The roll-call interpretation of the Shapley value with dependent cooperation

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Abstract

Consider a roll-call in a binary decision where the agents announce their vote one after the other. If agent's probability for "yes" is given by a common parameter $0 \le \gamma \le 1$, votes are independent, and the orderings of the agents are equiprobable, then the probabilities to cast the deciding vote equal the Shapley-Shubik index. The same remains true iff the underlying joint probability distribution is that of exchangeable random Bernoulli variables.

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1. Introduction

Consider a committee of heterogeneous agents who jointly decide on the acceptance of a proposal by voting either "yes" or "no". How important is each agent in the decision process? To that end, consider a *roll-call* where all agents row up in a line and declare their vote one after the other. In each ordering there is a unique agent whose declaration finalizes the decision, i.e., the outcome is fixed independent of the votes of the later agents. Calling such an agent *pivotal* we can ask for the probability of an agent to be pivotal. If all orderings are equiprobable and all agents independently vote "yes" with a common probability $0 \leq \gamma \leq 1$, then the vector of chances to be pivotal coincides with the Shapley-Shubik index.[1]

As an example consider a committee of three agents, where agent 1 with the support of at least two of the other agents can bring through a proposal while agents 2 and 3 cannot. Assume that there are always exactly two agents voting "yes" and all three cases occur with equal probability. If agents 1 and 2 vote "yes", then agent 1 is pivotal in all three orderings where agent 2 is prior and agent 2 is pivotal in the remaining three orderings. Performing the analysis for the two other cases gives $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ in the end, which also coincides with the Shapley-Shubik index despite the fact that the votes are not independent.

The aim of this note is to classify all probability distributions of the votes such that the vector of chances to be pivotal in the roll-call model with equiprobable orderings coincides with the Shapley-Shubik index. It turns out that this class consists of the joint probability distributions of exchangeable Bernoulli random variables. The very same is true if the model is generalized to coalitional games and the Shapley value.

2. Preliminaries

We denote the ith component of a vector $x \in \mathbb{R}^n$ by x_i and let $\chi_a(x) = \#\{1 \leq i \leq n : x_i = a\}$ count the number of entries that equal a.

Now, consider a set $N = \{1, \ldots, n\}$ of $n > 0$ agents who may cooperate in a given project. The creation of value is typically modeled as a *coalitional game* $v \colon 2^N \to \mathbb{R}$ mapping each *coalition*, i.e., each $S \subseteq N$ of cooperating agents, to a real number. As a normalization we assume $v(\emptyset) = 0$. How should the generated surplus $v(N)$ of the *grand coalition* be distributed among the agents? How important is each agent to the overall cooperation? *Values*, i.e., operators mapping coalitional games to $\mathbb{R}^n$, try to answer these questions. However, those operators are far too general to

[1] The coincidence was mentioned in (Mann and Shapley, 1964, fn. 3) and proven in Felsenthal and Machover (1996). See Shapley and Shubik (1954) for the original definition of the Shapley-Shubik index.

yield a reasonable solution so that additional properties may be required. A value ψ is called *linear* if $\psi(\alpha \cdot u + \beta \cdot v) = \alpha \cdot \psi(u) + \beta \cdot \psi(v)$ for all constants $\alpha, \beta \in \mathbb{R}$ and all coalitional games u, v on the same set N of agents, where $(\alpha \cdot u + \beta \cdot v)(S) = \alpha \cdot u(S) + \beta \cdot v(S)$ for all $S \subseteq N$. Given a value ψ we write $\psi_i(v)$ for the payoff for agent $i \in N$. With this, ψ is called *efficient* if $\sum_{i \in N} \psi_i(v) = v(N)$. An agent $i \in N$ satisfying $v(S) = v(S \cup \{i\})$ for all $S \subseteq N$ is called a *null* (in v). If $\psi_i(v) = 0$ for any coalitional game v and any null i in v, then ψ satisfies the *null player property.* Two agents $i, j \in N$ satisfying $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$ are called *equivalent.* With this, ψ is called *symmetric* if $\psi_i(v) = \psi_j(v)$ for any coalitional game v and any two equivalent agents $i, j \in N$.

