Low complexity bound on irregular LDPC belief-propagation decoding thresholds using a Gaussian approximation

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Since *irregular* low-density parity-check (LDPC) codes are known to perform better than regular ones, and to exhibit, like them, the so called "threshold phenomenon", this letter investigates a low complexity upper bound on belief-propagation decoding thresholds for this class of codes on memoryless BI-AWGN (Binary Input - Additive White Gaussian Noise) channels, with sum-product decoding. We use a simplified analysis of the belief-propagation decoding algorithm, i.e., consider a Gaussian approximation for message densities under density evolution, and a simple algorithmic method, defined recently, to estimate the decoding thresholds for regular and irregular LDPC codes.

Introduction: As first noticed by Gallager in his introductory work to regular LDPC codes [1], these exhibit the so called "threshold phenomenon". Namely, an upper bound for the channel noise can be defined by the noise threshold so that, if the channel noise is maintained below this threshold, the probability of lost information can be made as small as desired. Later it was shown in [2] that *irregular* LDPC codes perform better than regular ones, and exhibit this phenomenon, too.

LDPC codes are capacity-approaching codes, which means that practical constructions exist that allow the noise threshold to be set very close to the theoretical maximum (the Shannon limit) for a symmetric memoryless channel. Thus, the problem of an easy evaluation of the threshold, and, in general, of the performance of belief propagation decoding (see, e.g., [3] and [4]) is important to allow the design of capacity-approaching codes, based on noise threshold maximization.

Maximum Likelihood decoding of LDPC codes is in general not feasible [3]. Instead, Gallager proposed an iterative soft decoding algorithm, also called belief propagation [5]. Gallager also noted that, for any given channel conditions, it is possible to evaluate the performance of belief propagation by following the evolution of the distribution of the messages. This idea was extended in [6], where it was shown how to apply density evolution efficiently. One difficulty encountered when applying density evolution is given by the continuous nature of the messages which makes them hard to analyze. As an alternative, in [7] a Gaussian approximation for the message distribution was proposed, reducing the evolution of the infinite dimensional density space to the evolution of a single parameter. In this way, the mean value of a generic check node output message at the l-th iteration is simply described as a function of the check node output message mean value at the $(l-1)$-th iteration, thus obtaining a recurrent sequence. With this simplified description, the threshold can be calculated as the last value such that the recurrent sequence converges but no mathematical methods were provided in [7] to determine it.

In [8] it was presented a mathematical method to allow the noise thresholds evaluation using the quadratic degeneracy theory, thus transforming a recurrence relation convergence problem in a problem of mathematical analysis. Applying the result of [8] to the asymptotical behavior of the recurrent sequence thereby defined, a low complexity upper bound to the exact belief-propagation decoding thresholds can be derived. This analysis gives rise to a simple algebraic expression of the upper bound on irregular LDPC belief-propagation decoding thresholds using a Gaussian approximation, valid in the most common case (i.e., when the first non-zero coefficient of the d_l-uple $\{\lambda_i\}$ is λ_2), thus allowing its simple determination from the codes parameters.

Low complexity approximation of the exact belief-propagation decoding thresholds: Irregular LDPC codes [6] are defined by specifying the distribution of the node degrees in their Tanner graphs. In particular, in the edge-perspective degree distribution, λ_i (respectively ρ_j) is the fraction of edges in the Tanner graph connecting to a degree-i (respectively degree-j) variable (respectively check) node. To specify the degree distribution, the polynomials $\lambda(x)$ and $\rho(x)$ are defined, having degree d_l-1 and d_r-1, respectively. The d_l-uple $\{\lambda_i\}$ and d_r-uple $\{\rho_j\}$ both add up to 1.

Since in [7] it is shown that, without great sacrifice in accuracy, a one-dimensional quantity, namely the mean of Gaussian densities, can act as faithful surrogate for the message densities themselves, which is an infinite-dimensional vector, we assume that LDPC codes message distributions for AWGN channels are approximately Gaussian and denote the means of u and v by $m_u^{(l)}$ and $m_v^{(l)}$ at the l-th iteration, respectively. Moreover, the LLR message u_0 from the channel is assumed to be Gaussian with mean $m_{u_0}=2/\sigma_n^2$ and variance $4/\sigma_n^2$, where σ_n^2 is the variance of the channel noise. Defining, as in [8],

$$g_s(t)=\sum_{j=2}^{d_r}\rho_j g_{s,j}(t) \tag{1}$$

where

$$g_{s,j}(t)=\phi^{-1}\big(1-\big[1-\sum_{i=i_1}^{d_l}\lambda_i\phi(s+(i-1)t)\big]^{j-1}\big) \tag{2}$$

(being the function $\phi(x)$ defined in [7] and $s=m_{u_0}$), and applying the method therewith defined, instead of searching the last value of the parameter s granting the convergence of the sequence

$$t_l=g_s(t_{l-1}) \tag{3}$$

where t_l corresponds to $m_u^{(l)}$, we solve a problem of quadratic degeneracy which can be assigned to a standard software.

