Boolean Function Analysis on High-Dimensional Expanders

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Abstract

We initiate the study of Boolean function analysis on high-dimensional expanders. We describe an analog of the Fourier expansion and of the Fourier levels on simplicial complexes, and generalize the FKN theorem to high-dimensional expanders.

Our results demonstrate that a high-dimensional expanding complex X can sometimes serve as a sparse model for the Boolean slice or hypercube, and quite possibly additional results from Boolean function analysis can be carried over to this sparse model. Therefore, this model can be viewed as a derandomization of the Boolean slice, containing $|X(k)| = O(n)$ points in comparison to $\binom{n}{k+1}$ points in the $(k+1)$-slice (which consists of all n-bit strings with exactly $k+1$ ones).

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1 Introduction

Boolean function analysis is an essential tool in theory of computation. Traditionally, it studies functions on the Boolean cube $\{-1,1\}^n$. Recently, the scope of Boolean function analysis has extended further, encompassing groups [10, 9, 29, 11], association schemes [27, 13, 14, 16, 15, 6, 23], error-correcting codes [1], and quantum Boolean functions [26]. Boolean function analysis on extended domains has led to progress in learning theory [27] and on the unique games conjecture [7, 22, 2, 23].

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Another essential tool in theory of computation is expander graphs. Recently, high-dimensional expanders (HDXs), originally constructed by Lubotzky et al. [25, 24], have been introduced into theory of computation, with applications to property testing [5] and lattices [20]. Just as expander graphs are sparse models of the complete graph, so are high-dimensional expanders sparse models of the complete *hypergraph*, and hence can be used both for derandomization and to improve constructions of objects such as PCPs. *The goal of this work is to connect these two threads of research, introducing Boolean function analysis on high-dimensional expanders.*

We study Boolean functions on simplicial complexes. A pure d-dimensional simplicial complex X is a set system consisting of an arbitrary collection of sets of size $d+1$ together with all their subsets. The sets in a simplicial complex are called *faces*, and it is standard to denote by $X(i)$ the faces of X whose cardinality is $i+1$. Our simplicial complexes are *weighted* by a probability distribution Π_d on the top-level faces, which induces in a natural way probability distributions Π_i on $X(i)$ for all i. Our main object of study is the space of functions $f\colon X(d) \to \mathbb{R}$ (which are often called $\mathbb{R}$-cochains), and in particular, Boolean functions $f\colon X(d) \to \{0,1\}$.

While much of our work applies to arbitrary complexes, our goal is to study complexes which are *high-dimensional expanders*. There are several different non-equivalent ways to define high-dimensional expanders, generalizing various properties of expander graphs. The notion most appropriate to us is the two-sided[4] spectral definition of high-dimensional expanders, due to Dinur and Kaufman [5], who also show how to construct such complexes using the Ramanujan complexes of Lubotzky, Samuels and Vishne [25, 24].

There are several works on random walks on high dimensional expanders which naturally lead to analyzing both real-valued and Boolean valued functions on $X(d)$, for example see [20, 5, 21]. The most related work is by Kaufman and Oppenheim [21], which we discuss in Section 4.3.

Every function on the Boolean cube $\{-1,1\}^n$ has a unique representation as a multilinear polynomial, known as its *Fourier expansion*. The multilinear monomials can be partitioned into "levels" according to their degree, and this corresponds to an orthogonal decomposition of a function into a sum of its homogeneous parts, $f = \sum_{i=0}^{\deg f} f^{=i}$, a decomposition which is a basic concept in Boolean function analysis.

These concepts have known counterparts for the *complete complex*, which consists of all subsets of $[n]$ size at most $d+1$, where $d+1 \le n/2$. The facets (top-level faces) of this complex comprise the *slice* (as it is known for computer scientists) or the *Johnson scheme* (as it is known for coding theorists), whose spectral theory has been elucidated by Dunkl [8]. For $|t| \le d+1$, let $y_t(s) = 1$ iff $t \subseteq s$ (these are the analogs of monomials). Every function on the complete complex has a unique representation as a linear combination of monomials $\sum_t \tilde{f}(t) y_t$ (of various degrees) satisfying the *harmonicity* condition: for all $i \le d$ and all $t \in X(i)$,

$$\sum_{a \in [n] \setminus t} \tilde{f}(t \cup \{a\}) = 0.$$

(If we identify y_t with the product $\prod_{i \in t} x_i$ of "variables" x_i, then harmonicity of a multilinear polynomial P translates to the condition $\sum_{i=1}^n \frac{\partial P}{\partial x_i} = 0$.) As in the case of the Boolean

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[4] A related and slightly weaker notion of one-sided spectral expansion has already appeared earlier in [19, 12].

cube, this unique representation allows us to orthogonally decompose a function into its homogeneous parts (corresponding to the contribution of monomials y_t with fixed $|t|$), which plays the same essential part in the complete complex as its counterpart does in the Boolean cube. Moreover, this unique representation allows extending a function from the "slice" to the Boolean cube (which can be viewed as a superset of the "slice"), thus implying further results such as invariance principle [15, 16].

We generalize these concepts for complexes satisfying a technical condition we call *properness*, which is satisfied by both the complete complex as well as high-dimensional expanders. We show that the results on unique decomposition for the complete complex hold for arbitrary proper complexes, with a generalized definition of harmonicity which incorporates the distributions Π_i. In contrast to the case of the complete complex (and the Boolean cube), the homogeneous parts are only approximately orthogonal.

The homogeneous components in our decomposition are "approximate eigenfunctions" of the Laplacian, and this allows us to derive an approximate identity relating the total influence (defined through the Laplacian) to the norms of the components in our decomposition, in complete analogy to the same identity in the Boolean cube (expressing the total influence in terms of the Fourier expansion). All of this is summarized in Theorem 4.1.

As a demonstration of the power of this setup, we generalize the fundamental result of Friedgut, Kalai, and Naor [17] on Boolean functions almost of degree 1. We view this as a first step toward developing a full-fledged theory of Boolean functions on high-dimensional expanders. An easy exercise shows that a Boolean degree 1 function on the Boolean cube is a *dictator*, that is, depends on at most one coordinate; we call this the *exact FKN theorem*. The FKN theorem states that a Boolean function on the Boolean cube which is *close* to a degree 1 function is in fact close to a dictator, where closeness is measured in L_2.

The exact FKN theorem holds for the complete complex as well. Recently, the second author [13] extended the FKN result to the complete complex. Surprisingly, the class of approximating functions has to be extended beyond just dictators.

We prove an exact FKN theorem for arbitrary proper complexes, and an FKN theorem for high-dimensional expanders. In contrast to the complete complex, Boolean degree 1 functions on arbitrary complexes correspond to *independent sets* rather than just single points, and this makes the proof of the exact FKN theorem non-trivial. Our proof of the FKN theorem for high-dimensional expanders is very different from existing proofs. It follows the same general plan as our recent work on the biased Kindler–Safra theorem [4]. The idea is to view a high-dimensional expander as a convex combination of small sub-complexes, each of which is isomorphic to the complete k-dimensional complex on $O(k)$ vertices. We can apply the known FKN theorem separately on each of these, and deduce that our function is approximately well structured on each sub-complex. Finally, we apply the agreement theorem of Dinur and Kaufman [5] to show that the same thing is true on a global level.

1.1 Results

Given a positive integer n and $d < n/2$, let X be the d-dimensional simplex consisting of all subsets of $\{1, \dots, n\}$ of size at most $d+1$. For $\ell \leq d$, let $X(\ell)$ denote the set of subsets of size exactly $\ell+1$. In the following, we will view any function $f\colon X(\ell) \to \mathbb{R}$ as a function mapping n-bit strings $(y_1, \dots, y_n) \in \{0,1\}^n$ satisfying $y_1 + \cdots + y_n = \ell + 1$ to the reals. A classical result (see for example [16]) states that every such function has a unique representation $f = f_{-1} + \cdots + f_\ell$ satisfying the following properties:

- f_i is a homogeneous multilinear polynomial of degree $i+1$.

- Each f_i is in the kernel of $\sum_{j=1}^n \frac{\partial}{\partial y_j}$.
- $\|f\|^2 = \|f_{-1}\|^2 + \cdots + \|f_\ell\|^2$.
- $\langle DUf, f\rangle = \sum_{i=-1}^{\ell} \left(1 - \frac{i+1}{d-\ell}\right) \frac{\ell-i+1}{\ell+2} \|f_i\|^2$.