A well-known and commonly applied value is the *Shapley value* φ, see Shapley (1953). By $\mathcal{S}_n$ we denote the set of all permutations of N and by P_i^π the set of all agents preceding i in order $\pi \in \mathcal{S}_n$. With this, we have

$$\varphi_i(v) = \frac{1}{n!} \cdot \sum_{\pi \in \mathcal{S}_n} \left[v(P_i^\pi \cup \{i\}) - v(P_i^\pi) \right] \tag{1}$$

for all $i \in N$, which may also be rewritten as

$$\varphi_i(v) = \sum_{S \subseteq N \backslash \{i\}} \frac{|S|! \cdot (n - |S| - 1)!}{n!} \cdot \left[v(S \cup \{i\}) - v(S) \right]. \tag{2}$$

In Shapley (1953) it was shown that the Shapley value is the unique value that satisfies efficiency, linearity, symmetry and the null player property. Besides this axiomatic characterization, Equation (1) allows a different interpretation within the roll-call model. Assume that all agents row up in a line and declare their cooperation one after the other. Given the ordering $\pi \in \mathcal{S}_n$, at the time when agent i declares his or her cooperation, the corresponding marginal contribution amounts to $v(P_i^\pi \cup \{i\}) - v(P_i^\pi)$. Considering all orderings to be equiprobable gives Equation (1), cf. the bargaining model in Shapley (1953).

If an agent declares not to cooperate, then the formation of the grand coalition is blocked, which also reflects some kind of importance of an agent. The corresponding marginal contribution may be measured as

$$v(N\backslash P_i^\pi) - v(N\backslash (P_i^\pi \cup \{i\})) = v^*(P_i^\pi \cup \{i\}) - v^*(P_i^\pi)\,,$$

where $v^*(S) := v(N) - v(N\backslash S)$ for all $S \subseteq N$ defines the *dual game*. With this, we consider a generalized model. Each agent i can either declare to cooperate, modeled as $d_i = 1$, or not to cooperate, modeled as $d_i = -1$. An instance $R = (\pi, d)$ in the roll-call model consists of an ordering $\pi \in \mathcal{S}_n$ of the agents and a vote $d \in \{-1, 1\}^n$. By $\mathcal{Y}(R, i)$ we denote the set of agents $j \in N$ that precede agent i and said "yes" to a cooperation, i.e., $d_j = 1$. Similarly, $\mathcal{N}(R, i)$ denotes the set of agents $j \in N$ that precede agent i and said "no" to a cooperation, i.e., $d_j = -1$. With this we can define the marginal contribution as

$$M(v, R, i) = \begin{cases} v(\mathcal{Y}(R, i) \cup \{i\}) - v(\mathcal{Y}(R, i)) & \text{if } d_i = 1, \\ v^*(\mathcal{N}(R, i) \cup \{i\}) - v^*(\mathcal{N}(R, i)) & \text{if } d_i = -1. \end{cases} \tag{3}$$

Assuming some probability distribution $p\colon \{-1, 1\}^n \to \mathbb{R}$, i.e., $p(d) \geq 0$ for all $d \in \{-1, 1\}^n$ and $\sum_{d \in \{-1,1\}^n} p(d) = 1$, we can define a generalized value by

$$\varphi_i^p(v) = \frac{1}{n!} \cdot \sum_{\pi \in \mathcal{S}_n} \sum_{d \in \{-1,1\}^n} p(d) \cdot M(v, (\pi, d), i) \tag{4}$$

for all agents $i \in N$. In words, we average over equiprobable orderings and votes according to some given probability distribution.

In voting theory the subclass of *simple games*, i.e., $\operatorname{im}(v) = \{0, 1\}$ and $v(S) \leq v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$, is a well-studied restriction of coalitional games. Being part of a coalition here means voting "yes" on a proposal. The group decision is to accept the proposal iff $v(S) = 1$ for the set S of supporters. An even narrower subclass of simple games are unanimity games u_T for $\emptyset \neq T \subseteq N$ defined via $v(S) = 1$ iff $T \subseteq S$. For a simple game v we have $M(v, R, i) \in \{0, 1\}$ and $M(v, R, i) = 1$ iff agent i is pivotal in R.