When the second derivative $g_s''(t)\neq 0$ the problem of quadratic degeneracy is the system of equations

$$\begin{cases} g_s(t)=t \\ g_s'(t)=1 \end{cases} \tag{4}$$

Its solution (t^*,s^*) determines an approximation σ^* of the exact belief-propagation decoding threshold defined as $\sigma^*:=\sqrt{\frac{2}{s^*}}$.

To find the solution of (4), an explicitly invertible approximation of $\phi(x)$ is needed. In [7], an "ad hoc" approximation by elementary functions is given (see Eq. (8) in [7]) for $0<x<10$, which is explicitly invertible. A graph of $\phi(x)$ may be found in [9], where the approximation of $\phi(x)$ and its inverse were derived using the analysis outlined in [10].

Upper bound: To determine the upper bound on threshold we need first of all to determine the asymptotical behaviour of (4) for $t\to\infty$. For this purpose, we have to compute $\phi(x)$ in (4) with $x\geq 10$. To this end, we add a further invertible approximation of the function $\phi(x)$, the derivation of which is given in the Appendix. With this approximation, defining $z_s(t):=\frac{s+(i_1-1)t}{2}$ and $A_j:=\frac{1}{(j-1)^2\lambda_{i_1}^2}$, we can write the following

Lemma: As $t\to\infty$, $g_s(t)$ becomes:

$$g_s(t)=2\sum_{j=2}^{d_r}\rho_j \mathrm{W}(A_j z_s(t)e^{z_s(t)}) \tag{5}$$

Proof: When $t\to\infty$, since, as shown in [9], the function $\phi(x)$ has a rapid decrease in x, only the first terms of $\sum_{i=i_1}^{d_l}\lambda_i\phi(s+(i-1)t)$ are important in Eq. (1). Thus, we can simplify that sum to:

$$\sum_{i=i_1}^{d_l}\lambda_i\phi(s+(i-1)t)=\lambda_{i_1}\phi(s+(i_1-1)t)+O(\lambda_{i_2}\phi(s+(i_2-1)t)) \tag{6}$$

being λ_{i_1} and λ_{i_2} the first and second nonzero λ_i's, respectively, and observing that, in most practical cases, never λ_{i_2} is largely greater than λ_{i_1}, and that they are both very significant. By the development

$$(1-x)^n=1-nx+O(x^2),\quad x\to 0 \tag{7}$$

$$g_{s,j}(t)=\phi^{-1}((j-1)\lambda_{i_1}\phi(s+(i_1-1)t)[1+O(\phi(s+(i_1-1)t))]) \tag{8}$$

where we used $\phi(s+(i_2-1)t)\ll\phi(s+(i_1-1)t)$. Then

$$\phi(g_{s,j}(t))=(j-1)\lambda_{i_1}\phi(s+(i_1-1)t)[1+O(\phi(s+(i_1-1)t))] \tag{9}$$

Using the approximation (21) and ignoring the $O(\cdot)$ term:

$$\phi(g_{s,j}(t))=\sqrt{\frac{\pi}{s+(i_1-1)t}}e^{-\frac{s+(i_1-1)t-4\log((j-1)\lambda_{i_1})}{4}} \tag{10}$$

and, applying (22),

$$g_{s,j}(t) = 2\mathrm{W}\Big(\frac{s+(i_1-1)t}{2}e^{\frac{s+(i_1-1)t-4\log((j-1)\lambda_{i_1})}{2}}\Big) \tag{11}$$

This can be rewritten as

$$g_{s,j}(t) = 2\mathrm{W}(A_j z_s(t)e^{z_s(t)}) \tag{12}$$

and applying (1) we get the result.

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With $g_s(t)$ given in (5) and remembering that $\frac{\mathrm{d}W(x)}{\mathrm{d}x} = \frac{1}{x+e^{\mathrm{W}(x)}}$:

$$g_s'(t) = 2\sum_{j=2}^{d_r}\rho_j \frac{z_s'(t)e^{z_s(t)}(1+z_s(t))}{z_s(t)e^{z_s(t)} + e^{\mathrm{W}(A_j z_s(t)e^{z_s(t)})-\log A_j}} \tag{13}$$

Applying (4) to (5) and (13) we get:

$$\begin{cases} 2\sum_{j=2}^{d_r}\rho_j \mathrm{W}(A_j z_s(t)e^{z_s(t)}) = t \\ 2\sum_{j=2}^{d_r}\rho_j \frac{z_s'(t)e^{z_s(t)}(1+z_s(t))}{z_s(t)e^{z_s(t)}+e^{\mathrm{W}(A_j z_s(t)e^{z_s(t)})-\log A_j}} = 1 \end{cases} \tag{14}$$

Its solution (s^*, t^*), obtained applying the instruction set produced in [8], determines the bound $\sigma^* = \sqrt{\frac{2}{s^*}}$, which is valid $\forall i_1$.

