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Symmetric representations of elements of finite groups

Abeir Mikhail Kasouha

SYMMETRIC REPRESENTATIONS OF ELEMENTS OF FINITE GROUPS

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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ABSTRACT

The Lowest degree of a permutation representation of most of the sporadic group is either inconvenient, for example the smallest Janko group J_1 is a permutation group on 266 letters, or unmanageable, for example the Monster group is a permutation group on 10^{20} letters. We will demonstrate a method, concise and informative, to represent elements of finite groups which is particularly useful for the sporadic groups

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CHAPTER ONE

INTRODUCTION

Motivation

The smallest Janko group J_1 contains copies of the projective special linear group $PSL_2(11)$ to index 266, and Livingstone, (see Livingstone [2]), followed by Conway (see Conway [4]) constructed a graph, on 266 vertices, of which J_1 is the group of symmetries. In fact, the index, 266, of $PSL_2(11)$ in J_1 turns out to be the lowest index of any subgroup and so elements of J_1 are usually represented as permutations on 266 letters. MAGMA, (see Cannon[5]), and other group theoretic packages, handle permutations of this size with immense ease but recording and transmitting particular elements (other than electronically) is rather inconvenient. Conway pointed out that J_1 could be generated by 11 involutions which are permuted, by conjugation within J_1, by a subgroup isomorphic to $L_2(11)$, and that each element of J_1 can be written (not necessarily uniquely) as a permutation of $L_2(11)$ followed by a word of length less than or equal to four of these generators. We refer to this as a symmetric representation of an element of the group and to an element represented in this manner as symmetrically represented element. Curtis and Hasan (see Curtis and Hasan [6]) have written a computer program to manipulate elements of J_1 represented in this manner. In fact elements of any finite group that is generated by a conjugacy class of involutions, and in particular any finite non-abelian simple group, can be represented symmetrically.

In general, if we wish to multiply and invert elements in a straightforward manner we must represent them as either permutations or as matrices. The two operations are particularly easily performed on permutations. Moreover, the cycle shape of an element immediately yields its order, and often its conjugacy class. However, for the larger sporadic groups the lowest degree of a permutation representation is unmanageable (the Monster group is at best a permutation group on 10^{20} letters). Operations on matrices are much more difficult and time-consuming, and basic information about an element is not readily recovered from its matrix representation. Group elements can, of course, be expressed as words in any generating set, but even recognizing the identity element can be a formidable task. Again, given a short sequence of letters whose stabilizer is trivial, a permutation is uniquely defined by its image. This can be remarkably concise, but again does not readily admit the basic operations.

It is the main purpose of this thesis to demonstrate an alternative, concise but informative, method for representing group elements, which will prove particularly useful for the sporadic groups. We explain the theory behind symmetric presentations, and describe the algorithm for working with elements represented in this manner. Our method, which requires that the group be given as a homomorphic image of an infinite semi-direct product, combines conciseness with acceptable ease of manipulation. Inversion is as straightforward as for permutations, and multiplication can be performed manually or mechanically by means of a short recursive algorithm. In this thesis we represent elements of various finite groups including the groups $U_3(3):2$, J_1, $J_2:2$ and $G_2(4):2$. An algorithm for multiplying elements represented in this way is described.

Symmetric Generation of a Group

Let G be a group and let $T = \{t_1, t_2, \ldots, t_n\} \subseteq G$. Define $\overline{T} = \{T_1, T_2, T_3, \ldots, T_n\}$, where $T_i = \langle t_i \rangle$ for $i \in \{1,2,3,..,n\}$ which is the cyclic subgroup of order m generated by t_i. Let N be the control subgroup where $N = N_G(\overline{T})$.

T is a symmetric generating set for G iff

(i) $< T >$

(ii) N acts transitively on $\overline{T}$

Note that both i, and ii yields that G is a homomorphic image of the progenitor $m^{*n}:N$, which is an infinite semi-direct product of m^{*n} by N, where m^{*n} represents the free product of n copies of the cyclic group of order m.

The automorphisms of m^{*n} are permutations of the free generators. If m = 2 then N will simply act by conjugation as permutations of the n involuntary symmetric generators, $n^{-1}t_i n = t_i^n = t_{n(i)}$.

For example: $t_1^{(1\ 2)} = t_{(1\ 2)(1)} = t_2$

Now, since by the above, elements of N can be gathered on the left, every element of the progenitor can be represented as πw, where $\pi \in N$ and w is a word in the symmetric generators. Indeed this representation is unique provided w is simplified so that adjacent symmetric generators are distinct.

let $\pi t_i t_j \alpha \in 2^{*n}:N$ where $\pi, \alpha \in N$

$$\pi \ \alpha \alpha^{-1} \ t_i \ \alpha \alpha^{-1} \ t_j \alpha = \pi \alpha \, t_i^{\alpha} t_j^{\alpha}$$

$$= \pi \alpha \, t_{\alpha(i)} \ t_{\alpha(j)}$$

For example, $t_1(1,2)t_3$

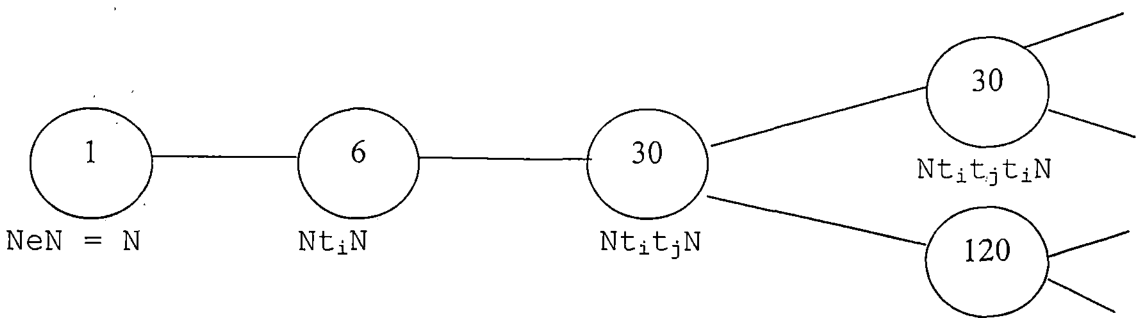
$= (1,2)(1,2)\ t_1(1,2)t_3$

$= (1,2)t_1^{(1,2)}t_3$

$= (1,2)t_2t_3$

The progenitor $m^{*n}:N$, is an infinite group.

For example, $2^{*6}: S_6$ is infinite as shown below.



Nt_iN

$=\{Nt_i\ n \mid n \in N\}$

$=\{N\ n\ n^{-1}\ t_i\ n \mid n \in N\}$

$=\{N\ n\ t_i^n \mid n \in N\}$

$=\{N\ t_i^n \mid n \in N\}$

$= \{N\ t_{n(i)} \mid n \in N\}$

$= \{Nt_1, Nt_2, \ldots, Nt_6\}$ (since S_6 is 6-transitive)

In order to obtain finite homomorphic images of a progenitor, we must factor it by a relation of the form $\pi = w$, where $\pi \in N$ and w is a word in the t_i's.

In order to find such a relation we proceed as follows:

1) Find the conjugate classes of the control group N. A relation would be of the form $(xw)^n$ where x is an element of a conjugate class of N and w a word in the t_i's and n is a positive integer.

For example, we may factor the progenitor $2^{*4}:S_4$ by the relation $(2,3)(t_0t_1)^2 = 1$.

2) We use the following Lemma by Curtis (see Curtis [6]) frequently to find suitable relations:

Lemma: $< t_i , t_j > \cap N \leq C_N(N^{ij})$

(N^{ij} is the point-wise stabilizer of {i,j} in N)

Proof:

Let $n \in < t_i , t_j > \cap N$. Say $n = t_i t_j t_i$

Need to show that $n \in C_N(N^{ij})$.

Let $g \in N^{ij}$

$$n^g = g^{-1} n g = g^{-1}(t_i t_j t_i) g$$

$$= t_i^g t_j^g t_i^g \text{ (since } g \in N^{ij})$$

$$= t_i t_j t_i$$

$$= n$$

Therefore $n^g = n$

Or $g^{-1} n g = n \Rightarrow ng = gn$

Thus $< t_i , t_j > \cap N \le C_N (N^{ij})$.

Similarly $< t_i , t_j, \ldots, t_k> \cap N \le C_N (N^{ij..k})$

□

We will only be considering the semi-direct product 2^{*n}: N, where 2^{*n} is the free product of n copies of the cyclic group C_2 of order 2. Here we are seeking homomorphic images of the progenitor 2^{*n} : N, where N is a transitive permutations group on n letters, Which act faithfully on N and on the generators of the free product. It is convenient to identify the n free progenitors and N with their respective images. Thus $\frac{2^{*n}:N}{\pi_1 w_1,\ldots,\pi_s w_s} \cong < N, T \mid N_p, t_i^2 = 1, t_i^{\pi} = t_{\pi(i)}, \pi_1 w_1 = \ldots = \pi_s w_s = 1>$,

where N_p represents the presentations of the control group N. The relation $t_i^{\pi} = t_{\pi(i)}$ is replaced by $[N^i, t_i] = 1$, that is; t_i commutes with the generators of the point-wise stabilizer N^i.

Example:

In order to compute the order of the progenitor $\frac{2^{*5}:S_5}{t_0 t_1 t_0 = (0,1)}$, we proceed as follows.

1: We use MAGMA to find the presentation of the control Group S_5 as follows.

```
F <x,y>:=FPGroup(N);
Print F;
```

where x = (1,2,3,4,5), and y = (1,2) are the generators for the control group S_5.

2: Find the point-wise stabilizer in S_5 of one point, say 1. The following MAGMA commands compute this.

```
N1:=Stabilizer (N,1);
Print N1;
```

3: The Symmetric generator t will commute with the generators of the point-wise stabilizer N1 that we found in step 2.

In order to obtain the index of N in G we shall perform a manual double coset enumeration; thus we must find all double cosets [w] and work out how many single cosets each of the double cosets contains. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication. Define $N^{(w)} = \{\pi \in N : N w\pi = N w\}$, for w a word in the symmetric generators and $N^w = \{\pi \in N : w\pi = \pi w\}$. Clearly $N^w \leq N^{(w)}$ (since $\pi w = w \pi$, for $\pi \in N$, $\Rightarrow N\pi w = N w \pi$)

The number of cosets in the double coset [w] = N w N is given by $[N: N^{(w)}] = \frac{|N|}{|N^{(w)}|}$, since

$N w \pi_1 \neq N w \pi_2 \Leftrightarrow N w \pi_1 \pi_2^{-1} \neq N w \Leftrightarrow \pi_1 \pi_2^{-1} \notin N^{(w)} \Leftrightarrow N^{(w)} \pi_1 \neq N^{(w)} \pi_2$.

Moreover, the completion test above is best performed by obtaining the orbits of $N^{(w)}$ on the symmetric generators. We need only identify for each [w], the double coset to which N wt_i belongs for one symmetric generator t_i from each orbit.

We give an easy example to explain our procedure.

<u>Example One:</u>

The progenitor

$$2^{*5} : S_5,$$

which is a free product of 5 copies of the cyclic group C_2 of order 2 extended by S_5.

2^{*5}: S_5 is an infinite group with a presentation given by:

$$<x,y,t \mid x^5,\ y^2,\ (x * y)^4,\ x * y * x^3 * y * x^2 * y * x^{-2} * y * x,\ t^2,\ (t,y),\ (t,y * x * y * x^3),\ (t,((y^x)^y)^x)>$$

Where x ~ (0 1 2 3 4)

and y ~ (1 2)

We need to factor this progenitor by a suitable relation. In order to do this we apply Curtis' Lemma and get:

$C_N\ (N^{0\ 1}) = \langle(0\ 1)\rangle$

Thus we try the relation $t_0t_1t_0 = (0\ 1)$. We use MAGMA to compute the index of N in G, where

$$G \cong \frac{2^{*5} : S_5}{t_0t_1t_0 = (0,1)},$$

which turns out to be 6. We will verify this below:

$t_0t_1t_0 = (0,1)$

$\Rightarrow Nt_0t_1t_0 = N(0,1)$

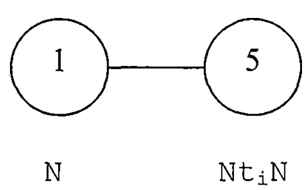
$\Rightarrow Nt_0t_1t_0 = N$ (since $(0,1) \in N$)

$\Rightarrow Nt_0t_1 = Nt_0$

So the double coset $Nt_iN = [i]$ contains five single cosets (t_0, t_1, t_2, t_3, t_4).

The double coset $[ij] = [i]$ since $Nt_0t_1 = Nt_0$ and S_5 is 5 transitive which means that $Nt_it_j = Nt_i$ for all $i,j \in \{0,1,2,3,4\}$

So the Cayley graph of G will be:



From the Cayley graph we can see that there are 6 single cosets each of order 120, thus Order of G, $|G| = (1 + 5)120 = 720$. Therefore $G \cong S_6$

Theorem One:

$$G \cong \frac{2^{*n} :S_n}{t_0t_1t_0 = (0,1)} \cong S_{n+1}$$

Proof:

$t_0t_1t_0 = (0,1)$

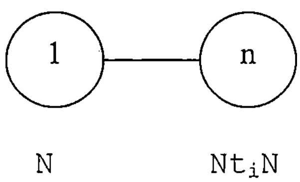
$Nt_0t_1t_0 = N(0,1)$

$Nt_0t_1t_0 = N$ (since $(0,1) \in N$)

$Nt_0t_1 = Nt_0$

So the double coset [i]contains n single cosets (t_0, t_1, t_2, t_3, t_4 … t_n)

The double coset [ij] = [i] since $Nt_0t_1 = Nt_0$ and S_n is n-transitive which means that $Nt_it_j = Nt_i$ for all $i,j \in \{0,1,2,3,4,\ldots, n\}$. So the Cayley graph of G will be:



From the Caley Graph we can see that there is n+1 single cosets each of order n!, thus Order of G, $|G| = (1 + n)n! = (n+1)!$ Therefore $G \cong S_{n+1}$.