3. Results

Proposition 1 *For any probability distribution p the value φ^p is linear, efficient, and satisfies the null player property.*

Proof The null player property is obvious from the definition. The same is true for linearity since $v^*(\mathcal{N}(R,i)\cup\{i\}) - v^*(\mathcal{N}(R,i)) = v(N\backslash\mathcal{N}(R,i)) - v(N\backslash(\mathcal{N}(R,i)\cup\{i\}))$. For efficiency we observe

$$\sum_{i=1}^{n} M(v,R,i) = v(\mathcal{Y}) - v(\emptyset) + v^*(\mathcal{N}) - v^*(\emptyset) = v(N) - v(\emptyset) = v(N),$$

where $\mathcal{Y} = \{i \in N : d_i = 1\}$ and $\mathcal{N} = \{i \in N : d_i = -1\}$, for any $R \in \mathcal{S}_n \times \{-1,1\}^n$ due to the telescope sum behavior. The observations $|\mathcal{S}_n| = n!$ and $\sum_{d\in\{-1,1\}^n} p(d) = 1$ finish the proof. □

So, in order to make φ^p coincide with the Shapley value φ, just "symmetry" is missing, which can obviously be achieved by additionally requiring $p(d) = p(d')$ for all $d, d' \in \{-1,1\}^n$ with $\chi_1(d) = \chi_1(d')$, where $\chi_a(x) = \#\{1 \le i \le n : x_i = a\}$ for $x \in \mathbb{R}^n$. In probability theory this condition is known as *exchangeability*. In words, the probability $p(d)$ just depends on the number $\chi_1(d)$ of "yes" votes in d. This goes in line with de Finetti's theorem stating that exchangeable observations are conditionally independent relative to some latent variable.[2] So, the Shapley value is also the appropriate answer in a roll-call model with dependent agents if the underlying random variables are exchangeable, see Hu (2006) for a combinatorial proof.

Of course, it would be nice to know whether that is the end of the road. And indeed, it is. Certainly, the comprehensive class of coalitional

[2]The coincidence result mentioned in Footnote 1 together with de Finetti's theorem also directly implies $\varphi^p = \varphi$ for all joint probability distributions p of exchangeable random Bernoulli variables.

games may cause severe restrictions on the set of admissible probability distributions p, so that we consider the rather narrow class of unanimity games here.

Proposition 2 *Any probability measure p on $\{-1,1\}^n$ that satisfies $\varphi^p(u_T) = \varphi(u_T)$ for all $\emptyset \neq T \subseteq N$ is the joint probability distribution of exchangeable random Bernoulli variables.*

For a proof we inductively infer a set of equations for the values of p, which finally yield Proposition 2, see Section A in the appendix. As a direct implication of our propositions we obtain:

Theorem 1 *Let $\mathcal{V}$ be a subclass of coalitional games containing unanimity games. We have $\varphi^p(v) = \varphi(v)$ for all $v \in \mathcal{V}$ if and only if p is the joint probability distribution of exchangeable random Bernoulli variables.*

To close we reconsider the simple game v from the introduction.[3] Now assume that agent 1 always disagrees with agent 2 and that the feasible four vote vectors are equiprobable. The changes of being pivotal in the roll-call model, with equiprobable orderings, are given by $(\frac{1}{2}, \frac{1}{8}, \frac{3}{8}) \neq (\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$.

[3]The voting procedure v can be represented by weights 2, 1, 1 and quota 3.

Appendix

A. Proof of Proposition 2

In order to prove Proposition 2 we introduce further notation and reformulate the statement so that it better fits to an inductive proof. Given a probability measure p on $\{-1,1\}^n$ we write

$$p(x) = \sum_{d \in \{-1,1\}^n \,:\, d_i = x_i \vee x_i = 0 \, \forall i \in N} p(d)$$

for all $x \in \{-1,0,1\}^n$, i.e., we sum the probabilities of all $\{-1,1\}$ vectors that match the -1s and 1s in x, where a 0 in x is a wildcard. We set $\chi_a(x) = \#\{1 \le i \le n \,:\, x_i = a\}$ for $x \in \mathbb{R}^n$ and write $\mathbf{1}_k$, $\mathbf{0}_k$ for the vectors of k ones and zeros, respectively.

