Enumerating Minimal Transversals of Hypergraphs without Small Holes

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Abstract

We give a polynomial delay algorithm for enumerating the minimal transversals of hypergraphs without induced cycles of length 3 and 4. As a corollary, we can enumerate, with polynomial delay, the vertices of any polyhedron $\mathcal{P}(A, \underline{1}) = \{x \in \mathbb{R}^n \mid Ax \geq \underline{1},\ x \geq \underline{0}\}$, when A is a balanced matrix that does not contain as a submatrix the incidence matrix of a cycle of length 4. Other consequences are a polynomial delay algorithm for enumerating the minimal dominating sets of graphs of girth at least 9 and an incremental delay algorithm for enumerating all the minimal dominating sets of a bipartite graph without induced 6 and 8-cycles.

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1 Introduction

The task of an *enumeration algorithm* is generating all the feasible solutions of a given property, such as enumerating all the maximal cliques of a graph or all the triangulations of a given set of points in a d-dimensional space. In enumeration algorithms the size of the output is often exponential in the size of the input, therefore, to define the tractability of enumeration problems, the complexity is measured based on the needed total time of the algorithm depending on the size of the input and the size of the output. If the total running time of the algorithm is bounded by a polynomial on the size of the input and the output, the algorithm is called *output-polynomial.* For a good survey of various combinatorial

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enumeration problems and their known time complexities see [32]. It is worth noticing that, unless P=NP, there are enumeration problems where no output polynomial enumeration algorithm exist [22, 23].

In the area of enumeration algorithms, enumerating all the inclusion-wise minimal transversals of a hypergraph[2], known as Hypergraph Dualisation, is a long-standing open problem which arises in different areas of computer science such as data mining [3], game theory [29, 19], artificial intelligence [12, 31], databases [8, 18], learning theory [1] and integer programming [5, 13]. Despite all the attempts, the complexity of the problem is not settled yet. The best-known algorithm for solving Hypergraph Dualisation in the general case is by Fredman and Khachiyan [14] (see also [24]) which solves the equivalent problem of monotone Boolean duality in quasi-polynomial time. Nevertheless, for several well-structured hypergraph classes, output-polynomial algorithms for Hypergraph Dualisation is known, *e.g.*, [11, 25, 22] to cite a few. It was also proved in [20] that the Hypergraph Dualisation is equivalent to the enumeration of minimal dominating sets in graphs, allowing to tackle this old problem in the realm of graph theory (see for instance [20, 16]). In this paper, we investigate the Hypergraph Dualisation problem in the class of hypergraphs without induced small cycles. A *k-cycle* in a hypergraph $\mathcal{H}$ is a sequence $(x_0, E_0, x_1, E_1, x_2, \cdots, x_{k-1}, E_{k-1}, x_0)$ where the x_i's belong to $V(\mathcal{H})$, the E_i's are in $\mathcal{H}$, $x_0 \in E_{k-1} \cap E_0$ and, for $1 \leq i \leq k-1$, $x_i \in E_i \cap E_{i-1}$. A *chord* in a k-cycles is a pair (x_i, E_j) where $j \notin \{i, (i-1) \mod k\}$. An enumeration algorithm is said to be of *polynomial delay* if the time between two outputs is bounded a polynomial on the input. Our main theorem is the following.

▶ **Theorem 1.** *Let $\mathcal{H}$ be a hypergraph without chordless 3 and 4-cycles. Then, one can enumerate with polynomial delay the minimal transversals of $\mathcal{H}$.*

The first consequence of our main theorem is a polynomial delay algorithm for listing the minimal dominating sets of graphs with girth at least 9, answering a question from [17]. The girth of a graph in G is the shortest chordless cycle[3] in G. Notice that the recent paper [26], proposes an enumeration algorithm, with a constant time delay, which enumerates all the (not necessarily minimal) dominating sets of a given graph of girth at least 9.

▶ **Corollary 2.**

a. *There is a polynomial delay algorithm that enumerates all the minimal dominating sets of a given input graph G of girth at least 9. More precisely, the result holds if G does not contain induced $(4, 5, 6, 7, 8)$-cycles.*

b. *There is an incremental delay algorithm that enumerates all the minimal dominating sets of a given bipartite graph without chordless 6 and 8-cycles.*

Enumeration algorithms also appear in computational geometry. The famous Minkowski-Weyl theorem states that every convex polyhedron can be represented as the intersection of finitely many affine half spaces, known as *H-representation*, and by the Minkowski sum of a polytope and a finitely generated cone, known as *V-representation*, while the two representations are equivalent. There are various open enumeration problems in this area, such as *facet enumeration* and *convex hull problem*. We refer to [15] for more study. One of the important enumeration problems in computational geometry is *vertex enumeration problem*

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[2] A hypergraph $\mathcal{H}$ is a collection of subsets of a ground set $V(\mathcal{H})$ and a *transversal* of $\mathcal{H}$ is a subset T of $V(\mathcal{H})$ that intersects all sets in $\mathcal{H}$.

[3] A k-cycle in a graph G is a sequence $(v_1, v_2, \ldots, v_k)$ where v_i and v_{i+1} are adjacent in G and v_1 is adjacent with v_k. A k-cycle $(v_1, \ldots, v_k)$ is chordless if there are no other edges between the v_i's.

which asks for generating all the vertices of a polyhedron given by its H-representation. Khachiyan et al. [23] proved that enumerating all the vertices of a rational polyhedron, given as an intersection of finitely many half spaces is an NP-hard enumeration problem. It's worth mentioning that the complexity of the problem in the case of polytopes (bounded polyhedrons), yet remains open. The hardness of enumerating the vertices of a polyhedron is more interesting when it comes true even for 0/1 polyhedrons (polyhedrons with vertices in $\{0,1\}^n$) [6] which is in contrast with the fact that all the vertices of a 0/1 polytope are enumerable in *polynomial delay* [9]. Regardless of the hardness of vertex enumeration for general polyhedrons, it is interesting to ask for which classes of polyhedrons the problem is tractable.

As a consequence of our main theorem, we also obtain a polynomial delay algorithm for enumerating the vertices of a large subclass of 0/1 polyhedrons given by *balanced* matrices. For doing so we use the known equivalent characterisation in terms of a Hypergraph Dualisation problem. Let us define the problem formally. Let $A \in \{0,1\}^{m\times n}$ and $\underline{1}$ and $\underline{0}$ be respectively all ones and all zeros vectors with appropriate size and $\mathcal{P}(A,\underline{1}) = \{x \in \mathbf{R}^n \mid Ax \geq \underline{1},\ x \geq \underline{0}\}$ be a polyhedron with only integral vertices. In other words, $\mathcal{P}(A,\underline{1})$ is the *set covering* polyhedron with the 0/1 *ideal* matrix A. It is well-known and not hard to see that the vertices of $\mathcal{P}(A,\underline{1})$ are in bijection with the minimal transversals of the hypergraph $\mathcal{H}[A]$, where the columns of A correspond to vertices of $\mathcal{H}[A]$ and the rows of A are incident vectors of the hyperedges of $\mathcal{H}[A]$ [27]. This gives an equivalence between the vertex enumeration problem for $\mathcal{P}(A,\underline{1})$ and the Hypergraph Dualisation problem for such hypergraphs $\mathcal{H}[A]$. The existence of the quasi-polynomial algorithm for enumerating the minimal transversals of a hypergraph [14] suggests that the complexity of vertex enumeration for $\mathcal{P}(A,\underline{1})$ is unlikely to be in NP.