Definitions

Here will be listing some basic definitions. G-Set: Let X be a set and G be a group, then X is a G-Set if there is a function $\alpha: G \times X \rightarrow X$, denoted by $\alpha(g,X) \rightarrow gx$, such that:

i) $1x = x$ for all $x \in X$; and

ii) $g(hx) = (gh)x$ for all $g,h \in G$ and $x \in X$

<u>K-Transitive</u>: Let X be a G-Set of degree n and let $k \leq n$ be a positive integer then X is K-transitive if, for every pair of k-tuples having distinct entries in X, say $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$, there is $g \in G$ with $gx_i = y_i$ for all i in $\{1, 2, \ldots, k\}$.

<u>Complement</u>: Let K be a subgroup of a group G. Then a subgroup $Q \leq G$ is a complement of K in G if $K \cap Q = 1$ and $KQ = G$.

<u>Semi-direct Product</u>: A group G is a semi-direct product of K by Q, if k is normal subgroup of G and K have a complement $Q_1 \equiv Q$.

Free Group: If X is a set of a group F, then F is a free Group with basis X if, for every group G and every function $f:X \rightarrow G$ extending f.

Word: A word on X is a sequence $w = (a_1, a_2, \ldots)$, where $a_i \in X \cup X^{-1} \cup \{1\}$ for all I, such that all $a_i = 1$ from some point on; that is, there is an integer $n \geq 0$ with $a_i = 1$ for all $I > n$. In particular, the constant sequence $(1,1,1,\ldots)$ is a word, called the empty word, and it is also denoted by 1.

Commutator: If $a, b \in G$, the commutator of a and b, denoted by $[a,b] = aba^{-1}b^{-1}$.

CHAPTER TWO

SYMMETRIC REPRESENTATION OF $PGL_2(7)$ ELEMENTS

Introduction

In this chapter we will show that the elements of the group $PGL_2(7)$ can be written as a permutation of the group S_4 followed by at most three of the symmetric generators. The symmetric presentations of the progenitor $2^{*4}:S_4$ are given by:

$$<x,y,t|x^4, y^2, (y * x)^3, t^2, (t,y), (t,(x*y)^{(x^3)})>,$$

where the control group $N = S_4 \cong <x,y| x^4, y^2, (y * x)^3>$

and $x \sim (0\ 1\ 2\ 3)$,

and $y \sim (1\ 2)$.

Consider the conjugate classes of the control group S_4, and try to write elements of S_4 in terms of the t_i's, say $(xt_0)^6$, and by applying Curtis' Lemma we get:

$$C_N (N^{0\ 1}) = <(0\ 1), (2\ 3)>.$$

So we factor the progenitor by the following relations $(xt_0)^6 = 1, (2,3) = (t_0t_1)^2$ to get G. The Index of N in G, the homomorphic image of

$$G \cong \frac{2^{*4}:S_4}{(xt_0)^6, (2,3) = (t_0t_1)^2}$$

turns out to by 14.

Double Coset Enumeration

Here we will show manual double coset enumeration of G over N:

The relator

$(xt_0)^6 = 1$

$\Rightarrow ((0\ 1\ 2\ 3)t_0)^6 = 1$

$\Rightarrow (0\ 1\ 2\ 3)t_0(0\ 1\ 2\ 3)t_0(0\ 1\ 2\ 3)t_0(0\ 1\ 2\ 3)t_0(0\ 1\ 2\ 3)t_0$
$(0\ 1\ 2\ 3)t_0 = 1$

$\Rightarrow (0\ 2)(1\ 3)t_1t_0(0\ 2)(1\ 3)t_1t_0(0\ 2)(1\ 3)t_1t_0 = 1$

$\Rightarrow (0\ 2)(1\ 3)t_1t_0t_3t_2t_1t_0 = 1$

$\Rightarrow (0\ 2)(1\ 3) = t_0t_1t_2t_3t_0t_1$

Note that $Nt_1t_0t_3 = N\ t_0t_1t_2$ (1)

and the relator $(2,3) = (t_0t_1)^2$

$\Rightarrow (2,3) = t_0t_1t_0t_1$

$\Rightarrow Nt_1t_0 = Nt_0t_1$ (2)

Therefore the double coset [i j] contain 6 single cosets since N is doubly transitive on T and since $Nt_0t_1 = Nt_1t_0$ (by 2).

Now $Nt_0t_1t_0 = Ntt_1t_0t_0$ (by 2) implies that [i j i]= [i]

Furthermore, $Nt_0t_1t_2 = Nt_1t_0t_2$ (by 1)

$= Nt_1t_0t_3$ (by 2)

$= Nt_0t_1t_3$ (by 1)

$= Nt_3t_2t_0$

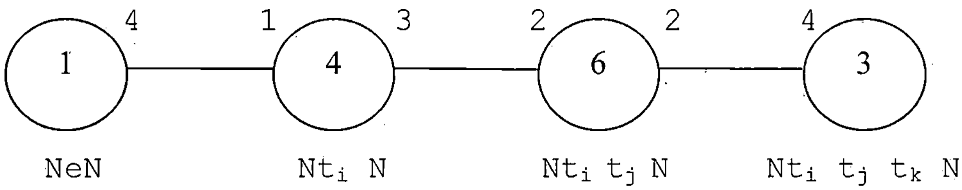
$= Nt_2t_3t_0$

$= Nt_3t_2t_1$

$= Nt_2t_3t_1$

Therefore the double coset [i j k] contains 3 single cosets since every 8 of them have the same name.

The Cayley graph of $PGL_2(7)$ over S_4 is given below:



At this point we can calculate the action of the four symmetric generators on the cosets of $PGL_2(7)$ over S_4. In order to calculate this for the symmetric generator t_0, we start with the identity coset N or * and then multiply on the right by t_0, the result is Nt_0 or 0. Then repeat the process by multiplying again on the right by t_0, the result now is $N\, t_0\, t_0 = N$. Now start with a new single coset and repeat the process. (Note that the permutation for the symmetric generators t_i's will be a product of two cycles).

We obtain:

t_0: (* 0)(1 10)(2 20)(3 30)(12 120)(13 130)(23 230)

t_1: (* 1)(2 21)(3 31)(0 01)(23 231)(20 201)(30 301)

t_2: (* 2)(1 12)(3 32)(0 02)(13 132)(10 102)(30 302)

t_3: (* 3)(1 13)(2 23)(0 03)(12 123)(10 103)(20 203)

We re-label the cosets according to the following scheme:

*	0	1	2	3	12	13	14	23	24	34	142	123	421
5	4	1	2	3	6	7	8	9	10	11	12	13	14

The four symmetric generators under this re-labeling become:

t_0 = (5 4)(1 8)(2 10)(3 11)(6 13)(7 14)(9 12)

t_1 = (5 1)(2 6)(3 7)(4 8)(9 12)(10 14)(11 13)

t_2 = (5 2)(1 6)(3 9)(4 10)(7 14)(8 12)(11 13)

t_3 = (5 3)(1 7)(2 9)(4 11)(6 13)(8 12)(10 14)

Every element g of G can written as $g = \pi w$ where π is a permutation of S_4 and w is a product of at most three of the symmetric generators t_i's.

We call this representation of G, the symmetric representation of elements of G.

As we have seen above, G has a permutation representation of degree 14, therefore every element g of $G \in S_{14}$.

Finding Symmetric Representation

Let $\alpha \in G$ be a permutation on 14 letters.

We find a symmetric representation for α; that is, find a $\pi \in N$ and a word w in the t_i's such that $\alpha = \pi w$ according to the following algorithm.

Now $N^{\alpha} = N\alpha$ (since the action is right multiplication)

$\Rightarrow N\alpha = Nt_it_j$ (Since $\alpha \in G$ and $G = \langle t_1, t_2, \dots, t_4 \rangle$)

$\Rightarrow N \alpha t_j^{-1} t_i^{-1} = N$

$\Rightarrow N \alpha t_j t_i = N$

$\Rightarrow \alpha t_j t_i \in N$

We now need to compute the action of $\alpha t_j t_i$ on the set $\{t_1, t_2, \dots, t_4\}$.

We explain the process through the following example.

Example:

Let $\alpha = (1\ 7\ 3\ 9\ 2\ 10\ 4\ 8)(5\ 13)(6\ 12\ 11\ 14) \in PGL_2(7)$.

Then $N^{\alpha} = N\alpha$

$\Rightarrow 5^{\alpha} = 13$ (since N or * is labeled as 5 above)

$\Rightarrow N\alpha = Nt_1t_2t_3$ (since $t_1t_2t_3$ is labeled as 13)

$\Rightarrow N \alpha t_3t_2t_1 = N$

$\Rightarrow \alpha t_3t_2t_1 = n$, say, belongs to N.

Now

$\alpha t_3t_2t_1 = (1\ 7\ 3\ 9\ 2\ 10\ 4\ 8)(5\ 13)(6\ 12\ 11\ 14)$

$(5\ 3)(1\ 7)(2\ 9)(4\ 11)(6\ 13)(8\ 12)(10\ 14)$

$(5\ 2)(1\ 6)(3\ 9)(4\ 10)(7\ 14)(8\ 12)(11\ 13)$

$(5\ 1)(2\ 6)(3\ 7)(4\ 8)(9\ 12)(10\ 14)(11\ 13)$

$= (1\ 2\ 3)(6\ 9\ 7)(8\ 10\ 11)(12\ 14\ 13)$

$\Rightarrow$

$Nt_1 = 1 \rightarrow 2 = Nt_2$

$Nt_2 = 2 \rightarrow 3 = Nt_3$

$Nt_3 = 3 \rightarrow 1 = Nt_1$

$Nt_4 = 4 \rightarrow 4 = Nt_4$

$\Rightarrow n = (1\ 2\ 3)$.

Therefore $\alpha = (1\ 2\ 3)\, t_1 t_2 t_3$.

Finding Permutation Representation of an Element Whose Symmetric Representation is Given

Algorithm:

We can recover the symmetric representation of elements of G from their permutation representation. Let $\pi w = \pi\ t_i t_j t_k$ be an element of a symmetrically presented group G. Then

$N^{\pi} = N\pi = N$

$(Nw_1)^{\pi} = Nw_1\pi = Nw\pi_{(1)}$

....

The following example illustrates the process.

Example:

Let $\beta = (1\ 2)(3\ 4)t_3$ be the symmetric representation of the element $\beta \in G$. Then the action of $(1\ 2)(3\ 4) \in N$ on the cosets of N in G is computed as follows:

$* \rightarrow *,\ 5 \rightarrow 5$
$1 \rightarrow 2,\ 1 \rightarrow 2$
$2 \rightarrow 1,\ 2 \rightarrow 1$
$3 \rightarrow 4,\ 3 \rightarrow 4$

$4 \to 3,\ 4 \to 3$
$12 \to 21,\ 6 \to 6$
$13 \to 24,\ 7 \to 10$
$14 \to 23,\ 8 \to 9$
$23 \to 14,\ 9 \to 8$
$24 \to 13,\ 10 \to 7$
$34 \to 43,\ 11 \to 11$
$142 \to 231,\ 12 \to 12$
$123 \to 214,\ 13 \to 13$
$421 \to 312,\ 14 \to 14$

$$\Rightarrow \beta = (1\ 2)(3\ 4)(7\ 10)(8\ 9)t_3$$

$$= (1\ 2)(3\ 4)(7\ 10)(8\ 9)(5\ 3)(1\ 7)(2\ 9)(4\ 11)(6\ 13)(8\ 12)(10\ 14)$$

$$= (1\ 9\ 12\ 8\ 2\ 7\ 14\ 10)(3\ 11\ 4\ 5)(6\ 13)$$

CHAPTER THREE

SYMMETRIC REPRESENTATION OF ELEMENTS OF $PGL_2(11)$

Introduction

In this chapter we will show that the elements of the group $PGL_2(11)$ can be represented as a permutation on six letters followed by a word in the symmetric generators of length at most three.

The symmetric presentations of the progenitor $2^{*6} : L_2(5)$ is given by:

$$<x,y,z,t \mid x^5, y^2, z^2, (x^{-1} * y)^2, (y * z)^2, (z * x)^3, y * x^3 * z * x^2 * z * x^{-2} * z, t^2, (t,x), (t,y)>,$$

where the action of x, y, and z on the symmetric generators is given by:

$x \sim (0\ 1\ 2\ 3\ 4)$

$y \sim (1\ 4)(2\ 3)$

$z \sim (0\ \infty)(1\ 4)$.

Consider the conjugate classes of the control group $L_2(5)$, and try to write elements of $L_2(5)$ in terms of the t_i's.

Let $(xt_0)^4 = (0\ 2)(1\ 3)t_1t_0t_3t_2t_1t_0 = 1$

We factor the progenitor by the following relation

$$(x\ t_0)^4 = 1$$

The Index of N in G, the homomorphic image

$$G \cong \frac{2^{*6} :L_2(5)}{(xt_0)^4}$$

is 22.

Double Coset Enumeration

Manual Double Coset enumeration of G over N:

The relator:

$(x\ t_0)^4 = 1$

$\Rightarrow ((0\ 1\ 2\ 3\ 4)\ t_0)^4 = 1$

$\Rightarrow (0\ 1\ 2\ 3\ 4)t_0\ (0\ 1\ 2\ 3\ 4)t_0\ (0\ 1\ 2\ 3\ 4)t_0\ (0\ 1\ 2\ 3\ 4)t_0=1$

$\Rightarrow (0\ 2\ 4\ 1\ 3)t_1t_0\ (0\ 2\ 4\ 1\ 3)\ t_1t_0 = 1$

$\Rightarrow (0\ 4\ 3\ 2\ 1)\ t_3t_2t_1t_0 = 1$

$\Rightarrow t_3t_2t_1t_0 = (0\ 1\ 2\ 3\ 4)$ (1)

Note that $Nt_3t_2 = Nt_0t_1$ (by 1)

$\Rightarrow Nt_0t_1\ (2\ 4)(3\ \infty) = Nt_3t_2\ (2\ 4)(3\ \infty)$

$\Rightarrow N\ (2\ 4)(3\ \infty)(t_0t_1)^{(2\ 4)(3\ \infty)} = N\ (2\ 4)(3\ \infty)(t_3t_2\ )^{(2\ 4)(3\ \infty)}$

$\Rightarrow N\ t_0t_1 = N\ t_\infty\ t_4$

Therefore $t_0t_1 \sim t_3t_2 \sim t_\infty\ t_4$

Since $L_2(5)$ is doubly transitive on {0,1,2,3,4,5, ∞}, the double coset [i j] contains 10 distinct single cosets because every coset has three names.