Lemma 1 *For any probability measure p on $\{-1,1\}^n$ that satisfies $\varphi^p(u_T) = \varphi(u_T)$ for all $\emptyset \neq T \subseteq N$ we have $p(x) = p(x')$ for all $x, x' \in \{-1,0,1\}^n$ with $\chi_{-1}(x) = \chi_{-1}(x')$, $\chi_1(x) = \chi_1(x')$, and $\{i \in N \,:\, x_i = 0\} = \{i \in N \,:\, x'_i = 0\}$.*

Proof For any given positive integer n we prove the statement by induction on $m := \chi_{-1}(x) + \chi_1(x)$, where $0 \le m \le n$. If $\chi_1(x) = 0$ or $\chi_{-1}(x) = 0$ we have $x = x'$, which implies the statement. Thus, we can assume $n \ge m \ge 2$, $\chi_1(x) \ge 1$, and $\chi_{-1}(x) \ge 1$ in the following.

For $T = \{i \in N \,:\, x_i \neq 0\}$ we consider the unanimity game u_T. To ease the notation we assume $T = \{1, 2, \ldots, m\} \neq \emptyset$ w.l.o.g. Let us first compute $\varphi_h^p(u_T)$ for $h \in T$. If $\overline{z} \in \{-1,1\}^{n-m}$ and $z \in \{-1,1\}^m$ with $z_h = -1$, then agent h is pivotal in exactly $1/\chi_{-1}(z)$ of the $n!$ roll-calls $(\pi, (z, \overline{z}))$, since it has to be the first -1 among the agents in T. If $z_h = 1$, then agent h is pivotal in $(\pi, (z, \overline{z}))$ iff $\pi_h \ge \pi_l$ and $z_l = 1$ for all $l \in T$, since it has to be the last among the agents in T. Thus, we compute

$$\varphi_h^p(u_T) = \frac{1}{m} \cdot p(\mathbf{1}_m, \mathbf{0}_{n-m}) + \sum_{z \in \{-1,1\}^m \,:\, z_h = -1} \frac{1}{\chi_{-1}(z)} \cdot p(z, \mathbf{0}_{n-m}).$$

For any $i, j \in T$ we have $\varphi_i^p(u_T) = \varphi_j^p(u_T)$ since $\varphi_i(u_T) = \varphi_j(u_T)$. To ease the notation we assume $i = 1$, $j = 2$, for a moment. Inserting into $\varphi_1^p(u_T) = \varphi_2^p(u_T)$ and canceling out equal summands gives

$$\sum_{y \in Y} f(y) \cdot p(-1, 1, y, \mathbf{0}_{n-m}) = \sum_{y \in Y} f(y) \cdot p(1, -1, y, \mathbf{0}_{n-m}),$$

where $Y = \{-1, 1\}^{m-2}$ and $f(y) = 1/\left(1 + \chi_{-1}(y)\right)$.

For $m = 2$ this equation is equivalent to the statement of the lemma so that we assume $m \geq 3$ in the following. For a given $y \in Y$ and an index l with $y_l = -1$ let y' and $\overline{y}$ arise from y by replacing y_l with 1 and 0, respectively, so that the induction hypothesis gives

$$\begin{aligned} & p(-1, 1, y, \mathbf{0}_{n-m}) + p(-1, 1, y', \mathbf{0}_{n-m}) = p(-1, 1, \overline{y}, \mathbf{0}_{n-m}) \\ = \; & p(1, -1, \overline{y}, \mathbf{0}_{n-m}) = p(1, -1, y, \mathbf{0}_{n-m}) + p(1, -1, y', \mathbf{0}_{n-m}). \qquad \square \end{aligned}$$

Setting $a(y) = p(-1, 1, y, \mathbf{0}_{n-m})$ and $b(y) = p(1, -1, y, \mathbf{0}_{n-m})$ for all $y \in Y$ we can apply Lemma 2 to deduce $p(-1, 1, y, \mathbf{0}_{n-m}) = p(1, -1, y', \mathbf{0}_{n-m})$ for all $y, y' \in Y$ with $\chi_{-1}(y) = \chi_{-1}(y')$.

Choosing $i, j \in T$ arbitrarily, we have $p(y, \mathbf{0}_{n-m}) = p(y', \mathbf{0}_{n-m})$ for all $y, y' \in \{-1, 1\}^m$ arising from each other by swapping a -1 and a 1. Thus, by a sequence of swaps we can show $p(y, \mathbf{0}_{n-m}) = p(y', \mathbf{0}_{n-m})$ for any $y, y' \in \{-1, 1\}^m$ with $\chi_1(y) = \chi_1(y')$, which inductively proves the statement for all $m \geq 2$. $\square$

Of course Lemma 1 implies Proposition 2 as the special case where $\chi_{-1}(x) + \chi_1(x) = n$.