In the Recent paper [13], Elbassioni and Makino have given an incremental polynomial time algorithm for enumerating the vertices of $\mathcal{P}(A,\underline{1})$ when A is a 0/1 totally unimodular matrix. Totally unimodular matrices are an important class of matrices for integer programming with the property that every square submatrix of it has determinant 0 or 1. The generating method in [13] is based on enumerating the transversals of the associated hypergraph and Seymour's fundamental decomposition theorem for totally unimodular matrices [30]. As an interesting open problem, one may ask about the existence of an output polynomial algorithm for enumerating the vertices of $\mathcal{P}(A,\underline{1})$ when A is a *balanced* matrix [13]. A 0/1 matrix is *balanced* if it does not contain a submatrix that is the incidence matrix of a cycle of odd length (see [30, Chapter 21] or [10]). Totally unimodular matrices are a proper subset of balanced matrices. A consequence of our main theorem is the following.

▶ **Theorem 3.** *There is a polynomial delay algorithm for listing the vertices of any given 0/1 polyhedron $\mathcal{P}(A,\underline{1})$ whenever A is a balanced matrix without any submatrix that is the incident matrix of a 4-cycle.*

As the algorithm needs some technical definitions, we postpone the details of the algorithm to Section 2. The main technical part of the paper is in Section 3 where we prove the main theorem.

2 Definitions and Preliminaries

The power set of a set V is denoted by 2^V, and for two sets A and B, we let $A \setminus B$ denote the set $\{x \in A \mid x \notin B\}$.

A hypergraph $\mathcal{H}$ is a collection of subsets of a finite ground set. The elements of $\mathcal{H}$ are called the *hyperedges* of $\mathcal{H}$ and the *vertex set* of $\mathcal{H}$ is $V(\mathcal{H}) := \bigcup_{E\in\mathcal{H}} E$. Given $S \subseteq V(\mathcal{H})$, we denote by $\mathcal{H}[S]$ the *hypergraph induced by* S, that is, $\mathcal{H}[S] := \{E \cap S \mid E \in \mathcal{H}\}$. Any subset $\mathcal{H}'$ of $\mathcal{H}$ is called a *sub-hypergraph* of $\mathcal{H}$. Notice that if there exists $E \in \mathcal{H}$ such that $E \subseteq V(\mathcal{H}) \setminus S$, then $\emptyset \in \mathcal{H}[S]$.

Given a hypergraph $\mathcal{H}$ and a subset $S \subseteq V(\mathcal{H})$ of its vertex set, we denote by $\mathcal{H}(S)$ the sub-hypergraph $\{E \in \mathcal{H} \mid S \cap E \neq \emptyset\}$; and for $v \in V(\mathcal{H})$, we write $\mathcal{H}(v)$ instead of $\mathcal{H}(\{v\})$. Notice that $\mathcal{H}(v)$ is the set of hyperedges containing v.

We assume that each hypergraph is given with an ordering $\leq$ of its set of vertices.

A *k-hole* in a hypergraph is a chordless cycle of length k.

A *transversal* (or hitting set) of a hypergraph $\mathcal{H}$ is a set $T \subseteq V(\mathcal{H})$ that has a nonempty intersection with every hyperedge $E \in \mathcal{H}$. A transversal T is said *minimal* if no proper subset of T is a transversal. For $T \subseteq V(\mathcal{H})$ and $x \in T$, we let $\mathcal{P}_T(x) := \{E \in \mathcal{H} \mid E \cap T = \{x\}\}$, and call it the set of *privates* of x with respect to T (we may drop the "with respect to T" when T is clear from the context). We say that $B \subseteq V(\mathcal{H})$ *breaks* $x \in T$, if for every $E \in \mathcal{P}_T(x)$, $E \cap B \neq \emptyset$ and if $B = \{b\}$, we say that b breaks x, for short.

We call $T \subseteq V(\mathcal{H})$ *irredundant* if $\mathcal{P}_T(x) \neq \emptyset$ for all $x \in T$. It is well-known that T is a minimal transversal if and only if T is a transversal and is irredundant. We denote by $\mathsf{tr}(\mathcal{H})$ the set of minimal transversals of $\mathcal{H}$. Observe that if all hyperedges of $\mathcal{H}$ are non-empty, then $\mathsf{tr}(\mathcal{H}) \neq \emptyset$. For more definitions and details on hypergraphs, we refer to [4].

▶ **Definition 4.** For $\ell \in V(\mathcal{H})$ and $E \in \mathcal{H}$ such that $\ell \in E$, we let $S(\ell, E)$, called a *double star*, be the sub-hypergraph $\mathcal{H}(\ell)[E]$. A double star is called *valid* if $(\mathcal{H} \setminus \mathcal{H}(\ell))[V(\mathcal{H}) \setminus E]$ does not contain the empty set.

The notion of double star is defined and used in [10] for the decomposition of balanced matrices. We rephrase it in terms of hypergraphs. Observe also that if a hypergraph is *Sperner* (no hyperedge contains another hyperedge), then every double star is valid. Even though for the Hypergraph Dualisation problem, it is enough to consider Sperner hypergraphs, we prefer giving the definition above for general hypergraphs for a better readability of our algorithms as we manipulate induced sub-hypergraphs. We often use the notation S for the double star $S(\ell, E)$ in customary whenever ℓ and E are clear from the context.

▶ **Fact 5.** *If $\emptyset \notin \mathcal{H}$, then $\mathcal{H}$ has a valid double star and it can be found in polynomial time*

Proof. Let $T \in \mathsf{tr}(\mathcal{H})$, which exists because $\emptyset \notin \mathcal{H}$. Let $\ell \in T$ and $E \in \mathcal{P}_T(\ell)$. Since T is a transversal and $E \cap T = \{\ell\}$, then each hyperedge in $\mathcal{H} \setminus \mathcal{H}(\ell)$ has a nonempty intersection with $T \setminus \{\ell\}$. Now, since $T \setminus \{\ell\} \subseteq V(\mathcal{H}) \setminus E$ because $E \cap T = \{\ell\}$, we can conclude that $(\mathcal{H} \setminus \mathcal{H}(\ell))[V(\mathcal{H}) \setminus E]$ does not contain the $\emptyset$ as a hyperedge. ◀

The algorithm for enumerating minimal transversals uses the standard technique which consists, for a hypergraph $\mathcal{H}$, in choosing a vertex ℓ and enumerate the minimal transversals that do not contain ℓ, denoted by $\mathbf{Inc}(\mathcal{H}, \ell)$, and those that do contain ℓ, denoted by $\mathbf{Exc}(\mathcal{H}, \ell)$. For enumerating the minimal transversals that do not contain ℓ, it suffices to make a recursive call to $\mathcal{H}[V(\mathcal{H}) \setminus \{\ell\}]$, once we ensure that one exists (which can be checked in polynomial time). But, enumerating the minimal transversals containing ℓ is a tough task and is exactly what makes the enumeration of $\mathsf{tr}(\mathcal{H})$ difficult because such a strategy causes to ask at each step the following NP-complete problem [7]: Given $X \subseteq V(\mathcal{H})$, does there exist a minimal transversal including X? In order to avoid this NP-complete problem, we use the following strategy, which depends heavily on the fact that 3-holes and 4-holes are forbidden:

1. We always choose ℓ to be in a valid double star $S(\ell, E)$.
2. We secondly show that, for any minimal transversal T of $\mathcal{H} \setminus S(\ell, E)$, $T \cup \{\ell\}$ is a minimal transversal of $\mathcal{H}$. Such minimal transversals are called *basic*.
3. We then show that any minimal transversal containing ℓ is either basic or can be obtained from a basic one by successively applying a *flipping method*. The flipping method yields a parent-child relation between the minimal transversals containing ℓ.
4. We finally use this parent-child relation to enumerate (with polynomial delay) the minimal transversals containing ℓ.