  Here $DU\colon \mathbb{R}^{X(\ell)} \to \mathbb{R}^{X(\ell)}$ is the (upper) Laplacian, given by the formula

  $$(DUf)(s) = \mathop{\mathbb{E}}_{j\notin s} \mathop{\mathbb{E}}_{k\in s\cup\{j\}} f\big((s\cup\{j\})\setminus\{k\}\big).$$

Our first result is an analogous decomposition for functions on any high-dimensional expander (not necessarily the complete complex):

▶ **Theorem 1.1** (Decomposition theorem for functions on HDX). *Let X be a d-dimensional expander. Every function $f\colon X(\ell) \to \mathbb{R}$ for $\ell \le d$, can be written uniquely as $f = f_{-1} + \cdots + f_\ell$ such that:*

- *f_i is a linear combination of the functions $y_s(t) = [t \supseteq s]$ for $s \in X(i)$.*
- *Interpreted as a function on $X(i)$, f_i lies in the kernel of the "Down" operator.*
- *$\|f\|^2 \approx \|f_{-1}\|^2 + \cdots + \|f_\ell\|^2$.*
- *If $\ell < k$ then $\langle DUf, f\rangle \approx \sum_{i=-1}^{\ell} \frac{\ell-i+1}{\ell+2} \|f_i\|^2$.*

The "Down" operator and the Laplacian are defined formally in Section 2. They differ from their counterparts in the complete complex by taking into account the measure Π_d.

Equipped with this decomposition, we prove the following exact and approximate FKN theorem on high dimensional expanders.

▶ **Theorem 1.2** (exact FKN theorem on HDX). *Let X be a d-dimensional expander. If $f\colon X(d) \to \{0,1\}$ has degree 1, then f is the indicator of either intersecting or not intersecting an independent set of X.*

▶ **Theorem 1.3** (FKN theorem on HDX). *Let X be a d-dimensional expander. If $F\colon X(d) \to \{0,1\}$ is ε-close (in L_2^2) to a degree-1 function then there exists a degree-1 function g on $X(d)$ such that $\Pr[F \ne g] = O(\varepsilon)$.*

Paper organization

We describe our general setup in Section 2. We describe the property of properness and its implications – a unique representation theorem and decomposition of functions into homogeneous parts – in Section 3, discussing the issue of orthogonality of the homogeneous parts in Section 4. Theorem 4.1 summarizes these results. We discuss high-dimensional expanders from our perspective in Section 5. We prove our exact FKN theorem in Section 6, and our FKN theorem in Section 7.

Theorem 1.1 is a combination of Theorem 3.2 (first two items), Theorem 4.1 (other two items), Lemma 4.7 (calculation of the coefficients in the last item) and Theorem 5.1 (showing that high-dimensional expanders satisfy the prerequisites of the preceding results). Theorem 1.2 is a restatement of Theorem 6.1. Theorem 1.3 is a restatement of Theorem 7.2.

2 Basic setup

A d-dimensional complex X is a non-empty collection of sets of size at most $d+1$. We call a set of size $i+1$ an *i-dimensional face* (or i-face for short), and denote the collection of all i-faces by $X(i)$. A d-dimensional complex X is *pure* if every i-face is a subset of some d-face. We will only be interested in pure complexes.

Let X be a pure d-dimensional complex. Given a probability distribution Π_d on its top-dimensional faces $X(d)$, for each $i < d$ we define a distribution Π_i on the i-faces using the following experiment: choose a top-dimensional face according to Π_d, and remove $d-i$ points at random. We can couple all of these distributions to a random vector $\vec{\Pi} = (\Pi_d, \dots, \Pi_{-1})$ of which the individual distributions are marginals.

Let $C^i = \{f \colon X(i) \to \mathbb{R}\}$ be the space of functions on $X(i)$. It is convenient to define $X(-1) := \{\emptyset\}$, and we also let $C^{-1} = \mathbb{R}$. We turn C^i to an inner product space by defining $\langle f, g\rangle := \mathbb{E}_{\Pi_i}[fg]$ and the associated norm $\|f\|^2 := \mathbb{E}_{\Pi_d}[f^2]$.

For $-1 \le i < d$, we define the Up operator $U_i \colon C^i \to C^{i+1}$ as follows:[5]

$$U_i g(s) := \frac{1}{i+2} \sum_{x \in s} g(s \setminus \{x\}) = \mathop{\mathbb{E}}_{t \subset s}[g(t)] ,$$

where t is obtained from s by removing a random element. Note that if $s \sim \Pi_{i+1}$ then $t \sim \Pi_i$.

Similarly, we define the Down operator $D_{i+1} \colon C^{i+1} \to C^i$ for $-1 \le i < d$ as follows:

$$D_{i+1} f(t) := \frac{1}{(i+2) \cdot \Pi_i(t)} \sum_{x \notin t \colon t \cup \{x\} \in X(i+1)} \Pi_{i+1}(t \cup \{x\}) \cdot f(t \cup \{x\}) = \mathop{\mathbb{E}}_{s \supset t}[f(s)] ,$$

where s is obtained from t by conditioning the vector $\vec{\Pi}$ on $\Pi_i = t$ and taking the $(i+1)$th component.

The operators U_i, D_{i+1} are adjoint to each other. Indeed, if $f \in C^{i+1}$ and $g \in C^i$ then

$$\langle g, D_{i+1} f\rangle = \mathop{\mathbb{E}}_{(t,s) \sim (\Pi_i, \Pi_{i+1})}[g(t) f(s)] = \langle U_i g, f\rangle .$$

When the domain is understood, we will use U, D instead of U_i, D_{i+1}. This will be especially useful when considering powers of U, D. For example, if $f \colon X(i) \to \mathbb{R}$ then

$$U^t f \equiv U_{i+t-1} \dots U_{i+1} U_i f.$$

The function y_s is the indicator function of containing s. Our definition of the Up operator guarantees the correctness of the following lemma.

▶ **Lemma 2.1.** *Let $s \in X(i)$. We can think of y_s as a function in C^j for all $j \ge i$. Using this convention, $U_j y_s = (1 - \frac{i+1}{j+2}) y_s$.*

Proof. Direct calculation shows that

$$(U_j y_s)(t) = \frac{1}{j+2} \sum_{x \in t} y_s(t \setminus \{x\}) = \frac{|t| - |s|}{j+2} y_s(t) ,$$

and so $U_j y_s = (1 - \frac{i+1}{j+2}) y_s$. ◀

For $0 \le i \le k$, the space of harmonic functions on $X(i)$ is

$$H^i := \ker D_i = \{f \in C^i : D_i f = 0\} .$$

We also define $H^{-1} = C^{-1}$. We are interested in decomposing C^k, so let us define for each $-1 \le i \le k$,

$$V^i := U^{k-i} H^i = \{U^{k-i} f \ : \ f \in H^i\} .$$

We can describe V^i, a sub-class of functions of c^k, in more concrete terms.

[5] The Up and Down operators differ from the boundary and coboundary operators of algebraic topology, which operate on linear combinations of *oriented* faces.

▶ **Lemma 2.2.** *Every function $h \in V^i$ has a unique representation of the form*

$$h = \sum_{s \in X(i)} \tilde{h}(s) y_s ,$$

where the coefficients $\tilde{h}(s)$ satisfy the following harmonicity *condition: for all $t \in X(i-1)$,*

$$\sum_{s \supset t} \Pi_i(s) \tilde{h}(s) = 0 .$$

Proof. First, let us show that every function of the form given above is in V^i. Lemma 2.1 shows that a function h is of the form $\sum_{s \in X(i)} \tilde{h}(s) y_s$ where $\tilde{h}$ satisfies the harmonicity condition if and only if $h = U^{k-i} r$, where $r \in C^i$ is of similar form $r = \sum_{s \in X(i)} \tilde{r}(s) y_s$, where $\tilde{r}$ satisfies the harmonicity condition. In fact, it is easy to see that $r = \tilde{r}$, and so the fact that $r \in H^i$ follows directly from the definition of the Down operator.

We have shown above that if $h = U^{k-i} r$ then the coefficients $\tilde{h}$ are a constant multiple of the values of r, hence this representation is also unique. ◀

3 Decomposition of the space C^k and a convenient basis

Our decomposition theorem relies on a crucial property of complexes, *properness*.

▶ **Definition 3.1.** A k-dimensional complex is *proper* if it is pure and $\ker U_i$ is trivial for $-1 \le i \le k-1$.

The complete k-dimensional complex on n points is proper iff $k+1 \le n/2$. A pure one-dimensional complex (i.e., a graph) is proper iff it is not bipartite. Unfortunately, we are not aware of a similar characterization for higher dimensions. However, in Section 5 we show that high-dimensional expanders are proper.

We can now state our decomposition theorem.

