The Cover Time of a Biased Random Walk on a Random Regular Graph of Odd Degree

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Abstract

We consider a random walk process, introduced by Orenshtein and Shinkar [10], which prefers to visit previously unvisited edges, on the random r-regular graph G_r for any odd $r \geq 3$. We show that this random walk process has asymptotic vertex and edge cover times $\frac{1}{r-2} n \log n$ and $\frac{r}{2(r-2)} n \log n$, respectively, generalizing the result from [7] from $r = 3$ to any larger odd r. This completes the study of the vertex cover time for fixed $r \geq 3$, with [3] having previously shown that G_r has vertex cover time asymptotic to $\frac{rn}{2}$ when $r \geq 4$ is even.

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1 Introduction

We consider a biased random walk on the random r-regular n-vertex graph G_r for any odd fixed $r \geq 5$, i.e. a graph chosen uniformly at random from the set of r-regular graph on an even number n of vertices. In short, this is a random walk which chooses a previously unvisited edge whenever possible, see Section 2 for a precise definition. This process was introduced by Orenshtein and Shinkar [10]. In [7] it is shown that with high probability, G_3 is such that the expected vertex cover time $C_V^b(G_3)$ and expected edge cover time $C_E^b(G_3)$ of the biased random walk satisfy[2]

$$C_V^b(G_3) \sim n \log n, \quad C_E^b(G_3) \sim \frac{3}{2} n \log n.$$

We generalize this result as follows.

▶ **Theorem 1.** *Suppose $r \geq 3$ is odd, and let G_r be chosen uniformly at random from the set of r-regular graphs on n vertices. Then with high probability, G_r is such that*

$$C_V^b(G_r) \sim \frac{1}{r-2} n \log n, \quad C_E^b(G_r) \sim \frac{r}{2(r-2)} n \log n.$$

With this the asymptotic leading term of $C_V^b(G_r)$ is known for all $r \geq 3$, with Berenbrink, Cooper and Friedetzky [3] having previously shown that $C_V^b(G_r) \sim \frac{rn}{2}$ for any even $r \geq 4$. They also showed that for even r, $C_E^b(G_r) = O(\omega n)$ for any ω tending to infinity with n, with the ω factor owing to the w.h.p.[3] existence of cycles of length up to ω.

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[2] We say that $a_n \sim b_n$ if $\lim a_n/b_n = 1$.
[3] An event $\mathcal{E}$ holds *with high probability* (w.h.p.) if $\Pr\{\mathcal{E}\} \to 0$ as $n \to \infty$.

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Cooper and Frieze [5] considered the simple random walk on G_r, showing that for any $r \geq 3$, $C_V^s(G_r) \sim \frac{r-1}{r-2} n \log n$ and $C_E^s(G_r) \sim \frac{r(r-1)}{2(r-2)} n \log n$, and we see that the biased random walk speeds up the cover time by a factor of $1/(r-1)$ for odd r. Cooper and Frieze [6] also consider the non-backtracking random walk, i.e. the walk which at no point reuses the edge used in the previous step, showing that $C_V^{nb}(G_r) \sim n \log n$ and $C_E^{nb}(G_r) \sim \frac{r}{2} n \log n$. Here, the biased random walk gains a factor of $1/(r-2)$ for odd r.

Theorem 1 will follow from the following theorem. For a graph G let $C_V^b(G;s)$ ($C_E^b(G;t)$) denote the expected time taken for the biased random walk to visit s vertices (t edges) of G. Note that $C_\cdot^b(G;\cdot)$ is defined as an expectation over the space of random walks on the fixed graph G, and that $\mathbb{E}\left(C_\cdot^b(G_r;\cdot)\right)$ takes the expectation of $C_\cdot^b(G;\cdot)$ when G is chosen uniformly at random from the set of r-regular graphs.

▶ **Theorem 2.** *Suppose $r \geq 3$ is odd, and suppose G_r is chosen uniformly at random from the set of r-regular graphs on an even number n of vertices. Let $n - n\log^{-1/2} n \leq s \leq n$ and $(1 - \log^{-1/2} n)\frac{rn}{2} \leq t \leq rn/2$, and let $\varepsilon > 0$. Then*

$$\mathbb{E}\left(C_V^b(G_r;s)\right) = \frac{1 \pm \varepsilon}{r-2} n \log\left(\frac{n}{n-s+1}\right) + o(n \log n),$$
$$\mathbb{E}\left(C_E^b(G_r;t)\right) = \frac{r \pm \varepsilon}{2(r-2)} n \log\left(\frac{rn}{rn-2t+1}\right) + o(n \log n).$$

We take $a = b \pm c$ to mean that $b - c < a < b + c$. The $(1 - \log^{-1/2} n)$ factor in the lower bounds for s, t is a fairly arbitrary choice, and the proof here is valid for any $(1 - 1/\omega)$ factor with ω tending to infinity sufficiently slowly. The specific choice of $\log^{-1/2} n$ is made to aid readability.

Applying Theorem 2 with $s = n$ and $t = rn/2$ gives $\mathbb{E}\left(C_V^b(G_r)\right) \sim \frac{1}{r-2} n \log n$ and $\mathbb{E}\left(C_E^b(G_r)\right) \sim \frac{r}{2(r-2)} n \log n$. A little extra work is needed to conclude that w.h.p. G_r is such that $C_V^b(G_r), C_E^b(G_r)$ have the same asymptotic values. We refer to the full paper version of [7], where this is done in detail.

2 Proof outline

The random r-regular graph G_r is chosen according to the *configuration model*, introduced by Bollobás [4]. Each vertex $v \in [n]$ is associated with a set $\mathcal{P}(v)$ of r *configuration points*, and we let $\mathcal{P} = \cup_v \mathcal{P}(v)$. We choose u.a.r. (*uniformly at random*) a perfect matching μ of the points in $\mathcal{P}$. Each μ induces a multigraph G on $[n]$ in which u is adjacent to v if and only if $\mu(x) \in \mathcal{P}(v)$ for some $x \in \mathcal{P}(u)$, allowing parallel edges and self-loops. Any simple r-regular graph is equally likely to be chosen under this model.

We study a *biased random walk*. On a fixed graph G, this process is defined as follows. Initially, all edges are declarded *unvisited*, and we choose a vertex v_0 uniformly at random as the *active* vertex. At any point of the walk, the walk moves from the active vertex v along an edge chosen uniformly at random from the unvisited edges incident to v, after which the edge is permanently declared *visited*. If there are no unvisited edges incident to v, the walk moves along a visited edge chosen uniformly at random. The other endpoint of the chosen edge is declared active, and the process is repeated.