Let $\mathcal{H}$ be a hypergraph with $n := |V(\mathcal{H})| + \sum_{F \in \mathcal{H}} |F|$. An *enumeration algorithm* for $\mathsf{tr}(\mathcal{H})$ is an algorithm that lists all the minimal transversals without repetitions. An enumeration algorithm $\mathbb{A}$ for $\mathsf{tr}(\mathcal{H})$ which terminates in time $p(n, |\mathsf{tr}(\mathcal{H})|)$ for some polynomial $p(x, y)$ is called *output-polynomial* and it is called *polynomial space* if it uses a space bounded by a polynomial in n. Assume now that $T_1, \ldots, T_m$ are the elements of $\mathsf{tr}(\mathcal{H})$ enumerated in the order in which they are generated by $\mathbb{A}$. Let us denote by $T(\mathbb{A}, i)$ the time $\mathbb{A}$ requires until it outputs T_i, also $T(\mathbb{A}, m+1)$ is the time required by $\mathbb{A}$ until it stops. Let $delay(\mathbb{A}, 1) = T(\mathbb{A}, 1)$ and $delay(\mathbb{A}, i) = T(\mathbb{A}, i) - T(\mathbb{A}, i-1)$. The *delay* of $\mathbb{A}$ is $\max\{delay(\mathbb{A}, i)\}$. Algorithm $\mathbb{A}$ is a *polynomial delay* algorithm if there is a polynomial $p(x)$ such that the delay of $\mathbb{A}$ is at most $p(n)$.

The remainder of this paper is as follows. In Section 3 we define the flipping method, the basic minimal transversals and the resulting parent-child relation. We also prove that if $S(\ell, E)$ is a valid double star, then any minimal transversal containing ℓ can be obtained from a basic minimal transversal by following the parent-child relation. The algorithm for enumerating the children of a minimal transversal containing ℓ is given in Section 3.2.

3 Enumeration of minimal transversals including ℓ from a valid double star $S(\ell, E)$

3.1 Basic transversals, flipping operation and parent-child relation

In this section, we introduce the family of minimal transversals $\mathcal{B}$, called *basic transversals*, which will be used as a *base* for generating all the minimal transversals T containing the vertex ℓ. In the first step, a valid double star $S(\ell, E)$ is fixed for $\mathcal{H}$.

▶ **Fact 6.** *For every minimal transversal T of $(\mathcal{H} \setminus \mathcal{H}(\ell))[V(\mathcal{H}) \setminus E]$, $T \cup \{\ell\}$ is a minimal transversal of $\mathcal{H}$.*

Proof. Since T is a minimal transversal of $(\mathcal{H} \setminus \mathcal{H}(\ell))[V(\mathcal{H}) \setminus E]$, $T \subseteq V(\mathcal{H}) \setminus E$ and each vertex of T has a private in $\mathcal{H} \setminus \mathcal{H}(\ell)$. As $\{\ell\}$ is a minimal transversal of $\mathcal{H}(\ell)$ and $E \cap T = \emptyset$, we can conclude that $T \cup \{\ell\}$ is a minimal transversal of $\mathcal{H}$. ◀

We denote by $\mathcal{B}(\ell, E)$ the set $\{T \cup \{\ell\} \mid T \in \mathsf{tr}((\mathcal{H} \setminus \mathcal{H}(\ell))[V(\mathcal{H}) \setminus E])\}$ and call it the set of *basic transversals* of $\mathcal{H}$. We will show that one can generate all the minimal transversals of $\mathcal{H}$ that contain ℓ by doing flipping operations, starting from $\mathcal{B}(\ell, E)$.

Recall that $S(\ell, E)$ is a fixed double star of a fixed hypergraph $\mathcal{H}$. The objective is to define a way to generate the set of all minimal transversals containing the vertex ℓ, starting with the basic transversals. We first define a parent-child relation based on a flipping operation which results in removing one vertex, from E, at a time. As a consequence, each minimal transversal containing ℓ will be reachable, by following the parent-child relation,

Algorithm 1: GreedyPair.

Input: $T \subseteq V$ and the largest succedent vertex x in T
Output: $(Y, (Z_y)_{y \in Y})$
Function *GreedyPair*$(\mathcal{H}, T, x)$
  $Y := \emptyset$;
  while $\mathcal{P}_T(x)$ *is not empty* **do**
    Choose the smallest $y \in V(\mathcal{P}_T(x)) \setminus E$ such that $\exists\ F \in \mathcal{P}_T(x)$, $F \setminus E \subseteq \{v \in V(\mathcal{H}) \mid v \leq y\}$;
    $Y := Y \cup \{y\}$;
    $Z_y := \{z \in T \mid \mathcal{P}_T(z) \subseteq \mathcal{H}(y)\}$;
    $T := (T \cup \{y\}) \setminus Z_y$;

from a basic transversal. In a second step, we explain how to generate the children of any minimal transversal containing ℓ. Let's denote by $\mathbf{Inc}(\mathcal{H}, \ell, E)$ the set of minimal transversals of $\mathcal{H}$ containing ℓ where $S(\ell, E)$ is a valid double star of $\mathcal{H}$.

▶ **Definition 7.** Let T be an irredundant set of $\mathcal{H}$. A vertex x in T is called a *succedent vertex* if $x \in E \setminus \{\ell\}$.

Observe that a minimal transversal is basic if and only if it does not contain any succedent vertex. Also, if x is a succedent vertex of $T \in \mathbf{Inc}(\mathcal{H}, \ell, E)$, then $\mathcal{P}_T(x) \subseteq \mathcal{H} \setminus \mathcal{H}(\ell)$ because $\ell \in T$.

▶ **Definition 8.** Let T be an irredundant set containing succedent vertices and let $x \in T$ be the largest succedent vertex of T with respect to the ordering $\leq$. We call $(Y, (Z_y)_{y \in Y})$ a *greedy pair of* $\mathcal{P}_T(x)$ if

1. $Y \subseteq V(\mathcal{P}_T(x)) \setminus E$ and $Z_y \subseteq T$ for each $y \in Y$,
2. Y is a minimal transversal of $\mathcal{P}_T(x)$,
3. for each $y \in Y$, there is a hyperedge $F \in \mathcal{P}_{T_y}(x)$ such that $y \in F$ and $F \setminus E \subseteq \{v \in V(\mathcal{H}) \mid v \leq y\}$, where $T_y := (T \cup \{y' \in Y \mid y' < y\}) \setminus (\cup_{y' < y} Z_{y'})$ for each $y \in Y$,
4. for each $y \in Y$, $Z_y := \{z \in T \mid \mathcal{P}_{T_y}(z) \subseteq \mathcal{H}(y)\}$ with T_y as defined above.

▶ **Fact 9.** *The greedy pair of $\mathcal{P}_T(x)$ is unique and is computed in polynomial time by the function* GreedyPair *depicted in Algorithm 1.*

Proof. It is easy to see that the function GreedyPair in algorithm 1 runs in polynomial time and its output, $(Y, (Z_y)_{y \in Y})$, is a greedy pair. Assume that there is another greedy pair $(Y', (Z'_y)_{y \in Y'})$. It is enough to show that $Y = Y'$ since $(Z_y)_{y \in Y}$ and $(Z'_y)_{y \in Y'}$ are determined completely by Y and Y', respectively. Let us enumerate Y and Y' as $y_1 < y_2 < \cdots < y_k$ and $y'_{i_1} < y'_{i_p} < \cdots < y'_{i_p}$, respectively. Let j the smallest such that $y'_{i_j} \neq y_j$. If y'_{i_j} does not exist, then Y' cannot be a transversal of $\mathcal{P}_T(x)$ and similarly if y_j does not exist. So, let us assume that such a j exists. If $y'_{i_j} < y_j$, then there is an edge $F_{i_j} \in \mathcal{P}_T(x)$ which does not intersect Y because F_{i_j} does not intersect $\{y_1, \ldots, y_{j-1}\}$ and $F_{i_j} \setminus E \subseteq \{w \in V(\mathcal{H}) \mid w < y_{i_j}\}$ (Condition (3) of Definition 8). Similarly, if $y_j < y_{i_j}$, there is an edge $F_j \in \mathcal{P}_T(x)$ which does not intersect Y' as F_j does not intersect $\{y_1, \ldots, y_{j-1}\}$ and $F_j \setminus E \subseteq \{w \in V(\mathcal{H}) \mid w < y_j\}$. In both cases, we contradict the fact that Y or Y' is a transversal of $\mathcal{P}_T(x)$. ◀

If $(Y, (Z_y)_{y \in Y})$ is the greedy pair of $\mathcal{P}_T(x)$, for a minimal transversal T, then Y is intended to replace x in T, but even though $(T \setminus \{x\}) \cup Y$ is a transversal, it is not necessarily minimal.