▶ **Theorem 3.2.** *If X is a proper k-dimensional complex then we have the following decomposition of C^k:*

$$C^k = V^k + V^{k-1} + \cdots + V^{-1} .$$

In other words, for every function $f \in C^k$ there is a unique choice of $h_i \in H^i$ such that the functions $f_i = U^{k-i} h_i$ satisfy $f = f_{-1} + f_0 + \ldots + f_k$.

Proof. We first prove by induction that every function $f \in C^\ell$ has a representation $f = \sum_{i=-1}^{\ell} U^{\ell-i} h_i$, where $h_i \in H^i$. This trivially holds when $\ell = -1$. Suppose now that the claim holds for some $\ell < k$, and let $f \in C^{\ell+1}$. Since $D^{\ell+1} \colon C^{\ell+1} \to C^\ell$ is a linear operator, we have $C^{\ell+1} = \ker D_{\ell+1} + \operatorname{im} D_{\ell+1}^* = \ker D_{\ell+1} + \operatorname{im} U_\ell$, and therefore we can write $f = h_{\ell+1} + Ug$, where $h_{\ell+1} \in H^{\ell+1}$ and $g \in C^\ell$. Applying induction, we get that $g = \sum_{i=-1}^{\ell} U^{\ell-i} h_i$, where $h_i \in H^i$. Substituting this in $f = h_{\ell+1} + Ug$ completes the proof.

It remains to show that the representation is unique. Since $\ker U_{i-1} = \ker D_i^*$ is trivial, $\dim H^i = \dim C^i - \dim C^{i-1}$ for $i \ge 0$. This shows that $\sum_{i=-1}^{k} \dim H^i = \dim C^k$. Therefore the operator $\varphi \colon H^{-1} \times \cdots \times H^k \to C^k$ given by $\varphi(h_{-1}, \ldots, h_k) = \sum_{i=-1}^{k} U^{k-i} h_i$ is not only surjective but also injective. In other words, the representation of f is unique. ◀

▶ **Corollary 3.3.** *If X is a proper k-dimensional complex then every function $f \in C^k$ has a unique representation of the form*

$$f = \sum_{s \in X} \tilde{f}(s) y_s ,$$

where the coefficients $\tilde{f}(s)$ *satisfy the following* harmonicity *conditions: for all* $0 \leq i \leq k$ *and all* $t \in X(i-1)$*:*

$$\sum_{\substack{s \in X(i) \\ s \supset t}} \Pi_i(s) \tilde{f}(s) = 0 \; .$$

Proof. Follows directly from Lemma 2.2. ◀

We can now define the degree of a function.

▶ **Definition 3.4.** The *degree* of a function f is the maximal cardinality of a face s such that $\tilde{f}(s) \neq 0$ in the unique decomposition given by Corollary 3.3.

Thus a function has degree d if its decomposition only involves faces whose dimension is less than d. The following lemma shows that the functions y_s, for all $(d-1)$-dimensional faces s, span the space of all functions of degree at most d.

▶ **Lemma 3.5.** *If* X *is a proper* k*-dimensional complex then the space of functions on* $X(k)$ *of degree at most* $d+1$ *has the functions* $\{y_s : s \in X(d)\}$ *as a basis.*

Proof. The space of functions on $X(k)$ of degree at most $d+1$ is spanned, by definition, by the functions y_t for $t \in X(-1) \cup X(0) \cup \cdots \cup X(d)$. This space has dimension $\sum_{i=-1}^{d} \dim H^i$. Since X is proper, $\dim H^i = \dim C^i - \dim C^{i-1}$ for $i > 0$, and so $\sum_{i=1}^{d} \dim H^i = \dim C^d = |X(d)|$.

Given the above in order to complete the proof it suffices to show that every y_t, $t \in X(i), i \leq d$, can be written as a linear combination of y_s for $s \in X(d)$. This will show that $\{y_s : s \in X(d)\}$ spans the space of functions of degree at most $d+1$. Since this set contains $|X(d)|$ functions, it forms a basis.

Recall that $y_t(r) = 1_{r \supset t}$, where $r \in X(k)$. If r contains t then it contains exactly $\binom{k+1-|t|}{d+1-|t|}$ many d-faces containing r, and so

$$y_t = \frac{1}{\binom{k+1-|t|}{d+1-|t|}} \sum_{\substack{s \supset t \\ s \in X(d)}} y_s.$$

This completes the proof. ◀

We call f_i the "level i" part of f, and denote the weight of f above level i by

$$wt_{>i}(f) := \sum_{j>i} \|f_j\|_2^2.$$

4 Orthogonality of decomposition

When X is the complete k-dimensional complex, the decomposition in Theorem 3.2 is orthogonal. For a general complex, this no longer need be the case. However, under certain conditions, the decomposition is *almost* orthogonal, as we show in this subsection, proving the following result:

▶ **Theorem 4.1.** *Let* $\vec{r}, \vec{\delta}$ *be vectors such that pointwise* $\vec{r} > \vec{0}$ *and* $\vec{\delta} < \vec{1}$*.*

Let X *be a proper* k*-dimensional complex, and define*

$$\gamma := \max_{0 \leq i \leq k-1} \|D_{i+1} U_i - (1-\delta_i) U_{i-1} D_i - r_i\|.$$

For every function f *on* C^ℓ *for* $\ell \leq k$*, the decomposition* $f = f_{-1} + \cdots + f_\ell$ *of Theorem 3.2 satisfies the following properties:*

- *For* $i \neq j$, $|\langle f_i, f_j\rangle| = O(\gamma)\|f_i\|\|f_j\|$.
- $\|f\|^2 = (1 \pm O(\gamma))(\|f_{-1}\|^2 + \cdots + \|f_\ell\|^2)$, *and for all* i, $\|f\|^2 = (1 \pm O(\gamma))(\|f_{\leq i}\|^2 + \|f_{>i}\|^2)$.
- *If* $\ell < k$ *then* $\langle DUf, f\rangle = (1 \pm O(\gamma)) \sum_{i=-1}^{\ell} \lambda_i \|f_i\|^2$, *where* $\vec{\lambda}$ *depends only on* $\ell, \vec{r}, \vec{\delta}$.

In other words, the decomposition of Theorem 3.2 is almost orthogonal, and its parts are almost eigenfunctions of the Laplacian operator DU. The (unnormalized) Laplacian operator is used in classical Boolean function analysis to define both the total influence $\mathrm{Inf}[f] = \langle DUf, f\rangle$ and the noise operator (in the semigroup formulation, $T_t = e^{-tDU}$).

As we show in Section 4.4 and Section 5, for both the complete complex and high-dimensional expanders we can take $r_i = \delta_i = \frac{1}{i+2}$, and given $\ell < k$, we get $\lambda_i = 1 - \frac{i+1}{\ell+2}$.

Since the first two properties for arbitrary ℓ follow from the case $\ell = k$ by truncating the complex, we concentrate below on proving this special case.

4.1 Sequentially differential posets

Let us first discuss *sequentially differential posets* [30, 31], using the example of the *unnormalized complete complex* whose top-level faces consists of all subsets of $[n]$ of size $k+1$, weighted according to the *counting measure*. The top-level faces of this complex form the "slice" $\binom{[n]}{k+1}$ (as it is known by computer scientists) of the *Johnson scheme* $J(n, k+1)$ (as it is known by coding theorists). The Up operator $\tilde{U}_i \colon C^i \to C^{i+1}$ is given by

$$\tilde{U}_i g(s) := \sum_{x \in s} g(s \setminus \{x\}),$$

and the Down operator $\tilde{D}_{i+1} \colon C^{i+1} \to C^i$ is given by

$$\tilde{D}_{i+1} f(t) := \sum_{y \notin t} f(t \cup \{y\}).$$

A simple calculation shows that

$$\tilde{D}_{i+1}\tilde{U}_i g(t) = \sum_{y \notin t} \sum_{x \in t \cup \{y\}} g(t \cup \{y\} \setminus \{x\}) = (n-i-1)g(t) + \sum_{r \colon |r \setminus t| = |t \setminus r| = 1} g(r),$$
$$\tilde{U}_{i-1}\tilde{D}_i g(t) = \sum_{x \in t} \sum_{y \notin t \setminus \{x\}} g(t \setminus \{x\} \cup \{y\}) = (i+1)g(t) + \sum_{r \colon |r \setminus t| = |t \setminus r| = 1} g(r).$$

Comparing the two expressions, we see that

$$\tilde{D}_{i+1}\tilde{U}_i - \tilde{U}_{i-1}\tilde{D}_i = r_i I, \text{ where } r_i = n - 2(i+1),$$

where I is the identity operator. Posets satisfying this property, for an arbitrary vector $\vec{r} = r_0, \ldots, r_{k-1}$, are known as *sequentially differential posets*. An important example is the *Grassmann lattice* of all subspaces of a finite-dimensional vector space over a finite field.