A biased random walk on the random r-regular graph can be seen as a random walk on the configuration model, where we expose μ along with the walk as follows. Initially choosing some point $x_0 \in \mathcal{P}$ u.a.r., we walk to $x_1 = \mu(x_0)$, chosen u.a.r. from $\mathcal{P} \setminus \{x_1\}$. Suppose $x_1 \in \mathcal{P}(v_1)$. From x_1 the walk moves to some unvisited $x_2 \in \mathcal{P}(v_1)$. In general, if

$W_k = (x_0, x_1, \ldots, x_k)$ then (i) if k is odd, the walk moves to $x_{k+1} = \mu(x_k)$ (chosen u.a.r. from $\mathcal{P} \setminus \{x_0, \ldots, x_k\}$ if x_k is previously unvisited), and (ii) if k is even, the walk moves from $x_k \in \mathcal{P}(v_k)$ to $x_{k+1} \in \mathcal{P}(v_k)$, chosen u.a.r. from the unvisited points of $\mathcal{P}(v_k)$ if such exist, otherwise chosen u.a.r. from all of $\mathcal{P}(v_k)$.

We define $C(t)$ to be the number of steps taken immediately before the walk exposes its tth distinct edge. To be precise, if $W_k = (x_0, \ldots, x_k)$ denotes the walk after k steps, then

$$C(t) = \min\{k : |\{x_0, x_1, \ldots, x_k\}| = 2t - 1\}.$$

Note that this set consists of exactly one k, as the walk will immediately go to $x_{C(t)+1} = \mu(x_k)$, which has not been visited before. We also let $W(t) = W_{C(t)}$. Note that $C(t)$ is a random variable over the combined probability space of random graphs and random walks, as opposed to $C_V^b(G_r)$ and $C_E^b(G_r)$ which are variables over the space of random graphs only. We will show (Lemma 8) that if $t_1 = (1 - \log^{-1/2} n)\frac{rn}{2}$ then

$$\mathbb{E}\left(C(t_1)\right) = o(n \log n),$$

which does not contribute significantly to the cover time. The main part of the proof is calculating $\mathbb{E}\left(C(t+1) - C(t)\right)$ when $t \geq t_1$. We define the random graph $G(t) \subseteq G_r$ as the graph spanned by the first t distinct edges visited by the walk. If, immediately after discovering its tth edge, the biased random walk inhabits a vertex incident to no unvisited edges, then a simple random walk commences on $G(t)$, and $C(t+1) - C(t)$ is the number of steps taken for this random walk to hit a vertex incident to an unvisited edge.

We construct from $G(t)$ a graph $G^*(t)$ by contracting all vertices incident to at least one unvisited edge into one "supervertex" x. Thus, conditioning on $W(t)$, the graph $G^*(t)$ is a fixed graph, i.e. one with no random edges. We will show that when $t \geq t_1$, w.h.p. x lies on "few" cycles of "short" length and has the appropriate number of self-loops (to be made precise in Section 4), which will imply that the expected hitting time of x for a simple random walk on $G^*(t)$ is

$$\mathbb{E}\left(H(x)\right) \sim \frac{1}{r-2}\frac{rn}{rn-2t}.$$

The paper is laid out as follows. Sections 3, 4 and 5 respectively discuss properties of the random regular graph, hitting times of simple random walks, and a uniformity lemma for biased random walks, and may be read in any order. Section 6 contains the calculation of the cover time. Appendix A and B are devoted to bounding the sizes of certain sets appearing in the calculations.

3 Properties of G_r

Here we collect some properties of random r-regular graphs, chosen according to the configuration model.

▶ **Lemma 3.** *Let $r \geq 3$. Let G_r denote the random r-regular graph on vertex set $[n]$, chosen according to the configuration model. Let ω tend to infinity arbitrarily slowly with n. Its value will always be small enough so that where necessary, it is dominated by other quantities that also go to infinity with n.*

(i) *With high probability, the second largest in absolute value of the eigenvalues of the transition matrix for a simple random walk on G_r is at most 0.99.*
(ii) *With high probability, G_r contains at most ωr^ω cycles of length at most ω,*
(iii) *The probability that G_r is simple is $\Omega(1)$.*

Friedman [8] showed that for any $\varepsilon > 0$, the second eigenvalue of the transition matrix is at most $2\sqrt{r-1}/r + \varepsilon$ w.h.p., which gives (i). Property (ii) follows from the Markov inequality, given that the expected number of cycles of length $k \leq \omega$ can be bounded by $O(r^k)$. For the proof of (iii) see Frieze and Karoński [9], Theorem 10.3. Note that (iii) implies that any property which holds w.h.p. for the configuration multigraph holds w.h.p. for simple r-regular graphs chosen uniformly at random.

Let $G(t)$ denote the random graph formed by the edges visited by $W(t)$. Let $X_i(t)$ denote the set of vertices incident to i unvisited edges in $G(t)$ for $i = 0, 1, \ldots, r$. Let $\overline{X}(t) = X_1(t) \cup \cdots \cup \cup X_r(t)$ denote the set of vertices incident to at least one unvisited edge. Let $G^*(t)$ denote the graph obtained from $G(t)$ by contracting the set $\overline{X}(t)$ into a single vertex, retaining all edges. Define $\lambda^*(t)$ to be the second largest eigenvalue of the transition matrix for a simple random walk on $G^*(t)$.

We note that by [2, Corollary 3.27], if Γ is a graph obtained from G by contracting a set of vertices, retaining all edges, then $\lambda(\Gamma) \leq \lambda(G)$. This implies that $\lambda^*(t) = \lambda(G^*(t)) \leq \lambda(G) \leq 0.99$ for all t. Initially, for small t, we find that w.h.p. $G^*(t)$ consists of a single vertex. In this case there is no second eigenvalue and we take $\lambda^*(t) = 0$. This is in line with the fact that a random walk on a one vertex graph is always in the steady state.

We define $C(t)$ to be the number of steps the biased random walk takes to traverse t distinct edges of G_r. Of course, if G_r is disconnected and the random walk starts in a connected component of less than t edges, then $C(t) = \infty$. We resolve this by defining a stopping time $T^* = \min\{t : \lambda^*(t) > 0.99\}$, and setting $C^*(t) = C(\min\{t, T\})$. Strictly speaking, the estimates of $C(t)$ in the upcoming sections are estimates of $C^*(t)$, but we do not make any explicit distinction between the two, noting that by Lemma 3 (i), w.h.p. $T^* = \infty$ which implies that $C^*(t) = C(t)$ for all t.

