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Raychaudhuri equation, big bang and accelerating universe†

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Abstract In this paper, we show that with appropriate inputs the Raychaudhuri equation can provide physically rational clues to certain cosmological phenomena

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1. Introduction

Relativistic cosmology has been plagued by the problem of singularity (big bang) corresponding to scale factor tending to zero at a finite time in the past Tolman and Eddington had held the view that the singularity might have resulted from the simplifying assumptions of homogeneity and isotropy Later investigations have, however, shown that this view is not valid It has, rather, been observed that a singularity in classical cosmology is inevitable

A theoretical confirmation of the inevitability of a singularity has been provided by the Raychaudhuri equation [1, 2]

$$v^{\mu}_{,\mu} + 2\omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2 - \theta_{\alpha} v^{\alpha} - 4\pi G(\rho + 3p) = 0 \qquad (1)$$

where,

v^{μ} = unit velocity vector

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† Dedicated in respectful memory of Professor A K Raychaudhuri

$\dot{v}^{\mu}$ = acceleration (departure of v^{μ} from geodesicity). It arises from the pressure gradient in the case of a perfect fluid.

θ = expansion

ω^2 = vorticity

σ^2 = shear

ρ = matter (energy) density

p = fluid pressure

The last term in eq. (1) represents the gravity effect and is abetted by shear.

With $\theta,_{\alpha} v^{\alpha} = d\theta/ds$, where s parametrises a geodesic, eq. (1) can be written in the form

$$\frac{d\theta}{ds} + \frac{1}{3}\theta^2 = \dot{v}^{\mu}_{;\mu} + 2\omega^2 - 2\sigma^2 - 4\pi G\left(\rho + 3p\right). \tag{2}$$

For a general metric, defining a scale function, R, such that $\theta = 3\left(\dot{R}/R\right)$, Raychaudhuri *et al* [3] have written eq. (2) in the alternative form

$$\frac{\ddot{R}}{R} = \frac{1}{3}\dot{v}^{\mu}_{;\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}\left(\rho + 3p\right). \tag{3}$$

In the absence of $v^{\mu}_{;\mu}$ and ω^2, from eqs. (2) and (3)

respectively we may write

$$\frac{d\theta}{ds} + \frac{1}{3}\theta^2 \leq 0 \tag{4}$$

and

$$\ddot{R} \leq 0. \tag{5}$$

Wald [4] has shown from eq.(4) and Raychaudhuri *et al* from eq. (5) that a collapse to a singularity is inevitable.

Therefore, as is well-known, on the basis of the Raychaudhuri equation, a number of singularity theorems have been derived [5]. The purpose of this paper is, however, to uncover some new facets of this equation. We show here that with appropriate inputs, this equation may provide clues to certain important cosmological developments.

2 (a) Time-dependent Λ

If we accept the view that the universe was born with a big bang from a singular state, then we have to answer the question What triggered the big bang ? Guth [6] recently advanced a plausible answer Dark energy had done it ! Indeed it is estimated [7] that the energy of the universe is 65% dark energy, 30% dark matter and 5% known matter (visible and invisible) showing that the dark energy is the most dominant form of energy in the universe

When Raychaudhuri formulated his equation, dark energy was unknown and the cause of the big bang was an unsolved mystery It would therefore be reasonable, according to the above statistics, to incorporate dark energy into the equation But how to do it ? Turner [7] emphasizes the role of Λ, introduced by Einstein as a repulsive " cosmological constant", in the context of the present-day accelerating universe One may therefore regard $\Lambda(t)$ as a candidate for dark energy It may be noted that Sussman *et al* [8] and Fukui [9] have also considered a time dependent Λ as representing dark energy (see also [19]) We can then incorporate dark energy into the Raychaudhuri equation through the following enlarged law of gravitation [2, 10, 11]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = -8\pi G T_{\mu\nu} \tag{6}$$

with a more general conservation equation

$$\left(8\pi G T^{\mu\nu} + g^{\mu\nu} \Lambda(t)\right)_{;\mu} = 0 \tag{7}$$

Using eq (6), the Raychaudhuri equation (3) is generalized into the form

$$\frac{R}{R} = \frac{1}{3} v^{\mu}_{;\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}(\rho + 3p) + \frac{1}{3}\Lambda(t) \tag{8}$$

Obviously $\Lambda(t)$ opposes the gravity effect due to $4\pi G(\rho + 3p)$ in eq (8) It may be mentioned in passing that this interpretation of $\Lambda(t)$ as representing dark energy distinguishes it from Λ introduced with an opposite sign by some other authors [3] and [12] in the enlarged law of gravitation

Now, if $\frac{1}{3}\Lambda(t)$ is so large at $R \to 0$ that it makes the right-hand side of eq (8) equal $\left(\frac{R}{R}\right)_{R \to 0}$, a big bang results and the universe is born Further, so long as the right-hand side remains positive, the universe continues to accelerate.Thus, $\Lambda(t)$ causes both a big bang as well as the acceleration of the universe

We may check the possibility of a big bang in the simplest cosmological theory, *viz*, in Friedmann cosmology (with $8\pi G = 1$) The equations to be considered are