Lemma 2 *Let $a(z), b(z) \in \mathbb{R}$ for all $z \in \{-1, 1\}^n$ with*

$$\sum_{z \in \{-1,1\}^n} \frac{a(z)}{1 + \chi_{-1}(z)} = \sum_{z \in \{-1,1\}^n} \frac{b(z)}{1 + \chi_{-1}(z)}$$

and $a(z) + a(z') = b(z) + b(z')$ for all $z, z' \in \{-1, 1\}^n$ that differ in exactly one coordinate. Then, $a(z) = b(z)$ for all $z \in \{-1, 1\}^n$.

Proof Let $Z^{(i)} \subseteq \{-1,1\}^n$, $z^{(i)} \in Z^{(i)}$, $l \in \{1,\ldots,n\}$ with $z_l^{(i)} = -1$, and $\bar{z}^{(i)} \in Z^{(i)}$, where $\bar{z}_j^{(i)} = z_j^{(i)}$ for all $j \in \{1,\ldots,n\}\backslash\{l\}$ and $\bar{z}_l^{(i)} = -z_l^{(i)} = 1$. If $\sum_{z\in Z^{(i)}} c^{(i)}(z)a(z) = \sum_{z\in Z^{(i)}} c^{(i)}(z)b(z)$ for some $c^{(i)}(z) \in \mathbb{R}$, then subtracting $c^{(i)} \cdot \big(a\big(z^{(i)}\big) + a\big(\bar{z}^{(i)}\big)\big)$ on the left hand side and $c^{(i)} \cdot \big(b\big(z^{(i)}\big) + b\big(\bar{z}^{(i)}\big)\big)$ on the right hand side yields

$$\sum_{z\in Z^{(i-1)}} c^{(i-1)}(z)a(z) = \sum_{z\in Z^{(i-1)}} c^{(i-1)}(z)b(z)$$

for $Z^{(i-1)} = Z^{(i)}\backslash\left\{z^{(i)}\right\}$, $c^{(i-1)}(z) = c^{(i)}(z)$ for all $z \in Z^{(i)}\backslash\left\{z^{(i)}, \bar{z}^{(i)}\right\}$, and $c^{(i-1)}\big(\bar{z}^{(i)}\big) = c^{(i)}\big(\bar{z}^{(i)}\big) - c^{(i)}\big(z^{(i)}\big)$.

Starting with $Z^{(r)} = \{-1,1\}^n$ and $c^{(r)}(z) = \frac{1}{1+\chi_{-1}(z)}$ for all $z \in \{-1,1\}^n$, where $r = 2^n$, and choosing $z^{(i)} \in Z^{(i)}$ such that $\chi_{-1}(z^{(i)}) = \max\left\{\chi_{-1}(z) \,:\, z \in Z^{(i)}\right\}$ for all $2 \le i \le r$ yields $Z^{(1)} = \{\mathbf{1}_n\}$ and $c^{(1)}(\mathbf{1}_n)a(\mathbf{1}_n) = c^{(1)}(\mathbf{1}_n)b(\mathbf{1}_n)$ with

$$c^{(1)}(\mathbf{1}_n) = \sum_{z\in\{-1,1\}^n} \frac{(-1)^{\chi_{-1}(z)}}{1+\chi_{-1}(z)} = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k+1} = \frac{1}{n+1},$$

where the last equation is due to Lemma 3. Thus, $a(\mathbf{1}_n) = b(\mathbf{1}_n)$.

Now let $z^* \in \{-1,1\}^n\backslash\{\mathbf{1}_n\}$ be arbitrary. Again, we start with $Z^{(r)} = \{-1,1\}^n$ and $c^{(r)}(z) = \frac{1}{1+\chi_{-1}(z)}$ for all $z \in \{-1,1\}^n$. For each $3 \le i \le r$ we choose $z^{(i)} \in Z^{(i)}\backslash\{z^*\}$ such that $\chi_{-1}(z^{(i)}) = \max\left\{\chi_{-1}(z) \,:\, z \in Z^{(i)}\backslash\{z^*\}\right\}$. Moreover we choose $l \in \{1,\ldots,n\}$ such that $\bar{z}^{(i)} \neq z^*$. With this we end up with $Z^{(2)} = \{z^*, \mathbf{1}_n\}$ and the equation

$$\frac{a(z^*)}{1+\chi_{-1}(z^*)} + c^{(2)}(\mathbf{1}_n)a(\mathbf{1}_n) = \frac{b(z^*)}{1+\chi_{-1}(z^*)} + c^{(2)}(\mathbf{1}_n)b(\mathbf{1}_n).$$