Furthermore, $N\ t_0t_1t_2 = Nt_3t_2t_2$ (since $Nt_3t_2 = Nt_0t_1$)

$= Nt_3$

Similarly N $t_0t_1t_4$ = N $t_\infty t_4$ t_4 (since $Nt_\infty t_4$ = Nt_0t_1)

= Nt_∞.

Now N $t_\infty t_0 t_\infty$ = $Nt_\infty t_0 t_\infty$ $t_3t_1t_1t_3$

= N t_∞ (1 3 ∞ 0 2)t_1t_3

= N $t_0t_1t_3$

= N $t_3t_2t_3$ (since $t_0t_1 \sim t_3t_2$),

also N $t_\infty t_0 t_\infty$ = $Nt_\infty t_0 t_\infty$ $t_2t_4t_4t_2$

= N t_∞ (2 ∞ 0 3 4)t_4t_2

= N $t_0t_4t_2$

= N $t_2t_3t_2$ (since Nt_0t_4 = N t_2t_3)

⇒ $Nt_3t_2t_3$ = $Nt_2t_3t_2$.

⇒ $Nt_it_jt_i$ = $Nt_jt_it_j$ for i and j in {0,1,2,3,4,5, ∞} since $L_2(5)$ is doubly transitive on {0,1,2,3,4,5, ∞}.

In particular, N t_∞ t_0 t_∞ = N t_0 t_∞ t_0 .

Also, N $t_4t_1t_4$ = $Nt_4t_1t_4t_\infty$ t_2t_2 t_∞

= N t_4 (1 0 2 ∞ 4)t_2t_∞

= N $t_1t_2t_\infty$

= N t_∞ t_0 t_∞ (since $t_1t_2 \sim t_\infty$ t_0).

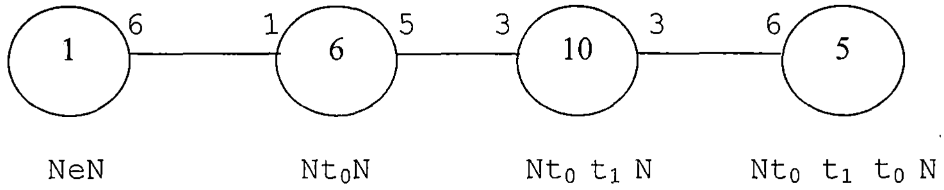
Thus we have $t_\infty t_0 t_\infty \sim t_0 t_\infty t_0 \sim t_3t_2t_3 \sim t_2t_3t_2$

$\sim t_4t_1t_4 \sim t_1t_4t_1$

Therefore the double coset [i j i] contains 5 single cosets since every 6 share the same name.

Also we have that [0 1 3]= [0 1 0].

The Cayley graph of $PGL_2(11)$ over $L_2(5)$ is given below:



For the action of the four symmetric generators on the cosets of $PGL_2(11)$ over $L_2(5)$ (see Appendix A)

1. []
2. [5]
3. [1]
4. [6]
5. [4]
6. [2]
7. [1, 5]
8. [6, 5]
9. [4, 5]
10. [3]
11. [3, 6]
12. [4, 6]
13. [6, 1]
14. [5, 6]
15. [6, 4]
16. [1, 6]
17. [2, 6]
18. [1, 6, 1]
19. [2, 6, 2]
20. [5, 6, 5]
21. [1, 5, 1]
22. [4, 6, 4]

Where,

t_1 = (1 3)(2 15)(4 13)(5 11)(6 9)(7 21)(8 22)(10 12)(14 19)(16 18)(17 20)

t_2 = (1 6)(2 12)(3 8)(4 7)(5 14)(9 22)(10 15)(11 18)(13 20)(16 21)(17 19)

t_3 = (1 10)(2 16)(3 14)(4 9)(5 8)(6 13)(7 18)(11 21)(12 19)(15 20)(17 22)

t_4 = (1 5)(2 13)(3 17)(4 15)(6 16)(7 10)(8 18)(9 19)(11 20)(12 22)(14 21)

t_5 = (1 2)(3 7)(4 8)(5 9)(6 11)(10 17)(12 18)(13 19)(14 20)(15 21)(16 22)

t_6 = (1 4)(2 14)(3 16)(5 12)(6 17)(7 19)(8 20)(9 21)(10 11)(13 18)(15 22)

Representation of Elements of $PGL_2(11)$

Every element of G can written as πw, where π is a permutation of $L_2(5)$ (on 6 letters) and w is a product of at most three of the symmetric generators t_i's.

We explain the process through the following example.

Let $\alpha \in PGL_2(11)$,

where α = (1 6 12 3)(2 7 19 9)(4 15)(5 14 22 11)(8 18 13 10)(16 20)(17 21).

Now $N^{\alpha} = N\alpha$

$\Rightarrow$ $1^{\alpha} = 6$ (since N or * is labeled as 1 above)

$\Rightarrow$ $N\alpha = Nt_2$ (since t_2 is labeled as 6)

$\Rightarrow$ $N\alpha t_2 = N$

$\Rightarrow$ $\alpha t_2 = n$

$\Rightarrow$ α = (1 2 5 6 3)t_2, where (1 2 5 6 3) $\in L_2(5)$.

Permutation Representation of an Element Whose Symmetric Representation is Given

Starting with an element of the progenitor $2^{*6}:L_2(5)$ which can be written as πw where π is a permutation of $L_2(5)$ and w is a word in the symmetric generators t_i's, we can find its permutation representation on 22 letters as we explain below.

Example:

Let $\pi = (1\ 3\ 5)(2\ 6\ 4)t_3 \in 2^{*6}:L_2(5)$.

Then the action of π on the twenty two cosets of N in $PGL_2(11)$ is given by:

$\pi = (2\ 3\ 10)(4\ 5\ 6)(7\ 12\ 16)(8\ 11\ 13)(14\ 17\ 15)(19\ 22\ 21)t_3.$

$= (2\ 3\ 10)(4\ 5\ 6)(7\ 12\ 16)(8\ 11\ 13)(14\ 17\ 15)(19\ 22\ 21)$

$(1\ 10)(2\ 16)(3\ 14)(4\ 9)(5\ 8)(6\ 13)(7\ 18)(11\ 21)(12\ 19)$

$(15\ 20)(17\ 22)$

$\pi = (1\ 10\ 16\ 18\ 7\ 19\ 17\ 20\ 15\ 3)(2\ 14\ 22\ 11\ 6\ 9\ 4\ 8\ 21\ 12)$

$(5\ 13\ )$.

CHAPTER FOUR

SYMMETRIC REPRESENTATION OF ELEMENTS OF THE JANKO GROUP J_1

Introduction

In this chapter we will discuss how elements of the smallest Janko group J_1 can be represented as a permutation of the group $L_2(11)$ which is a permutation on 11 letters followed by a word of at most four of the symmetric generators t_i's.

The progenitor 2^{*11}: $L_2(11)$ which is presented as follows:

$$\langle a,g,t \mid g^2, (a^{-1} * g)^3, a^{-11}, (g * a^3 * g * a^{-3})^2, a^{-2} * g * a * g * a^{-1} * g * a * g * a^{-1} * g * a * g * a^{-1} * g * a * g * a^{-2} * g, t^2, (t,g), (t,a^6 * g * a^9 * g * a^2), (t,(a^3 * g)^2 * a^5)\rangle,$$

where the action of N on the symmetric generators is:

a ~ (0 1 2 3 4 5 6 7 8 9 X)

g ~ (3 4)(2 X)(5 9)(6 7)

We factor the progenitor by the following relation

$(3,4)(0,1)(2,6)(7,X) = t_0t_1t_0t_1t_0$ and $(0,8,1)(2,7,9,X,6,5)(3,4) = t_0t_1t_8t_0t_1$

The index of N in G, the homomorphic image of

$$G \cong \frac{2^{*11} : L_2(11)}{(0,8,1)(2,7,9,X,6,5)(3,4) = 01801, (3,4)(0,1)(2,6)(7,X) = 01010},$$

turns out to be 266.

Double Coset Enumeration

Manual Double Coset enumeration of G over N:

The relator

$(3,4)(0,1)(2,6)(7,X) = t_0t_1t_0t_1t_0$ (Here X represents 10)

$\Rightarrow N\ t_0\ t_1\ t_0 = Nt_0\ t_1$

$L_2(11)$ is doubly transitive, therefore the double coset

[i j i] = [i j].

Also the relator

$(0,8,1)(2,7,9,X,6,5)(3,4) = t_0t_1t_8t_0t_1$

$\Rightarrow N\ t_0\ t_1\ t_8 = Nt_1\ t_0$

$N^{(0\ 1)}$ has orbits {5,8,9} and {2,3,4,6,7,X}

$\Rightarrow N\ t_0\ t_1\ t_i = Nt_1\ t_0$ for i in {5,8,9}

$N\ t_0\ t_1\ t_2 = Nt_0\ t_1t_8\ t_8\ t_2\ t_0\ t_0$

$= N\ (0,8,1)(2,7,9,x,6,5)(3,4)t_1\ t_0t_8\ t_2\ t_0\ t_0$

$= Nt_3\ t_8\ t_0$

$= Nt_5\ t_4\ t_3$

$= Nt_6\ t_9\ t_5$

$= Nt_2\ t_X\ t_6$

Therefore the double coset [0 1 j] for j in {2,3,4,6,7,X} contains 132 single cosets in it since every 5 of these single cosets have the same name.

$Nt_0\ t_1\ t_2\ t_c = Nt_it_j$ for all c in {0,2,3,5,6}

$Nt_0\ t_1\ t_2\ t_1 = Nt_0\ (1\ 2)(3\ 7)(4\ 5)(8\ 0)\ t_1\ t_2$

$= N\ t_8\ t_1\ t_2$

$Nt_0\ t_1\ t_2\ t_4 = Nt_6\ t_9\ t_5\ t_4$ (since $Nt_0\ t_1\ t_2 = N\ t_6\ t_9\ t_5$)

$= N\ t_6\ (1\ 8\ 0\ 2\ 3\ 7)(4\ 5\ 9)(6\ X)\ t_5\ t_9\ t_4$

$= N\ t_X\ t_5\ t_9\ t_4$

$= N\ t_X\ (1\ 7\ 3\ 2\ 0\ 8)(4\ 9\ 5)(6\ X) t_9\ t_5$

$= N\ t_6\ t_9\ t_5$

$Nt_0\ t_1\ t_2\ t_8 = Nt_3\ t_8\ t_0\ t_8$

$= N\ t_3\ (3\ 4)(5\ 7)(6\ 9)(8\ 0)\ t_8\ t_0$

$= N\ t_4\ t_8\ t_0$

$Nt_0\ t_1\ t_2\ t_9 = Nt_6\ t_9\ t_5\ t_9$

$= N\ t_6\ (1\ 0)(2\ 7)(5\ 9)(6\ x) t_9\ t_5$

$= N\ t_X\ t_9\ t_5$

$Nt_0\ t_1\ t_2\ t_X = Nt_2\ t_X\ t_6\ t_X$

$= N\ t_2\ (1\ 2)(3\ 8)(6\ X)(7\ 0) t_X\ t_6$

$= N\ t_1\ t_X\ t_6$

Now

$Nt_0\ t_1\ t_2\ t_7 = Nt_5\ t_4\ t_3\ t_7$

$= Nt_5\ t_4\ t_3\ t_7\ t_1\ t_1$

$= Nt_5\ t_4\ (1\ 7\ 3)(2\ 4\ X\ 6\ 9\ 8)(5\ 0)\ t_7\ t_3\ t_1$

$= Nt_0\ t_X\ t_7\ t_3\ t_1$

$= Nt_X t_0\ t_3\ t_1$

$Nt_0\ t_1\ t_2\ t_7 = Nt_5\ t_4\ t_3\ t_7$

$= Nt_5\ t_4\ t_3\ t_7\ t_2\ t_2$

$= Nt_5 t_4 (1\ 0\ 9\ 6\ X\ 5)(2\ 7\ 3)(4\ 8)\ t_7 t_3 t_2$

$= Nt_1 t_8 t_7 t_3 t_2$

$= Nt_8 t_1 t_3 t_2$

Now calculating the stabilizing group $N^{(0\ 1\ 2\ 7)}$ (see Appendix B(1)) which turns out to have order 55. Therefore the number of the single cosets in that double coset is $\frac{|N|}{|N^{(0127)}|} = \frac{660}{55} = 12$

Where $N^{(0\ 1\ 2\ 7)}$ is transitive or has orbit

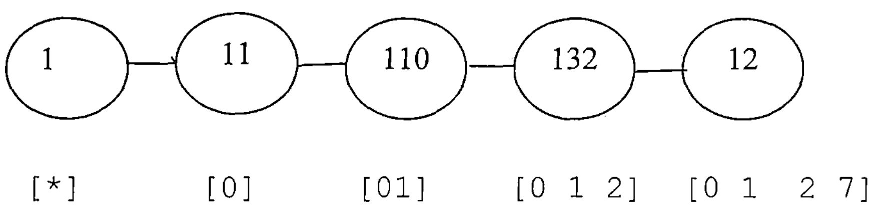
(0 1 2 3 4 5 6 7 8 9 X)

But $Nt_0 t_1 t_2 t_7 t_7 = Nt_0 t_1 t_2$

Therefore the double coset [0 1 2 7 i] = [0 1 2]

for all i in {0,1,2,3,4,5,6,7,8,9,X}

Thus the Cayley graph will be



This group is isomorphic to the Janko group J_1. The lowest index subgroup of this Janko group is of order 266.