The set $\cup_{y\in Y} Z_y$ is the set to remove to obtain a minimal transversal. The next lemma allows to prove that $(T \cup Y) \setminus ((\cup_{y\in Y} Z_y) \cup \{x\})$ is a minimal transversal.

▶ **Lemma 10.** *Let T be an irredundant set of $\mathcal{H}$ containing succedent vertices and let x be the largest succedent vertex of T. Let $(Y, (Z_y)_{y\in Y})$ be the greedy pair of $\mathcal{P}_T(x)$ and let $Z := \cup_{y\in Y} Z_y$. Then, the following properties hold:*

- **a.** *For each vertex x' of $T \cap E$ different from x, and each $F' \in \mathcal{P}_T(x')$, we have $F' \cap Y = \emptyset$.*
- **b.** *For each hyperedge $F \in \mathcal{H} \setminus \mathcal{H}(x)$, we have $|F \cap Y| \leq 1$.*
- **c.** *For each $z \in Z$, there is exactly one $y \in Y$ such that $\mathcal{H}(y) \cap \mathcal{P}_T(z) \neq \emptyset$.*
- **d.** *If there are $z_i, z_j \in Z$ and $y \in Y$ such that $\mathcal{H}(y) \cap \mathcal{P}_T(z_i) \neq \emptyset$ and $\mathcal{H}(y) \cap \mathcal{P}_T(z_j) \neq \emptyset$, then there is no hyperedge in $\mathcal{H} \setminus \mathcal{H}(y)$ which includes both z_i and z_j.*
- **e.** *If there are $z \in Z$ and $y \in Y$ such that $\mathcal{H}(y) \cap \mathcal{P}_T(z) \neq \emptyset$, then $\mathcal{H}(x) \cap \mathcal{H}(z) \subseteq \mathcal{H}(x) \cap \mathcal{H}(y)$.*

Proof. All the proofs are by contradicting the fact that $\mathcal{H}$ is $(3, 4)$-hole free.

- **a.** Let $x' \neq x$ be a vertex of $T \cap E$ and $F' \in \mathcal{P}_T(x')$ and $y \in Y \cap F'$. Also, assume that $F \in \mathcal{P}_T(x)$, containing y. By the definition of the double star $S(\ell, E)$, the hyperedge E contains all the succedent vertices and therefore, (y, F, x, E, x', F', y) is a 3-hole in $\mathcal{H}$.
- **b.** Let $F \in \mathcal{H} \setminus \mathcal{H}(x)$ be a hyperedge containing two vertices $y_i \leq y_j$, both from Y. Let $F_i \in \mathcal{P}_Y(y_i) \cap \mathcal{H}(x)$ and $F_j \in \mathcal{P}_Y(y_j) \cap \mathcal{H}(x)$, which exist by the definition of a greedy pair. Then $(y_i, F, y_j, F_j, x, F_i, y_i)$ constitutes a 3-hole in $\mathcal{H}$.
- **c.** Let $N_i \in \mathcal{H}(y_i) \cap \mathcal{P}_T(z)$ and $N_j \in \mathcal{H}(y_j) \cap \mathcal{P}_T(z)$ for two distinct vertices y_i and y_j of Y. Notice that $N_i \neq N_j$ by (b). Let $F_i \in \mathcal{P}_T(x) \cap \mathcal{P}_Y(y_i)$ and $F_j \in \mathcal{P}_T(x) \cap \mathcal{P}_Y(y_j)$. Then $(z, N_i, y_i, F_i, x, F_j, y_j, N_j, z)$ gives a 4-hole in $\mathcal{H}$.
- **d.** Let $z_i, z_j \in Z$ such that there is a vertex $y \in Y$, and $F_i \in \mathcal{H}(y) \cap \mathcal{P}_T(z_i)$ and $F_j \in \mathcal{H}(y) \cap \mathcal{P}_T(z_j)$. Suppose that there is a hyperedge $F \in \mathcal{H} \setminus \mathcal{H}(y)$ that contains both z_i and z_j. Then, $(z_j, F, z_i, F_i, y, F_j, z_j)$ induces a 3-hole in $\mathcal{H}$.
- **e.** Let $F_z \in \mathcal{H}(y) \cap \mathcal{P}_T(z)$ and $N_z \in \mathcal{H}(x) \cap \mathcal{H}(z)$ such that $y \notin N_z$. Let $N_y \in \mathcal{H}(x) \cap \mathcal{H}(y)$. Then, $(x, N_z, z, F_z, y, N_y, x)$ is a 3-hole in $\mathcal{H}$. ◀

From Lemma 10, we can deduce the following.

▶ **Proposition 11.** *Let $T \in \mathbf{Inc}(\mathcal{H}, \ell, E)$ be a non-basic minimal transversal and let x be the largest succedent vertex of T. Let $(Y, (Z_y)_{y\in Y})$ be the greedy pair of $\mathcal{P}_T(x)$. Then, $T^* := (T \cup Y) \setminus ((\cup_{y\in Y} Z_y) \cup \{x\})$ belongs to $\mathbf{Inc}(\mathcal{H}, \ell, E)$. Moreover, T^* has one less succedent vertices than T.*

Proof. Let $Z := \cup_{y\in Y} Z_y$. By definition of T^*, it is clear that $\ell \in T^*$ and it has less succedent vertices than T since x is removed from T. By the definition of the greedy pair, each vertex $y \in Y$ has a private with respect to T^*, and by Lemma 10(a), each vertex of $(T \cap E) \setminus \{x\}$ has a private with respect to T^*. Also, by the definition of the greedy pair, each vertex z of $T \setminus (E \cup Z)$ has a private with respect to T^*. It remains to show that T^* is a transversal. If there is $F \in \mathcal{H}$ such that $F \cap T^* = \emptyset$, then by the definition of Z and by Lemma 10(e), F is not the private of any vertex $z \in Z$ and then, there are at least two distinct vertices z and z' both contained in $F \cap Z$. By Lemma 10(d), $z \in Z_{y_i}$ and $z' \in Z_{y_j}$ for two distinct y_i and y_j in Y. Let y_j be the largest such that there is $z' \in Z_{y_j}$ and $z' \in F$. Then, F necessarily belongs to the private of z' with respect to $(T \cup \{y' \in Y \mid y' < y_j\}) \setminus (\cup_{y'<y_j} Z_{y'})$, which contradicts the fact that $z' \in Z_{y_j}$. This concludes the proof. ◀

We are now ready to define the parent-child relation.

Algorithm 2: $\mathsf{Enum}(\mathcal{H}, \leq)$.

