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Direct derivation of Sagnac effect

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Abstract Rizzi and Ruggiero and later Ruggiero independently have shown that it is possible to calculate Sagnac effect in flat, Schwarzschild, Lense-Thirring (slowly spinning sphere), Kerr and Godel metrics in analogy with Aharonov-Bohm effect

One may reasonably wonder Is it possible to derive this effect independently, *i e* by some direct method ? In this paper, we show that the answer to this question is indeed in the affirmative

Keywords Sagnac effect, general relativity

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1. Introduction

M.G. Sagnac predicted the so-called Sagnac effect [1] in 1905 and experimentally verified it in 1913 [2] for light. Since then, a lot of experimental and theoretical studies have been carried out on this effect [see Ref [3] and Ref. [4] and other references therein] It has been found that this effect holds for both luxons (light-like particles) and tardyons (material particles).

The subject of derivation of the Sagnac effect has attracted much attention. Recently Rizzi and Ruggiero [3] and later Ruggiero [4] have derived the effect in analogy with Aharonov-Bohm effect. One may reasonably wonder : Is it not possible to calculate the effect independently, *i.e.* by some direct method ? We show in this paper that it is indeed possible to derive this effect by a direct, simple method in different types of metrics *viz.,* flat, Schwarzschild, Lense-Thirring, Kerr and Gödel metrics.

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2. Direct derivation of Sagnac effect in some metrics

We first demonstrate the method of derivation in a flat metric and then apply it to some other metrics.

2.1. Flat metric :

We start with the following flat metric :

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin\theta^2 \, d\bar{\phi}^2 \tag{1}$$

For $r = R$ (constant) and $\theta = \pi/2$ (constant), eq.(1) becomes

$$ds^2 = c^2 dt^2 - R^2 d\bar{\phi}^2 \tag{2}$$

We apply the transformation

$$\bar{\phi} = \phi + \Omega \, t \tag{3}$$

where Ω is an angular speed.

Using eq. (3) in eq. (2),

$$ds^2 = \left(1 - \frac{R^2\Omega^2}{c^2}\right) c^2 dt^2 - R^2 d\phi^2 - 2R^2\Omega \, d\phi \, dt . \tag{4}$$

From eq. (4), the proper time-interval is

$$d\tau = \left(1 - \frac{R^2\Omega^2}{c^2}\right)^{1/2} dt$$

$$= \frac{\frac{ds}{dt}\frac{ds}{d\phi} \, d\phi}{c^2\left(1 - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \frac{R^2 \frac{d\phi}{dt} \, d\phi}{c^2\left(1 - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \frac{2R^2\Omega \, d\phi}{c^2\left(1 - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} . \tag{5}$$

From eq. (5), for a beam of particles co-propagating along a semicircular path, we have,

$$\tau_1 = \int^{\tau_1} d\tau$$

$$= \int_0^\pi \frac{\frac{ds}{dt}\frac{ds}{d\phi}\,d\phi}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \int_0^\pi \frac{R^2\frac{d\phi}{dt}\,d\phi}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \int_0^\pi \frac{2R^2\Omega\,d\phi}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}}$$

$$= A\ (1^{st}\text{ integral}) + B\ (2^{nd}\text{ integral}) + \frac{2\pi R^2\Omega}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} \tag{6}$$

Again, from eq. (5), for a similar beam of particles counterpropagating along a semicircular path, we have,

$$\tau_2 = \int_0^{\tau_2} d\tau$$

$$= \int_0^\pi \frac{\frac{ds}{dt}\frac{ds}{(-d\phi)}(-d\phi)}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \int_0^\pi \frac{R^2\frac{(-d\phi)}{dt}(-d\phi)}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \int_0^\pi \frac{2R^2\Omega\,(-d\phi)}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}}$$

$$= \text{A}\ (1^{st}\text{ integral}) + \text{B}\ (2^{nd}\text{ integral}) - \frac{2\pi R^2\Omega}{c^2\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} \tag{7}$$

The proper time-difference between the two beams therefore is

$$\Delta\tau = \tau_1 - \tau_2 = \frac{4\pi}{c^2}\frac{R^2\Omega}{\left(1-\frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}}\,. \tag{8}$$

This is the same as eq.(33) in [3]. Obviously eq. (8) is true for all types of particles — — luxons as well as tardyons.