At this point we can calculate the symmetric generators t_i (see Appendix B(2) for the labeling)

t_1=(1, 3)(2, 13)(4, 23)(5, 11)(7, 55)(8, 64)(9, 20)
(10, 77)(12, 94)(14,108)(15, 67)(16, 42)(17, 125)
(18, 37)(19, 134)(22, 29)(24, 116)(25,84)(26, 86)
(28, 130)(30, 91)(31, 60)(32, 79)(33, 192)(34, 135)

(35,113)(36, 199)(38, 70)(39, 75)(40, 69)(41, 211)
(43, 81)(44, 54)(45,98)(46, 183)(47, 49)(50, 141)
(51, 227)(52, 197)(53, 102)(56, 88)(57, 198)(58, 97)
(59, 145)(61, 242)(62, 167)(63, 217)(65, 123)(66, 105)
(68,142)(71, 147)(73, 139)(74, 253)(76, 124)(78, 166)
(80, 151)(82, 168)(83,117)(85, 100)(87, 155)(89, 159)
(92, 162)(93, 209)(95, 225)(96, 214)(99,208)(101, 254)
(103, 243)(104, 177)(106, 174)(107, 132)(109, 263)
(110,164)(111, 241)(112, 146)(114, 201)(115, 189)
(118, 216)(119, 252)(120,215)(121, 170)(122, 194)
(126, 236)(128, 235)(129, 184)(131, 233)(133,187)
(136, 204)(137, 205)(138, 220)(140, 188)(143, 195)
(144, 206)(148,231)(150, 212)(152, 232)(153, 218)
(154, 181)(156, 265)(157, 171)(158,266)(160, 224)
(161, 219)(163, 260)(165, 229)(169, 240)(172, 223)
(173,222)(175, 186)(176, 228)(178, 200)(179, 221)
(180, 238)(182, 210)(185,203)(190, 239)(193, 230)
(202, 237)(207, 264)(213, 251)(226, 257)(234, 259)
(244, 249)(245, 262)(246, 248)(247, 256)(250, 258)
(255, 261)

t_2 =(1, 5)(2, 25)(3, 27)(4, 70)(6, 86)(7, 143)(8, 184)
(9, 22)(12, 153)(13,104)(14, 159)(15, 28)(16, 154)
(17, 61)(18, 39)(19, 132)(20, 144)(21,23)(24, 92)
(26, 216)(29, 189)(30, 51)(31, 122)(32, 102)
(33, 241)(34,135)(35, 115)(36, 203)(37, 200)(40, 65)
(41, 97)(42, 161)(43, 55)(44,82)(46, 56)(47, 106)
(48, 76)(50, 232)(52, 233)(53, 71)(54, 188)(57,262)
(58, 205)(59, 246)(60, 149)(62, 221)(63, 119)(64, 72)
(66, 173)(67,178)(68, 157)(69, 171)(73, 141)(74, 133)
(75, 130)(77, 255)(78, 258)(79,146)(80, 151)(81, 167)
(83, 150)(84, 242)(85, 207)(87, 116)(88, 245)
(90,174)(91, 181)(93, 164)(94, 208)(95, 138)(96, 163)
(98, 126)(99, 192)(100,172)(101, 198)(103, 123)
(105, 128)(107, 137)(108, 127)(109, 229)(110,158)
(111, 218)(112, 240)(113, 206)(114, 238)(117, 239)
(120, 182)(121,136)(124, 251)(125, 177)(131, 175)
(134, 211)(139, 265)(140, 201)(142,243)(145, 209)
(147, 169)(148, 215)(152, 160)(155, 166)(156, 224)
(162,250)(165, 186)(168, 180)(170, 210)(176, 202)
(179, 195)(183, 254)(185,260)(187, 244)(190, 217)
(191, 236)(193, 230)(196, 261)(197, 263)(199,259)
(204, 231)(212, 252)(214, 234)(219, 227)(220, 264)
(222, 237)(223,225)(226, 247)(228, 235)(248, 266)
(249, 257)(253, 256)

t_3 = (1, 9)(2, 132)(3, 48)(4, 157)(5, 45)(6, 217)(7, 95)
(8, 117)(10, 25)(11,153)(12, 150)(13, 93)(14, 26)
(15, 73)(16, 181)(17, 187)(18, 43)(21,224)(23, 218)
(24, 123)(27, 175)(28, 128)(29, 160)(30, 140)(31, 53)
(32,79)(33, 62)(34, 75)(35, 193)(36, 198)(37, 90)
(38, 70)(39, 210)(40,89)(41, 266)(42, 158)(44, 215)
(46, 253)(47, 249)(49, 103)(50, 204)(51,77)(52, 154)
(54, 135)(55, 244)(56, 234)(57, 219)(58, 263)(59, 229)
(60,172)(61, 170)(63, 220)(64, 72)(65, 202)(66, 167)
(67, 209)(69, 238)(71,141)(74, 246)(76, 239)(78, 155)
(80, 104)(82, 180)(84, 106)(85, 127)(86,233)(87, 97)
(88, 163)(91, 190)(94, 254)(96, 183)(98, 171)(99, 240)
(100,134)(101, 130)(102, 133)(105, 122)(107, 139)
(108, 257)(109, 146)(110,164)(111, 159)(112, 166)
(113, 191)(114, 227)(115, 213)(116, 189)(118,173)
(119, 162)(120, 182)(121, 245)(124, 185)(125, 223)
(126, 262)(129,226)(131, 184)(136, 208)(137, 228)
(142, 186)(143, 194)(144, 247)(145,260)(147, 169)
(148, 221)(149, 205)(151, 161)(152, 248)(156, 216)
(165, 259)(168, 241)(174, 201)(177, 199)(178, 222)
(179, 258)(188, 203)(192,195)(196, 252)(197, 211)
(200, 255)(206, 250)(207, 242)(212, 235)(214,232)
(225, 243)(231, 256)(236, 264)(237, 251)(261, 265)

t_4 = (1, 18)(2, 198)(3, 90)(4, 82)(5, 89)(6, 215)(7, 12)
(8, 137)(9, 81)(10,163)(11, 241)(13, 50)(14, 74)(15,80)
(16, 42)(17, 147)(19, 132)(20,48)(21, 220)(22, 217)
(23, 111)(24, 235)(25, 190)(26, 144)(27, 162)(28,49)
(29, 135)(30, 262)(31, 136)(32, 155)(33, 231)(34, 142)
(35, 100)(38,70)(40, 180)(41, 87)(45, 238)(46, 194)
(47,185)(51, 116)(52, 263)(53,199)(54, 92)(55, 96)
(56, 200)(57, 205)(58, 97)(59, 216)(60, 149)(61,118)
(62, 152)(63, 119)(64, 141)(65, 145)(66, 257)(67, 127)
(68, 252)(69,157)(71, 85)(72, 102)(73, 240)(75, 138)
(76, 123)(77, 264)(78, 158)(79,109)(83, 223)(84, 265)
(86, 207)(88, 150)(91, 95)(93, 230)(94, 179)(98,218)
(99, 117)(101, 167)(103, 246)(104, 228)(105, 225)
(106, 237)(107,172)(108, 156)(110, 266)(112, 166)
(113, 219)(114, 203)(115, 234)(120,175)(121, 174)
(122, 250)(124, 242)(125, 161)(126, 255)(128, 247)
(129,221)(130, 229)(131, 178)(133, 209)(134, 208)
(139, 191)(140, 213)(143,212)(146, 181)(148, 259)
(153, 171)(154, 164)(159, 168)(165, 201)(170,248)
(173, 256)(176, 187)(177, 193)(182, 224)(183, 236)

(184, 214)(188,232)(189, 244)(192, 249)(195, 233)
(196,210)(197, 211)(202, 243)(206,260)(222, 227)
(226,245)(239, 253)(251, 254)(258, 261)

t_5 = (1, 15)(2, 79)(3, 127)(4, 201)(5, 49)(6, 13)(7, 139)
(8, 211)(9, 138)(10,231)(11, 155)(12, 210)(14, 257)
(16,258)(17, 56)(18, 151)(19, 132)(20,185)(21, 23)
(22,181)(24, 145)(25, 91)(26, 189)(27, 94)(29, 259)
(30,51)(31, 120)(33, 62)(34, 206)(35, 195)(36, 221)
(37, 90)(38, 172)(39,89)(40, 135)(41, 220)(42, 183)
(43, 241)(44, 214)(45, 152)(46, 134)(47,215)(48, 168)
(50, 159)(52, 82)(53, 92)(54, 191)(55, 72)(57, 149)
(58,97)(59, 209)(60, 87)(61, 156)(63, 264)(64, 203)
(65, 239)(66, 247)(68,158)(69, 105)(70, 208)(71, 219)
(74, 176)(75, 109)(76, 245)(77, 116)(78,222)(81, 229)
(83, 117)(84, 230)(85, 161)(86, 110)(88, 143)(93, 148)
(95,146)(98, 262)(99, 236)(100, 178)(101, 217)
(102,160)(103, 192)(104,124)(106, 128)(107, 250)
(108, 140)(111, 205)(112, 163)(113, 150)(115,213)
(118, 121)(119, 218)(122, 194)(123, 243)(125, 212)
(126, 255)(129,240)(130, 198)(131, 147)(133, 254)
(136, 263)(137, 186)(141, 204)(142,188)(144, 256)
(153, 253)(154, 235)(157, 251)(162, 169)(164, 171)
(165,175)(166, 238)(167, 190)(170, 260)(173, 242)
(174, 234)(177, 232)(180,265)(184, 224)(187, 237)
(193, 248)(196, 199)(200, 233)(202, 207)(216,225)
(223, 261)(226, 266)(227, 246)(228, 244)(249, 252)

t_6 = (1, 31)(2, 246)(3, 149)(4, 255)(5, 194)(6, 240)
(7,212)(8, 106)(9, 92)(10,25)(11, 27)(12, 244)(13, 121)
(14, 26)(15, 182)(16, 42)(17, 164)(18,204)(19, 234)
(20, 258)(21, 242)(22, 87)(23, 266)(24, 155)(28, 49)
(29,206)(30, 51)(32, 251)(33, 256)(34, 180)(35, 166)
(36, 220)(37, 90)(38,104)(39, 145)(40, 249)(41, 225)
(43, 111)(44, 82)(45, 86)(46, 224)(47,93)(48, 159)
(50, 205)(52, 173)(54, 154)(55, 222)(56, 105)(57, 141)
(58,185)(61, 215)(62, 190)(63, 153)(64, 170)(65, 175)
(66, 247)(67, 158)(68,263)(69, 211)(70, 209)(71, 127)
(72, 75)(73, 138)(74, 143)(76, 169)(77,231)(78, 162)
(79, 98)(80, 181)(81, 183)(83, 192)(84, 200)(85, 241)
(88,117)(89, 146)(91, 140)(94, 95)(96, 172)(97, 229)
(99, 260)(100, 160)(101,119)(102, 118)(103, 237)
(107,207)(108, 213)(109, 245)(113, 250)(114,201)
(115, 116)(123, 199)(125, 196)(128, 171)(129, 178)
(130, 177)(131,254)(132, 233)(133, 203)(134, 156)

(135, 188)(137, 243)(139, 186)(142,253)(144, 189)
(147, 236)(148, 184)(150, 232)(151, 219)(152, 197)
(157,163)(161, 168)(165, 228)(167, 257)(174, 248)
(176, 187)(179, 208)(191,216)(193, 230)(195, 262)
(198, 202)(210, 261)(214, 226)(217, 265)(218,259)
(221, 235)(223, 239)(227, 238)(252, 264)

t_7 = (1, 35)(2, 247)(3, 191)(4, 123)(5, 213)(6, 228)
(7, 12)(8, 119)(9, 230)(10,239)(11, 263)(13, 148)
(14,26)(15, 179)(16, 261)(17, 97)(18, 160)(19,175)
(20, 48)(21, 23)(22, 45)(24, 162)(25, 264)(27, 192)
(28, 49)(29,87)(30, 145)(31, 112)(32, 79)(33, 217)
(34, 238)(36, 262)(37,237)(38,141)(39, 266)(40, 197)
(41, 152)(42, 168)(43, 116)(44, 154)(46, 181)(47,82)
(50, 86)(51, 219)(52, 260)(53, 92)(54, 243)(55, 170)
(56, 96)(57,249)(59, 246)(60, 202)(61, 248)(62, 85)
(64, 109)(67, 231)(68, 105)(69,207)(70, 99)(71, 131)
(72, 133)(73, 206)(74, 159)(75, 235)(76, 128)(77,232)
(78, 157)(80, 91)(81, 153)(83, 172)(84, 163)(88, 200)
(89, 95)(90,171)(93, 139)(94, 140)(98, 223)(101, 199)
(102, 218)(103, 236)(104,225)(106, 177)(107, 209)
(108, 208)(110, 233)(111, 120)(114, 121)(117,155)
(118, 150)(122, 220)(124, 234)(125, 196)(126, 183)
(127, 142)(129,184)(130, 255)(132, 250)(134, 194)
(135, 178)(136, 204)(137, 186)(138,205)(143, 176)
(144, 258)(146, 224)(147, 169)(149, 167)(151, 242)
(156,182)(158, 259)(161, 212)(164, 257)(165, 227)
(173, 211)(174, 253)(180,214)(185, 198)(187, 189)
(188, 201)(190, 222)(203, 265)(210, 244)(215,254)
(216, 229)(221, 256)(226, 240)(241, 245)(251, 252)

t_8 = (1, 17)(2, 135)(3, 196)(4, 62)(5, 118)(6, 13)(7, 12)
(8, 51)(9, 176)(10,25)(11, 165)(14, 206)(15, 96)
(16,241)(18, 169)(19, 208)(20, 180)(21,146)(22, 262)
(23, 226)(24, 79)(26, 156)(27, 78)(28, 243)(29, 152)
(31,110)(32, 218)(35, 58)(36, 177)(37, 238)(38, 107)
(39, 89)(40, 199)(41,198)(42, 46)(43, 81)(44, 234)
(45, 154)(47, 128)(48, 256)(49, 101)(50,138)(52, 132)
(53, 92)(54, 173)(55, 116)(57, 70)(59, 192)(60, 149)
(63,171)(64, 229)(65, 237)(66, 242)(67, 257)(68, 157)
(69, 134)(71, 247)(72,105)(73, 264)(74, 86)(75, 126)
(76, 185)(77, 183)(80, 189)(82, 87)(83,215)(84, 111)
(85, 178)(88, 112)(90, 248)(91, 166)(93, 186)(94, 153)
(95,139)(98, 260)(99, 104)(100, 160)(102,188)(103,265)
(106, 124)(108,235)(109, 251)(113, 191)(114, 201)