With $a_j := -2\log((j-1)\lambda_{i_1})$, Eq. (11) can be rewritten as

$$g_{s,j}(t) = 2\mathrm{W}\Big(z_s(t)e^{z_s(t)+a_j}\Big) \tag{15}$$

Assuming that the following simplified approximation holds

$$g_{s,j}(t) \simeq 2\mathrm{W}\Big((z_s(t)+a_j)e^{z_s(t)+a_j}\Big), \tag{16}$$

calling $x := z_s(t) + a_j$, and being $W(xe^x) \equiv x$ for $x > 0$, we find another asymptotical expression for $g_{s,j}(t)$ which is much simpler than the one obtained in (12):

$$g_{s,j}(t) \simeq 2x = 2(z_s(t)+a_j) = s + (i_1-1)t - 4\log((j-1)\lambda_{i_1}) \tag{17}$$

Applying (1) and getting its first derivative, we may rewrite (14) as:

$$\begin{cases} s + (i_1-1)t - 4\log\lambda_{i_1} - 4\sum_{j=2}^{d_r}\rho_j\log(j-1) = t \\ i_1 - 1 = 1 \end{cases} \tag{18}$$

Its solution is:

$$s^*_{\mathrm{approx}} = 4\log\lambda_{i_1} + 4\sum_{j=2}^{d_r}\rho_j\log(j-1) \tag{19}$$

Using the Jensen's inequality $\prod_{j=2}^{d_r}(j-1)^{\rho_j} \le \sum_{j=2}^{d_r}\rho_j(j-1)$, we may write the following upper bound on (19):

$$s^*_{\mathrm{approx}} \le s^*_{\mathrm{Jensen}} = 4\log\lambda_{i_1} + 4\log\Big(\sum_{j=2}^{d_r}(j-1)\rho_j\Big) \tag{20}$$

Taking the simple algebrical expressions (19) and (20), $\sigma^* = \sqrt{\frac{2}{s^*}}$ gives two further bounds on threshold, both valid only for $i_1 = 2$.

Numeric results: For the irregular rate-1/2 LDPC codes reported in [6], we report in Table 1 the σ^* values found with density evolution in [6] together with the upper bounds for σ we found in the present work, namely $\sigma^*_{\mathrm{approx}} = \sqrt{2/s^*_{\mathrm{approx}}}$ (from (19)), $\sigma^*_{\mathrm{Jensen}} = \sqrt{2/s^*_{\mathrm{Jensen}}}$ (from (20)), and $\sigma^* = \sqrt{2/s^*}$ (solution of (14)).

Conclusions: In this letter, the derivation of low complexity upper bounds on belief-propagation decoding thresholds was addressed, using the algorithmic method proposed in [8]. The results showed good agreement with the threshold values found with density evolution in [6].

Table 1: Bounds on decoding thresholds of good rate-$1/2$ codes listed in [6].

d_l	5	15	20	30
i	λ_i	λ_i	λ_i	λ_i
2	0.32660	0.23802	0.21991	0.19606
3	0.11960	0.20997	0.23328	0.24039
4	0.18393	0.03492	0.02058	
5	0.36988	0.12015		
6			0.08543	0.00228
7		0.01587	0.06540	0.05516
8			0.04767	0.16602
9			0.01912	0.04088
10				0.01064
14	0.00480			
15		0.37627		
19			0.08064	
20			0.22798	
28				0.00221
30				0.28636
j	ρ_j	ρ_j	ρ_j	ρ_j
6	0.78555			
7	0.21445			
8		0.98013	0.64854	0.00749
9		0.01987	0.34747	0.99101
10			0.00399	0.00150
σ^*	0.9194	0.9622	0.9649	0.9690
$\sigma^*_{\mathrm{approx}}$	0.971728	0.987091	1.02193	1.05493
$\sigma^*_{\mathrm{Jensen}}$	0.969081	0.986917	1.01966	1.05485
σ^*	0.97173	0.987094	1.02193	1.05493

Appendix: Approximation of $\phi(x)$ for $x \ge 10$: In [7], the following approximation (called here $\hat\phi(x)$) was used for $\phi(x)$ when x is large

$$\phi(x) \simeq \hat\phi(x) := \sqrt{\frac{\pi}{x}}e^{-\frac{x}{4}} \tag{21}$$

which we found invertible by means of the Lambert-W function. Being $\mathrm{W}(\cdot)$ the Lambert-W function, the inverse of $\hat\phi(x)$ is

$$\phi^{-1}(y) \simeq \hat\phi^{-1}(y) = 2\mathrm{W}\Big(\frac{\pi}{2y^2}\Big) \tag{22}$$

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