In the setup considered in this paper, the top-level faces are weighted by a distribution rather than an arbitrary measure. We therefore consider the *(normalized) complete complex*, in which the top-level faces are weighted by the uniform distribution. It is easy to check that all distributions Π_i are uniform, and therefore

$$U_i g(s) = \frac{1}{i+2} \sum_{x \in s} g(s \setminus \{x\}),$$
$$D_{i+1} f(t) = \frac{1}{n-i-1} \sum_{y \notin t} f(t \cup \{y\}).$$

As before, we can calculate

$$\begin{aligned}D_{i+1}U_i g(t) &= \frac{1}{(n-i-1)(i+2)} \sum_{y \notin t} \sum_{x \in t \cup \{y\}} g(t \cup \{y\} \setminus \{x\}) \\ &= \frac{1}{i+2} g(t) + \frac{1}{(n-i-1)(i+2)} \sum_{r \colon |r \setminus t| = |t \setminus r| = 1} g(r), \\ U_{i-1}D_i g(t) &= \frac{1}{(i+1)(n-i)} \sum_{x \in t} \sum_{y \notin t \setminus \{x\}} g(t \setminus \{x\} \cup \{y\}) \\ &= \frac{1}{n-i} g(t) + \frac{1}{(i+1)(n-i)} \sum_{r \colon |r \setminus t| = |t \setminus r| = 1} g(r).\end{aligned}$$

Comparing the two expressions, we see that

$$D_{i+1}U_i - (1-\delta_i)U_{i-1}D_i = r_i I, \text{ where } r_i = \delta_i = \frac{1}{i+2} - \frac{i+1}{i+2} \cdot \frac{1}{n-i-1}. \tag{1}$$

Since U_{i-1} and D_i are adjoint, $U_{i-1}D_i$ is positive semidefinite. When $2(i+1) < n$, the constant r_i is positive, and so $D_{i+1}U_i$ is positive definite, which in particular implies that U_i has trivial kernel. This shows that the complete complex is proper when $k < n/2$.

4.2 Almost sequentially differential posets

High-dimensional expanders do not satisfy an identity of the form (1). However, they satisfy an *approximate* version for of this identity, as we show in Section 5. The classical theory of sequentially differential posets shows that the decomposition of Theorem 3.2 is orthogonal. We will now show that an approximate version of (1) suffices for approximate orthogonality.

Given a positive vector $\vec{r}$, a vector $\vec{\delta}$ with coordinates less than 1, and a parameter γ, let us say that a k-dimensional complex is $(\vec{r}, \vec{\delta}, \gamma)$-*almost sequentially differential*, or $(\vec{r}, \vec{\delta}, \gamma)$-ASD for short, if for $0 \le i \le k-1$,

$$\|D_{i+1}U_i - (1-\delta_i)U_{i-1}D_i - r_i\| \le \gamma,$$

where the norm is the spectral norm.

We start by showing that in such a complex, any two distinct parts in the decomposition of Theorem 3.2 are approximately orthogonal, in some sense.

▶ **Lemma 4.2.** *Suppose that X is a proper k-dimensional complex which is $(\vec{r}, \vec{\delta}, \gamma)$-ASD, and let $f \in C^k$ have the decomposition $f = f_{-1} + \cdots + f_k$ for $f_i = U^{k-i}h_i$, as in Theorem 3.2. For $i \neq j$,*

$$\langle f_i, f_j \rangle = O(\gamma)\|h_i\|\|h_j\|.$$

where the hidden constant depends only on $k, \vec{r}, \vec{\delta}$ but not on $n = |X(0)|$.

Proof. Recall that $h_i \in H^i = \ker D_i$. Given this, it is easy to see that f_k is orthogonal to f_{k-1}, indeed $\langle f_k, f_{k-1} \rangle = \langle h_k, Uh_{k-1} \rangle = \langle Dh_k, h_{k-1} \rangle = 0$ because $Dh_k = 0$. To warm up let us first prove the claim for $f_{k-1} = Uh_{k-1}$ and $f_{k-2} = U^2 h_{k-2}$. In this case $\langle f_{k-1}, f_{k-2} \rangle = \langle Uh_{k-1}, U^2 h_{k-2} \rangle = \langle D^2 U h_{k-1}, h_{k-2} \rangle$. If we could replace $D^2 U h_{k-1}$ by $(1-\delta_{k-1})DUDh_{k-1} + r_{k-1}Dh_{k-1} = 0$ as in a sequentially differential poset, we would be done. However, this is not necessarily true. We instead use the property of being $(\vec{r}, \vec{\delta}, \gamma)$-ASD and replace $DDUh_{k-1}$ with $(1-\delta_{k-1})DUDh_{k-1} + r_{k-1}Dh_{k-1} = 0$, incurring an error of $O(\gamma)$ and completing the argument in this case.

Now move to general i, j and suppose without loss of generality that $j < i$, and so $k-j > k-i$. We have $\langle f_i, f_j\rangle = \langle D^{k-j}U^{k-i}h_i, h_j\rangle$. Let us denote $E_i = D_{i+1}U_i - U_{i-1}D_i - r_i$. We expand $D^{k-j}U^{k-i}$ into a sum of terms, using the following algorithm. We start with the sum containing one term, $D^{k-j}U^{k-i}$. At each step, we pick a term not containing E and not of the form U^aD^b, and isolate one of the occurrences of DU, say the term is $c\alpha DU\beta$ (here c is a real number and α, β are products of operators). We replace this term with the terms $c(1-\delta_d)\alpha UD\beta$, $cr_d\alpha\beta$ (for the appropriate d), and $c\alpha E\beta$ (this corresponds to the identity $D_{i+1}U_i = (1-\delta_i)U_{i-1}D_i + r_i + E_i$). This process clearly terminates eventually, with an expression that depends on $k, \vec{r}, \vec{\delta}$ but not on n.

Since $k-j > k-i$, all terms either contain E or end with D. The terms ending with D vanish since $h_i \in H^i$. All other terms are of the form $c\langle \alpha E\beta h_i, h_j\rangle = c\langle E\beta h_i, \alpha^* h_j\rangle$, where β, α^* are products of Ds and Us. Since D and U are contractions, we can estimate

$$|\langle E\beta h_i, \alpha^* h_j\rangle| \le \|E\beta h_i\|\|\alpha^* h_j\| \le \gamma\|h_i\|\|h_j\|.$$

The lemma immediately follows. ◀

The preceding lemma gives an error estimate in terms of the norms $\|h_i\|$. The following lemma will enable us to express the error in terms of the norms $\|f_i\|$.

▶ **Lemma 4.3.** *Suppose that X is a proper k-dimensional complex which is $(\vec{r}, \vec{\delta}, \gamma)$-ASD, and let $f \in C^k$ have the decomposition $f = f_{-1} + \cdots + f_k$ for $f_i = U^{k-i}h_i$, as in Theorem 3.2. For every i there exists a constant ρ_i, depending only on $\vec{r}, \vec{\delta}$, such that*

$$\|f_i\| = (1 \pm O(\gamma))\rho_i\|h_i\|,$$

where the hidden constant depends only on $k, \vec{r}, \vec{\delta}$.

Proof. The argument is very similar to that of Lemma 4.2. This time we are computing $\|f_i\|^2 = \langle D^{k-i}U^{k-i}h_i, h_i\rangle$. Executing the same algorithm as before, we will be left with many terms of the form $\langle h_i, h_i\rangle$, with various coefficients depending only on $\vec{r}, \vec{\delta}$. The end result will be that $\|f_i\|^2 = \rho_i^2\|h_i\|^2 \pm O(\gamma)\|h_i\|^2$ for some ρ_i (note that the coefficient is positive since $\vec{r}$ is positive and all entries of $\vec{\delta}$ are less than 1). The lemma follows. ◀

Combining both lemmata, we obtain the following corollary.

▶ **Corollary 4.4.** *Suppose that X is a proper k-dimensional complex which is $(\vec{r}, \vec{\delta}, \gamma)$-ASD, and let $f \in C^k$ have the decomposition $f = f_{-1} + \cdots + f_k$, as in Theorem 3.2. If γ is small enough (as a function of $k, \vec{r}, \vec{\delta}$) then for $i \neq j$,*

$$\langle f_i, f_j\rangle = O(\gamma)\|f_i\|\|f_j\|,$$

where the hidden constant depends only on $k, \vec{r}, \vec{\delta}$.

