. \end{aligned}$$

Since η' was chosen arbitrarily

$$\mathrm{Dim}^A(e \cdot z) \geq \min\{s, 1\} - \frac{\varepsilon'}{4}\,.$$

As ε' was chosen arbitrarily, by the point-to-set principle

$$\begin{aligned} \dim_P(\mathrm{proj}_e E) &\geq \sup_{z\in E} \mathrm{Dim}^A(e \cdot z) \\ &\geq \sup_{\varepsilon>0} \mathrm{Dim}^A(e \cdot z_{\varepsilon'}) \\ &= \min\{s, 1\}\,, \end{aligned}$$

and the proof is complete. ◀

4.2 Marstrand's Projection Theorem

We now give a new, algorithmic information theoretic proof of Marstrand's projection theorem. Recall that

▶ **Theorem 1.** *Let $E \subseteq \mathbb{R}^n$ be analytic with $\dim_H(E) = s$. Then for almost every $e \in S^{n-1}$,*

$$\dim_H(\mathrm{proj}_e E) = \min\{s, 1\}\,.$$

Note the order of the quantifiers. To use the point-to-set principle, we must first choose a direction $e \in S^{n-1}$. We then must show that for every oracle A and $\varepsilon > 0$, there is some $z \in E$ such that

$$\dim^A(e \cdot z) \geq \dim_H(E) - \varepsilon\,.$$

In order to apply Theorem 12, we must guarantee that (A, e) does not significantly change the complexity of z. To ensure this, we will use the point-to-set principle of J. Lutz and Hitchcock (Theorem 6). While this result is less general than the principle of J. Lutz and N. Lutz, the oracle characterizing the dimension of a Σ^0_2 set is easier to work with.

To take advantage of this, we use the following lemma.

▶ **Lemma 13.** *Let $E \subseteq \mathbb{R}^n$ be analytic with $\dim_H(E) = s$. Then there is a $\boldsymbol{\Sigma}^0_2$ set $F \subseteq E$ such that $\dim_H(F) = s$.*

Proof. It is well known that if $E \subseteq \mathbb{R}^n$ is analytic, then for every $\varepsilon \in (0, s]$, there is a compact subset $E_\varepsilon \subseteq E$ such that $\dim_H(E_\varepsilon) = s - \varepsilon$ (see e.g. Bishop and Peres [2]). Thus, the set

$$F = \bigcup_{i=\lceil 1/s \rceil}^{\infty} E_{1/i}$$

is a $\boldsymbol{\Sigma}^0_2$ set with $\dim_H(F) = s$. ◀

We will also use the following observation, which is a consequence of the well-known fact from descriptive set theory that Σ classes are closed under computable projections.

▶ **Observation 14.** *Let $E \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{N}$ be such that E is a Σ^0_2 set relative to A. Then for every $e \in S^{n-1}$, $\operatorname{proj}_e E$ is a Σ^0_2 set relative to (A, e).*

Finally, we must ensure that e does not significantly change the complexity of z. For this, we will use the following definition and theorem due to Calude and Zimand [3]. We rephrase their work in terms of points in Euclidean space. Let $n \in \mathbb{N}$, $z \in \mathbb{R}^n$ and $e \in S^{n-1}$. We say that z and e are *independent* if, for every $r \in \mathbb{N}$, $K^e_r(z) \geq K_r(z) - O(\log r)$ and $K^z_r(e) \geq K_r(e) - O(\log r)$.

▶ **Theorem 15** ([3]). *For every $z \in \mathbb{R}^n$, for almost every $e \in S^{n-1}$, z and e are independent.*

With these ingredients we can give a new proof Marstrand's projection theorem using algorithmic information theory.

Proof of Theorem 1. Let $E \subseteq \mathbb{R}^n$ be analytic with $\dim_H(E) = s$. By Lemma 13, there is a $\boldsymbol{\Sigma}^0_2$ set $F \subseteq E$ such that $\dim_H(F) = s$. Let $A \subseteq \mathbb{N}$ be an oracle such that F is Σ^0_2 relative to A. Using Theorem 6, for every $k \in \mathbb{N}$ we may choose a point $z_k \in F$ such that

$$\dim^A(z_k) \geq s - 1/k\,.$$

Let $e \in S^{n-1}$ be a point such that, for every $k \in \mathbb{N}$, the following hold.

- For every r and $t < r$, $K^{A,z_k,e_3\dots,e_n}_t(e) \geq t - O(1)$.
- For every r, $K^{A,e}_r(z_k) \geq K^A_r(z_k) - O(\log r)$.

A basic fact of algorithmic randomness states that almost every e satisfies the first item. By Theorem 15, almost every e satisfies the second item. So almost every e satisfies these requirements.

Fix $k \in \mathbb{N}$. Let $\eta' \in \mathbb{Q} \cap (0,1) \cap (0, \dim^A(z_k))$ and $\varepsilon' > 0$. It is clear, by our choices of e and z_k, that the conditions of Theorem 12 are satisfied for all sufficiently large r. We may therefore apply Theorem 12, resulting in

$$K_r^{A,e}(e \cdot z_k) \geq \eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1-\eta'} r - K(2\varepsilon') - K(\eta') - O_z(\log r)\,.$$

Hence,

$$\begin{aligned}
\dim^{A,e}(e \cdot z_k) &= \liminf_{r\to\infty} \frac{K_r^{A,e}(e \cdot z_k)}{r} \\
&\geq \liminf_{r\to\infty} \frac{\eta' r - \varepsilon' r - \frac{2n\varepsilon'}{1-\eta'} r - K(2\varepsilon') - K(\eta') - O_z(\log r)}{r} \\
&= \eta' - \varepsilon' - \frac{2n\varepsilon'}{1-\eta'}\,.
\end{aligned}$$

Since both η' and ε' were chosen independently and arbitrarily, we see that

$$\begin{aligned}
\dim^{A,e}(e \cdot z_k) &\geq \dim^{A,e}(z_k) \\
&\geq \min\{s, 1\} - 1/k\,.
\end{aligned}$$

As k was chosen arbitrarily, Observation 14 and Theorem 6 give

$$\begin{aligned}
\dim_H(\operatorname{proj}_e E) &\geq \dim_H(\operatorname{proj}_e F) \\
&= \sup_{z\in F} \dim^{A,e}(e \cdot z) \\
&\geq \sup_{k\in\mathbb{N}} \dim^{A,e}(e \cdot z_k) \\
&= \min\{s, 1\}\,,
\end{aligned}$$

and the proof is complete. ◀

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