Input: A $(3,4)$-hole free hypergraph $\mathcal{H}$ and a linear ordering $\leq$ of $V(\mathcal{H})$.

begin
  Let $S(\ell, E)$ be a valid double star of $\mathcal{H}$
  foreach $T \in \mathsf{Enum}\ ((\mathcal{H} \setminus \mathcal{H}(\ell))[V(\mathcal{H}) \setminus E], \leq$ **do**
    output $(T \cup \{\ell\})$
    $\mathsf{Enum\text{-}Children}\ (T \cup \{\ell\})$
  $\mathsf{Enum}\ (\mathcal{H}[V(\mathcal{H}) \setminus \{\ell\}], \leq)$

▶ **Definition 12.** Let T be a non-basic minimal transversal T in $\mathbf{Inc}(\mathcal{H}, \ell, E)$. Let x be the largest succedent vertex of T and let $(Y, (Z_y)_{y \in Y})$ be the greedy pair of $\mathcal{P}_T(x)$. We call $T^* := (T \cup Y) \setminus ((\cup_{y \in Y} Z_y) \cup \{x\})$ the *parent* of T, and call T the *child* of T^* with respect to $(x, Y, (Z_y)_{y \in Y})$.

The following will be used to characterise the children of any minimal transversal in $\mathbf{Inc}(\mathcal{H}, \ell, E)$.

▶ **Fact 13.** *If T is a child of T^* with respect to $(x, Y, (Z_y)_{y \in Y})$ then for every $y \in Y$, $\mathcal{P}_T(x) \cap \mathcal{P}_{T^*}(y) \neq \emptyset$ and for all $z \in Z_{\min(Y)}$, $\mathcal{P}_T(z) \subseteq \mathcal{P}_{T^*}(\min(Y))$.*

▶ **Lemma 14.** *Let T be a child of T^* with respect to $(x, Y, (Z_y)_{y \in Y})$ and let $y_1 := \min(Y \setminus Y_0)$ where $Y_0 := \{y \in Y \mid \mathcal{P}_{T^*}(y) \subseteq \mathcal{H}(x)\}$. Let $Z' := \cup_{y \in Y \setminus \{y_1\}} Z_y$ and $C_{y_1} := V(\mathcal{H}) \setminus (T^* \cup E \cup \{w \in V(\mathcal{H}) \mid \exists t \in T \setminus Z_{y_1}$ s.t. $\mathcal{P}_{T \setminus Z_{y_1}}(t) \subseteq \mathcal{H}(w)\})$. Then, $Z_{y_1} \cup \{x\}$ is a minimal transversal of $\mathcal{P}_{(T^* \cup Z') \setminus (Y \setminus \{y_1\})}(y_1)[C_{y_1} \cup \{x\}]$.*

Proof. Notice first that by Fact 13, for each $z \in Z_{y_1}$, $\mathcal{P}_T(z) \subseteq \mathcal{P}_{T^*}(y_1)$ and since $T \setminus Z_{y_1} = (T^* \setminus Y) \cup (Z' \cup \{x\})$ and T is a minimal transversal, we can conclude that $Z_{y_1} \cup \{x\}$ is an irredundant set of $\mathcal{P}_{(T^* \cup Z') \setminus (Y \setminus \{y_1\})}(y_1)$ by Lemma 10(c). Now, because T is a minimal transversal of $\mathcal{H}$, no $z \in Z_{y_1}$ breaks the private of some $t \in T \setminus Z_{y_1}$. So, $Z_{y_1} \subseteq C_{y_1}$. Assume that there is $F \in \mathcal{P}_{(T^* \cup Z') \setminus (Y \setminus \{y_1\})}(y_1)$ such that $F \cap (Z_{y_1} \cup \{x\}) = \emptyset$. Because $T = (T^* \cup Z' \cup Z_{y_1} \cup \{x\}) \setminus Y$, we would have $T \cap F = \emptyset$, contradicting the fact that T is a minimal transversal. ◀

The proofs of the following are trivial from the definitions.

▶ **Lemma 15.** *For every non-basic minimal transversal T in $\mathbf{Inc}(\mathcal{H}, \ell, E)$, the parent of T can be computed in polynomial time.*

▶ **Proposition 16.** *The directed graph with vertex set $\mathbf{Inc}(\mathcal{H}, \ell, E)$ and arc set the pairs (T^*, T) such that T^* is the parent of T is acyclic.*

The algorithm consists now in doing a DFS traversal of the directed graph of Proposition 16. The description is given in Algorithm 2. In order to prove that it runs with polynomial delay, it remains to show that for each non-basic minimal transversal, its children can be enumerated with polynomial delay.

We will now prove in the next section that the children of any $T \in \mathbf{Inc}(\mathcal{H}, \ell, E)$ can be enumerated with polynomial delay and polynomial space.

3.2 Enumerating the children of $T \in \mathrm{Inc}(\mathcal{H}, \ell, E)$

Remember that $\mathcal{H}$ is given with an ordering $\leq$ of $V(\mathcal{H})$. Also, $S(\ell, E)$ is a valid double star of $\mathcal{H}$. For a minimal transversal $T^* \in \mathbf{Inc}(\mathcal{H}, \ell, E)$, let $\mathbf{Cov}(T^*) := \{x \in E \setminus T^* \mid$ for each $x' \in T^* \cap E,\ x' \leq x$ and $\mathcal{P}_{T^* \cup \{x\}}(x') \neq \emptyset\}$. The set $\mathbf{Cov}(T^*)$ is the set of vertices in E that can be added to T^* without breaking the privates of any $x' \in E \cap T^*$ and are therefore candidates for generating the children of T^*.

Let $x \in \mathbf{Cov}(T^*)$ and let $Y_0 := \{y \in T^* \setminus E \mid \mathcal{P}_{T^*}(y) \subseteq \mathcal{H}(x)\}$.

▶ **Lemma 17.** *If $Y_0 \neq \emptyset$, then $T_0 := (T^* \setminus Y_0) \cup \{x\}$ is a minimal transversal of $\mathcal{H}$.*

Proof. By the definition of Y_0 and of T_0, we can conclude that T_0 is an irredundant set. If there is $N \in \mathcal{H}$ not intersected by T_0, then $N \in \mathcal{H}(y) \cap \mathcal{H}(y')$ for two distinct vertices in Y_0. Let $F \in \mathcal{P}_{T^*}(y)$ and $F' \in \mathcal{P}_{T^*}(y')$. Then, (x, F, y, N, y', F', x) is a 3-hole in $\mathcal{H}$, a contradiction. ◀

▶ **Lemma 18.** *If T is a child of T^* with respect to $(x, Y, (Z_y)_{y \in Y})$, then $Y_0 \subseteq Y$.*

Proof. Let T be a child of T^* with respect to $(x, Y, (Z_y)_{y \in Y})$. We first claim that $Y_0 \cap T = \emptyset$. Suppose that there is $y_0 \in T \cap Y_0$. Since T is a child of T^*, then $T^* = (T \cup Y) \setminus (Z \cup \{x\})$ for some greedy pair $(Y, (Z_y)_{y \in Y}$ with $Z := \cup_{y \in Y} Z_y$. Therefore, $y_0 \notin Y$ and thus $y_0 \in T \setminus (Z \cup \{x\})$. Now, because $\mathcal{P}_{T^*}(y_0) \subseteq \mathcal{H}(x)$, we can conclude that there is $y \in Y$ and $F \in \mathcal{H}(y) \cap \mathcal{H}(y_0)$ with $F \in \mathcal{P}_T(y_0)$. Because $y \in Y$, then there is $F' \in \mathcal{P}_T(x)$ such that $F' \in \mathcal{P}_{T^*}(y_1)$. Therefore, $(x, F_0, y_0, F, y_1, F', x)$ is a 3-hole, contradicting that $\mathcal{H}$ is $(3, 4)$-hole free, for some $F_0 \in \mathcal{P}_{T^*}(y_0)$.