As a consequence, we obtain an *approximate L2 mass formula*:

▶ **Corollary 4.5.** *Under the conditions of Corollary 4.4, for every $i \le j$ we have*

$$\|f_i + \cdots + f_j\|^2 = (1 \pm O(\gamma))(\|f_i\|^2 + \cdots + \|f_j\|^2),$$

where the hidden constant depends only on $k, \vec{r}, \vec{\delta}$.

In particular,

$$\|f\|^2 = (1 \pm O(\gamma))(wt_{\le i}(f) + wt_{>i}(f)) = (1 \pm O(\gamma))(\|f_{\le i}\|^2 + \|f_{>i}\|^2).$$

Proof. Expanding $\|f_i + \cdots + f_j\|^2$, we obtain

$$\|f_i + \cdots + f_j\|^2 - \|f_i\|^2 - \cdots - \|f_j\|^2 = 2 \sum_{i \le a < b \le j} \langle f_a, f_b \rangle = O(\gamma) \sum_{i \le a < b \le j} \|f_a\|\|f_b\| \le$$
$$O(\gamma)\left(\|f_i\| + \cdots + \|f_j\|\right)^2 \le O(\gamma)(\|f_i\|^2 + \cdots + \|f_j\|^2),$$

swallowing a factor of k in the last inequality. ◀

4.3 Laplacian

The (upper) Laplacian is the operator DU, used to define a random walk on a specific level $\ell < k$ of the complex (the lower Laplacian UD defines another random walk). In the complete complex, indeed in arbitrary complexes satisfying (1), the functions in the decomposition $f = f_{-1} + \cdots + f_\ell$ are eigenfunctions of the Laplacian. The same holds approximately for almost sequentially differential posets, using arguments very similar to the foregoing.

An analog of Lemma 4.2 shows that for $i \neq j$,

$$\langle Uf_i, Uf_j \rangle = O(\gamma)\|h_i\|\|h_j\|,$$

which as in Corollary 4.4 shows that

$$\langle Uf_i, Uf_j \rangle = O(\gamma)\|f_i\|\|f_j\|.$$

The argument of Corollary 4.5 shows that

$$\langle DUf, f \rangle = \|Uf\|^2 = (1 \pm O(\gamma))(\|Uf_{-1}\|^2 + \cdots + \|Uf_\ell\|^2).$$

Conversely, an analog of Lemma 4.3 shows that there are positive constants λ_i, depending only on $\vec{r}, \vec{\delta}$, such that for each i,

$$\|Uf_i\|^2 = (1 \pm O(\gamma))\lambda_i \rho_i^2 \|h_i\|^2 = (1 \pm O(\gamma))\lambda_i \|f_i\|^2,$$

using Lemma 4.3. Putting everything together, we get the following result:

▶ **Lemma 4.6.** *Suppose that X is a proper k-dimensional complex which is $(\vec{r}, \vec{\delta}, \gamma)$-ASD, let $\ell < k$, and let $f \in C^\ell$ have the decomposition $f = f_{-1} + \cdots + f_\ell$, as in Theorem 3.2. For every i there exists a constant λ_i, depending only on $\vec{r}, \vec{\delta}$, such that*

$$\langle DUf, f \rangle = (1 \pm O(\gamma)) \sum_{i=-1}^{\ell} \lambda_i \|f_i\|^2,$$

where the hidden constant depends only on $k, \vec{r}, \vec{\delta}$.

This result is analogous to [21, Theorem 6.2], in which a similar decomposition is obtained. However, whereas our decomposition is to functions $f_{-1}, \dots, f_\ell$ in C^ℓ, the decomposition of [21] is analogous to our functions $h_{-1}, \dots, h_\ell$, which live in different spaces.

4.4 Eigenvalues of the Laplacian

Section 4.1 shows that the complete d-dimensional complex is $(\vec{r}, \vec{\delta}, \gamma)$-ASD, where $r_i = \delta_i = \frac{1}{i+2}$ and $\gamma = O(\frac{1}{n-d-1})$, the same parameters as high-dimensional expanders, as we show in Section 5. This allows us to compute the eigenvalues $\vec{\lambda}$ of the Laplacian in Lemma 4.6 for both the complete complex and high-dimensional expanders.

The classical theory of the "slice" (see for example [14, 16]) shows that V^i is spanned by functions of the form $\prod_{j=1}^{i+1}(y_{a_j} - y_{b_j})$, where the $2i$ indices a_j, b_j are distinct. (Recall that in the decomposition of f, the component f_i belongs to V^i.)

▶ **Lemma 4.7.** *Let $\varphi_i \in C^\ell$ be the function*

$$\varphi_i = (y_1 - y_2)(y_3 - y_4) \cdots (y_{2i+1} - y_{2i+2}).$$

On the complete complex,

$$DU\varphi_i = \left(1 - \frac{i+1}{n-\ell-1}\right)\left(1 - \frac{i+1}{\ell+2}\right)\varphi_i.$$

Proof. Let $s \in X(\ell)$, and recall that

$$DU\varphi_i(s) = \mathop{\mathbb{E}}_{a \notin s} \mathop{\mathbb{E}}_{b \in s \cup \{a\}} \varphi_i(s \cup \{a\} \setminus \{b\}).$$

For brevity, we use $r_{a,b} := s \cup \{a\} \setminus \{b\}$.

We consider several cases. Suppose first that $2j-1, 2j \in s$ for some j, so that $\varphi_i(s) = 0$. If $b \neq 2j-1, 2j$ then $\varphi_i(r_{a,b}) = 0$. Conversely, $\varphi_i(r_{a,2j-1}) = -\varphi_i(r_{a,2j})$ for any choice of a. Since $\Pr[b = 2j-1] = \Pr[b = j]$, we see that $DU\varphi_i(s) = 0$.

Suppose next that $2j-1, 2j \notin s$ for some j, so that again $\varphi_i(s) = 0$. If $a \notin 2j-1, 2j$ then $\varphi_i(r_{a,b}) = 0$. Conversely, $\varphi_i(r_{2j-1,b}) = -\varphi_i(r_{2j,b})$ for any $b \in s$, and $\varphi_i(r_{2j-1,2j-1}) = \varphi_i(r_{2j,2j}) = 0$. Once again, this shows that $DU\varphi_i(s) = 0$.

Finally, suppose that $1, 3, \dots, 2i+1 \in s$ and $2, 4, \dots, 2i+2 \notin s$, so that $\varphi_i(s) = 1$. If $a = 2j$ for some j then $\varphi_i(r_{a,2j-1}) = -1$, $\varphi_i(r_{a,2j}) = 1$, and $\varphi_i(r_{a,b}) = 0$ for $b \neq 2j-1, 2j$. Thus $\varphi_i(r_{2j,b})$ vanishes in expectation. When $a \neq 2, 4, \dots, 2i+2$, we have $\varphi_i(r_{a,b}) = 1$ if $b \neq 1, 3, \dots, 2i+1$, and $\varphi_i(r_{a,b}) = 0$ otherwise. Thus $DU\varphi_i(s)$ is the probability that $a \neq 2, 4, \dots, 2i+2$ and that $b \neq 1, 3, \dots, 2i+1$, which is $(1 - \frac{i+1}{n-\ell-1})(1 - \frac{i+1}{\ell+2})$. ◀

When $n \to \infty$, both γ and $\frac{i+1}{n-\ell-1}$ tend to zero. Comparing Lemma 4.6 and Lemma 4.7, we see that the value of λ_i in Lemma 4.6, which doesn't depend on n, is

$$\lambda_i = 1 - \frac{i+1}{\ell+2}.$$

5 High-dimensional expanders

Let X be a d-dimensional complex with an associated probability distribution Π_d on $X(d)$, which induces probability distributions on $X(-1), \dots, X(d-1)$ as we have described above. For every i-dimensional face $s \in X(i)$ for $i < d-1$, the *link* of s is the weighted graph X_s defined as follows:

- The vertices are points $x \notin s$ such that $s \cup \{x\} \in X(i+1)$ is a face.
- The edges are pairs of points $\{x, y\}$ such that $s \cup \{x, y\} \in X(i+1)$ is a face. (Since the complex is pure, x and y are vertices.)
- The weight of the *directed* edge (x, y) is

$$w_s(x, y) := \frac{1}{2} \Pr_{t \sim \Pi_{i+1}} [t = s \cup \{x, y\} \mid t \supset s].$$