4 Hitting times in simple random walks

We are interested in calculating $C(t+1) - C(t)$, i.e. the time taken between discovering the tth and the $(t+1)$th edge. Between the two discoveries, the biased random walk can be coupled to a simple random walk on the graph induced by $W(t)$, and in this section we derive the hitting time of a certain type of expanding vertex set.

▶ **Definition 4.** Let $G = (V, E)$ be an r-regular graph. A set $S \subseteq V$ is a *root set of order* ℓ if (i) $|S| \geq \ell^5$, (ii) the number of edges with both endpoints in S is between $|S|/2$ and $(1/2 + \ell^{-3})|S|$, and (iii) there are at most $|S|/\ell^3$ paths of length at most ℓ between vertices of S which use no edges fully contained in S.

The following lemma establishes the hitting time of root sets.

▶ **Lemma 5.** *Let ω tend to infinity arbitrarily slowly with n. Suppose G is an r-regular graph on n vertices whose transition matrix has second largest eigenvalue $\lambda \leq 0.99$, containing at most ωr^ω cycles of length at most ω. If S is a root set of order ω and a simple random walk is initiated at a uniformly random vertex of G, then the expected number of steps needed to reach S is*

$$\mathbb{E}\left(H(S)\right) \sim \frac{r}{r-2}\frac{n}{|S|}.$$

The full proof of Lemma 5 is omitted in this extended abstract. The proof is based on the following (see e.g. [2, Lemma 2.11]). If $|S| = n/\omega$ for some ω tending to infinity with n, then

$$\mathbb{E}\left(H(S)\right) = \frac{n}{|S|} Z_{SS}.$$

Here Z_{SS} is a constant which can be approximated by the expected number of times a walk starting in S visits S in its first ω steps. We show that this expectation is approximately $r/(r-2)$ for root sets of order ω.

The following lemma is an important step in generalizing Theorem 1 from $r = 3$ to larger r. It follows from reversibility properties of random walks on regular graphs, and the proof is omitted in this extended abstract.

▶ **Lemma 6.** *Let G be an r-regular graph with positive eigenvalue gap. Let $R \subseteq S \subseteq V$ be vertex sets. Suppose a simple random walk is initiated at a uniformly random vertex $y \in R$, and ends as soon as it hits $S \setminus \{y\}$. Then there is a constant $B > 0$ such that for any $x \in S$, the probability that the walk ends at x is at most $B/|R|$.*

5 The structure of $\overline{X}$

The walk $W(t)$ induces a colouring on the edges and vertices of G_r as follows. An edge is coloured red, green or blue if it has been visited zero, one or at least two time(s), respectively. A vertex is (i) green if it is incident to exactly $r-1$ green edges and one red edge, (ii) red if it is incident to red edges only, and (iii) blue otherwise.

Recall that $X_i(t)$ denotes the set of vertices incident to exactly i red edges in $W(t)$. We let $X_1^g(t)$, $X_1^b(t)$ denote the green and blue vertices of $X_1(t)$, respectively, and set

$$Z(t) = X_1^b(t) \cup \bigcup_{i=2}^{r} X_i(t).$$

The green edges and vertices are of particular interest. Suppose $e_1 = (u, v), e_2 = (v, w)$ are consecutive green edges in the walk $W(t)$, meeting at a vertex v. Let $p, q \in \mathcal{P}(v)$ denote the configuration points in v of e_1, e_2, respectively. We call the pair (p, q) a *green link* if v is a green vertex.

Given a walk $W(t)$, we form the *contracted walk* $\langle W(t)\rangle$ as follows. For any green link (p, q), replace the corresponding edges $(u, v), (v, w)$ by the edge (u, w) (coloured green), freeing the configuration points p, q. This is repeated until there are no green links left. Note that if $e_1, e_2, \ldots, e_k$ are green edges visited sequentially by the walk where e_i, e_{i+1} share a green link, then at the end of the process the entire path is replaced by one green edge.

Let $L(W)$ denote the set of green links in the walk W, so $L(W) \subseteq \mathcal{P} \times \mathcal{P}$ is a set of ordered pairs of configuration points. Say that two walks W_1, W_2 are equivalent if $\langle W_1\rangle = \langle W_2\rangle$ and $L(W_1) = L(W_2)$. The equivalence class is denoted $[W] = (\langle W\rangle, L(W))$. The next lemma shows that equivalent walks are equiprobable.

▶ **Lemma 7.** *If W is such that $Pr\{[W(t)] = [W]\} > 0$, then*

$$Pr\{W(t) = W \mid [W(t)] = [W]\} = \frac{1}{|[W]|}.$$

Proof. Let W be a walk with $\Pr\{W(t) = W\} > 0$. We can calculate the probability of $W(t) = W$ exactly. There are two different types of steps a walk can take. Suppose the walk has visited t distinct edges.

- If the walk occupies a vertex incident to no red edges, it chooses an edge with probability r^{-1}.
- If the walk occupies a vertex incident to k red edges, it chooses one of the k red edges with probability k^{-1}. The other endpoint of the red edge is chosen uniformly at random from $rn - 2t - 1$ configuration points.

The probability of $W(t) = W$ is

$$\Pr\{W(t) = W\} = \frac{1}{rn} \prod_{k=2}^{r} k^{-i_k} \prod_{s=0}^{t} \frac{1}{rn - 2s - 1},$$

for some integers $i_2, \ldots, i_r \geq 0$, counting the number of steps of the different types. The $1/rn$ factor accounts for the starting point of the walk. Now, if $W_1 \sim W_2$, then W_1 and W_2 contain the same number of edges, and $i_k(W_1) = i_k(W_2)$ for $k = 2, \ldots, r$. Indeed, W_1 and W_2 only disagree in which order they visit the links in L. ◀

We can now view the biased random walk as a walk on the equivalence class $[W(t)]$. Any time a green edge in $[W(t)]$ is visited, the probability that the edge corresponds to a green link in a randomly chosen $W(t) \in [W(t)]$ is about $L(t)/\Phi(t)$, where $L(t)$ is the number of green links in $[W(t)]$ and $\Phi(t)$ the number of green edges in $W(t)$. This along with bounds for $X_1^g(t)$ and $Z(t)$ provides a precise recursion for $\mathbb{E}(\Phi(t))$, which we use to prove the following. W.h.p., if $t = (1-\delta)\frac{rn}{2}$,

$$|X_1^g(t)| \sim rn\delta \quad \text{when } \delta \leq \log^{-1/2} n, \tag{1}$$

$$|Z(t)| = O(n\delta^{3/2}) \quad \text{when } \delta \leq \log^{-1/2} n, \tag{2}$$

$$\Phi(t) \geq n\delta^{1-\alpha} \quad \text{when } n^{-4/5+\beta} \leq \delta \leq \log^{-1/2} n, \tag{3}$$

where $a > 0$ and $0 < \beta < 1/20$ are constants. Note in particular that $Z(t) \ll X_1^g(t) \ll \Phi(t)$ in the ranges where these bounds apply. Details are found in Appendix A and B.