(115, 142)(117, 236)(119, 212)(120,159)(121, 162)
(122, 258)(123, 131)(127, 145)(129, 184)(130, 253)
(133,246)(136, 170)(137, 244)(140, 204)(141, 259)
(143, 233)(144, 254)(148,224)(150, 219)(151, 217)
(155, 207)(158, 261)(161, 213)(163, 205)(167,214)
(168, 225)(172, 190)(174, 250)(175, 232)(179, 195)
(181, 249)(182,239)(193, 231)(194, 203)(197, 211)
(200, 230)(202, 221)(209, 245)(210,266)(216, 227)
(220, 222)(223, 255)(228, 240)(252, 263)

t_9 = (1, 8)(2, 42)(3, 72)(4, 26)(5, 129)(6, 13)(7, 170)
(9,83)(10, 205)(11,27)(12, 88)(15, 197)(17, 30)(18,186)
(19, 174)(20, 48)(21, 84)(22,227)(23, 181)(24, 199)
(25, 252)(28, 229)(29, 149)(31, 124)(32, 237)(33,101)
(34, 243)(35, 63)(36, 198)(37, 233)(38, 177)(39, 109)
(40, 145)(41,45)(43, 257)(44, 155)(46, 105)(47, 58)
(49,236)(50, 121)(52, 175)(53,188)(54, 245)(55, 183)
(56,96)(57, 202)(59, 246)(60, 190)(61, 118)(62,148)
(65,123)(66, 141)(67, 131)(68, 157)(69, 135)(70, 219)
(71,265)(73,138)(74, 179)(75, 178)(76, 91)(77, 127)
(78, 163)(79, 116)(80, 151)(81,218)(82, 173)(85, 92)
(86, 217)(87, 253)(89, 261)(90, 98)(93, 171)(94,204)
(95, 166)(97, 159)(99, 195)(100, 160)(102, 223)
(103,248)(104,189)(107, 191)(108, 132)(110, 164)
(111, 235)(112, 180)(113, 154)(114,162)(115, 213)
(120, 244)(122, 212)(125, 239)(126, 255)(128, 169)
(130,200)(133, 172)(134, 196)(136, 256)(139, 153)
(140, 185)(142, 158)(143,259)(144, 242)(146, 232)
(147,258)(150, 216)(152, 192)(156, 215)(161,251)
(165,263)(167, 260)(168, 234)(176, 224)(182, 241)
(187,203)(193,266)(194, 206)(201, 228)(207, 210)
(208, 222)(209, 230)(214, 247)(220,226)
(221,262)(225,231)(238, 254)(240, 264)(249, 250)

t_{10} = (1, 4)(2, 12)(3, 21)(5, 38)(6, 47)(8, 14)(9, 68)
(10,76)(11, 27)(13, 99)(15,114)(16, 85)(17, 33)(18, 44)
(19, 131)(20, 140)(22, 45)(24, 30)(25,111)(28, 171)
(29,178)(31, 126)(32, 87)(34, 135)(35, 65)(36, 198)
(37,103)(39, 89)(40, 159)(41, 185)(42, 78)(43, 216)
(46,160)(48, 221)(49,167)(50, 97)(51, 95)(52, 164)
(53, 219)(54, 236)(55, 213)(56, 96)(57,73)(58, 109)
(59, 152)(60, 243)(61, 218)(63, 119)(64, 188)(66, 247)
(67,127)(69, 122)(71, 128)(72, 130)(74, 145)(75, 206)
(77, 81)(79, 223)(80,212)(83, 117)(84, 177)(86, 179)
(88, 155)(90, 143)(91, 259)(92, 173)(93,222)(94, 192)

(98, 238)(100, 133)(101, 162)(102, 249)(104, 242)
(105,141)(106, 124)(107, 234)(108, 142)(110, 139)
(112,181)(113, 191)(115,168)(116, 265)(118, 174)
(120, 182)(121, 217)(125, 208)(129, 200)(132,156)
(134,245)(136, 204)(137, 209)(138, 226)(144, 194)
(146,237)(147,264)(148, 193)(149, 189)(150, 215)
(151,190)(153, 241)(154, 183)(158,257)(161, 260)
(163, 186)(165, 246)(166, 228)(169, 250)(170, 230)
(172,214)(175, 244)(176, 187)(180, 184)(195, 261)
(196, 251)(197, 199)(202,224)(203, 220)(205, 254)
(207, 229)(210, 239)(211, 266)(225, 233)(227,262)
(231, 252)(232, 258)(235, 248)(240, 253)(256, 263)

t_{11} = (1, 2)(3, 6)(4, 7)(5, 10)(8, 16)(9, 19)(11, 24)
(14,29)(15, 32)(17, 34)(18,36)(20, 40)(21, 41)(22, 45)
(23,46)(26, 50)(27, 52)(28, 54)(30, 57)(31,59)(33, 62)
(35, 66)(37, 69)(38, 71)(39, 74)(43, 81)(44, 82)(47,87)
(48,91)(49, 93)(51, 94)(53, 101)(55, 105)(56, 107)
(58,111)(60, 116)(61,118)(63, 121)(64, 72)(65, 123)
(67,127)(68, 128)(70, 133)(73, 138)(75,95)(76, 131)
(77,145)(78, 147)(80, 150)(83, 103)(84, 96)(85, 152)
(86,154)(88, 90)(89, 158)(92, 161)(97, 165)(98, 119)
(99,169)(100, 156)(102,173)(104, 176)(106, 124)
(108,136)(109, 157)(110, 153)(112, 166)(113,183)
(114,185)(115, 188)(117, 190)(120, 148)(122, 194)
(125,196)(126,178)(129, 202)(130, 193)(134, 199)
(137,186)(139, 206)(140, 207)(141,170)(142, 163)
(143,203)(144, 151)(146, 187)(149, 214)(155, 215)
(159,222)(160, 223)(162, 225)(164, 226)(167, 229)
(168, 182)(171, 232)(172,235)(174, 237)(175, 239)
(177,208)(179, 195)(180, 240)(181, 217)(184,233)
(189,245)(191, 227)(192, 241)(197, 248)(200, 249)
(201,250)(204,252)(205, 243)(209, 231)(210, 230)
(211,256)(212, 220)(213, 244)(216,257)(218, 242)
(219, 253)(221, 258)(224, 260)(228, 238)(234, 251)
(236,259)(254, 266)(255, 264)(261, 262)(263, 265).

Representation of Elements of J_1

Every Element of J_1 can be presented as a permutation of N or $L_2(11)$ in our case followed by at most four of the symmetric representations t_i's.

For Example:

Let $\beta \in J_1$, where x is a permutation on 266 letters

β = (1, 6, 55, 104, 139, 93, 172, 115, 70, 91, 263, 126, 10, 199, 264, 248, 140, 223, 12)(2, 13, 211, 259, 100, 48, 152, 5, 40, 235, 255, 52, 54,265, 213, 238, 110, 87, 4)(3, 183, 167, 128, 20, 260, 39, 103, 98, 242, 151, 136, 233, 266, 178, 60, 234, 114, 7)(8, 116, 200, 161, 219, 158, 205, 236, 157, 88, 141, 228, 118, 212, 50, 84, 221, 156, 198)(9, 69, 22, 43, 243, 244, 208, 195, 241, 49, 163, 124, 194, 14, 59, 145,117, 252, 76)(11, 130, 112, 257, 83, 108, 174, 73, 182, 71, 37, 204, 202, 253, 102, 162, 89, 190, 29)(15, 46, 94, 209, 106, 122, 119, 226,82, 64, 187, 144, 45, 81, 239, 169, 21, 66, 135)(16, 77, 137, 131, 18, 72, 147, 27, 210, 177, 201, 41, 23, 227, 192, 120, 133, 57, 44)(17, 127, 65, 34, 79, 51, 155, 63, 153, 231, 184, 173, 74, 175, 176, 150, 170, 99, 247)(19, 134, 189, 188, 166, 218, 138, 28, 232, 85, 31, 24, 168, 146, 220, 217, 164, 193, 38)(25, 56, 250, 256, 249, 75, 47, 35, 196, 203, 86, 33, 32, 105, 240, 159, 160, 191, 165)(26, 214, 207,142, 129, 225, 53, 222, 68, 36, 42, 197, 262, 58, 90, 181, 61, 206, 215)(30, 230, 180, 101, 254, 251, 179, 229, 224, 92, 95, 148, 78, 246, 96, 185, 113, 111, 132)(62, 67, 123, 125, 258, 171, 109, 107, 143, 121, 216, 186, 237, 80, 154, 97, 149, 245, 261)

$N^{\beta} = N\beta$

$1^{\beta} = 6$

$N\beta = Nt_1t_{11}$

$N\beta\, t_{11}t_1 = N$

$\Rightarrow \beta\ t_{11}t_1 \in N$

$n = xt_{11}t_1 =$ (3, 35, 17, 15, 4)(5, 9, 18, 8, 31)(6, 66, 34, 32, 7)(10, 19, 36, 16, 59)(11, 230, 169, 211,126)(12, 13, 247, 135, 79)(14, 60, 213, 176,151)(20, 160, 51, 120, 38)(21, 113, 97, 96, 201)(22, 43, 137, 124, 194)(23, 191, 58, 56, 114)(24, 210, 99, 256, 178)(25, 132, 198, 42, 246)(26, 149, 115, 187, 80)(27, 193, 147, 197, 255)(28, 157, 90, 63, 164)(29, 116, 244, 104, 144)(30, 182, 70, 48, 100)(33, 67, 123, 196, 195)(37, 119, 110, 49, 68)(39, 117, 136, 129, 92)(40, 223, 94, 148, 71)(41, 183, 165, 84, 250)(44, 64, 112, 118, 138)(45, 81, 186, 106,122)(46, 227, 111, 107, 185)(47, 105, 238, 218, 139)(50, 214, 188, 146, 150)(52, 130, 78, 248, 264)(53, 89, 83, 204,184)(54, 109, 88, 121, 226)(55, 228, 242, 206, 87)(57, 168, 133, 91, 156)(61, 73, 82, 72, 166)(62, 127, 65, 125, 179)(69, 98, 153, 93, 128)(74, 190, 108, 202, 161)(75, 155, 170, 240, 173)(76, 134, 262, 241, 209)(77, 175, 177, 258, 152)(85, 145, 239, 208, 221)(86, 167, 142, 237, 212)(95, 215, 141, 180, 102)(101, 158, 103, 252, 233)(131, 199, 261, 192, 231)(140, 224, 219, 159, 172)(143, 217, 257, 243, 251)(154, 229, 163, 174, 220)(162, 266, 236, 263, 200)(181, 216, 205, 234, 203)(207, 260, 253, 222, 235)(225, 254, 259, 265, 249)

$\Rightarrow x = nt_{11}t_1$

We need to find n. To do that we need to find the images of the symmetric generators in β

$Nt_1 = 3 \rightarrow 35$
$Nt_2 = 5 \rightarrow 9$
$Nt_3 = 9 \rightarrow 18$
$Nt_4 = 18 \rightarrow 8$
$Nt_5 = 15 \rightarrow 4$
$Nt_6 = 31 \rightarrow 5$
$Nt_7 = 35 \rightarrow 17$
$Nt_8 = 17 \rightarrow 15$
$Nt_9 = 8 \rightarrow 31$
$Nt_{10} = 4 \rightarrow 3$
$Nt_{11} = 2 \rightarrow 2$

Therefore n = (1, 7, 8, 5, 10)(2, 3, 4, 9, 6)

$\Rightarrow$ x = (1, 7, 8, 5, 10)(2, 3, 4, 9, 6)$t_{11}t_1$

Finding Permutation Representation of an Element Whose Symmetric Representation is Given

Every Element of the symmetrically presented group G represents an element of the Janko group J_1.

Let x = n $t_it_jt_k$ where n $\in$ N and i,j,k $\in$ {1,..11}

x is a permutation in J_1. To find out what is x as a permutation on 266 letters, we find the image N under x, call it $Nt_it_jt_k$. We will show the following example to illustrate the process.

Example: Let x = (1,2,3,4,5,6,7,8,9,10,11)$t_1t_2t_3$

Then the action of the element (0 1 2 3 4 5 6 7 8 9 x) on the 266 single coset is as follows:

λ = (2 3 5 9 18 15 31 35 17 8 4)(6 11 22 43 80 120 166 97 51 26 12)(7 13 27 45 81 151 182 112 58 30 14)(10 20 39 73 136 195 164 119 62 42 21)(16 23 25 48 89 138 204 179 110 63 33)(19 37 28 53 100 56 106 123 135 72 38)(24 29 55 104 175 238 229 219 156 88 47)(32 60 115 187 137 201 246 191 118 83 44)(34 64 70 132 90 49 92 160 96 124 65)(36 67 122 193 147 211 255 247 196 129 68)(40 75 141 208 233 171 101 85 46 84 76)(41 77 144 210 240 263 262 257 212 148 78)(50 94 86 153 217 241 181 111 91 159 95)(52 98 167 161 224 163 185 145 206 170 99)(54 102 172 234 237 243 188 133 107 174 103)(57 108 143 93 162 152 183 242 239 180 109)(59 113 61 117 82 79 149 213 176 186 114)(66 125 184 157 198 127 194 230 169 197 126)(69 130 71 134 200 128 199 178 105 177 131)(74 139 121 192 154 218 190 168 146 205 140)(87 116 189 244 228 165 227 216 150 215 155)(142 203 209 250 248 236 173 223 214 251 202)(158 220 231 258 266 264 256 261 226 252 221)(207 253 265 245 249 235 259 222 225 232 254)

$\Rightarrow x = \lambda\ t_1t_2t_3$

= (1 175 241 181 62 52 5 247 265 219 107 249 122 35 199 90 84 237 24)(2 45 244 65 75 214 115 246 264 129 225 29 194 230 166 149 185 74 71)(3 153 162 204 33 151 61 12 233 68 165 140 231 119 81 161 248 171 96) (4 80 221 152 234 176 184 238 142 145 20 101 60 160 56 37 34 131 202)(6 9 255 46 132 201 41 25 239 227 173 134 209 155 112 266 127 53 242)(7 10 22 66 170 254 63 240 59 213 212 50 11 116 193 31 250 229 158) (8 224 124 49 206 208 27 262 144 182 32 105 93 123 54 141 195 42 218)(13 48 26 136 95 15 205 135 72 157 126 28 79 172 177 154 150 120 179)(14 18 222 138 245 17 64 70 228 146 139 39 261 47 97 57 111 77 191)(16 38 197 113 106 89 236 251 178 118 76 98 148 78 100 121 168 99 58)(19 210 169 86 159 125 226 220 44 109 130 147 87 92 216 196 117 82 133)(21 200 137 69 73 256 51 217 23 207 102 243 174 186 30 85 94 156 253) (36 128 188 55 223 88 103 180 211 252 192 190 215 189 108 258 164 235 232)(40 43 104 259 167 114 67 143 260 183 187 263 163 198 257 83 203 110 91)

(See Appendix B(3) for magma work).