Note that the weights define a probability distribution w_s on the (directed) edges. We denote the marginal of w_s on its first coordinate by

$$w_s(x) := \sum_{y \neq x} w_s(x, y) = \Pr_{t \sim \Pi_{i+1}} [t \supset s \cup \{x\} \mid t \supset s] = \Pr_{(u,v) \sim (\Pi_{i+1}, \Pi_i)} [u = s \cup \{x\} \mid v = s].$$

This is also the marginal of w_s on its second coordinate. We turn the space of function on vertices into an inner product space by defining

$$\langle f,g\rangle := \mathop{\mathbb{E}}_{x\sim w_s}[f(x)g(x)].$$

We define an operator A_s on functions on vertices by the matrix $A_s(x,y) := w_s(x,y)/w_s(x)$, which corresponds to the quadratic form

$$\langle f, A_s g\rangle = \sum_{x,y} w_s(x,y)f(x)g(y).$$

By definition, A_s fixes constant functions, and so it is a Markov operator. Since $w_s(x,y) = w_s(y,x)$, it is also self-adjoint with respect to the inner product above. Thus A_s has eigenvalues $\lambda_1 = 1, \lambda_2, \ldots, \lambda_m$, where m is the number of vertices. We define $\lambda(A_s) = \max(|\lambda_2|, |\lambda_m|)$. Orthogonality of eigenspaces guarantees that

$$|\langle f, A_s g\rangle - \mathbb{E}[f]\,\mathbb{E}[g]| \le \lambda(A_s)\|f\|\|g\|.$$

We say that X is a *γ-two-sided high-dimensional expander* (called γ-HD expander in [5]) if *every* link X_s of X satisfies $\lambda(A_S) \le \gamma$.

Let f be a function on $X(i)$, where $i < d-1$. We have

$$\langle DUf, f\rangle = \langle Uf, Uf\rangle = \mathop{\mathbb{E}}_{t\sim\Pi_{i+1}} \mathop{\mathbb{E}}_{x,y\in t}[f(t\setminus\{x\})\cdot f(t\setminus\{y\}].$$

The probability that x is equal to y is exactly $1/(i+2)$, and so

$$\langle DUf, f\rangle = \frac{1}{i+2}\mathop{\mathbb{E}}_{t\sim\Pi_{i+1}} \mathop{\mathbb{E}}_{x\in t}[f(t\setminus\{x\})^2] + \frac{i+1}{i+2}\mathop{\mathbb{E}}_{t\sim\Pi_{i+1}} \mathop{\mathbb{E}}_{x\neq y\in t}[f(t\setminus\{x\})\cdot f(t\setminus\{y\}].$$

If $t\sim\Pi_{i+1}$ and $x\in t$ is chosen at random, then $t\setminus\{x\}\sim\Pi_i$. Therefore the first term is equal to $\frac{1}{i+2}\|f\|^2$. To compute the second term, let $s = t\setminus\{x,y\}$. Since $t\sim\Pi_{i+1}$ and $x\neq y\in t$ are chosen at random, we have $s\sim\Pi_{i-1}$. Given such an s, the probability to get specific (t,x,y) is exactly $w_s(x,y)$ (the factor $1/2$ accounts for the relative order of x,y), and so

$$\langle DUf, f\rangle = \frac{1}{i+2}\|f\|^2 + \frac{i+1}{i+2}\mathop{\mathbb{E}}_{s\sim\Pi_{i-1}} \mathop{\mathbb{E}}_{(x,y)\sim w_s}[f(s\cup\{x\})f(s\cup\{y\})].$$

We now note that

$$\mathop{\mathbb{E}}_{x\sim w_s}[f(s\cup\{x\})] = (Df)(s).$$

Therefore we have

$$|\mathop{\mathbb{E}}_{(x,y)\sim w_s}[f(s\cup\{x\})f(s\cup\{y\})] - (Df)(s)^2| \le \lambda(A_s)\mathop{\mathbb{E}}_{x\sim w_s}[f(s\cup\{x\})^2].$$

If X is a γ-two-sided high-dimensional expander then $\lambda(A_s)\le\gamma$ for all s, and so (using I for the identity operator)

$$|\langle (DU - \tfrac{1}{i+2}I - \tfrac{i+1}{i+2}UD)f, f\rangle| \le \gamma \mathop{\mathbb{E}}_{s\sim\Pi_{i-1}} \mathop{\mathbb{E}}_{x\sim w_s}[f(s\cup\{x\})^2] = \gamma\|f\|^2.$$

We have proved the following result.

▶ **Theorem 5.1.** *Suppose that X is a d-dimensional γ-two-sided high-dimensional expander. Then its $(d-1)$-skeleton Y (consisting of all faces of X of dimension at most $d-1$) is $(\vec{r},\vec{\delta},\gamma)$-ASD, where $r_i = \delta_i = \frac{1}{i+2}$. Moreover, if $\gamma < \frac{1}{d}$ then Y is proper.*

Proof. The first part follows from the foregoing. For the second part, note that $U_{i-1}D_i$ is positive semidefinite, and so all eigenvalues of $\frac{i+1}{i+2}U_{i-1}D_i + \frac{1}{i+2}$ are at least $\frac{1}{i+2}$. This implies that all eigenvalues $D_{i+1}U_i$ are at least $\frac{1}{i+2} - \gamma > 0$, and in particular U_i has trivial kernel. ◀

Dinur and Kaufman [5] proved that such expanders do exist.

▶ **Theorem 5.2** ([5, Lemma 1.5]). *For every $\lambda > 0$ and every $d \in \mathbb{N}$ there exists an explicit infinite family of bounded degree d-dimensional complexes which are λ-two-sided high-dimensional expanders.*

Theorem 5.1 shows that high-dimensional expanders are almost sequentially differential. In the full version of the paper [3], we show that the converse also holds, given the additional assumption that all links are connected. Our proof uses the local Garland method [18], as substantiated by Oppenheim [28].

6 Boolean degree 1 functions

In this section we characterize all Boolean degree 1 functions in nice complexes.

▶ **Theorem 6.1.** *Suppose that X is a proper k-dimensional complex, where $k \geq 2$. A function $f \in C^k$ is a Boolean degree 1 function if and only if there exists an independent set I such that f is the indicator of intersecting I or of not intersecting I.*

Proof. If f is the indicator of intersecting an independent set I then $f = \sum_{v \in I} y_v$, and so $\deg f \leq 1$. If f is the indicator of not intersecting an independent set I then $f = \sum_{v \in X(0)} y_v/(k+1) - \sum_{v \in I} y_v$, and so again $\deg f \leq 1$.

Suppose now that f is a Boolean degree 1 function. If $|X(0)| \leq 2$ then the theorem clearly holds, so assume that $|X(0)| > 2$. Lemma 3.5 shows that f has a unique representation of the form

$$f = \sum_{v \in X(0)} c_v y_v.$$

Since f is Boolean, it satisfies $f^2 = f$. Note that

$$f^2 = \sum_{\{u,v\} \in X(1)} 2c_u c_v y_{\{u,v\}} + \sum_{v \in X(0)} c_v^2 y_v.$$

Moreover, since every input x to f which contains v contains exactly k other members of $X(0)$, and since $X(1)$ contains all pairs of points from x, we have

$$y_v = \sum_{u \in X(0) \setminus \{v\}} \frac{y_{\{u,v\}}}{k}.$$

This shows that

$$0 = f^2 - f = \sum_{\{u,v\} \in X(1)} 2c_u c_v y_{\{u,v\}} + \frac{1}{k} \sum_{v \in X(0)} (c_v^2 - c_v) \sum_{u \in X(0) \setminus \{v\}} y_{\{u,v\}} =$$
$$\frac{1}{k} \sum_{\{u,v\} \in X(1)} (2k c_u c_v + c_u^2 - c_u + c_v^2 - c_v) y_{\{u,v\}}.$$

Lemma 3.5 shows that the coefficients of all $y_{\{u,v\}}$ must vanish, that is, for all $u \neq v$ we have

$$2kc_u c_v = c_u(1 - c_u) + c_v(1 - c_v).$$

Consider now a triple of points u, v, w such that $\{u, v, w\} \in X(2)$, and the corresponding system of equations:

$$\begin{aligned}
2kc_u c_v &= c_u(1 - c_u) + c_v(1 - c_v),\\
2kc_u c_w &= c_u(1 - c_u) + c_w(1 - c_w),\\
2kc_v c_w &= c_v(1 - c_v) + c_w(1 - c_w).
\end{aligned}$$

Subtracting the second equation from the first, we obtain

$$2kc_u(c_v - c_w) = c_v(1 - c_v) - c_w(1 - c_w) = (c_v - c_w) - (c_v^2 - c_w^2) = (c_v - c_w)(1 - c_v - c_w).$$

This shows that either $c_v = c_w$ or $2kc_u = 1 - c_v - c_w$.