Thus $a(z^*) = b(z^*)$ since $\frac{1}{1+\chi_{-1}(z^*)} \neq 0$ and $a(\mathbf{1}_n) = b(\mathbf{1}_n)$. $\square$

Lemma 3 *For all $n \in \mathbb{N}_{>0}$ we have* $\sum\limits_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{k+1} = \frac{1}{n+1}$.

Proof

$$\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{k+1} = \sum_{k=0}^{n} \binom{n+1}{k+1} \cdot \frac{(-1)^k}{n+1} = \frac{1}{n+1} \cdot \sum_{k=1}^{n+1} \binom{n+1}{k} \cdot (-1)^{k-1}$$

$$= \frac{1}{n+1} \cdot \left(1 - \sum_{k=0}^{n+1} \binom{n+1}{k} \cdot (-1)^k \right) = \frac{1}{n+1} \cdot \left(1 - (1-1)^{n+1} \right) = \frac{1}{n+1} \quad \square$$

B. Alternative proof

Lemma 4 *For all $n \in \mathbb{N}_{>0}$ we have*

(a) $\sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k = 0$*;*

(b) $\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{k+1} = \frac{1}{n+1}$*;*

(c) $\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{k+1+x} = \frac{n!}{\prod_{k=0}^{n}(1+x+k)}$ *for all* $x \in \mathbb{R}_{>-1}$*;*

(d) $\sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{k+x} = \frac{n!}{\prod_{k=0}^{n}(x+k)}$ *for all* $x \in \mathbb{R}_{>0}$*.*

For $n = 0$ the first sum is equal to 1. The three other formulas are also valid for $n = 0$.

Proof For part (a) the Binomial theorem gives $0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k$. Note that $n \geq 1$ is necessary since $\sum_{k=0}^{0} \binom{0}{k} \cdot (-1)^k = \binom{0}{0} = 1$.

For part (c) we consider the polynomial

$$f(x) = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k \cdot \prod_{0 \leq j \leq n \,:\, j \neq k} (x+1+j)$$

of degree at most n. For every integer $0 \leq i \leq n$ we have

$$f(-i-1) = \binom{n}{i} \cdot (-1)^i \cdot \prod_{0 \leq j \leq n \,:\, j \neq i} (x+1+j) = \binom{n}{i} \cdot (-1)^i \cdot (-1)^i i! \cdot (n-i)! = n!.$$

Thus $f(x) = n!$ and part (c) follows via division by $\prod_{k=0}^{n}(1+x+k)$. Setting $x = 0$ we obtain part (b). Part (d) follows from (c) by a simple variable transformation. $\square$

Lemma 5 *For a non-negative integer n let M be the $2^n \times 2^n$ matrix defined by $M_{T,S} = \frac{1}{1+|T\backslash S|}$ for all $S, T \subseteq \{1, \dots, n\}$. The inverse matrix is given by $M^{-1}_{T,S} = \binom{n+1}{n+|T\backslash S|} \cdot (-1)^{|S \Delta T|}$.*

Proof Since $(M \cdot M^{-1})_{T,S} = \sum_{U \in \{1,\dots,n\}} M_{T,U} \cdot M^{-1}_{U,S}$, we have

$$\begin{aligned}
& \left(M \cdot M^{-1}\right)_{T,S} \\
= & \sum_{U \subseteq S} \frac{n+1}{1+|T\backslash U|} \cdot (-1)^{|S\backslash U|} + \sum_{U \subseteq N \,:\, |U\backslash S|=1} \frac{-1}{1+|T\backslash U|} \cdot (-1)^{|S\backslash U|} \\
= & \sum_{U \subseteq S} \frac{n+1}{1+|T\backslash U|} \cdot (-1)^{|S\backslash U|} + \sum_{U \subseteq S} \sum_{l \in N\backslash S} \frac{-1}{1+|T\backslash(U \cup \{l\})|} \cdot (-1)^{|S\backslash U|}
\end{aligned}$$