CHAPTER FIVE

SOME INTERESTING CASES

$S_7 : 2$

The progenitor $2^{*6}:S_6$ has the following presentations:

$<x,y,t | x^6, y^2, x * y * x^4 * y * x^2 * y * x^{-2} * y * x, (y * x)^5, x^2 * y * x^3 * y * x^{-3} * y * x^{-3} * y * x, t^2, (t,y), (t, y*x^4*y*x^4*y*x^3*y*x^2*y), (t,x^2 * y * x^4 * y * x^5 * y * x^2), (t, y*x^4 *y * x^3 * y * x^4), (t, y^{(x\wedge 3)}), (x * t)^6>$,

where the action of N on the symmetric generators is given by: $x \sim (1\ 2\ 3\ 4\ 5\ 6)$

$y \sim (1\ 2)$.

We factor the progenitor $2^{*6}:S_6$, as shown below, to obtain the finite homomorphic image

$$G \cong D_7 : 2 \cong \frac{2^{*6}:S_6}{(t_1t_2t_1\,(132))^2 = 1, t_3t_2 = (132)\,t_3t_1}$$

Manual Double Coset Enumeration:

Before we do the double coset enumeration of G over S_6, we note that we can write only the even permutations of S_6 in terms of the t_i's. Therefore, G will not be generated by $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ and the set $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ will generate a subgroup of G of index 2.

Consider the relation $(t_1\ t_2\ t_1\ (132))^2 = 1$.

Let $x = (1\ 3\ 2)$. Then $(t_1\ t_2\ t_1\ x)^2 = 1$

$\Rightarrow\ t_1\ t_2\ t_1\ x\ t_1\ t_2\ t_1\ x = 1$

$\Rightarrow\ t_1\ t_2\ t_1\ x^2\ (t_1\ t_2\ t_1)^x = 1$

$\Rightarrow t_1\ t_2\ t_1\ x^2\ (t_1\ t_2\ t_1)^{(1\ 3\ 2)} = 1$

$\Rightarrow t_1\ t_2\ t_1\ x^2\ t_3\ t_1\ t_3 = 1$

$\Rightarrow\ x^2\ (t_1\ t_2\ t_1)^{x\hat{}2}\ t_3\ t_1\ t_3 = 1$

$\Rightarrow\ x^2\ (t_1\ t_2\ t_1)^{(1\ 2\ 3)}\ t_3\ t_1\ t_3 = 1$

$\Rightarrow\ (1\ 2\ 3) t_2\ t_3\ t_2\ t_3\ \ t_1\ t_3\ =\ 1$

$\Rightarrow\ t_2\ t_3\ t_2\ t_3\ \ t_1\ t_3\ =\ (132)$.

Also

$(t_1\ t_2\ t_1\ x)^2 = 1$

$\Rightarrow\ t_1\ t_2\ t_1\ x\ t_1\ t_2\ t_1\ x = 1$

$\Rightarrow\ x\ t_3\ t_1\ t_3\ t_1\ t_2\ t_1 = x^{-1}$

$\Rightarrow\ t_3\ t_1\ t_3\ t_1\ t_2\ t_1 = x^{-2} = x = (\ 1\ 3\ 2)$.

Therefore,

$t_3\ t_1\ t_3\ t_1\ t_2\ t_1 = t_2\ t_3\ t_2\ t_3\ \ t_1\ t_3$.

$\Rightarrow\ t_1\ t_2 =\ \ t_3\ t_1\ t_3\ t_2\ t_3\ t_2\ t_3\ \ t_1\ t_3\ t_1$

$\Rightarrow\ Nt_1 =\ \ N\ t_1\ t_3\ t_2\ t_2\ t_3$

$=\ \ N\ t_1\ (132)\ t_3\ t_1\ t_2\ t_3$ (Since $t_3\ t_2\ = (132)\ t_3\ t_1$)

$= N\ (123)\ (132)\ t_1\ (132)\ t_3\ t_1\ t_2\ t_3$

$=\ \ N\ t_3\ t_3\ t_1\ t_2\ t_3$

$=\ \ N\ t_1\ t_2\ t_3$

$\Rightarrow Nt_1\, t_3 = N\, t_1\, t_2$

$\Rightarrow Nt_1\, t_2 = N\, t_1\, t_3 = Nt_1\, t_4 = N\, t_1\, t_5 = Nt_1\, t_6.$

Now,

$Nt_2\, t_1 = N\, t_3\, t_3\, t_2\, t_1$

$= N\, t_3\, (132)\, t_3\, t_1\, t_1$

$= N\, t_2\, t_3.$

Therefore,

$Nt_2\, t_1 = N\, t_2\, t_3 = Nt_2\, t_4 = N\, t_2\, t_5 = Nt_2\, t_6.$

Also,

$Nt_3\, t_1 = N\, t_3\, t_2\, t_2\, t_1$

$= N\, (132)\, t_3\, t_1\, t_2\, t_1$

$= N\, t_3\, t_1\, t_2\, t_1$

$\Rightarrow Nt_3\, t_1\, t_1 = N\, t_3\, t_1\, t_2$

$\Rightarrow Nt_3 = N\, t_3\, t_1\, t_2$

$\Rightarrow Nt_3\, t_2 = N\, t_3\, t_1.$

Therefore,

$Nt_3\, t_1 = N\, t_3\, t_2 = Nt_3\, t_4 = N\, t_3\, t_5 = Nt_3\, t_6.$

Similarly, we have

$Nt_4\, t_1 = N\, t_4\, t_2 = Nt_4\, t_3 = N\, t_4\, t_5 = Nt_4\, t_6$

$Nt_5\, t_1 = N\, t_5\, t_2 = Nt_5\, t_3 = N\, t_5\, t_4 = Nt_5\, t_6$

$Nt_6\, t_1 = N\, t_6\, t_2 = Nt_6\, t_3 = N\, t_6\, t_4 = Nt_6\, t_5.$

Thus the double coset $Nt_i N$ contains 6 single cosets.

Now,

$Nt_1\ t_2 = N\ t_1 t_3 = Nt_1\ t_4 = N\ t_1 t_5 = Nt_1\ t_6$

$\Rightarrow\ Nt_1\ t_i\ t_j = Nt_1$ for all i,j in {2,3,4,5,6}

and

$Nt_2\ t_1 = N\ t_2 t_3 = Nt_2\ t_4 = N\ t_2 t_5 = Nt_2\ t_6$

$\Rightarrow\ Nt_2\ t_i\ t_j = Nt_2$ for all i,j in {1,3,4,5,6}

…

Therefore $Nt_i\ t_j\ t_k N = Nt_i$ for all $i \neq j \neq k$.

Now we consider the double coset $N\ t_i\ t_j\ t_i N$ where $i \neq j$.

$N\ t_1 t_2 = N\ t_1 t_3$

$\Rightarrow N\ t_1 t_2\ t_1 = N\ t_1\ t_3\ t_1$.

Therefore, since

$Nt_1\ t_2 = N\ t_1 t_3 = Nt_1\ t_4 = N\ t_1 t_5 = Nt_1\ t_6$,

$\Rightarrow Nt_1\ t_2\ t_1 = N\ t_1 t_3 t_1 = Nt_1\ t_4 t_1 = N\ t_1 t_5 t_1 = Nt_1\ t_6 t_1$.

Also

$Nt_2\ t_1 = N\ t_2 t_3 = Nt_2\ t_4 = N\ t_2 t_5 = Nt_2\ t_6$

$\Rightarrow Nt_2\ t_1 t_2 = N\ t_2 t_3 t_2 = Nt_2\ t_4 t_2 = N\ t_2 t_5 t_2 = Nt_2\ t_6 t_2$,

and

$Nt_3\ t_1 = N\ t_3 t_2 = Nt_3\ t_4 = N\ t_3 t_5 = Nt_3\ t_6$

$\Rightarrow Nt_3\ t_1 t_3 = N\ t_3 t_2 t_3 = Nt_3\ t_4 t_3 = N\ t_3 t_5 t_3 = Nt_3\ t_6 t_3$,

again

$Nt_4\ t_1 = N\ t_4 t_2 = Nt_4\ t_3 = N\ t_4 t_5 = Nt_4\ t_6$

$\Rightarrow Nt_4\ t_1\ t_4 = N\ t_4\ t_2\ t_4 = Nt_4\ t_3\ t_4 = N\ t_4\ t_5\ t_4 = Nt_4\ t_6\ t_4,$

and

$Nt_5\ t_1 = N\ t_5\ t_2 = Nt_5\ t_3 = N\ t_5\ t_4 = Nt_5\ t_6$

$\Rightarrow Nt_5\ t_1\ t_5 = N\ t_5\ t_2\ t_5 = Nt_5\ t_3\ t_5 = N\ t_5\ t_4\ t_5 = Nt_5\ t_6\ t_5,$

finally

$Nt_6\ t_1 = N\ t_6\ t_2 = Nt_6\ t_3 = N\ t_6\ t_4 = Nt_6\ t_5$

$\Rightarrow Nt_6\ t_1\ t_6 = N\ t_6\ t_2\ t_6 = Nt_6\ t_3\ t_6 = N\ t_6\ t_4\ t_6 = Nt_6\ t_5\ t_6.$

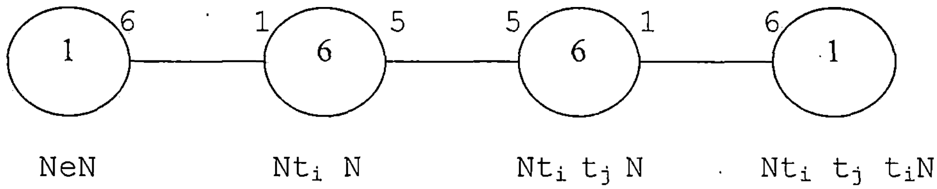
$$
\begin{aligned}
N\ t_1\ t_2\ t_1 &= N\ t_1\ t_2\ t_1\ t_3\ t_1\ t_3\ t_3\ t_1\ t_3 \\
&= N(123)\ t_3\ t_1\ t_3 \\
&= N\ t_3\ t_1\ t_3 \quad .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& Nt_1\ t_2\ t_1 = N\ t_1\ t_3\ t_1 = Nt_1\ t_4\ t_1 = N\ t_1\ t_5\ t_1 = Nt_1\ t_6\ t_1 \\
= \ & Nt_2\ t_1\ t_2 = N\ t_2\ t_3\ t_2 = Nt_2\ t_4\ t_2 = N\ t_2\ t_5\ t_2 = Nt_2\ t_6\ t_2 \\
= \ & Nt_3\ t_1\ t_3 = N\ t_3\ t_2\ t_3 = Nt_3\ t_4\ t_3 = N\ t_3\ t_5\ t_3 = Nt_3\ t_6\ t_3 \\
= \ & Nt_4\ t_1\ t_4 = N\ t_4\ t_2\ t_4 = Nt_4\ t_3\ t_4 = N\ t_4\ t_5\ t_4 = Nt_4\ t_6\ t_4 \\
= \ & Nt_5\ t_1\ t_5 = N\ t_5\ t_2\ t_5 = Nt_5\ t_3\ t_5 = N\ t_5\ t_4\ t_5 = Nt_5\ t_6\ t_5 \\
= \ & Nt_6\ t_1\ t_6 = N\ t_6\ t_2\ t_6 = Nt_6\ t_3\ t_6 = N\ t_6\ t_4\ t_6 = Nt_6\ t_5\ t_6.
\end{aligned}
$$

Therefore the double coset $N\ t_i\ t_j\ t_i\ N$ where $i \neq j$ contains 1 single coset.

The Cayley Graph of G over N is given below:



We now calculate the action of the symmetric generators on the cosets of N in G. As explained previously to compute the action of the symmetric generator t_0, we start with the identity coset N or * and multiply on the right by t_0 to get Nt_0 or 0. Then multiply again on the right by t_0, the result now is $N\ t_0\ t_0 = N$. Now start with a new single coset and repeat the process.

The action of the symmetric generators is given below:

t_0: (* 0)(1 12)(2 20)(3 30)(4 40)(5 50)(01 010)

t_1: (* 1)(0 01)(2 21)(3 31)(4 41)(5 51)(12 121)

t_2: (* 2)(0 02)(1 12)(3 32)(4 42)(5 52)(21 212)

t_3: (* 3)(0 03)(1 13)(2 23)(4 43)(5 53)(31 313)

t_4: (* 4)(0 04)(1 14)(2 24)(3 34)(5 54)(41 414)

t_5: (* 5)(0 05)(1 15)(2 25)(3 35)(4 45)(51 515)

We re-label the 14 cosets as follows:

1 → *
2 → 0
3 → 1
4 → 2
5 → 3
6 → 4
7 → 5

8 → 01
9 → 12
10 → 21
11 → 31
12 → 41
13 → 51
14 → 123

Under the above re-labeling the action of the symmetric generators becomes:

t_0 = (1 2)(3 9)(4 10)(5 11)(6 12)(7 13)(8 14)

t_1 = (1 3)(2 8)(4 10)(5 11)(6 12)(7 13)(9 14)

t_2 = (1 4)(2 8)(3 9)(5 11)(6 12)(7 13)(10 14)

t_3 = (1 5)(2 9)(3 9)(4 10)(6 12)(7 13)(11 14)

t_4 = (1 6)(2 8)(3 9)(4 10)(5 11)(7 13)(12 14)

t_5 = (1 7)(2 8)(3 9)(4 10)(5 11)(6 12)(13 14)

$5^2:D_6$

The progenitor $2^{*3}:S_3$ has the following symmetric presentation:

$<x,y,t|y^2, x^{-3}, (y * x^{-1})^2, t^2, (t,y), (y*t)^{10},$

$(y\hat{}(x*y)*t)^6, (y\hat{}(x*y)*t*t^{(x\hat{}2)})^{10}>$,

where the action of N on the symmetric generators is given by:

x ~ (1 2 3)

y ~ (1 2)

We will show that a finite homomorphic image of the above progenitor is given by:

$$G = 5^2:D_6 \cong \frac{2^{*3}:S3}{[(0,1,2)t_0]^{10}, t_it_jt_i = t_jt_it_j, t_0t_1t_2t_0t_2t_1t_0 = t_1t_0t_2t_1t_2t_0t_1}$$

(for $i,j \in \{0,1,2\}$) (see Bray[1]).