Suppose $n^{-4/5+\beta} \leq \delta \leq \log^{-1/2} n$. As $L(t) = \frac{r-1}{2} X_1^g(t) = o(\Phi(t))$, when $W(t) \in [W(t)]$ is chosen uniformly at random, the links of $L(t)$ are sprinkled into the much larger set of green edges, and are expected to be spread far apart. This will imply that $X_1^g(t)$ is a root set of order ω, and as $X_1^g(t)$ makes up almost all of $\overline{X}(t)$ by (1) and (2), the set $\overline{X}(t)$ is also a root set of order ω. When $\delta \leq n^{-4/5+\beta}$, the same technique can be applied with a little more work.

6 Calculating the cover time

Define

$$\delta_0 = \frac{1}{\log\log n}, \quad \delta_1 = \frac{1}{\log^{1/2} n}, \quad \delta_2 = \frac{1}{\log^2 n}, \quad \delta_3 = n^{-3/4}, \quad \delta_4 = n^{-1}\log n, \tag{4}$$

and $t_i = (1-\delta_i)\frac{rn}{2}$ for $i = 0, 1, 2, 3$. From this point on we will use t and δ interchangeably to denote time, and the two are always related by $t = (1-\delta)\frac{rn}{2}$. We begin by showing that the time taken to find the first t_1 edges contributes insignificantly to the cover time.

▶ **Lemma 8.**

$$\mathbb{E}(C(t_1)) = o(n \log n).$$

This is proved in Section 6.1. We then move on to estimating the expected cover time increment for larger t.

▶ **Lemma 9.** *For $t_1 \leq t \leq t_4$ and any $\varepsilon > 0$,*

$$\mathbb{E}(C(t+1) - C(t)) = \left(\frac{r}{r-2} \pm \varepsilon\right) \frac{n}{rn - 2t}.$$

The time to discover the final $O(\log n)$ edges can be bounded as follows:

$$\mathbb{E}\left(C\left(\frac{rn}{2}\right) - C(t_4)\right) \le \sum_{t=t_4}^{rn/2-1} O\left(\frac{n}{rn-2s}\right) = o(n\log n).$$

The proof of Lemma 9 is based on the following calculation. Define events

$$\mathcal{A}(t) = \left\{|X_1^g(t) - (rn-2t)| \le \frac{rn-2t}{\omega}\right\},$$
$$\mathcal{B}(t) = \{\overline{X}(t) \text{ is a root set of order } \omega\},$$

and set $\mathcal{E}(t) = \mathcal{A}(t) \cap \mathcal{B}(t)$. Then for any $\varepsilon > 0$, $\mathbb{E}\left(C(t+1) - C(t)\right)$ can be calculated as

$$\left(\frac{r}{r-2} \pm \varepsilon\right)\frac{n}{rn-2t}\Pr\left\{\mathcal{E}(t)\right\} + O\left(\frac{n}{rn-2t}\Pr\left\{\overline{\mathcal{E}(t)}\right\}\right) + O(\log n).$$

Indeed, suppose $\mathcal{E}(t)$ holds. As $X_1(t)$ contains almost all unvisited configuration points, edge t is attached to some $v \in X_1(t)$ w.h.p., and a simple random walk commences at v, ending once it hits $\overline{X} \setminus \{v\}$. As the vertices of $\overline{X}$ are spread far apart, it is unlikely that this happens within $O(\log n)$ steps. After a logarithmic number of steps, the random walk has mixed to within ε of the stationary distribution π in total variation. Lemma 5 shows that after this point, the expected time taken to hit $\overline{X}$ is $(r/(r-2) \pm \varepsilon)n/|\overline{X}|$, and as $\mathcal{A}(t)$ holds we have $|\overline{X}| \sim (rn-2t)$. If $\mathcal{E}(t)$ does not hold, then we use the fact that the hitting time in a regular graph with positive eigenvalue gap is $O(n/|\overline{X}|) = O(n/(rn-2t))$ (as $|\overline{X}| \ge (rn-2t)/r$) as long as the graph has a positive eigenvalue gap. We refer to the discussion in Section 3 justifying our assumption that the second largest eigenvalue stays at most 0.99 throughout the process. Lemma 9 will now follow from proving that $\Pr\left\{\mathcal{E}(t)\right\} = 1 - o(1)$ for any fixed $t_1 \le t \le t_4$. This is done in Section 6.2.

6.1 Phase one: Proof of Lemma 8

With t_1 as in (4), we show that $\mathbb{E}\left(C(t_1)\right) = o(n\log n)$. Suppose $W(t) = (x_0, x_2, \ldots, x_k)$ for some t, k. If $x_k \in \mathcal{P}(\overline{X}(t))$ then $x_{k+1} = \mu(x_k)$ is uniformly random inside $\mathcal{P}(\overline{X}(t)) \setminus \{x_k\}$, and since $C(t+1) = C(t) + 1$ in the event of $x_{k+1} \in \mathcal{P}(X_2 \cup \cdots \cup X_r)$, we have

$$\mathbb{E}\left(C(t+1) - C(t)\right) \le 1 + \mathbb{E}\left(C(t+1) - C(t) \mid x_{k+1} \in \mathcal{P}(X_1)\right)\Pr\left\{x_{k+1} \in \mathcal{P}(X_1)\right\}, \quad (5)$$

We use the following theorem of Ajtai, Komlós and Szemerédi [1] to bound the expected change when $x_{k+1} \in \mathcal{P}(X_1)$.

▶ **Theorem 10.** *Let $G = (V, E)$ be an r-regular graph on n vertices, and suppose that each of the eigenvalues of the adjacency matrix with the exception of the first eigenvalue are at most λ_G (in absolute value). Let A be a set of cn vertices of G. Then for every ℓ, the number of walks of length ℓ in G which avoid A does not exceed $(1-c)n((1-c)r + c\lambda_G)^\ell$.*

The set A of Theorem 10 is fixed. In our case we choose a point x_{k+1} uniformly at random from $\mathcal{P}(X_1(t))$, so we consider a simple random walk initiated at a uniformly random vertex $u \in X_1(t)$. The subsequent walk now begins at vertex u and continues until it hits a vertex of $Y_u = \overline{X}(t) \setminus \{u\}$. Because the vertex u is random, the set Y_u differs for each possible exit vertex $u \in X_1(t)$. To apply Theorem 10, we split $X_1(t)$ into two disjoint sets A, A' of (almost) equal size. For $u \in A$, instead of considering the number of steps needed to hit Y_u, we can upper bound this by the number of steps needed to hit $B' = A' \cup X_2 \cup \cdots \cup X_r$, and vice versa. Suppose without loss of generality that $u \in A$.