Let us use the abbreviations $a = |T \cap S|$ and $b = |S\backslash T|$, so that $a + b = |S|$. With this we compute

$$\begin{aligned}
& \sum_{U \subseteq S} \frac{n+1}{1+|T\backslash U|} \cdot (-1)^{|S\backslash U|} \\
= & \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} \cdot \frac{n+1}{1+|T|-i} \cdot (-1)^{|S|-i-j} \\
= & \sum_{i=0}^{a} \binom{a}{i} \cdot \frac{n+1}{1+|T|-i} \cdot (-1)^{|S|-i} \cdot \left(\sum_{j=0}^{b} \binom{b}{j} \cdot (-1)^j \right)
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{U \subseteq S} \sum_{l \in N\backslash S} \frac{-1}{1+|T\backslash(U \cup \{l\})|} \cdot (-1)^{|S\backslash U|} \\
= & \sum_{l \in N\backslash S} \sum_{i=0}^{a} \sum_{j=0}^{b} \binom{a}{i} \binom{b}{j} \cdot \frac{-1}{1+|T\backslash\{l\}|-i} \cdot (-1)^{|S|-i-j} \\
= & -\sum_{l \in N\backslash S} \sum_{i=0}^{a} \binom{a}{i} \cdot \frac{1}{1+|T\backslash\{l\}|-i} \cdot (-1)^{|S|-i} \cdot \left(\sum_{j=0}^{b} \binom{b}{j} \cdot (-1)^j \right),
\end{aligned}$$

so that $(M \cdot M^{-1})_{T,S} = 0$ if $b > 0$ due to Lemma 4.(a). Thus, we assume $b = 0$, i.e., $S \subseteq T$ in the following and use the abbreviation $x = |T\backslash S|$.

With this we have $|N\backslash S \cap T| = |T\backslash S| = x$, $|N\backslash(S \cup T)| = n - |S| - x$, and

$$\begin{aligned}
&\left(M \cdot M^{-1}\right)_{T,S} \\
= \quad & \sum_{i=0}^{|S|} \binom{|S|}{i} \cdot \frac{(n+1)\cdot(-1)^{|S|-i}}{1+x+|S|-i} - \sum_{i=0}^{|S|} \binom{|S|}{i} \cdot \frac{(n-|S|-x)\cdot(-1)^{|S|-i}}{1+x+|S|-i} \\
& - \sum_{i=0}^{|S|} \binom{|S|}{i} \cdot \frac{x \cdot (-1)^{|S|-i}}{x+|S|-i} \\
= \quad & \sum_{i=0}^{|S|} \binom{|S|}{i} \cdot \frac{(|S|+x+1)\cdot(-1)^{|S|-i}}{1+x+|S|-i} - \sum_{i=0}^{|S|} \binom{|S|}{i} \cdot \frac{x \cdot (-1)^{|S|-i}}{x+|S|-i} \\
= \quad & (|S|+x+1) \cdot \sum_{i=0}^{|S|} \frac{\binom{|S|}{i}\cdot(-1)^i}{1+x+i} - x \cdot \sum_{i=0}^{|S|} \frac{\binom{|S|}{i}\cdot(-1)^i}{x+i}. \qquad \square
\end{aligned}$$

For $x = 0$, i.e., $S = T$, Lemma 4.(b) then gives $\left(M \cdot M^{-1}\right)_{T,T} = 1$. For $x > 0$, i.e., $S \subsetneq T$, parts (c) and (d) of Lemma 4 give

$$\left(M \cdot M^{-1}\right)_{T,S} = (|S|+x+1) \cdot \frac{n!}{\prod_{k=0}^{|S|}(1+x+k)} - x \cdot \frac{n!}{\prod_{k=0}^{|S|}(x+k)} = 0.$$

As an example we have

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

with

$$M_2^{-1} = \begin{pmatrix} 3 & -3 & -3 & 3 \\ -1 & 3 & 1 & -3 \\ -1 & 1 & 3 & -3 \\ 0 & -1 & -1 & 3 \end{pmatrix}.$$