If $c_u \neq c_v, c_w$ then $2kc_w + c_u + c_v = 2kc_v + c_u + c_w = 1$, which implies that $c_v = c_w$. Thus c_u, c_v, c_w can consist of at most two values. If $c := c_u = c_v = c_w$ then $2kc^2 = 2c(1 - c)$, and so $c \in \{0, 1/(k + 1)\}$. If $c := c_v = c_w \neq c_u$ then $2c^2 = 2c(1 - c)$, and so $c \in \{0, 1/(k + 1)\}$ as before. We also have $2kc_u c = c_u(1-c_u)+c(1-c)$. If $c = 0$ then this shows that $c_u(1-c_u) = 0$, and so $c_u = 1$. If $c = 1/(k + 1)$ then one can similarly check that $c_u = 1/(k + 1) - 1$.

Summarizing, one of the following two cases must happen:

1. Two of c_u, c_v, c_w are equal to 0, and the remaining one is either 0 or 1.
2. Two of c_u, c_v, c_w are equal to $1/(k + 1)$, and the remaining one is either $1/(k + 1)$ or $1/(k + 1) - 1$.

Let us say that a vertex $v \in X(0)$ is of type A if $c_v \in \{0, 1\}$, and of type B if $c_v \in \{1/(k + 1), 1/(k + 1) - 1\}$. Since the complex is pure and at least two-dimensional, every vertex must participate in a triangle (two-dimensional face), and so every vertex is of one of the types. In fact, all vertices must be of the *same* type. Otherwise, there would be a vertex v of type A incident to a vertex w of type B. However, since the complex is pure, $\{v, w\}$ must participate in a triangle, contradicting the classification above.

Suppose first that all vertices are type A, and let $I = \{v : c_v = 1\}$. Note that f indicates that the input face intersects I. Clearly I must be an independent set, since otherwise f would not be Boolean. When all vertices are type B, the function $1-f = \sum_{v \in X(0)}(1/(k+1)-c_v)y_v$ is of type A, and so f must indicate not intersecting an independent set. ◀

7 FKN-theorem on high dimensional expanders

In this section, we prove an analog of the classical result of Friedgut, Kalai and Naor [17] to high dimensional expanders. The FKN theorem that states that any Boolean function F on the hypercube that is close to a degree-1 function f (not necessarily Boolean) in the L_2^2-sense must agree with some Boolean degree-1 function (which must be a dictator) at most points. This result for the Boolean hypercube can be easily extended to functions on k-slices of the hypercube provided $k = \Theta(n)$.

▶ **Theorem 7.1** (FKN theorem on the slice [13]). *Let $n, k \in \mathbb{Z}_{\geq 0}$ and $\varepsilon \in (0, 1)$ such that $n/4 \leq k \leq n/2$. Let $F\colon \binom{[n]}{k} \to \{0, 1\}$ be a Boolean function such that $\mathbb{E}[(F - f)^2] < \varepsilon$ for some degree-1 function $f\colon \binom{[n]}{k} \to \{0, 1\}$. Then there exists a degree-1 function $g\colon \binom{[n]}{k} \to \mathbb{R}$ such that*

$$\Pr[F \neq g] = O(\varepsilon).$$

Furthermore, $g \in \{0, 1, y_i, 1 - y_i\}$, that is, g is a Boolean dictator (1-junta).

▶ Remark.

1. The function g promised by the theorem satisfies $\mathbb{E}[(g-F)^2] = Pr[g \neq F] = O(\varepsilon)$ and hence, by the L_2^2-triangle inequality we have $\mathbb{E}[(f-g)^2] \le 2\,\mathbb{E}[(f-F)^2] + 2\,\mathbb{E}[(g-F)^2] = O(\varepsilon)$. This is the way that the FKN theorem is traditionally stated, but we prefer the above formulation as this is the one we are able to generalize to the high-dimensional expander setting.
2. The function 1 can also be written as $\sum_j (1/k) y_j$. The function $1 - y_i$ can also be written as $\sum_{j \neq i} (1/k) y_j + (1/k - 1) y_i$.
3. The result of [13] is quite a bit stronger: for every $k \le n/2$, it promises the existence of a function $g \colon \binom{[n]}{k} \to \mathbb{R}$, not necessarilly Boolean, such that $\mathbb{E}[(f-g)^2] = O(\varepsilon)$. Moreover, either g or $1 - g$ is of the form $\sum_{i \in S} y_i$ for $|S| \le \max(1, \sqrt{\varepsilon} \cdot n/k)$. The bound on the size of S ensures that $\Pr[g \in \{0,1\}] = 1 - O(\varepsilon)$.

Our main theorem is an extension of the above theorem to k-faces of a high-dimensional expander.

▶ **Theorem 7.2** (FKN theorem for high dimensional expanders). *Let X be a d-dimensional λ-two-sided high-dimensional expander, and let $4k^2 < d$ and $\lambda < 1/d$. Let $F \colon X(k) \to \{0,1\}$ be a function such that $\mathbb{E}[(F-f)^2] < \varepsilon$ for some degree-1 function $f \colon X(k) \to \mathbb{R}$. Then there exists a degree-1 function $g \colon X(k) \to \mathbb{R}$ such that*

$$\Pr[F \neq g] = O_\lambda(\varepsilon).$$

Furthermore, the degree-1 function g can be written as $g(y) = \sum_i d_i y_i$, where $d_i \in \{0, 1, \frac{1}{k+1}, \frac{1}{k+1} - 1\}$.

The high-dimensional analog of the FKN theorem is obtained from the FKN theorem for the slice using the agreement theorem of Dinur and Kaufman [5].

7.1 Agreement theorem for high dimensional expanders

Dinur and Kaufman [5] prove an agreement theorem for high-dimensional expanders. The setup is as follows. For each k-face s we are given a local function $f_s \colon s \to \Sigma$ that assigns values from an alphabet Σ to each point in s. Two local functions $f_s, f_{s'}$ are said to agree if $f_s(v) = f_{s'}(v)$ for all $v \in s \cap s'$. Let $\mathcal{D}_{k,2k}$ be the distribution on pairs (s_1, s_2) obtained by choosing a random $t \sim \Pi_{2k}$ and then independently choosing two k-faces $s_1, s_2 \subset t$. The theorem says that if a random pair pair of faces $(s, s') \sim \mathcal{D}_{k,2k}$ satisfies with high probability that f_s agrees with $f_{s'}$ on their intersection, then there must be a global function $g \colon X(0) \to \Sigma$ such that almost always $g|_s \equiv f_s$. Formally:

▶ **Theorem 7.3** (Agreement theorem for high-dimensional expanders [5]). *Let X be a d-dimensional λ-two-sided high-dimensional expander, and let $k^2 < d$ and $\lambda < 1/d$ and Σ some fixed finite alphabet. Let $\{f_s \colon s \to \Sigma\}_{s \in X(k)}$ be an ensemble of local functions on $X(k)$, i.e. $f_s \in \Sigma^s$ for each $s \in X(k)$. If*

$$\Pr_{(s_1,s_2)\sim\mathcal{D}_{k,2k}}[f_{s_1}|_{s_1 \cap s_2} \equiv f_{s_2}|_{s_1 \cap s_2}] > 1 - \varepsilon$$

then there is a $g \colon X(0) \to \Sigma$ such that

$$\Pr_{s \sim \Pi_k}[f_s \equiv g|_s] \ge 1 - O_\lambda(\varepsilon).$$

While Dinur and Kaufman state the theorem for a binary alphabet, the general version follows in a black box fashion by applying the theorem for binary alphabets $\lceil \log_2 |\Sigma| \rceil$ many times.

7.2 Proof of Theorem 7.2

Let $f, F \in C^k$, where F is a Boolean function and f is a degree-1 function as in the hypothesis of Theorem 7.2. Let

$$\varepsilon_f := \mathop{\mathbb{E}}_{s}[\mathrm{dist}(f(s), \{0,1\})^2]. \tag{2}$$

We have $\varepsilon_f \leq \varepsilon$. Since f is a degree-1 function, Lemma 3.5 guarantees that there exist $a_i \in \mathbb{R}$ such that $f(y) = \sum_{i \in n} a_i y_i$. Note that here we view the inputs of f as n-bit strings with exactly $k+1$ ones, the rest being zero.

