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Chaotic behaviour of population on a square lattice

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Abstract : Coverage of occupied sites on a square lattice is allowed to evolve according to a set of rules The rules imply an attractive interaction for growth of new members, the original members 'die', and the new 'population' multiplies and redistributes randomly over the lattice We show that this scenario leads to a steady coverage, cycles with a finite number of points and ultimately chaos as model parameters vary. The calculated results are verified by computer simulation. An immobile situation, where migration or redistribution over the lattice is restricted is also simulated.

Keywords Bifurcation, chaos, square lattice, computer simulation

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1. Introduction

Simple cellular automata algorithms are known to generate a variety of spatio-temporal patterns [1-3]. We present here an algorithm for evolution of population on a square lattice, which gives rise to chaotic behaviour. The term 'population' represents here, a collection of some kind of entity, physical, biological or otherwise, distributed on a square lattice. The probabilities for growth or death of these entities are site specific, *i.e.* dependent on the number of occupied nearest neighbours. We call the fractional number of occupied sites the 'coverage' and study evolution of the coverage under certain rules.

Several versions of this model are possible, we study one version in detail, since it gives particularly interesting results. On varying model parameters the population distribution or coverage evolves to a steady state upto a certain parameter value, then shows repeated bifurcations and period doubling leading to chaotic behaviour interspersed with periodic windows. The bifurcation diagram looks remarkably like that of the celebrated logistic map [4].

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Many non-linear equations and iterated maps are known to generate chaos [5, 6], but we have not come across such reports for the type of system discussed here.

2. Model

Our model is as follows. We consider a square lattice, with unit spacing. The sites may be occupied or vacant. To start with – a fraction p_0 of sites are randomly occupied. Next – the vacant sites can be filled up with a certain probability. We assume this growth probability to be dependent on the nearest neighbourhood environment. Then the original sites are evacuated and the newly added population multiplies by a certain constant factor A. This next generation population with concentration p_1 are now randomly redistributed over the square lattice. This is the 'migration' process. These steps are repeated over and over to see how the population p_n at the nth stage evolves.

The neighbourhood dependent growth probability is B^k, where k is the number of vacant nearest neighbour (n.n.) sites surrounding a vacant site, and B is a constant such that $0 < B < 1$. The situation is illustrated in Figure 1.

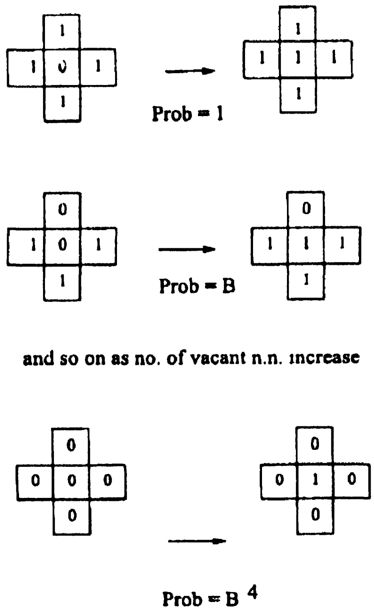

Figure 1. Schematic illustration for the growth rule on the square lattice.

The evolution process can be represented by the following iterative map

$$p_{n+1} = A(1-p_n)\left[p_n + B(1-p_n)\right]^4 . \tag{1}$$

Here, the term $(1 - p_n)$ on the right represents the probability of a site being vacant. The fourth power binomial expression, when expanded, gives the probabilites of this vacant site having 1 to 4 vacant n.n. with the correct weightage. If all neighbours are occupied, the growth probability is 1, and it decreases with the number of vacant n.n. This in effect represents an attractive interaction between the entities occupying the sites. The factor

A, multiplies the new population, the original generation dies out and the right hand side becomes the $(n + 1)$th generation population p_{n+1}. If we now assume these new generation members to randomly redistribute themselves over the whole lattice, we are ready to repeat the process to get p_{n+2} and so on.

The probability parameter B can take values between zero to one The multiplying parameter A can take any positive value but obviously, if it becomes so large that the product on the right exceeds 1, the situation becomes unphysical As we vary A and B within the permitted range, very interesting behaviour for the coverage (or population concentration) is observed. For $B = 0.5$, when we vary A, a steady coverage is observed upto $A = 4.82$ approximately At $A = 4.824$ (we vary A in steps of 0.001) the steady state bifurcates to a 2-point cycle. Here the coverage oscillates continuously between two values As A is increased further there are repeated bifurcations and finally 'chaos' where the coverage has an infinite period never repeating itself. There are intermittent periods where finite cycles return and period-3 cycles can also be seen. Finally at $A = 6\ 102$ the range of p covers the whole range of physically meaningful values Further increase in A leads to unphysical negative values or values higher than 1. All these results are independent of the initial coverage p_0. Figure 2 shows the bifurcation diagram with $B = 0$.