As explained in the previous section, the symmetric generator will generate a subgroup of G of index 2.

Double Coset Enumeration:

Manual double coset enumeration of G over N:

The relation $[(0,1,2)t_0]^{10} = 1$

$\Rightarrow (0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0(0,1,2)t_0 = 1$

$\Rightarrow (0,1,2)(0,1,2)\underline{(0,2,1)t_0(0,1,2)}t_0(0,1,2)(0,1,2)\underline{(0,2,1)t_0(0,1,2)}t_0(0,1,2)(0,1,2)\underline{(0,2,1)t_0(0,1,2)}t_0(0,1,2)(0,1,2)\underline{(0,2,1)t_0(0,1,2)}t_0(0,1,2)(0,1,2)\underline{(0,2,1)t_0(0,1,2)}t_0 = 1$

$\Rightarrow (0,2,1)t_1t_0(0,2,1)t_1t_0(0,2,1)t_1t_0(0,2,1)t_1t_0(0,2,1)t_1t_0 = 1$

$\Rightarrow (0,2,1)(0,2,1)\underline{(0,1,2)t_1t_0(0,2,1)}t_1t_0(0,2,1)(0,2,1)\underline{(0,1,2)t_1t_0(0,2,1)}t_1t_0(0,2,1)t_1t_0 = 1$

$\Rightarrow (0,1,2)t_0t_2t_1t_0\underline{(0,1,2)t_0t_2t_1t_0(0,2,1)t_1t_0} = 1$

$\Rightarrow (0,1,2)t_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = 1$

$\Rightarrow Nt_0t_2t_1t_0t_2t_1t_0t_2t_1t_0 = N$

Note that $Nt_0t_2t_1t_0t_2 = Nt_0t_1t_2t_0t$ (1)

The relations $t_it_jt_i = t_jt_it_j$ (for $i,j \in \{0,1,2\}$)

give: $t_1t_0t_1 = t_0t_1t_0$

$t_1t_2t_1 = t_2t_1t_2$

$t_2t_0t_2 = t_0t_2t_0$ (2)

Also from above $Nt_0t_1t_2t_0t_2t_1t_0 = Nt_1t_0t_2t_1t_2t_0t_1$ (3)

Note that the double coset [i] has three single cosets, Nt_0, Nt_1 and Nt_2, the double coset [ij] has six single, and the double coset [ijk] has six single cosets in it

Cosets as well.

Now, $Nt_0t_1t_0 = Nt_1t_0t_1$.

Therefore the double coset [iji] has three single cosets since each two have the same name.

Now,

$Nt_0t_1t_0 = Nt_1t_0t_1 \Rightarrow Nt_0t_1t_0t_2 = Nt_1t_0t_1t_2$.

Therefore the double coset [ijik] has three single cosets since every two have the same name.

The double coset [ijki] has six single cosets.

$Nt_0t_1t_2t_0t_1 = Nt_0t_2t_1t_0t_2$ (by (1))

Therefore the double coset [ijkij] have three single cosets in it since every three two have the same name.

$Nt_0t_1t_2t_0t_2 = Nt_0t_1t_0t_2t_0$ (since $t_it_jt_i = t_jt_it_j$)

$= Nt_1t_0t_1t_2t_0$.

Therefore the double coset [ijkik]=[ijiki]=[ijikj] and it contains six single cosets.

$Nt_0t_1t_2t_0t_2t_0 = Nt_0t_1t_0t_2$ (since $t_it_jt_i = t_jt_it_j$

$\Rightarrow t_it_jt_it_j = t_it_j$).

Therefore the double coset [ijkiki]= [ijik].

Now,

$Nt_0t_1t_2t_0t_2t_1 = Nt_0t_1t_0t_2t_0t_1$

$= Nt_1t_0t_1t_2t_0t_1$.

Therefore [ijkikj]=[ijikij]=[ijikji].

But $Nt_0t_1t_2t_0t_1 = Nt_0t_2t_1t_0t_2$ (by (1))

$\Rightarrow Nt_0t_1t_2t_0 = Nt_0t_2t_1t_0t_2t_1$

$\Rightarrow Nt_0t_1t_2t_0t_2t_1 = Nt_0t_2t_1t_0t_2t_1t_2t_1$

$\Rightarrow Nt_0t_1t_2t_0t_2t_1 = Nt_0t_2t_1t_0t_1t_2$ (since $t_it_jt_it_j = t_jt_i$).

Therefore the double coset [ijkikj] has three single cosets since every two cosets have the same name.

Also $Nt_0t_1t_2t_0t_2t_1t_0 = Nt_0t_2t_1t_0t_1t_2t_0$ (follow from previously)

$= Nt_0t_2t_0t_1t_0t_2t_0 = Nt_2t_0t_2t_1t_0t_2t$

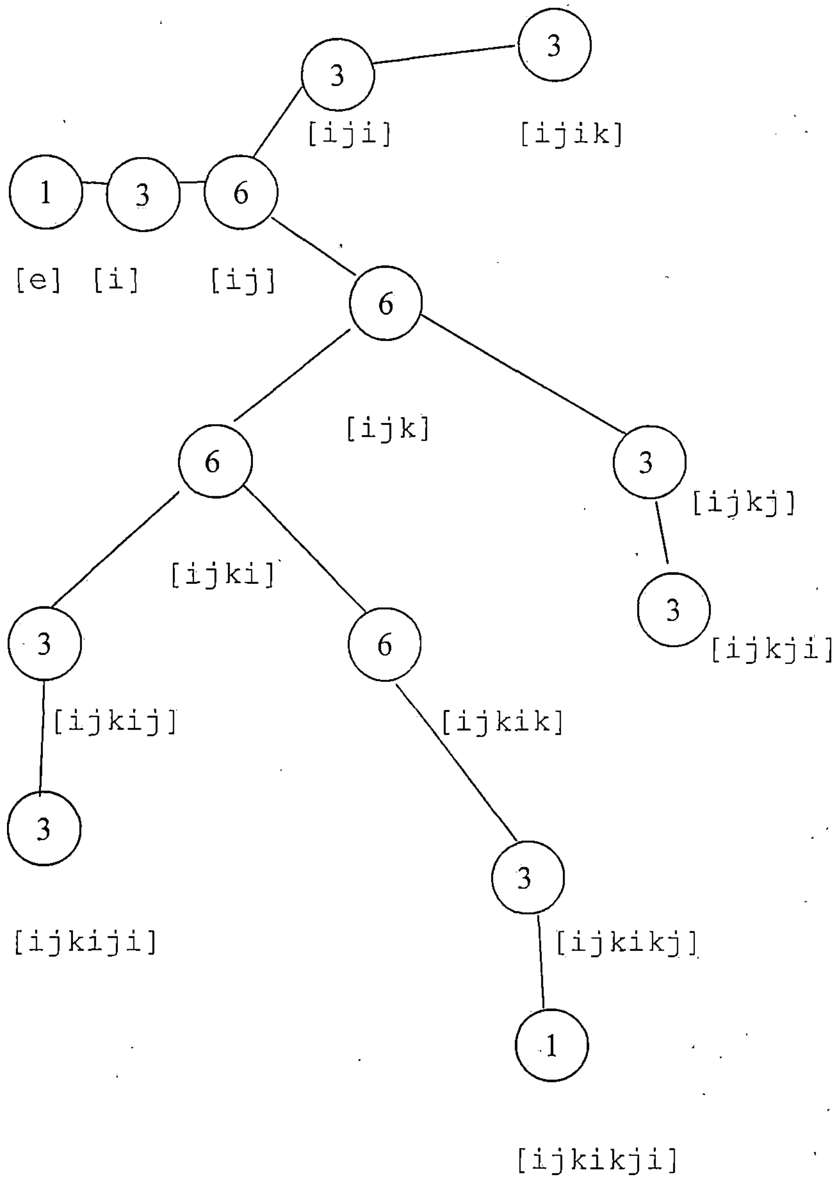
$= Nt_2t_0t_2t_1t_2t_0t_2 = Nt_2t_0t_1t_2t_1t_0t_2$.

Then $Nt_0t_1t_2t_0t_2t_1t_0 = Nt_0t_2t_1t_0t_1t_2t_0 = Nt_2t_0t_1t_2t_1t_0t_2$.

Similarly $Nt_1t_0t_2t_1t_2t_0t_1 = Nt_1t_2t_0t_1t_0t_2t_1 = Nt_2t_1t_0t_2t_0t_1t_2$.

But $Nt_0t_1t_2t_0t_2t_1t_0 = Nt_1t_0t_2t_1t_2t_0t_1$. Therefore the double coset [ijkikji] contains only one single coset since all the six single coset have the same name.

The Cayley Graph of $5^2:D_6$ over S_3:



The action of the symmetric generators is given below:

t_0: (* 0)(1 10)(2 20)(01 010)(02 020)(12 120)
(21 210)(121 1210)(0102 01020)(0201 02101)
(012 0120)(021 0210)(102 1020)(201 2010)(1021 10210)
(1201 12010)(2012 20120)(2102 21020)(01201 012010)
(10212 102120)(20121 201210)(012021 0120210)
(0121 01210)(102101 10201)(201202 20102)

t_1: (* 1)(0 01)(2 21)(02 021)(12 121)(10 101)
(20 201)(020 0201)(0102 10212)(1210 12010)
(012 0121)(102 1021)(120 1201)(210 2101)(0120 01201)
(0210 02101)(2012 20121)(2102 20120)(10210 102101)
(01202 012021)(21020 210201)(102120 1021201)
(1020 10201)(01210 012101)(20102 201202)

t_2: (* 2)(0 02)(1 12)(01 012)(10 102)(20 202)
(21 212)(010 0102)(0201 20121)(1210 21020)
(021 0212)(120 1202)(201 2012)(210 2102)(0120 01202)
(0210 02102)(1021 10212)(1201 12012)(20120 201202)
(02101 012021)(12010 102120)(210201 2102012)
(2010 20102)(01210 012010)(10201 102101)

We re-name the singles cosets below:

1 → *
2 → 0
3 → 1
4 → 10
5 → 2
6 → 20
7 → 01
8 → 010
9 → 02
10 → 020
11 → 12
12 → 120
13 → 21
14 → 210
15 → 121
16 → 1210
17 → 0102
18 → 01020
19 → 0201
20 → 02101
21 → 012
22 → 0120
23 → 021
24 → 0210
25 → 102
26 → 1020
27 → 201
28 → 2010

$29 \rightarrow 1021$
$30 \rightarrow 10210$
$31 \rightarrow 1201$
$32 \rightarrow 12010$
$33 \rightarrow 2012$
$34 \rightarrow 20120$
$35 \rightarrow 2102$
$36 \rightarrow 21020$
$37 \rightarrow 01201$
$38 \rightarrow 012010$
$39 \rightarrow 10212$
$40 \rightarrow 102120$
$41 \rightarrow 20121$
$42 \rightarrow 201210$
$43 \rightarrow 012021$
$44 \rightarrow 0120210$
$45 \rightarrow 0121$
$46 \rightarrow 01210$
$47 \rightarrow 102101$
$48 \rightarrow 10201$
$49 \rightarrow 201202$
$50 \rightarrow 20102$

Therefore the action of the symmetric generators is:

t_0 = (1 2)(3 4)(5 6)(7 8)(9 10)(11 12)(13 14)(15 16)(17 18)
(19 20)(21 22)(23 24)(25 26)(27 28)(29 30)(31 32)
(33 34)(35 36)(37 38)(39 40)(41 42)(43 44)(45 46)
(47 48)(49 50)

t_1 = (1 3)(2 7)(5 13)(9 23)(11 15)(4 8)(6 27)(10 19)(17 39)
(16 32)(21 45)(25 29)(12 31)(14 28)(22 37)(24 20)
(33 41)(35 34)(30 47)(18 43)(36 42)(40 44)(26 48)
(46 38)(50 49)

t_2 = (1 5)(2 9)(3 11)(7 21)(4 25)(6 10)(13 15)(8 17)(19 41)
(16 36)(23 45)(12 26)(27 33)(14 35)(22 18)(24 37)
(29 39)(31 30)(34 49)(20 43)(32 40)(42 44)(28 50)
(46 38)(48 47)

Symmetric Presentation for Some Groups

Here is some of the groups we considered:

The progenitor $2^{*3}:S_3$

Factored by relations	Order of the homomorphic image G	Index of S_3 in G
$[(0,1,2)t_0]^{10}, t_i t_j t_i = t_j t_i t_j$, $t_0 t_1 t_2 t_0 t_2 t_1 t_0 = t_1 t_0 t_2 t_1 t_2 t_0 t_1$	300	50
$(t_1 t_2)^2 = t_0 t_1 t_2 t_0$, $t_i t_j t_k = t_k t_j t_i$,	192	34
$[(1,2)t_0]^{10}, [(0,1)t_0]^6, [(0,1)t_0 t_2]^{10}$	900	150
$[(0,1,2)t_0]^{10}, [(0,2)t_0]^{10}, [(0,2)t_0 t_2]^{10}$, $[(0,1,2)t_0 t_1 t_0]^{10}$	249600	41600

As well as the progenitor $2^{*6}:S_6$

Factored by relations	Order of the homomorphic image G	Index of S_6 in G
$(t_1 t_2 t_1 t_2\ (1,5))^2, (t_1 t_2 t_1\ (1,5))^2$	92160	128
$(t_1 t_2 t_1\ (132))^2 = 1$, $t_3 t_2 = (132)\ t_3 t_1$	10080	14
$((1,2,3,4,5,6)t_0)^6$	23040	32

Also the progenitor $2^{*7}:L_2(7)$

Factored by relation(s)	Order of the homomorphic image G	Index of $L_2(7)$ in G
$(t_6\ t_7)^2$	43008	256
$(t_6\ t_7\ (1,2,4)(3,6,5))^3$	2688	16
$(t_6\ t_7\ (1,2,3,4,5,6,7))^3$, $((1,2,3,4,5,6,7)\ t_1)^8$	40320	240
$((t_2\ t_6)^2(2,5)(4,6))^2$	22020096	131072
$(t_1 t_0)^2$, $(t_1t_0(2,5)(4,6))^2$, $(t_0t_3(2,6)(4,5))^2$, $((2,6)(4,5)\ t_2)^4$	21504	128
$((1,2,3,4,5,6,7)\ t_1)^6$	15120	90

And finally the Progenitor 2^{*11}: $L_2(11)$

Factored by relation(s)	Order of the homomorphic image G	Index of 2^{*11}: $L_2(11)$ in G
$(t_{11}t_{10}(2,6)(3,7)(4,10)(8,9))^4$	1351680	2048
$(t_0t_1(1,5)(3,8)(4,10)(7,9))^3$	351120	532

CHAPTER SIX

SYMMETRIC REPRESENTATION OF ELEMENTS OF $U_3(3): 2$

Introduction

In this chapter we will show that every element of $U_3(3): 2$ can be written as a permutation on 14 letters followed by a word, in the symmetric generators t_i's, of length at most two.