We begin by defining an ensemble of pairs of local functions $\{(f|_t, F|_t)\}_{t \in X(2k)}$, $\{(f|_u, F|_u)\}_{u \in X(4k)}$ which are the restrictions of (f, F) to the $2k$-face t and $4k$-face u. Formally, for any $t \in X(2k)$ and $u \in X(4k)$, consider the restriction of f to t and u defined as follows:

$$f|_t, F|_t \colon \binom{t}{k} \to \mathbb{R}, \qquad f|_t(y) = f(y) = \sum_{i \in t} a_i y_i, \qquad F|_t(y) = F(y),$$

$$f|_u, F|_u \colon \binom{u}{k} \to \mathbb{R}, \qquad f|_u(y) = f(y) = \sum_{i \in u} a_i y_i, \qquad F|_u(y) = F(y).$$

Observe that the $f|_t$'s are degree-1 functions while the $F|_t$'s are Boolean functions (similarly for $f|_u$'s and $F|_u$'s).

Now, define the following quantities

$$\varepsilon_t := \mathop{\mathbb{E}}_{s\colon s \subset t}[\mathrm{dist}(f|_t(s), F|_t(s))^2], \qquad \delta_u := \mathop{\mathbb{E}}_{s\colon s \subset u}[\mathrm{dist}(f|_u(s), F|_u(s))^2].$$

Clearly, $\mathbb{E}_t[\varepsilon_t] = \mathbb{E}_u[\delta_u] = \varepsilon_f \leq \varepsilon$, where ε_f is as in (2).

Let $\alpha_k = \frac{1}{k+1}$. Applying Theorem 7.1 (along with Remark 7) to the functions $(f|_t, F|_t)$ for each $t \in X(2k)$ we have the following claim:

▶ **Claim 7.4.** *For every $t \in X(2k)$, there exists a degree-1 dictator $g_t \colon \binom{t}{k} \to \{0,1\}$ such that*

$$\mathop{\mathbb{E}}_{s\colon s \subset t}[(f|_t - g_t)^2] = O(\varepsilon_t).$$

Furthermore, there exists a function $d_t \colon t \to \{0, 1, \alpha_k, \alpha_k - 1\}$ such that $g_t(y) = \sum_{i \in t} d_t(i) y_i$.

Similarly for each $u \in X(4k)$ we have:

▶ **Claim 7.5.** *For every $u \in X(4k)$, there exists a degree-1 dictator $h_u \colon \binom{u}{k} \to \{0,1\}$ such that*

$$\mathop{\mathbb{E}}_{s\colon s \subset u}[(f|_u - h_u)^2] = O(\delta_u).$$

Furthermore, there exists a function $e_u \colon u \to \{0, 1, \alpha_k, \alpha_k - 1\}$ such that $h_u(y) = \sum_{i \in u} e_u(i) y_i$.

We will now prove that the collection of local functions $\{d_t\}_t$ typically agree with each other. We will then be able to use the agreement theorem Theorem 7.3 to sew these different local functions together to yield a single function $d : X(0) \to \{0, 1, \alpha_k, \alpha_k - 1\}$. This d will determine a global degree-1 function g defined as follows: $g(y) = \sum_{i \in X(0)} d(i) y_i$.

▶ **Claim 7.6.** *There exists a function* $d\colon X(0) \to \{0, 1, \alpha_k, \alpha_k - 1\}$ *such that* $\Pr_t[d_t \equiv d|_t] = 1 - O_\lambda(\varepsilon)$.

Proof. To sew the various d_t together via the agreement theorem, we would like to first bound the probability

$$\Pr_{(t_1,t_2)\sim\mathcal{D}_{2k,4k}}[d_{t_1}|_{t_1\cap t_2} \not\equiv d_{t_2}|_{t_1\cap t_2}] .$$

Recall the definition of the distribution $\mathcal{D}_{2k,4k}$: we first pick a set $u \in X(4k)$ according to Π_{4k} and then two $2k$-faces t_1, t_2 of u uniformly and independently. Consider the three functions d_{t_1}, d_{t_2} and e_u. Clearly, if $d_{t_1}|_{t_1\cap t_2} \not\equiv d_{t_2}|_{t_1\cap t_2}$ then one of $e_u|_{t_1} \not\equiv d_{t_1}$ or $e_u|_{t_2} \not\equiv d_{t_2}$ must hold. Thus,

$$\Pr_{(t_1,t_2)\sim\mathcal{D}_{2k,4k}}[d_{t_1}|_{t_1\cap t_2} \not\equiv d_{t_2}|_{t_1\cap t_2}] \le 2 \cdot \Pr_{t,u}[e_u|_t \not\equiv d_t] . \tag{3}$$

Thus, it suffices to bound the probability $\Pr_{t,u}[e_u|_t \not\equiv d_t]$ where $u \sim \Pi_{4k}$ and t is a random $2k$-face of u.

For any fixed $t \subset u$, the L_2^2 triangle inequality shows that

$$\mathbb{E}[(h_u|_t - g_t)^2] \le 2\,\mathbb{E}[(h_u|_t - f|_t)^2] + 2\,\mathbb{E}[(f|_t - g_t)^2] = 2\,\mathbb{E}[(h_u|_t - f|_t)^2] + O(\varepsilon_t).$$

Taking expectation over $t \in X(2k)$ conditioned on $t \subset u$, we see that

$$\mathop{\mathbb{E}}_{t\subset u} \mathbb{E}[(h_u|_t - g_t)^2] \le 2\,\mathbb{E}[(h_u - f|_u)^2] + O\left(\mathop{\mathbb{E}}_{t\colon t\subset u} \varepsilon_t\right) = O(\delta_u) + O\left(\mathop{\mathbb{E}}_{t\colon t\subset u} \varepsilon_t\right).$$

Taking expectation over $u \sim \Pi_{4k}$, we now have

$$\mathop{\mathbb{E}}_{u}\mathop{\mathbb{E}}_{t\subset u} \mathbb{E}[(h_u|_t - g_t)^2] = O(\varepsilon).$$

For any fixed $t \subset u$, both $h_u|_t$ and g_t are Boolean degree-1 juntas. Hence either they agree, or $\mathbb{E}[(h_u|_t - g_t)^2] = \Omega(1)$. This shows that $h_u|_t$ disagrees with g_t with probability $O(\varepsilon)$, and so

$$\Pr_{t,u}[e_{u|_t} \not\equiv d_t] = O(\varepsilon).$$

We now return to (3), concluding that

$$\mathop{\mathbb{E}}_{(t_1,t_2)\sim\mathcal{D}_{2k,4k}}[d_{t_1}|_{t_1\cap t_2} \not\equiv d_{t_2}|_{t_1\cap t_2})] = O(\varepsilon).$$

We have thus satisfied the hypothesis of the agreement theorem (Theorem 7.3). Invoking the agreement theorem, we deduce that $\Pr_{t\sim\Pi_{2k}}[d_t \equiv d|_t] = 1 - O_\lambda(\varepsilon)$. ◀

The d's guaranteed by Claim 7.6 naturally correspond to a degree-1 function $g\colon X(k) \to \mathbb{R}$ as follows:

$$g(y) := \sum_{i\in X(0)} d(i) y_i.$$

We now show that this g is mostly Boolean.

▶ **Claim 7.7.** $\Pr_s[g(s) \in \{0, 1\}] = 1 - O_\lambda(\varepsilon)$.

Proof. Since g_t is Boolean valued,

$$\Pr_{s\sim\Pi_k}[g(s) \in \{0,1\}] \geq \Pr_t[g|_t = g_t] = \Pr_t[d|_t \equiv d_t] = 1 - O(\varepsilon). \blacktriangleleft$$

We now show that g in fact agrees pointwise with F most of the time.

▶ **Claim 7.8.** $\Pr_s[g \neq F] = O(\varepsilon)$.

Proof. Fix any $t \in X(2k)$. We compute $\Pr_{s\colon s\subset t}[F|_t \neq g_t]$ as follows

$$\begin{aligned}\Pr[F|_t \neq g_t] &= \|F|_t - g_t\|^2 && [\text{ Since } F|_t \text{ and } g_t \text{ are both Boolean }]\\ &\leq 2\cdot\|F|_t - f|_t\|^2 + 2\cdot\|f|_t - g_t\|^2\\ &= O(\varepsilon_t) + O(\varepsilon_t) = O(\varepsilon_t).\end{aligned}$$

We can now compute $\Pr_s[F \neq g]$ as follows:

$$\Pr[F \neq g] = \mathop{\mathbb{E}}_t \Pr[F|_t \neq g|_t] \leq \mathop{\mathbb{E}}_t \Pr[F|_t \neq g_t] + \Pr_t[g|_t \neq g_t] = O(\varepsilon) + \Pr_t[d|_t \not\equiv d_t] = O_\lambda(\varepsilon). \blacktriangleleft$$

This completes the proof of Theorem 7.2.

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