If T is a child of T^*, then by the previous claim, we know that $Y_0 \cap T = \emptyset$. Because $T^* = (T \cup Y) \setminus (\{x\} \cup Z)$, we can conclude that Y_0 is necessarily a subset of Y, otherwise it would not be a subset of T^*. ◀

Level-0-child. If $Y_0 \neq \emptyset$ and $T_0 := (T^* \setminus Y_0) \cup \{x\}$ is a child of T^* with respect to $(x, Y_0, \emptyset)$, we call T_0 the *level*-0 child of T^*.

Note. If $Y_0 = \emptyset$, we "symbolically" call $T^* \cup \{x\}$ the *level*-0 child of T^* with respect to $(x, \emptyset, \emptyset)$. Also, we say that $(\emptyset, \emptyset)$ is the greedy pair of $\mathcal{P}_{T^* \cup \{x\}}(x)$. Notice that $T^* \cup \{x\}$ is not a minimal transversal.

We will now characterise the other children of T^*. Before, let us first prove the following which is the base of the characterisation.

▶ **Lemma 19.** *Let T be an irredundant set containing succedent vertices and let x be the largest succedent vertex of T. Let $y_1 \in Y$ be the smallest such that $Z_{y_1} \neq \emptyset$. Then, $(Y, (Z_y)_{y \in Y})$ is the greedy pair of $\mathcal{P}_T(x)$ if and only if $(Y, (Z'_y)_{y \in Y})$ is the greedy pair of $\mathcal{P}_{T \setminus Z_{y_1}}(x)$ where*

$$Z'_y := \begin{cases} Z_y & \text{if } y \neq y_1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Let $(Y', (W_y)_{y \in Y'})$ be the greedy pair of $\mathcal{P}_{T \setminus Z_{\min(Y)}}(x)$. Because $(W_y)_{y \in Y'}$ is determined by Y', it is enough to prove that $Y = Y'$. Let us enumerate Y as $y_{t_1} < y_{t_2} < \cdots < y_{t_p} < y_1 < y_2 < \cdots < y_k$ with $Z_{y_{t_j}} = \emptyset$ for all $1 \leq j \leq p$. Then, Y' is the sequence $y_{t_1} < y_{t_2} < \cdots < y_{t_p} < y'_{i_1} < y'_{i_2} < \cdots < y'_{i_t}$. Let j be the smallest such that $y'_{i_j} \neq y_j$. Let $F_{i_j} \in \mathcal{P}_{T \setminus Z_{y_1}}(x)$ such that $F_{i_j} \setminus E \subseteq \{v \in V(\mathcal{H}) \mid v \leq y'_{i_j}\}$ and let $F_j \in \mathcal{P}_T(x)$ such that $F_j \setminus E \subseteq \{v \in V(\mathcal{H}) \mid v \leq y_j\}$. Observe that because $\mathcal{P}_T(x) \subseteq \mathcal{P}_{T \setminus Z_{y_1}}(x)$, then

$F_j \in \mathcal{P}_{T\setminus Z_{y_1}}(x)$. If $y_j < y'_{i_j}$, then y_j should necessarily belong to Y', contradicting the choice of j. So $y'_{i_j} < y_j$. Assuming that $F_{i_j} \in \mathcal{P}_T(x) \subseteq \mathcal{P}_{T\setminus Z_{y_1}}(x)$ contradicts the fact that $y'_{i_j} \notin Y$. So, $F_{i_j} \notin \mathcal{P}_T(x)$, *i.e.*, there is $z_1 \in Z_{y_1}$ such that $z_1 \in F_{i_j}$. Let $F_1 \in \mathcal{P}_T(z_1)$. Because $z_1 \in Z_{y_1}$, we have that $F_1 \in \mathcal{H}(y_1)$. Then, $(z_1, F_1, y_1, F_j, x, F_{i_j}, z_1)$ is a 3-hole in $\mathcal{H}$. ◀

Level-i children. Let $R := \{y \in T^* \setminus (E \cup Y_0) \mid \mathcal{P}_{T^*}(y) \cap \mathcal{H}(x) \neq \emptyset\}$. The high-level idea for generating the children of T^* with respect to x consists in, recursively, deleting a *correct* set of vertices $Y_0 \subseteq Y \subseteq R$ from T^* and assigning the privates of vertices in Y to x. The choice of Y is determined by checking whether there is a $Z \in \mathsf{tr}(\cup_{y\in Y}\mathcal{P}_{T^*}(y) \setminus \mathcal{H}(x))$ containing x such that $T := (T^* \setminus Y) \cup Z$ is a minimal transversal and child of T^*, which can be checked. As we will see, this step is independent of the choice of Z. In a second step, we will enumerate the set of suitable minimal transversals Z of $\cup_{y\in Y}\mathcal{P}_{T^*}(y) \setminus \mathcal{H}(x)$.

A collection $\mathcal{I}$ of subsets of a ground set V is an *accessible system* if for each $I \in \mathcal{I}$, there is an $i \in I$ such that $I \setminus \{i\} \in \mathcal{I}$. The two following lemmas show that the set of children of T^* is like an accessible system following the Y-parts of the greedy pairs.

▶ **Lemma 20.** *Suppose that $Y_0 = \emptyset$. Let T be a child of T^* with respect to $(x, Y, (Z_y)_{y\in Y})$, with $Y \subseteq R$. Then, $T' := (T \cup \{\min(Y)\}) \setminus Z_{\min(Y)}$ is a child of T^* with respect to $(x, Y \setminus \{\min(Y)\}, (Z_y)_{y\in Y\setminus\{\min(Y)\}})$.*

Proof. If $|Y| = 1$, then $T' = T^* \cup \{x\}$. That is a symbolic *level-0* child of T^* with respect to $(x, \emptyset, \emptyset)$. If $|Y| \geq 2$, let $y_1 := \min(Y)$. By Fact 13, $\mathcal{P}_T(z) \subseteq \mathcal{P}_{T^*}(y_1)$. Then using Lemma 10(d) and by definition, T' is a transversal of $\mathcal{H}$. In order to prove that it is minimal, we must show T' is irredundant. By the construction of Z_1, if y_1 breaks $t \in T$ then $t \in Z_1$. Also, since $|Y| \geq 2$, by Fact 13, $\mathcal{P}_T(x) \cap \mathcal{P}_{T^*}(y) \neq \emptyset$, when $y \in Y \setminus \{y_1\}$ and by the minimality of T^*, $\mathcal{H}(y_1) \cap \mathcal{P}_{T^*}(y) = \emptyset$. Therefore $\mathcal{P}_{T\cup\{y\}}(x) \neq \emptyset$ and T' is irredundant.

Now, by the GreedyPair Algorithm and uniqueness of the greedy pair, we can easily check that the greedy pair of $\mathcal{P}_{T'}(x)$ is exactly $(Y \setminus \{\min(Y)\}, (Z_y)_{y\in Y\setminus\{\min(Y)\}})$ and $T' \cup (Y \setminus \{\min(Y)\}) \setminus ((Z_y)_{y\in Y\setminus\{\min(Y)\}}) = T^*$ is the parent of T'. ◀

▶ **Lemma 21.** *Suppose that $Y_0 \neq \emptyset$. Let T be a child of T^* with respect to $(x, Y, (Z_y)_{y\in Y})$, with $Y \subseteq R \cup Y_0$ and $|Y \setminus Y_0| \geq 1$. Then, $T' := (T \cup \{\min(Y \setminus Y_0)\}) \setminus Z_{\min(Y\setminus Y_0)}$ is a child of T^* with respect to $(x, Y \setminus \{\min(Y \setminus Y_0)\}, (Z_y)_{y\in Y\setminus\{\min(Y\setminus Y_0)\}})$.*