2.2. Schwarzschild metric :

The Schwarzschild metric for a spherical object is

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \left(1 - \frac{2GM}{rc^2}\right) dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\bar{\phi}^2 \tag{9}$$

where M is the mass of the object.

For $r = R$ (constant) and $\theta = \frac{\pi}{2}$ (constant),

$$ds^2 = \left(1 - \frac{2GM}{Rc^2}\right) c^2 dt^2 - R^2 d\bar{\phi}^2 \tag{10}$$

Using the transformation (3) in eq. (10)

$$ds^2 = \left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right) c^2 dt^2 - R^2 d\phi^2 - 2R^2 \Omega\, d\phi\, dt\, . \tag{11}$$

From eq. (11), the proper time-interval is

$$d\tau = \left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}} dt$$

$$= \frac{\frac{ds}{dt}\frac{ds}{(d\phi)}\, d\phi}{c^2\left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \frac{R^2 \frac{d\phi}{dt}\, d\phi}{c^2\left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}}$$

$$+ \frac{2R^2\Omega\, d\phi}{c^2\left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}}\, . \tag{12}$$

From eq. (12), for a beam of particles co-propagating along a semicircular path, we have

$$\tau_1 = \int_0^{\tau_1} d\tau = \int_0^{\pi} \frac{\frac{ds}{dt}\frac{ds}{d\phi}\, d\phi}{c^2\left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}}$$

$$\int_0^{\pi} \frac{R^2 \frac{d\phi}{dt} d\phi}{c^2 \left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} + \int_0^{\pi} \frac{2R^2\Omega \, d\phi}{c^2 \left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}}$$

= A (1st integral) + B (2nd integral)

$$\frac{2\pi R^2 \Omega}{c^2 \left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)^{\frac{1}{2}}} \tag{13}$$

Similarly, from eq.(12), for a similar beam of particles counterpropagating in a semicircular path, we have,

$$\tau_2 = A + B - \frac{2\pi R^2 \Omega}{\left(1 - \frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}\right)} \tag{14}$$

Hence, the proper time-difference between the two beams is

$$\Delta\tau = \tau_1 - \tau_2 = \frac{4\pi}{c^2} \cdot \frac{R^2\Omega}{\frac{2GM}{Rc^2} - \frac{R^2\Omega^2}{c^2}} \tag{15}$$

This is the same as eq. (27) in [4].

2.3. Lense-Thirring metric (slowly spinning sphere):

The Lense-Thirring metric is [5]

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 \, dt^2 - \left(1 + \frac{2GM}{rc^2}\right)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right]$$

$$\frac{4GJ}{r\,c^3} \sin^2\theta \, d\bar{\phi} \; cdt \tag{16}$$

For $r = R$ (constant) and $\theta = \pi/2$ (constant), eq. (16) becomes

$$ds^2 = \left(1 - \frac{2GM}{Rc^2}\right) c^2\, dt^2 - \left(1 + \frac{2GM}{Rc^2}\right) R^2\, d\bar{\phi}^2 + \frac{4GJ}{Rc^3}\, d\bar{\phi}\, cdt\,. \tag{17}$$

Using the transformation (3) in eq. (17),

$$ds^2 = \left[1 - \frac{2GM}{Rc^2} - \left(1 + \frac{2GM}{Rc^2}\right)\frac{R^2\Omega^2}{c^2} + \frac{4GJ\Omega}{Rc^4}\right] c^2\, dt^2$$

$$- \left(1 + \frac{2GM}{Rc^2}\right) R^2\, d\phi^2 - \left[\left(1 + \frac{2GM}{Rc^2}\right) 2R^2\Omega - \frac{4GJ}{Rc^3}\right] d\phi\, dt\,. \tag{18}$$

Now, proceeding as before, we can find that the proper time-difference between two identical oppositely circulating beams of particles is

$$\Delta\tau = \frac{4\pi}{c^2}\, \frac{\left[\left(1 + \frac{2GM}{Rc^2}\right) R^2\Omega - \frac{2GJ}{Rc^2}\right]}{\left[1 - \frac{2GM}{Rc^2} - \left(1 + \frac{2GM}{Rc^2}\right)\frac{R^2\Omega^2}{c^2} + \frac{4GJ\Omega}{Rc^4}\right]^{\frac{1}{2}}} \tag{19}$$

This is the same as eq. (11) in [4].