Let $S(\ell)$ be a simple random walk of length ℓ starting from a uniformly chosen vertex of A. Thus $S(\ell)$ could be any of $|A|r^\ell$ uniformly chosen random walks. Let $c = |B'|/n$. The probability p_ℓ that a randomly chosen walk of length ℓ starting from A has avoided B' is, by Theorem 10, at most

$$p_\ell \leq \frac{1}{(|X_1(t)|/2)r^\ell}(1-c)n(r(1-c)+c\lambda_G)^\ell \leq \frac{2(1-c)n}{|X_1(t)|}((1-c)+c\lambda)^\ell,$$

where $\lambda \leq .99$ (see Lemma 3) is the absolute value of the second largest eigenvalue of the transition matrix of S. Thus

$$\mathbb{E}_A\left(H(C)\right) \leq \sum_{\ell \geq 1} p_\ell \leq \frac{2(1-c)n}{|X_1(t)|}\frac{1}{c(1-\lambda)}. \tag{6}$$

So,

$$\mathbb{E}\left(C(t+1) - C(t) \mid x_{2k} \in \mathcal{P}(X_1(t))\right) = O\left(\frac{(n-|B'|)n}{|X_1||B'|}\right). \tag{7}$$

Now, for any t we have $r^{-1}(rn-2t) \leq |B'| \leq rn - 2t$, so summing over $0 \leq t \leq t_1$, (5) gives $\mathbb{E}\left(C(t_1)\right) = o(n \log n)$.

6.2 Phase two: Proof of Lemma 9, $t_1 \leq t < t_3$

Let ω tend to infinity arbitrarily slowly with n and define for $t \geq t_1$,

$$\mathcal{A}(t) = \left\{|X_1^g(t) - (rn-2t)| \leq \frac{rn-2t}{\omega}\right\},$$

$$\mathcal{B}(t) = \{\overline{X}(t) \text{ is a root set of order } \omega\},$$

and set $\mathcal{E}(t) = \mathcal{A}(t) \cap \mathcal{B}(t)$. As discussed above, it remains to prove the following lemma.

▶ **Lemma 11.** *Fix $t_1 \leq t \leq t_4$. Then*

$$Pr\{\mathcal{E}(t)\} = 1 - o(1).$$

Proof. First fix $t_1 \leq t \leq t_3$. By (1) – (3), for some $\alpha > 0$, the following holds w.h.p.:

$$\Phi(t) \geq n\delta^{1-\alpha},$$

$$X_1^g(t) = rn\delta(1 - O(\delta^{1/2})),$$

$$Z(t) = O(n\delta^{3/2}).$$

Condition on some $[W(t)]$ satisfying these values. We will distribute the links $L(t)$ into the green edges to form $W(t)$. Suppose $\ell_1 \in L(t)$ is placed at some green edge e_1. As there are at most $Z(t)r^\omega$ green edges within distance ω of $Z(t)$, the probability that it is placed within distance ω of $Z(t)$ is $O(Z(t)r^\omega/\Phi(t)) = o(1)$. The probability that any particular $\ell_2 \in L(t)$ is placed on one of the $O(r^\omega)$ green edges within distance ω of e_1 is $O(r^\omega/\Phi(t))$. Let $D(\ell_1, \ell_2)$ be the distance in $[W(t)]$ between ℓ_1 and ℓ_2. Then

$$\sum_{\ell_1 \neq \ell_2} \Pr\left\{D(\ell_1, \ell_2) \leq \omega\right\} = O\left(\frac{|L(t)|^2 r^\omega}{\Phi(t)}\right) = O\left(n\delta^{2-\frac{1+\varepsilon}{r-1}}3^\omega\right) = o(n\delta).$$

This shows that all but $o(n\delta)$ vertices in $\overline{X}(t)$ are $v \in X_1^g(t)$ with $d(v, \overline{X}(t)) > \omega$. By Lemma 3, at most $\omega r^\omega = o(n\delta)$ vertices in G lie on cycles of length at most ω. This shows that w.h.p., $\overline{X}(t)$ is a root set of order ω.

For $t_3 \leq t \leq t_4$ we can no longer use the bound (3) for $\Phi(t)$, but instead we can show that w.h.p., the conditions of $\mathcal{E}(t_3)$ hold with enough room to spare that they must hold also for t. For example, $Z(t_3)$ is empty w.h.p., so $Z(t) \subseteq Z(t_3)$ must also be empty. ◀

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A Set sizes

Recall the definition

$$Z(t) = X_1^b(t) \cup \bigcup_{i=2}^{r} X_i(t),$$

where X_i denotes the set of vertices incident to i unvisited edges, and X_1^b is the set of vertices in X_1 which are incident to at least one edge which has been visited more than once.

▶ **Lemma 12.** *There exists a constant $B > 0$ such that for $t \geq t_0$ and $0 < \theta = o(1)$,*

$$\mathbb{E}\left(e^{\theta Z(t)}\right) \leq \exp\left\{\theta B n \delta^{3/2}\right\}.$$

Proof. We show that there exists a $B > 0$ such that for any $m \geq 1$,

$$\Pr\left\{[m] \subseteq Z(t)\right\} \leq (B\delta)^{3m/2},$$

beginning with $m = 1$ before the general statement. Let $\mathcal{L} = \mathcal{L}(r)$ denote the set of vectors $(\ell_1, \ell_2, \ldots, \ell_k)$ with $\ell_i \in \{1, 2\}$ such that $\sum \ell_i \leq r - 1$, including in $\mathcal{L}$ the empty vector $\emptyset$, excluding the vector $(2, 2, \dots, 2)$ consisting of $(r-1)/2$ copies of 2 (which corresponds to X_1^g, as we will see). We partition

$$Z(t) = \bigcup_{\ell \in \mathcal{L}} Z_\ell(t),$$

where $v \in Z_\ell(t)$ for $\ell = (\ell_1, \dots, \ell_k)$ if and only if there exists a sequence $0 < s_1 < s_2 < \cdots < s_k \le t$ such that v moves from $X_{r-\ell_1-\cdots-\ell_{j-1}}$ to $X_{r-\ell_1-\cdots-\ell_j}$ at time s_j for $j = 1, \dots, k$, and is in $X_{r-\ell_1-\cdots-\ell_k}$ at time t. If $v \in X_i$ at time s, the probability that v is chosen by random assignment is $i/(rn-2s)$, while Lemma 6 shows that the probability that v is at the end of a blue walk is $O(1/(rn-2s))$. In either case, the probability that v moves from one set to another is at most $B/(rn-2s)$ for some $B > 0$. For a fixed $\ell = (\ell_1, \dots, \ell_k) \in \mathcal{L}$, with $s_0 = 1$,