Proof. Recall by Lemma 18 that $Y_0 \subsetneq Y$. By the similar argument as in Lemma 20, we can conclude that T' is a minimal transversal of $\mathcal{H}$ as in addition $\mathcal{P}_T(x) \setminus (\cup_{y\in Y\setminus Y_0}\mathcal{P}_{T^*}(y)) \neq \emptyset$. Again, GreedyPair Algorithm shows that T' is a child of T^* with respect to $(x, Y \setminus \{\min(Y \setminus Y_0)\}, (Z_y)_{y\in Y\setminus\{\min(Y\setminus Y_0)\}})$. ◀

In the following, we want to characterise the other children of T^*. For $1 \leq i \leq |R|$, we call T a *level-i child* of T^* if T is a child with respect to $(x, Y, (Z_y)_{y\in Y})$ with $|Y \setminus Y_0| = i$. We have seen by Lemmas 20 and 21 that if T is a level-i child of T^* with respect to $(x, Y, (Z_y)_{y\in Y})$, then $(T \setminus Z_{\min(Y\setminus Y_0)}) \cup \{\min(Y \setminus y_0)\}$ is a level-$(i-1)$ child of T^* with respect to $(x, Y \setminus \{\min(Y \setminus Y_0)\}, (Z_y)_{y\in Y\setminus\{\min(Y\setminus Y_0)\}})$. We have already seen how to generate the level-0 child. It remains now to explain how to generate the level-i children from the level-$(i-1)$ children.

Let T be a level-$(i-1)$ child of T^* with respect to $(x, Y, (Z_y)_{y\in Y})$. Let $\mathbf{Correct}(Y) := \{y \in R \setminus Y \mid y < \min(Y \setminus Y_0)\}$. Recall that $\mathbf{Correct}(Y) \subseteq T$. For $y \in \mathbf{Correct}(Y)$, we let $C_y := V(\mathcal{P}_{T\setminus\{x\}}(y)) \setminus (T^* \cup E \cup \{w \in V(\mathcal{H}) \mid \exists t \in T \setminus \{y\} \text{ s.t. } \mathcal{P}_{T\setminus\{y\}}(t) \subseteq \mathcal{H}(w)\})$. The set C_y is the set of vertices that are candidates, other than x, for computing minimal transversals of $\mathcal{P}_{T^*}(y)$, and such that they do not break the privates of any other vertices in T^*, except those of y. The following characterises the level-i children from level-$(i-1)$ children.

▶ **Proposition 22.** *Let $i \geq 1$. T is a level-i child of T^* if and only if there is T' a level-$(i-1)$ child of T^* with respect to $(x, Y, (Z_y)_{y\in Y})$ and $y_1 \in \mathbf{Correct}(Y)$ such that $(Y \cup \{y_1\}, (Z_y)_{y\in Y})$ is the greedy pair of $\mathcal{P}_{T'\setminus\{y_1\}}(x)$ and $T := (T'\setminus\{y_1\})\cup Z_{y_1}$ with $Z_{y_1}\cup\{x\}$ a minimal transversal of $\mathcal{P}_{T'\setminus\{x\}}(y_1)[C_{y_1} \cup \{x\}]$.*

Proof. Let T be a level-i child of T^* with respect to $(x, Y, (Z_y)_{y\in Y})$ and let $y_1 := \min(Y \setminus Y_0)$. By Lemma 21, $T' := (T\cup\{y_1\})\setminus Z_{y_1}$ is a child of T^* with respect to $(x, Y\setminus\{y_1\}, (Z_y)_{y\in Y\setminus\{y_1\}})$ and by Lemma 19, $(Y, (Z'_y)_{y\in Y})$ is the greedy pair of $\mathcal{P}_{T'\setminus\{y_1\}}(x)$ where

$$Z'_y := \begin{cases} Z_y & \text{if } y \neq y_1, \\ \emptyset & \text{otherwise.} \end{cases}$$

By Lemma 14, $Z_{y_1} \cup \{x\}$ is a minimal transversal of $\mathcal{P}_{T'\setminus\{x\}}(y)[C_{y_1} \cup \{x\}]$.

Let T' be a level-$(i-1)$ child of T^* with respect to $(x, Y, (Z_y)_{y\in Y})$ and let $y_1 \in \mathbf{Correct}(Y)$ be such that $(Y \cup \{y_1\}, (Z_y)_{y\in Y})$ is the greedy pair of $\mathcal{P}_{T'\setminus\{y_1\}}(x)$. Let $Z_{y_1} \subseteq C_{y_1}$ such that $Z_{y_1} \cup \{x\}$ is a minimal transversal of $\mathcal{P}_{T'\setminus\{x\}}(y_1)$. Let $T := (T' \setminus \{y_1\}) \cup Z_{y_1}$, which by definition is a transversal of $\mathcal{H}$. By definition of C_{y_1}, no $z \in Z_{y_1}$ breaks the privates, with respect to $T' \setminus \{y_1\}$, of some vertex $t \in T' \setminus \{y\}$ and since $Z_{y_1} \cup \{x\}$ is a minimal transversal of $\mathcal{P}_{T'\setminus\{x\}}(y_1)$, each vertex in $Z_{y_1} \cup \{x\}$ has a private with respect to T. Now, if there are $z, z' \in Z_{y_1} \cup \{x\}$ that break the privates, with respect to T, of some $t \in T' \setminus (\{x\} \cup Z_{y_1})$, then by definition of C_{y_1}, there are $F_1, F_2 \in \mathcal{P}_{T'\setminus\{y_1\}}(t)$, $F \in \mathcal{P}_{Z_{y_1}\cup\{x\}}(z)$, $F' \in \mathcal{P}_{Z_{y_1}\cup\{x\}}(z')$, both belonging to $\mathcal{P}_{T'\setminus\{x\}}(y)$. But, then $(z, F_1, t, F_2, z', F', y, F, z)$ is a 4-hole in $\mathcal{H}$. So, T is a minimal transversal of $\mathcal{H}$. By Lemma 19, $(Y \cup \{y_1\}, (Z'_y)_{y\in Y\cup\{y_1\}})$ is the greedy pair of $\mathcal{P}_T(x)$ where

$$Z'_y := \begin{cases} Z_y & \text{if } y \neq y_1, \\ Z_{y_1} & \text{otherwise.} \end{cases}$$

Therefore, T is a level-i child of T^*. ◀

Combining Lemma 21 and Proposition 22, we are now ready to prove that the algorithm Enum given in Algorithm 2 runs with polynomial delay.

Proof of Theorem 1. We claim that the algorithm Enum depicted in Algorithm 2 and combined with the algorithm Enum-Children enumerates the minimal transversals of a $(3, 4)$-hole free hypergraph with polynomial delay.

For any minimal transversal $T \in \mathsf{tr}(\mathcal{H})$, either $T \in \mathbf{Inc}(\mathcal{H}, \ell, E)$ or T belongs to $\mathsf{tr}(\mathcal{H}[V(\mathcal{H}) \setminus \{\ell\}])$. Since we enumerate both in Lines 3-6, we can conclude that the algorithm enumerates all the minimal transversals of $\mathcal{H}$ because of Proposition 16 and also, each non-basic minimal transversal has a parent. It remains to show that we enumerate all the children of each minimal transversal in $\mathbf{Inc}(\mathcal{H}, \ell, E)$.

Now, the function given in Algorithm 3 first outputs the level-0-child if it exists, otherwise it stops. Then, it calls Algorithm 4, which does a DFS on the tree where you have an arc (T', T) if T' is a level-$(i-1)$ child and T is a level-i child obtained from T' as stated in Proposition 22. One can, therefore, conclude that the algorithm correctly outputs all the children of a minimal transversal in $\mathbf{Inc}(\mathcal{H}, \ell, E)$. We can, therefore, conclude that by combining Algorithms 2, 3 and 4 we output exactly the set of minimal transversals.