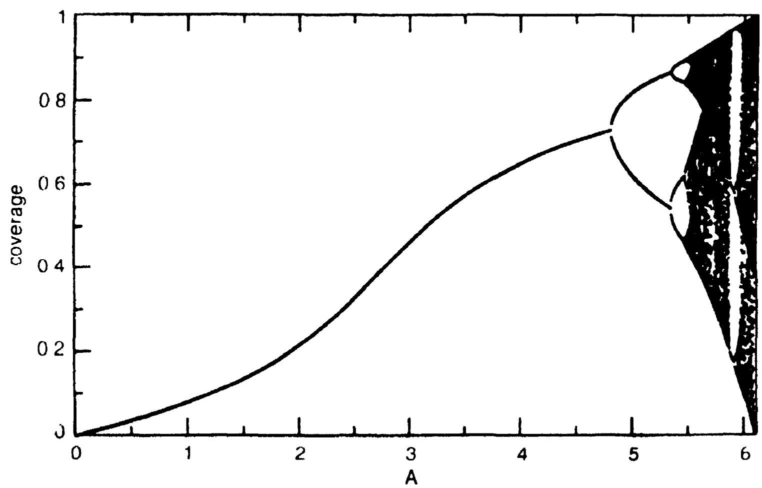

Figure 2. The variation of coverage with A shows repeated bifurcations and chaos

The fine structure of the figure is extremely intricate, as in the bifurcation diagram for the logistic map [4]. Figure 3 shows the details under higher magnification Emergence of the period three region is clearly visible here. So the occurrence of chaos is to be expected [7], according to the well known rule 'period three implies chaos'. A Misiurewicz point [8] where chaotic bands cross, is also visible. To quantify the chaotic behaviour, we have calculated the Lyapunov exponent according to the prescription in [8] around $A = 6.02$ and obtained a positive value of 0.568.

To verify these remarkable results, we have simulated the model on a computer. A square lattice of size 300 × 300 is generated. Sites on it are randomly occupied with an initial concentration p_i. The vacant sites are then scanned to note k, the number of vacant n.n. and they are filled up with a probability B^k. The new sites are counted and multiplied by the factor A and old occupied sites are evacuated. The existing sites are now redistributed over the whole lattice, and the process is repeated, several thousands of times. The behaviour of the coverage agrees with the calculated results shown in figure (2). It takes typically 60 – 70 steps to reach the characteristic evolution pattern which is independent of the initial coverage. For $B < 6.102$, p takes values from 0 to 1. Obviously the unphysical values $p > 1$ or $p < 0$ are not possible in the simulation.

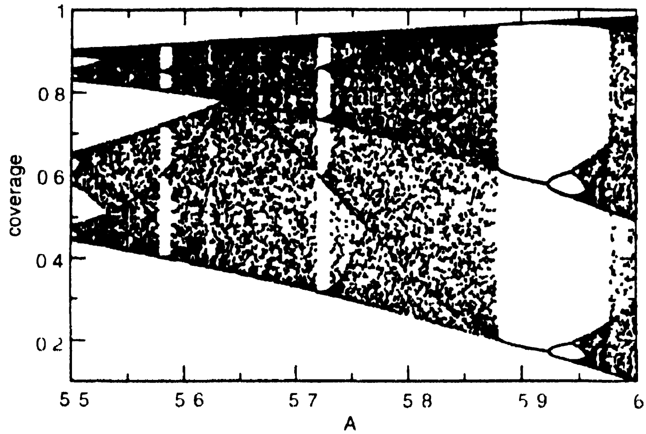

Figure 3. Details from a blow-up of Figure 2 show periodic windows with cycles of period 3 within the chaotic region.

It is interesting to note that when we remain confined to physically acceptable values of p in the range 0 to 1, full blown chaos can be observed only for B close to 0.5. For $B = 0.2$, a small steady state value of p is obtained upto $A \sim 11$, and after that the value oscillates between $\pm\infty$. On the other hand for larger B, *e.g.* with $B = 0.8$, steady state is obtained for $A \sim 2$, but no bifurcation is observed with $0 < p < 1$.

3. Applications

We discuss in brief some possible physical situations where this model may be applied.