For $n = 3$ we have

$$M_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 1 & 1 & 1 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 & 1 \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

with

$$M_3^{-1} = \begin{pmatrix} 4 & -4 & -4 & -4 & 4 & 4 & 4 & -4 \\ -1 & 4 & 1 & 1 & -4 & -4 & -1 & 4 \\ -1 & 1 & 4 & 1 & -4 & -1 & -4 & 4 \\ -1 & 1 & 1 & 4 & -1 & -4 & -4 & 4 \\ 0 & -1 & -1 & 0 & 4 & 1 & 1 & -4 \\ 0 & -1 & 0 & -1 & 1 & 4 & 1 & -4 \\ 0 & 0 & -1 & -1 & 1 & 1 & 4 & -4 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

Lemma 6 *Let $N = \{1, \ldots, n\}$. For any probability measure p on $\{M \subseteq N\}$ that satisfies $\varphi^p(u_T) = \varphi(u_T)$ for all $\emptyset \neq T \subseteq N$ we have $p(D) = p(D')$ for all $0 \subseteq D, D' \subseteq N$ with $|D| = |D'|$.*

Proof If $|D| = 0$ or $|\bar{D}| = 0$ we have $D = D'$, which implies the statement. Thus, we can assume $n \geq 2$, $|D| \geq 1$, and $|\bar{D}| \geq 1$ in the following.

For $T \subseteq N$ we consider the unanimity game u_T. Let us first compute $\varphi_h^p(u_T)$ for $h \in T$. If $h \notin S$, then agent h is pivotal in exactly $1/|T \backslash S|$ of the $n!$ roll-calls (π, S), since it has to be the first *decliner* among the agents in T. If $h \in S$, then agent h is pivotal in (π, S) iff $\pi_h \geq \pi_l$ and $l \in S$ for all $l \in T$, since it has to be the last among the agents in T. Thus, we compute

$$\varphi_h^p(u_T) = \sum_{T \subseteq S \subseteq N} \frac{1}{|T|} \cdot p(S) + \sum_{\emptyset \subseteq S \subseteq N \backslash \{h\}} \frac{1}{|T \backslash S|} \cdot p(S).$$

Let i and j be two different agents in N. For $h \in \{i,j\}$ and $\{i,j\} \subseteq T \subseteq N$ we have

$$\begin{aligned}\varphi_h^p(u_T) &= \sum_{T\subseteq S\subseteq N} \frac{1}{|T|} \cdot p(S) + \sum_{\emptyset\subseteq S\subseteq N\backslash\{i,j\}} \frac{1}{|T\backslash S|} \cdot p(S) \\ &\quad + \sum_{\emptyset\subseteq S\subseteq N\backslash\{i,j\}} \frac{1}{|T\backslash S|-1} \cdot p(S \cup \{i,j\}\backslash\{h\}). \qquad (5)\end{aligned}$$

For a given $\emptyset \subseteq X \subseteq N\backslash\{i,j\}$ we set

$$v_X = \sum_{\{i,j\}\subseteq T\subseteq N} \lambda_{T,X} \cdot u_T,$$

where $\lambda_{T,X} = M^{-1}_{T\backslash\{i,j\},X}$ using the notation from Lemma 5. Thus,

$$\sum_{\{i,j\}\subseteq T\subseteq N} \lambda_T \cdot \frac{1}{|T\backslash S|-1} = \begin{cases} 0 & : \quad S \neq X \\ 1 & : \quad S = X \end{cases} \qquad (6)$$

for any $\emptyset \subseteq S \subseteq N\backslash\{i,j\}$.

By construction we have $\varphi_i(v_X) = \varphi_j(v_X)$ and $\varphi_i^p(v_X) = \varphi_j^p(v_X)$, so that

$$\sum_{\{i,j\}\subseteq T\subseteq N} \lambda_{T,X} \cdot \varphi_i^p(u_T) \sum_{\{i,j\}\subseteq T\subseteq N} \lambda_{T,X} \cdot \varphi_j^p(u_T).$$

Now we cancel out equal summands on both sides. Due to Equation (5) for each T only the last sum multiplied by $\lambda_{T,X}$ remains. Equation (6) further reduces this to $p(X \cup \{j\}) = p(X \cup \{i\})$.

We get from set $X \cup \{i\}$ to $X \cup \{j\}$ by replacing agent i by agent j. Thus, by a sequence of such replacements we can show $p(D) = p(D')$ for any $\emptyset \subseteq D, D' \subseteq N$ with $|D| = |D'|$. □

Of course Lemma 6 implies Proposition 2.

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