(i) *Raychaudhuri equation (from (8))*

$$\frac{\ddot{R}}{R} = -\frac{1}{6}(\rho + 3p) + \frac{1}{3}\Lambda(t) \tag{9}$$

(ii) *Conservation equation (from (7))*

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) = -\dot{\Lambda} \, . \tag{10}$$

The solution is

$$R = R_0 t^n, \, \rho = \rho_0 / t^2, \;\; \Lambda = \Lambda_0 / t^2 \tag{11}$$

where

$$\Lambda_0 = \frac{3}{4}\rho_0^2 (1+\gamma)^2 - \rho_0$$

and

$$n = \frac{1}{2}\rho_0 (1+\gamma) \tag{12}$$

γ comes from the assumed relation

$$p = \gamma \rho \, . \tag{13}$$

Assuming a sufficiently large value for ρ_0, Λ_0 becomes larger still (see(12)). Then, we find that

$$\frac{\ddot{R}}{R} = \left[\frac{\rho_0(1+\gamma)}{2}\left(\frac{\rho_0(1+\gamma)}{2} - 1\right)\right]\frac{1}{t^2} \tag{14}$$

is positive and remains positive even at $t \to 0$ resulting in a big bang.

2. (b) Bulk Viscosity :

The bulk viscosity of a kind of viscous cosmic fluid (dark energy ?) is considered by some authors ([13], [14] and other references therein) to be responsible for the observed acceleration of the universe. It modifies pressure, p, into a pressure, p^*, given by

$$p^* = p + \pi \tag{15}$$

where π represents the viscous process. According to Isreal and Stewart ([15], [16]), π is given by the equation

$$\pi + \tau\,\dot{\pi} = -3\,\zeta\,\frac{\dot{R}}{R} - \frac{1}{2}\tau\left(3\frac{\dot{R}}{R} + \frac{\dot{\tau}}{\tau} - \frac{\dot{\zeta}}{\zeta}\frac{\dot{T}}{T}\right) \tag{16}$$

where T is the temperature and τ is the relaxation time If we neglect the second term on the right-hand side of eq. (16) and also τ, we obtain the Eckart result [17]

$$\pi = -3\,\zeta \frac{R}{R} \tag{17}$$

leading to

$$p^{*} = p - 3\,\zeta \frac{R}{R} = p - \zeta\,\theta \tag{18}$$

where θ is the expansion. The viscosity coefficient, ζ, must be positive so that the entropy increases according to the second law of thermodynamics

Now, substituting p^{*} for p in the Raychaudhuri equation (3)

$$\frac{R}{R} = \frac{1}{3} v^{\mu}_{;\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}\left[\rho + 3\left(p - \zeta\theta\right)\right]$$

i e

$$\frac{R}{R} = \frac{1}{3} v^{\mu}_{\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}(\rho + 3p) + 4\pi G\zeta\theta \tag{19}$$

Obviously, the bulk viscosity term in eq.(19) opposes the gravity effect due to $(4\pi G/3)(\rho + 3p)$. Now, if the viscosity term is so large that it makes the right-hand side in eq (19) positive, then it may cause the acceleration of the universe as envisaged by some authors referred to above

2. (c) *Negative Pressure :*

Let us replace p by $-p$ in eq. (3). Then, we have

$$\frac{R}{R} = \frac{1}{3} v^{\mu}_{;\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}(\rho - 3p)$$

i e,

$$\frac{R}{R} = \frac{1}{3} \dot{v}^{\mu}_{;\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}\rho + 4\pi G p \tag{20}$$

In eq.(20), the pressure term opposes the gravity effect due to $(4\pi G/3)\rho$. Now, if the pressure term is large enough to make the right-hand side in eq. (20) positive, then it may cause rapid expansion of the universe. That negative pressure causes rapid expansion has been much discussed in the literature.

2. (d) Scalar Potential :

Banerjee and Das [18] have recently considered the universe as matter-dominated as at present, and as embedded in a scalar field, ϕ, with a scalar potential, $V(\phi)$ They have shown that the acceleration of the universe may be caused by a simple trigonometrica potential Now, it can be shown from their field equations that

$$\frac{R}{R} = -\frac{4\pi G}{3}\left[\rho + 2\phi^2 - 2V(\phi)\right] \tag{21}$$

Taking note of eq (21), the Raychaudhuri equation (3) can be written in the form

$$\frac{R}{R} = \frac{1}{3}v^{\mu}_{\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}\left[\rho + 2\phi^2 - 2V(\phi)\right]$$

i e

$$\frac{R}{R} = \frac{1}{3}v^{\mu}_{\mu} + \frac{2}{3}\omega^2 - \frac{2}{3}\sigma^2 - \frac{4\pi G}{3}\left[\rho + 2\phi^2\right] + \frac{8\pi G}{3}V(\phi) \tag{22}$$

Eq (22) shows that if $V(\phi)$ is sufficiently large so as to make the right-hand side positive the acceleration of the universe should occur

Thus, from the above four examples, it is evident that with appropriate inputs, the Raychaudhuri equation indeed reveals interesting cosmological possibilities

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