$$\begin{aligned}\Pr\{1 \in Z_\ell(t)\} \le \sum_{s_1 < \cdots < s_k} \prod_{j=1}^{k} \left[\prod_{s=s_{j-1}+1}^{s_j - 1} \left(1 - \frac{r - (\ell_1 + \cdots + \ell_{j-1})}{rn - 2s}\right) \frac{B}{rn - 2s_j} \right] \\ \times \prod_{s=s_k+1}^{t} \left(1 - \frac{r - (\ell_1 + \cdots + \ell_k)}{rn - 2s}\right). \qquad (8)\end{aligned}$$

For $b \ge 1$ we use the bound

$$\prod_{s=t_0}^{t} \left(1 - \frac{b}{rn - 2s}\right) \le \left(\frac{rn - 2t}{rn - 2t_0}\right)^{b/2}. \qquad (9)$$

Combining (8) and (9), the probability that $1 \in Z_\ell(t)$ is bounded above by

$$\sum_{s_1 < \cdots < s_k} \left[\prod_{j=1}^{k} \frac{B}{rn - 2s_j} \left(\frac{rn - 2s_j}{rn - 2s_{j-1}}\right)^{(r-(\ell_1+\cdots+\ell_{j-1}))/2} \right] \left(\frac{rn - 2t}{rn - 2s_k}\right)^{(r-(\ell_1+\cdots+\ell_k))/2}. \qquad (10)$$

Collecting powers of $rn - 2s_j$ for $j = 1, \dots, k$, we have

$$\Pr\{1 \in Z_\ell(t)\} \le B^k \frac{(rn - 2t)^{(r-(\ell_1+\cdots+\ell_k))/2}}{(rn)^{r/2}} \sum_{s_1 < \cdots < s_k} \prod_{j=1}^{k} (rn - 2s_j)^{\ell_j/2 - 1}.$$

Let N denote the number of indices $j \in \{1, \dots, k\}$ with $\ell_j = 1$. Then

$$\sum_{s_1 < \cdots < s_k} \prod_{j=1}^{k} (rn - 2s_j)^{\ell_j/2 - 1} \le \prod_{j=1}^{k} \left(\sum_{s=0}^{t} (rn - 2s_j)^{\ell_j/2 - 1} \right) \le n^{k-N} (rn - 2t)^{N/2},$$

which implies that

$$\Pr\{1 \in Z_\ell(t)\} \le \frac{B^k}{r^{r/2}} (rn - 2t)^{(r+N-(\ell_1+\cdots+\ell_k))/2} n^{k-N-r/2}.$$

As $\ell_1 + \cdots + \ell_k = 2k - N$, we have $(r + N - (\ell_1 + \cdots + \ell_k))/2 = r/2 - k + N$. So

$$\Pr\{1 \in Z_\ell(t)\} \le \frac{B^k}{r^{k-N}} \delta^{r/2-k+N}.$$

We now argue that $r/2 - k + N \ge 3/2$, or equivalently $2(k - N) \le r - 3$, for all $\ell \in \mathcal{L}$. Firstly, if $\ell_1 + \cdots + \ell_k \le r - 3$ then we have $2(k - N) \le 2k - N = \ell_1 + \cdots + \ell_k \le r - 3$. Secondly, if

$\ell_1+\cdots+\ell_k = r-2$ then as $r-2$ is odd we have $N \geq 1$, so $2(k-N) \leq 2k-N-1 \leq r-3$. Finally, if $\ell_1+\cdots+\ell_k = r-1$ then (as $(2,2,\ldots,2) \notin \mathcal{L}$) we have $N \geq 2$, so $2(k-N) \leq 2k-N-2 \leq r-3$.

As $|\mathcal{L}(r)|$ is a function of r only, and therefore constant with respect to n, it follows that

$$\Pr\{1 \in Z(t)\} = \sum_{\ell\in\mathcal{L}(r)} \Pr\{1 \in Z_\ell(t)\} = O(\delta^{3/2}).$$

We turn to bounding the probability that $[m] \subseteq Z(t)$. We fix $\ell^{(1)},\ldots,\ell^{(m)} \in \mathcal{L}$ and bound the probability that $i \in Z_{\ell^{(i)}}(t)$ for $i = 1,\ldots,m$. Let $k(i) = \dim \ell^{(i)}$ denote the number of components of $\ell^{(i)}$. Then, summing over all choices $s_j^{(i)}$ for $1 \leq i \leq m$ and $1 \leq j \leq k(i)$,

$$\begin{aligned}
&\Pr\{i \in Z_{\ell^{(i)}}(t), i = 1,\ldots,m\} \\
&\leq \sum_{s_j^{(i)}} \prod_{i=1}^{m} B^{k(i)} \frac{(rn-2t)^{(r-\sum_j \ell_j^{(i)})/2}}{(rn)^{r/2}} \prod_{j=1}^{k(i)} (rn - 2s_j^{(i)})^{\ell_j^{(i)}/2-1} \\
&\leq \prod_{i=1}^{m} \left[B^{k(i)} \frac{(rn-2t)^{(r-\sum_j \ell_j^{(i)})/2}}{(rn)^{r/2}} \prod_{j=1}^{k(i)} \left(\sum_{s=0}^{t} (rn-2s)^{\ell_j^{(i)}/2-1} \right) \right] \\
&\leq B^{\sum k(i)} \delta^{3m/2} = O((B^r\delta)^{3m/2}).
\end{aligned}$$

Summing over all $O(m)$ choices of $\ell^{(i)}, i = 1,\ldots,m$, we have

$$\Pr\{[m] \subseteq Z(t)\} = O(m(B^r\delta)^{3m/2}) \leq (C\delta)^{3m/2}$$

for some constant $C > 0$. By symmetry the same bound holds for any vertex set of size m. It follows that for any m, writing $(n)_m = n(n-1)\ldots(n-m+1)$,

$$\mathbb{E}\left((Z(t))_m\right) \leq (n)_m \times (C\delta)^{3m/2} \leq (Cn\delta^{3/2})^m.$$