2.4. Kerr metric :

The usual form of Kerr metric is

$$ds^2 = \left(1 - \frac{2GMr}{\rho^2 c^2}\right) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2$$

$$- \left(r^2 + a^2 + \frac{2GM\, a^2 r \sin^2\theta}{\rho^2 c^2}\right) \sin^2\theta\, d\bar{\phi}^2 + \frac{4GMar \sin^2\theta}{\rho^2 c^2} d\bar{\phi}\, cdt \tag{20}$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta\,,$$

$$\Delta = r^2 - \frac{2GMr}{c^2} + a^2,\ Ma = J/c^2\,,$$

J being the absolute value of angular momentum.

For $r = R$ (constant) and $\theta = \frac{\pi}{2}$ (constant), (20) becomes

$$ds^2 = \left(1 - \frac{2GMr}{Rc^2}\right)c^2dt^2 - \left(R^2 + a^2 + \frac{2GMa^2}{Rc^2}\right)d\phi^2 + \frac{4GMa}{R^2c^2}\,d\phi\; cdt \tag{21}$$

Using the transformation (3) in (21),

$$ds^2 = \left[1 - \frac{2GM}{Rc^2} - \left(R^2 + a^2 + \frac{2GMa^2}{Rc^2}\right)\frac{\Omega^2}{c^2} + \frac{4GMa\Omega}{Rc^2}\right]c^2\,dt^2$$

$$-\left(R^2 + a^2 + \frac{2GMa^2}{Rc^2}\right)d\phi^2 - 2\left[\left(R^2 + a^2 + \frac{2GMa^2}{Rc^2}\right)\Omega - \frac{2GMa}{Rc}\right]d\phi\; dt \tag{22}$$

Now, proceeding as usual, we can find that the proper time-difference between two indentical oppositely circulating beams of particles is

$$\Delta\tau = \frac{4\pi}{c^2} \times \frac{\left[\left(R^2 + a^2 + \frac{2GMa^2}{Rc^2}\right)\Omega - \frac{2GMa}{Rc}\right]}{\left[1 - \frac{2GM}{Rc^2} - \left(R^2 + a^2 + \frac{2GMa^2}{Rc^2}\right)\frac{\Omega^2}{c^2} + \frac{4GMa\Omega}{Rc^2}\right]^{\frac{1}{2}}} \tag{23}$$

This is the same as eq. (17) in [4].

2.5. Gödel metric :

The Gödel metric is

$$ds^2 = c^2\,dt^2 - \frac{dr^2}{1 + \left(\frac{r}{2a}\right)^2} - r^2\left[1 - \left(\frac{r}{2a}\right)^2\right]d\bar{\phi}^2 - dz^2 + 2r^2\frac{1}{\sqrt{2}a}\,d\bar{\phi}\; cdt \tag{24}$$

where a is a constant > 0.

Now, for $r = R$ (constant) and z = constant,

$$ds^2 = c^2\,dt^2 - R^2\left[1 - \left(\frac{R}{2a}\right)^2\right]d\bar{\phi}^2 + \frac{2R^2}{\sqrt{2}a}\,d\bar{\phi}\; cdt \tag{25}$$

Using the transformation (3) in eq. (25)

$$ds^2 = \left[1 - \left\{R^2\left(1-\left(\frac{R}{2a}\right)^2\right)\Omega^2 - \frac{2R^2}{\sqrt{2}a}\,\Omega c\right\}\Big/c^2\right]$$

$$\times c^2dt^2 - R^2\left[1-\left(\frac{R}{2a}\right)^2\right]d\phi^2 - \left[2R^2\Omega\left(1-\left(\frac{R}{2a}\right)^2\right) - \frac{2R^2c}{\sqrt{2}a}\right]d\phi\, dt \tag{26}$$

One can see that in this case the proper time-difference between two identical oppositely circulating beams of particles is

$$\Delta\tau = \frac{4\pi}{c^2} \times \frac{\left[R^2\Omega\left(1-\left(\frac{R}{2a}\right)^2\right) - \frac{R^2c}{\sqrt{2}a}\right]}{\left[1 - \left\{R^2\left(1-\left(\frac{R}{2a}\right)^2\right)\Omega^2 - \frac{2R^2c}{\sqrt{2}a}\,\Omega\right\}\Big/c^2\right]^{\frac{1}{2}}} \tag{27}$$

This is the same as eq. (21) in [4].

3. Concluding remarks

Thus we find that it is possible, without having recourse to any analogy or any external aid, to derive Sagnac effect in a number of metrics in a straight-forward manner.

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