3.1. Ferro-magnetic particles in magnetic field :

Consider a 2-dimensional arrangement of spin-half particles, with a ferro-magnetic interaction between neighbours. When placed in an external magnetic field H, each particle can be in the ground state, the spin aligned in the direction of the field, or in the higher level with anti-parallel alignment. The difference in energy between the two levels is $2\mu H$, where μ is the magnetic moment. At a finite temperature more spins are in lower level. Spins can flip, depending on the temperature, keeping the total energy constant. This causes a spatial rearrangement, preserving the number of up and down spins. Let us now introduce a radio-frequency field, which acts as a 'pumping' source of energy, similar to that in a laser. The frequency of the r.f. field is v, where

$$hv = 2\mu H. \tag{2}$$

The present model may now be applied to this system as follows. We assign the state 1 *i e* 'occupied' to the spins in the upper energy level and 0 or 'vacant' to those in the lower. Assuming ferromagnetic interaction between nearest neighbours only, particles in the upper level, draw up nearest neighbours according to the rule illustrated in figure (1). The number of particles already in the upper level is reduced by a factor C, as they drop to the lower level, with emission of a photon. The number pushed up to the upper level are multiplied by factor A, due to the pumping. The temperature of the system is such as to redistribute or randomise the spatial arrangement of the spins with the total energy held constant. This sequence of events is represented by the equation below

$$p_{n+1} = Cp_n + A(1-p_n)\left[p_n + B(1-p_n)\right]^4. \tag{3}$$

This equation, with suitable choice of parameters A, B, C, leads to cycles with periods 2, 4, 8 and so on. We found that for $C = 0.6$ and $A = 4$, $B = 0.4$ gives a period-2 cycle, $B = 0.43$ a period-4 cycle and $B = 0.44$ a period-8 cycle. Values of p become unphysical for higher B. For $C = 0.4$, we get 2, 4 and 8 period cycles for B having values 0.53, 0.55 and 0.57 respectively. The full phase behaviour is yet to be explored. The percolation behaviour of a magnetic system with 'bootstrapping' was studied in [9].

3.2 Condensation and evaporation :

A somewhat similar model was suggested for condensation and evaporation of particles on a plane by Dutta *et al* [10]. The iterative map for this model can be represented in the present scheme as follows

$$p_{n+1} = p_n - Cp_n + (1-p_n)\left[(1-p_n) + Dp_n\right]^4 \tag{4}$$

Here C represents evaporation probability at an occupied site and D^k is the adsorption probability at a vacant site with k occupied n.n. The probability of growth is less for sites with more occupied neighbours, so this situation mimics a repulsion between the particles. C and D are related to temperature and pressure, and it was found that under certain conditions the steady coverage bifurcates to a 2-point cycle.

For the condensation / evaporation case Dutta *et al* [10] also studied an immobile particle model, through computer simulation, where the lateral movement of the particles is not allowed, so there is no randomisation after deposition. We also study an analogous version of the present model.

Dutta *et al* [11] previously studied a system with attractive interaction between particles condensing on a square lattice. Here the algorithm was different from the algorithm used in the present paper, and the results showed a phase transition and hysteresis, rather than bifurcation.

Equation (1) is not valid unless the p_n entities are randomly distributed on the square lattice So only computer simulation is applicable to study the immobile case, with no migration of the new generation members We start as before, with a random distribution of a concentration p of occupied sited, and occupy new sites according to the rule described already But, here we forbid lateral motion of the newly deposited members. The additional $(A - 1)\ p_n$ daugther members produced by the multiplying factor A are distributed randomly over vacant sites

The results for the immobile case are almost identical to the mobile case This is rather surprising considering that in the 'immobile' case the generation which grows according the environment-specific rule have to remain confined to their original sites whereas in the 'mobile' case the second generation particles are randomly redistributed over the lattice before the next deposition step starts.

4. Discussion

Nonlinear systems described by iterative maps are well known [5, 6], they give rise to fascinatingly complex diagrams However, these systems are of special interest when they can be associated with real situations, – physical, chemical or biological The present model though hypothetical, may be applicable to various problems such as adsorption/ desorption or reaction-diffusion [12] problems in chemistry, magnetic systems in physics [9] or population dynamics [13] studies

The number of neighbours considered for each site determines the order of the polynomial expression occurring in our map One can consider variations such as a linear chain, with two neighbours, or include second neighbours, to get a total of eight neighbouring sites on a square lattice

Different modifications are possible, to represent attractive or repulsive interactions between the participating entities The growth or evacuation probability can be made site specific according to the demands of the problem Allowing the parameters to take unphysical values also gives results which may be interesting from the point of view of mathematics, though not meaningful in a practical sense. We hope to report some such results in future. In conclusion we may say that we have presented an interesting iterative map, associated with growth/death or adsorption / desorption on a regular lattice

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