For $s > 1$ we apply the binomial theorem to obtain

$$\mathbb{E}\left(s^{Z(t)}\right) = \mathbb{E}\left((1+(s-1))^{Z(t)}\right) = \sum_{m\geq 0} \frac{\mathbb{E}\left((Z(t))_m\right)(s-1)^m}{m!}.$$

We set $s = e^\theta \leq 1 + 2\theta$ (as $\theta = o(1)$) to obtain

$$\mathbb{E}\left(e^{\theta Z(t)}\right) \leq \sum_{m\geq 0} \frac{(Cn\delta^{3/2})^m (2\theta)^m}{m!} \leq \exp\left\{\theta D n \delta^{3/2}\right\},$$

for some $D > 0$. ◀

▶ **Corollary 13.** *For $t = (1-\delta)\frac{rn}{2}$ with $\delta = o(1)$, and $0 < \theta = o(1)$,*

$$\mathbb{E}\left(e^{-\theta X_1^g(t)}\right) = \exp\{-\theta rn\delta(1-o(1))\}$$

The technique used to prove Lemma 12 can be strengthened to obtain concentration for the number of unvisited vertices $X_r(t)$.

▶ **Lemma 14.** *For $\theta > 0$,*

$$\mathbb{E}\left(e^{\theta X_r(t)}\right) \leq \exp\left\{2\theta n \delta^{r/2}\right\}. \tag{11}$$

Furthermore, if $t = (1-\delta)\frac{rn}{2}$ *with* $\delta = o(1)$ *and* $n\delta^{r/2} \to \infty$*, then for any* ω *tending to infinity arbitrarily slowly,*

$$Pr\left\{|X_r(t) - n\delta^{r/2}| > \frac{n\delta^{r/2}}{\omega^{1/2}}\right\} \leq \frac{1}{\omega}.$$

Finally, if $n\delta^{r/2} = o(1)$ *then* $X_r(t) = 0$ *w.h.p.*

Lemma 14, the proof of which is omitted here, relates the number of unvisited edges to the number of unvisited vertices: we expect $|X_r(t)| = n - s$ to occur when $t \approx \left(1 - \frac{s}{n}\right)^{2/r}$. This heuristically explains why $C_E^b(G_r) \sim \frac{r}{2} C_V^b(G_r)$. Detailed calculations for the vertex cover time are carried out for $r = 3$ in [7], and the calculations for larger r are identical.

B The green edges

Let $\Phi(t)$ denote the number of green edges in $W(t)$.

▶ **Lemma 15.** *Let* $0 < \varepsilon < r-2$ *and define*

$$\delta_\varepsilon = \left(\frac{\log^4 n}{n}\right)^{\frac{r-1}{r+\varepsilon}}, \quad t_\varepsilon = (1-\delta_\varepsilon)\frac{rn}{2}.$$

Then with high probability, $\Phi(t) \geq n\delta^{\frac{1+\varepsilon}{r-1}}$ *for all* $t_1 \leq t \leq t_\varepsilon$*.*

Proof. Firstly, let us see how $\Phi(t)$ changes with time. Fix $\varepsilon_1 > 0$ such that

$$\frac{1}{(1-\varepsilon_1)(r-1)} < \frac{1+\varepsilon}{r-1}, \tag{12}$$

and let

$$\mathcal{X}(t) = \{X_1^g(t) \geq (1-\varepsilon_1)(rn-2t)\}$$

and let $\mathbf{1}_t$ denote the indicator variable for $\mathcal{X}(t)$. We note that with $\lambda = 1/\log n$, by Corollary 13

$$\Pr\left\{\overline{\mathcal{X}(t)}\right\} \leq \frac{\mathbb{E}\left(e^{-\lambda X_1^g(t)}\right)}{e^{-\lambda(1-\varepsilon_1)(rn-2t)}} \leq \exp\left\{-\frac{\varepsilon_1 n\delta_\varepsilon}{\log n}\right\} =: \eta, \tag{13}$$

for any $t \leq t_\varepsilon$.

▶ Claim 1. For $0 < \theta \leq \delta_\varepsilon \log^{-2} n$, $\varepsilon_1 > 0$ and $t_0 \leq t \leq t_\varepsilon$,

$$\mathbb{E}\left(e^{-\theta(\Phi(t+1)-\Phi(t))}\mathbf{1}_t \mid [W(t)]\right) \leq \exp\left\{\frac{2\theta\Phi(t)}{(1-\varepsilon_1)(r-1)(rn-2t)}(1+O(\gamma))\right\}\mathbf{1}_t,$$

with $\gamma = o(\log^{-1} n)$.

Proof of Claim 1. Condition on a $[W(t)]$ such that $X_1^g(t) \geq (1-\varepsilon_1)(rn-2t)$. If the next edge is added without entering a blue walk, then $\Phi(t+1) = \Phi(t) + 1$. So,

$$\Pr\{\Phi(t+1) - \Phi(t) = 1 \mid [W(t)]\} = 1 - \frac{X_1(t)}{rn-2t}.$$

Suppose the new edge chooses a vertex of $X_1(t)$, thus entering a blue walk. We may view this as a walk on $[W(t)]$, and any time a green edge is traversed, we ask if the green edge in $[W(t)]$ contains a green link in $W(t)$, in which case the blue walk ends. If not, the green edge turns blue and Φ decreases by one.

There are $L(t) = \frac{r-1}{2} X_1^g(t)$ green links, distributed into the $\Phi(t)$ green edges by a Pólya urn process as discussed in Section 5. Suppose $e_1, e_2, \ldots, e_\ell$ are green edges in $[W(t)]$, and let $K_1, K_2, \ldots, K_\ell$ be the lengths of the corresponding paths in $W(t)$, corresponding to the first ℓ entries of a vector $(k_1, \ldots, k_\phi)$ drawn uniformly at random from all vectors with $k_i \geq 1$ and $\sum_{i=1}^{\phi} k_i = \Phi(t)$. The probability that none of the ℓ edges contains a green link is exactly

$$\Pr\{K_i = 1 \text{ for } i = 1, 2, \ldots, \ell\} = \prod_{i=1}^{\ell} \frac{\binom{\Phi-i-1}{\phi-i-1}}{\binom{\Phi-i}{\phi-i}} = \prod_{i=1}^{\ell} \left(1 - \frac{L(t)}{\Phi(t) - i}\right) \leq \left(1 - \frac{L(t)}{\Phi(t)}\right)^{\ell}.$$

This shows that the number of green edges visited before discovering a green link can be bounded by a geometric random variable. If a green edge is visited without a discovery, that edge turns blue. Note that the blue walk may also end when a vertex of X_i^b is found for some $i \geq 1$; we are upper bounding the number of green edges visited.