Algorithm 3: Enum-Children$(\mathcal{H}, \leq, \ell, E, T)$.

Input: A $(3,4)$-hole free hypergraph $\mathcal{H}$, a linear ordering $\leq$ of $V(\mathcal{H})$, a valid double star $S(\ell, E)$ and $T \in \mathbf{Inc}(\mathcal{H}, \ell, E)$.

begin
  foreach $x \in \mathbf{Cov}(T)$ **do**
    Let $Y_0 := \{y \in T \mid \mathcal{P}_T(y) \subseteq \mathcal{H}(x)\}$ and $T_0 := ((T \setminus Y_0) \cup \{x\})$
    if $(Y_0, \emptyset)$ *is the greedy pair of* $\mathcal{P}_{T \setminus Y_0}(x)$ **then**
      if $Y_0 \neq \emptyset$ **then**
        output T_0
      Let $R := \{y \in T \setminus (E \cup Y_0) \mid \mathcal{P}_T(y) \cap \mathcal{H}(x) \neq \emptyset\}$
      Enum-ChildrenAux $(\mathcal{H}, \leq, \ell, E, T_0, x, R, Y_0)$

Algorithm 4: Enum-ChildrenAux$(\mathcal{H}, \leq, \ell, E, T, x, R, Y)$.

Input: A $(3,4)$-hole free hypergraph $\mathcal{H}$, a linear ordering $\leq$ of $V(\mathcal{H})$, a valid double star $S(\ell, E)$, T a transversal of $\mathcal{H}$, a succedent $x \in T$, R the set of candidates and Y the already chosen candidates.

begin
  foreach $y \in \mathbf{Correct}(Y)$ **do**
    if $Y \cup \{y\}$ *is the* Y*-part of the greedy pair of* $\mathcal{P}_{T \setminus \{y\}}(x)$ **then**
      foreach Z_y *containing* x *in* Enum $(\mathcal{P}_{T \setminus \{x\}}(x)[C_y \cup \{x\}], \leq)$ **do**
        output $((T \cup Z_y) \setminus \{y\})$
        Enum-ChildrenAux $(\mathcal{H}, \leq, \ell, E, (T \cup Z_y) \setminus \{y\}), x, R, Y \cup \{y\})$

Let us now analyse its time complexity. Let $n := |V(\mathcal{H})| + \sum_{E \in \mathcal{H}} |E|$. We first notice that one can combine both algorithms Enum and Enum-ChildrenAux into a single one which will do a DFS traversal of the tree of recursive calls (see for instance [2, 21]). Second, each call of Enum or of Enum-ChildrenAux, after a pre-processing polynomial in n, either outputs a new minimal transversal or exits. Therefore, the tree of combined recursive calls of both algorithms has size bounded by $O(n^c) \cdot |\mathsf{tr}(\mathcal{H})|$, for some universal constant c, *i.e.*, the amortised time complexity of the algorithm is $O(n^c)$. By using the same technique as in [28], which consists in outputting a solution at the beginning when the depth of the recursive call is odd, and at the end when the depth is even, one obtains the desired delay per solution. ◀

4 Related results and Conclusion

Enumerating all the minimal dominating sets of a graph is another interesting task in the area of enumeration algorithms with numerous applications (see for instance [20]). Our algorithm readily can enumerate the minimal dominating sets of graphs of girth at least 9. We refer to [26] and [17] for related works. The result of this paper also, gives an incremental delay algorithm for enumerating all the minimal dominating sets in a bipartite graph without induced cycles of length 6 and 8.

▶ **Corollary 2.**

a. *There is a polynomial delay algorithm that enumerates all the minimal dominating sets of a given input graph G of girth at least 9. More precisely, the result holds if G does not contain induced $(4, 5, 6, 7, 8)$-cycles.*

b. *There is an incremental delay algorithm that enumerates all the minimal dominating sets of a given bipartite graph without chordless 6 and 8-cycles.*

Proof of Corollary 2.

a. Let G be a graph of girth at least 9. Let $\mathcal{N}[G]$ be the hypergraph $\{N[x] \mid x \in V(G)\}$ where $N[x]$ is the set $\{x\} \cup \{y \in V(G) \mid y \text{ a neighbour of } x\}$. It is well-known that D is a minimal dominating set of G if and only if D is a minimal transversal of $\mathcal{N}[G]$. One can easily check that if G does not contain a chordless cycle of length strictly smaller than 9 and greater than 3, then $\mathcal{N}[G]$ cannot contain a chordless 3 or 4-cycle, because a cycle of length 3 in $\mathcal{N}[G]$ can only be obtained by an induced cycle of length at most 6 in G and a cycle of length 4 in $\mathcal{N}[G]$ by an induced cycle of length at most 8 in G. By Theorem 1, one can enumerate with polynomial delay all the minimal transversals of $\mathcal{N}[G]$.

b. Let $G := (R, B, E)$ be a bipartite graph without chordless 6 and 8-cycles. By Theorem 1 one can enumerate all the minimal sets $D \subseteq R$ such that D dominates B (called *red-blue dominating sets* in [16]). By using the flipping method in [17], one reduces the existence of an incremental delay enumeration algorithm for the minimal dominating sets of G to the existence of a polynomial delay enumeration algorithm for the minimal red-blue dominating sets in induced subgraphs of G. This concludes the proof. ◀

As we have already discussed in the introduction, the vertices of the polyhedron $\mathcal{P}(A, \underline{1}) = \{x \in \mathbb{R}^n \mid Ax \geq \underline{1},\ x \geq \underline{0}\}$ are in bijection with the minimal transversals of the corresponding hypergraph $\mathcal{H}[A]$, where the columns of A correspond to the vertices of $\mathcal{H}[A]$ and the rows of A are incident vectors of the hyperedges of $\mathcal{H}[A]$ [27]. If the coefficient matrix A in the polyhedron is balanced, then the corresponding hypergraph does not contain any odd-hole.

▶ **Theorem 3.** *There is a polynomial delay algorithm for listing the vertices of any given $0/1$ polyhedron $\mathcal{P}(A, \underline{1})$ whenever A is a balanced matrix without any submatrix that is the incident matrix of a 4-cycle.*

Proof of Theorem 3. Let A be a balanced matrix without any 4-cycle submatrix. Then, $\mathcal{H}[A]$, the hypergraph corresponding to the matrix A as explained above, does not contain chordless cycles of length 3 or 4. By Theorem 1 one can enumerate with polynomial delay the minimal transversals of $\mathcal{H}[A]$, which by [27] correspond to the vertices of $\mathcal{P}(A, \underline{1})$. ◀

We conclude the paper by observing that even though our algorithm in Theorem 1 is a polynomial delay one, it uses exponential space and it should be interesting to know whether one can modify it in order to use polynomial space. However, there are more challenging questions, and in particular, it is still open whether there is an output-polynomial time algorithm for enumerating the vertices of a polyhedron $\mathcal{P}(A, \underline{1})$ when A is balanced. We just notice that one needs another technique to deal with balanced hypergraphs as our technique cannot avoid the requirement of the hypergraph to be without chordless 4-cycles. A more challenging question in this area asks for the existence of an output-polynomial time algorithm for the vertices of bounded polyhedron [13].

In many enumeration algorithms, like ours in this paper or [2, 17], the enumeration is reduced to traverse a graph with vertex set the set of solutions, and the difficulty is usually how to generate the neighbors of a given solution. In our paper, we solve this problem by a rather technical, but nice parent-child relation based on the structure of the hypergraphs.

However, the used techniques in almost all such papers are ad-hoc (despite the nice attempts in [2]) and the area still lacks a general theory on identifying a large family of combinatorial enumeration problems on which such a technique works finely.

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