So in distribution,

$$\Phi(t+1) - \Phi(t) \stackrel{d}{=} 1 - B\left(\frac{X_1(t)}{rn - 2t}\right) R_t$$

where $B(p)$ denotes a Bernoulli random variable taking value 1 with probability p, and R_t is stochastically dominated above by a geometric random variable with success probability $L(t)/\Phi(t)$. The two random variables on the right-hand side are independent. So

$$\mathbb{E}\left(e^{-\theta(\Phi(t+1)-\Phi(t))} \mid [W(t)]\right) = e^{-\theta}\left(1 - \frac{X_1(t)}{rn-2t} + \frac{X_1(t)}{rn-2t}\mathbb{E}\left(e^{\theta R_t} \mid [W(t)]\right)\right)$$

The map $x \mapsto e^{\theta x}$ is increasing for $\theta > 0$, so we can couple R_t to a geometric random variable S_t with success probability $L(t)/\Phi(t)$ in such a way that

$$\mathbb{E}\left(e^{\theta R_t} \mid [W(t)]\right) \leq \mathbb{E}\left(e^{\theta S_t} \mid [W(t)]\right).$$

As S_t is geometrically distributed and $X_1^g(t) \geq (rn-2t)/2$ by conditioning on $\mathcal{X}(t)$,

$$\mathbb{E}\left(e^{\theta S_t} \mid [W(t)]\right) = 1 + \theta\frac{\Phi(t)}{L(t)} - O\left(\frac{\theta^2\Phi(t)^2}{L(t)^2}\right) = 1 + \theta\frac{\Phi(t)}{L(t)}(1 + O(\gamma)).$$

Conditioning on $X_1^g(t) \geq (1-\varepsilon_1)(rn-2t)$ implies that $L(t) = \frac{r-1}{2}X_1^g(t) = \Omega(n\delta)$, so

$$\gamma := \theta\frac{\Phi(t)}{L(t)} \leq \delta_\varepsilon \log^{-2} n \frac{n}{\Omega(n\delta_\varepsilon)} = o(\log^{-1} n).$$

We also have $X_1^b(t) \leq rn - 2t - X_1^g(t)$, so

$$\frac{X_1(t)}{L(t)} = \frac{X_1^g(t)}{L(t)} + \frac{X_1^b(t)}{L(t)} \leq \frac{2}{r-1} + \frac{\varepsilon_1(rn-2t)}{(1-\varepsilon_1)\frac{r-1}{2}(rn-2t)} = \frac{2}{(1-\varepsilon_1)(r-1)}.$$

So for $[W(t)] \in \mathcal{X}(t)$,

$$\begin{aligned}
&\mathbb{E}\left(e^{-\theta(\Phi(t+1)-\Phi(t))}\mathbf{1}_t \mid [W(t)]\right) \\
&\leq e^{-\theta}\left(1 - \frac{X_1(t)}{rn-2t} + \frac{X_1(t)}{rn-2t}\left(1 + \theta\frac{\Phi(t)}{L(t)}(1+O(\gamma))\right)\right) \\
&\leq \exp\left\{\frac{2\theta\Phi(t)}{(1-\varepsilon_1)(r-1)(rn-2t)}(1+O(\gamma))\right\}.
\end{aligned}$$

◀

Define for $0 < \theta = o(1)$,

$$f_t(\theta) = \mathbb{E}\left(e^{-\theta\Phi(t)}\mathbf{1}_t\right).$$

As $\Phi(t) \geq L(t) = \frac{r-1}{2}X_1^g(t)$ we have for $0 < \theta = o(1)$, by Corollary 13,

$$f_{t_0}(\theta) \leq \mathbb{E}\left(e^{-\theta\Phi(t_0)}\right) \leq \mathbb{E}\left(e^{-\theta\frac{r-1}{2}X_1^g(t)}\right) = \exp\left\{-\theta\frac{r-1}{2}rn\delta_0(1+o(1))\right\}. \tag{14}$$

Claim 1 shows that for $t_0 \leq t < t_\varepsilon$,

$$f_{t+1}(\theta) \leq f_t\left(\theta\left(1 - \frac{2(1+O(\gamma))}{(1-\varepsilon_1)(r-1)(rn-2t)}\right)\right) + \eta$$

where $\eta = \exp\{-\varepsilon_1 n\delta_\varepsilon/\log n\}$ is an upper bound for $\Pr\left\{\overline{\mathcal{X}(t+1)}\right\}$, as defined in (13). As $\gamma = o(\log^{-1} n)$, we have

$$\prod_{s=t_0}^{t-1}\left(1 - \frac{2(1+O(\gamma))}{(1-\varepsilon_1)(r-1)(rn-2s)}\right) \sim \left(\frac{rn-2t}{rn-2t_0}\right)^{\frac{1}{(1-\varepsilon_1)(r-1)}}.$$

It follows by induction and from (14) that if $F(t) = n\delta^{\frac{1+\varepsilon}{r-1}}$,

$$\begin{aligned}
f_t(\theta) &\leq f_{t_0}\left(\theta\prod_{s=t_0}^{t-1}\left(1 - \frac{2(1+O(\gamma))}{(1-\varepsilon_1)(r-1)(rn-2s)}\right)\right) + (t-t_0)\eta \\
&\leq \exp\left\{-\theta rn\delta_0\left(\frac{\delta}{\delta_0}\right)^{\frac{1}{(1-\varepsilon_1)(r-1)}}\right\} + (t-t_0)\eta \\
&\leq \exp\left\{-r\theta F(t)\right\} + n\eta.
\end{aligned}$$

Here we used the fact that ε_1 was chosen in (12) to satisfy $1/(1-\varepsilon_1)(r-1) < (1+\varepsilon)/(r-1)$.

Now, setting $\theta = \delta_\varepsilon \log^{-2} n$, using the bound $\mathbf{1}_{\{X>a\}} \leq X/a$,

$$\begin{aligned}
\Pr\left\{\Phi(t) < F(t)\right\} &\leq \Pr\left\{\overline{\mathcal{X}(t)}\right\} + \Pr\left\{\Phi(t) < F(t),\ \mathcal{X}(t)\right\} \\
&\leq \eta + \mathbb{E}\left(\mathbf{1}_{\left\{e^{-\theta\Phi(t)} > e^{-\theta F(t)}\right\}}\mathbf{1}_t\right) \\
&\leq \eta + e^{\theta F(t)} f_t(\theta) \\
&= O(ne^{\theta F(t)}\eta) + e^{-\theta(r-1)F(t)} \\
&= o(n^{-1}).
\end{aligned}$$

It follows that $\Phi(t) \geq F(t)$ for all t in the given range with high probability. ◀