A presentation for the progenitor $2^{*(7+7)}:PGL_2(7)$ is given by:

$<x,y,t,s \mid x^7, y^2, t^2, (x^{-1} * t)^2, (y * x)^3, t * x^{-1} * y * x * t * y, x^2 * y * x^3 * y * x^{-4} * y * x^{-4} * y * x, s^2, (s^3,y),(s^4, x*y)>$,

where $N= PGL_2(7) \cong <x,y,t \mid x^7, y^2, t^2, (x^{-1} * t)^2, (y * x)^3, t * x^{-1} * y* x * t * y, x^2 * y * x^3 * y * x^{-4} * y * x^{-4} * y * x>$;

and the action of N on the symmetric generators is given by

x ~ (1 2 3 4 5 6 7)(14 13 12 11 10 9 8)

y ~ (2 6)(4 5)(14 10)(12 13)

t ~ (7 14)(1 8)(2 9)(3 10)(4 11)(5 12)(6 13)

We factor the progenitor by the following relations $t=s_7s_{14}s_7$, $y = (s_8s_7)^2$, $y = s_3s_8s_1s_7$ to get the finite homomorphic image:

$$G \cong \frac{2^{*(7+7)} :PGL_2(7)}{t = s_7 s_{14} s_7, y = (s_8 s_7)^2, y = s_3 s_8 s_1 s_7}.$$

The index of $PGL_2(7)$ in G is 36. $G \cong U_3(3):2$, The Unitary Group,(see Abdul Jabbar[3]).

Double Coset Enumeration

Manual double cosets enumeration of $U_3(3):2$ over $PGL_2(7)$:

$t = s_7 s_{14} s_7$

$\Rightarrow (7,14)(1,8)(2,9)(3,10)(4,11)(5,12)(6,13) = s_7\, s_{14}\, s_7$

$\Rightarrow N(7,14)(1,8)(2,9)(3,10)(4,11)(5,12)(6,13) = N s_7\, s_{14}\, s_7$

$\Rightarrow N\ s_7\ s_{14} = N\ s_7.$

$y = (s_8 s_7)^2$

$\Rightarrow (2,6)(4,5)(14,10)(12,13) = (s_8\, s_7)^2$

$\Rightarrow N\ (2,6)(4,5)(14,10)(12,13) = s_8\, s_7\, s_8\, s_7$

$\Rightarrow N\ s_7\ s_8 = N\ s_8\ s_7.$

$y = s_3\ s_8\ s_1\ s_7$

$\Rightarrow (2,6)(4,5)(14,10)(12,13) = s_3\ s_8\ s_1\ s_7$

$\Rightarrow N(2,6)(4,5)(14,10)(12,13) = N\ s_3\ s_8\ s_1\ s_7$

$\Rightarrow N\ s_7\ s_1 = N\ s_3\ s_8.$

Now,

$(2,6)(4,5)(14,10)(12,13) = s_3\ s_8\ s_1\ s_7$

$\Rightarrow (2,6)(4,5)(14,10)(12,13) s_7\ s_1 = s_3\ s_8$

$\Rightarrow ((2,6)(4,5)(14,10)(12,13) s_7\ s_1)^{(2,4)(3,7)(9,12)(11,13)}$

$= (s_3\ s_8)^{(2,\ 4)(3,\ 7)(9,\ 12)(11,\ 13)}$

$\Rightarrow\ (2\ 5)(4\ 6)(9\ 11)(10\ 14)\ s_3\ s_1 = s_7\ s_8$

$\Rightarrow\ (2\ 5)(4\ 6)(9\ 11)(10\ 14)\ s_3\ s_1 (2\ 5)(4\ 6)(9\ 11)(10\ 14)\ s_3\ s_1$

$= s_7\ s_8 s_7\ s_8$

$\Rightarrow s_3\ s_1\ s_3\ s_1 = y$

$\Rightarrow Ns_3\ s_1 = Ns_1\ s_3.$

But $N(s_3\ s_1)^{\pi} = N\ s_7\ s_1$, where $\pi = (3\ 7)(5\ 6)(9\ 13)(11\ 12)$

$\Rightarrow Ns_3\ s_1 \in [7\ 1] = Ns_7\ s_1N$

$\Rightarrow Ns_is_j = Ns_js_i$ for every single coset Ns_is_j in the double coset $[7\ 1] = Ns_7\ s_1N$. (1)

1 7 ~ 7 1 (by 1).

Also 1 7 = 7 13 13 1

~ 7 13 1

(since $t = s_7s_{14}s_7$ then $t^{(1\ 3)(4\ 5)(10\ 12)(13\ 14)}$

$= (s_7s_{14}s_7)^{(1\ 3)(4\ 5)(10\ 12)(13\ 14)}$

$\Rightarrow (1\ 12)(2\ 9)(3\ 8)(4\ 10)(5\ 11)(6\ 14)(7\ 13) = s_7s_{13}s_7$

$\Rightarrow\ N\ s_7\ s_{13} = Ns_7$)

= 7 13 1 13 13

= 7 (1 13)(2 14)(3 8)(4 9)(5 10)(6 11)(7 12) 13

(since $t = s_7s_{14}s\ t^{\pi} = (s_7s_{14}s_7)^{\pi}$,

where $\pi = (1\ 14)(2\ 8)(3\ 9)(4\ 10)(5\ 11)(6\ 12)(7\ 13)$

$\Rightarrow\ (1\ 13)(2\ 14)(3\ 8)(4\ 9)(5\ 10)(6\ 11)(7\ 12) = s_{13}s_1s_{13}$)

$=(1\ 13)(2\ 14)(3\ 8)(4\ 9)(5\ 10)(6\ 11)(7\ 12)7^{(1\ 13)(2\ 14)(3\ 8)(4\ 9)(5\ 10)(6\ 11)(7\ 12)}12$

$= (1\ 13)(2\ 14)(3\ 8)(4\ 9)(5\ 10)(6\ 11)(7\ 12)12\ 13$

$\sim 12\ 13$

$\sim 13\ 12$

$\sim 8\ 3$ (since $y = s_3\ s_8\ s_1\ s_7$

$\Rightarrow N\ s_7\ s_1 = Ns_3\ s_8$)

$\sim 3\ 8$.

But $8\ 3 \in [7\ 8]$ (since $N(s_8\ s_3)^{\pi} = Ns_7\ s_8$,

where $\pi=(1\ 9\ 5\ 12)(2\ 13\ 3\ 8\ 7\ 11\ 4\ 14)(6\ 10) \in N$).

Therefore $[7,8]= [7,1]$.

Now,

$7\ 1\ 2 \sim 13\ 12\ 2$ (since $Ns_7\ s_1 = Ns_{13}\ s_{12}$)

$= 13\ 12\ 2\ 12\ 12$

$\sim 13(1\ 11)(2\ 12)(3\ 13)(4\ 14)(5\ 8)(6\ 9)(7\ 10)12$

(since $t^{(1\ 13)(2\ 14)(3\ 8)(4\ 9)(5\ 10)(6\ 11)(7\ 12)}$

$= (s_7 s_{14} s_7)^{(1\ 13)(2\ 14)(3\ 8)(4\ 9)(5\ 10)(6\ 11)(7\ 12)}$

so $(1\ 11)(2\ 12)(3\ 13)(4\ 14)(5\ 8)(6\ 9)(7\ 10)= s_{12}s_2s_{12}$)

$\sim\ 3\ 12 \in [7\ 8]$

(Since $\exists$ $n= (1\ 4\ 2)(3\ 6\ 7)(8\ 12\ 9)(10\ 11\ 13)$ in N such that $[7,8]^n = [3,12]$)

$\Rightarrow\ 7\ 1\ 4 \sim 3\ 8$ (since $[7,1,2]^{\wedge(2\ 4)(5\ 6)(9\ 11)(12\ 13)} = [7,1,2]$), 7 1 5 ~ 3 8 ($[7, 1, 2]^{(2\ 5)(4\ 6)(9\ 11)(10\ 14)} = [7, 1, 5]$),

and 7 1 6 ~ 3 8 ($[7,1,2]^{(2\ 6)(4\ 5)(10\ 14)(12\ 13)} = [7, 1, 6]$)

Also,

7 1 9 ~ 8 3 9 (since $Ns_7\, s_1 = Ns_8\, s_3$)

= 8 3 9 3 3

= 8 (1 11)(2 13)(3 9)(4 10)(5 14)(6 12)(7 8)3

(since $t^n = (s_7 s_{14} s_7)^n$

where n = (1 7 3)(2 6 4)(9 12 14)(10 11 13)

then (1 11)(2 13)(3 9)(4 10)(5 14)(6 12)(7 8)= $s_3 s_9 s_3$)

~ 7 3 ∈ [7 1] (Since ∃ n = (1 3)(4 5)(10 12)(13 14)

in N such that $[7,1]^n = [7,3]$)

⇒ 7 1 11 ~ 7 3 (since $[7,1,9]^{\wedge\ (2\ 5)(4\ 6)(9\ 11)(10\ 14)} =$ [7,1,11]).

Also,

7 1 10 ~ 8 3 10 (since $Ns_7\, s_1 = Ns_8\, s_3$)

~ 8 (1 8)(2 11)(3 10)(4 9)(5 13)(6 12)(7 14)3

~ 1 3 ∈ [7 1]

(Since ∃ n = (1 3 7)(2 4 6)(9 14 12)(10 13 11) in N

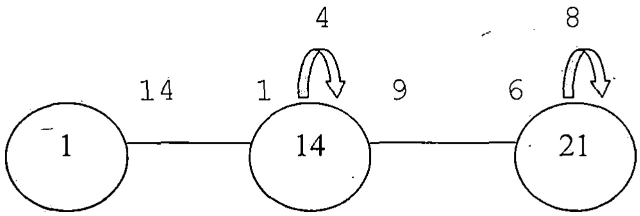
such that $[7,1]^n = [1,3]$)

⇒ 7 1 14 ~ 1 3 (since $[7,1,10]^{(2\ 6)(4\ 5)(10\ 14)(12\ 13)}$

= [7,1,14]).

The double cosets [w] = N w N in G.		
Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of Cosets
[*]	Since N is transitive on T	1
[7]	$N^{(7)} = N^{7}$ = < y,(1 4 3 5)(2 6)(8 11)(10 14 12 13),(1 2 3 6)(4 5)(8 9)(10 14 13 12), (2 5)(4 6)(9 11)(10 14)> ~ S_4 has orbits {7},{8,9,11}, {10,12,13,14} and {1,2,3,4,5,6} on T.	14
[14 10]=[14]		
	7 8 ~ 8 7 ~ 1 3 ~ 3 1 ~ 9 11 ~ 11 9 (Note that [3 8] = [7 8])	
[7 8] = [7 1]	for i in {1,3,7,8,9,11}	21
[7 8 i] = [7]	for j in {2,4,5,6,10,12,13,14,15,16,17,18,19,20,21, 22,23,24,25,26,27,28,29,30,31,32,33,34,35, 36}	
[7 8 j] = [7 8]		

The Cayley graph of $U_3(3):2$ over $PGL_2(7)$



At this point we can re-name our 36 single cosets and calculate the action of the symmetric generators, s_1, $s_2, \ldots, s_{36}$, on the cosets(See Appendix C for magma work).

```
1     []
2     [ 7 ]
3     [ 1 ]
4     [ 14 ]
5     [ 6 ]
6     [ 2 ]
7     [ 8 ]
8     [ 7, 1 ]
9     [ 13 ]
10    [ 10 ]
11    [ 6, 7 ]
12    [ 5 ]
13    [ 3 ]
14    [ 9 ]
15    [ 2, 7 ]
16    [ 1, 3 ]
17    [ 11, 12 ]
18    [ 6, 5 ]
19    [ 12 ]
20    [ 11 ]
21    [ 4 ]
22    [ 14, 12 ]
23    [ 10, 14 ]
24    [ 11, 8 ]
25    [ 9, 14 ]
26    [ 1, 6 ]
27    [ 10, 8 ]
28    [ 10, 11 ]
29    [ 4, 5 ]
```

30 [14, 11]
31 [10, 13]
32 [8, 13]
33 [9, 13]
34 [9, 12]
35 [9, 10]
36 [8, 12].

The action on the symmetric generators is given below:

s_1 = (1 3)(2 8)(4 27)(5 26)(6 17)(7 23)(10 18)(11 25)(12 34)(13 16)(15 35)(21 33)(22 30)(24 29)(28 31)(32 36)

s_2 = (1 6)(2 15)(3 17)(4 33)(5 24)(8 32)(9 25)(11 14)(12 36)(13 28)(16 29)(18 23)(21 27)(22 30)(26 31)(34 35)

s_3 = (1 13)(2 23)(3 16)(5 30)(6 28)(7 8)(9 36)(11 33)(12 25)(15 34)(17 22)(18 27)(19 32)(21 35)(24 29)(26 31)

s_4 = (1 21)(2 31)(3 33)(4 17)(5 32)(6 27)(8 36)(11 25)(12 29)(13 35)(15 34)(16 24)(18 23)(19 30)(20 22)(26 28)

s_5 = (1 12)(2 22)(3 34)(5 18)(6 36)(8 32)(9 28)(10 26)(11 33)(13 25)(15 35)(16 24)(17 30)(20 31)(21 29)(23 27)

s_6 = (1 5)(2 11)(3 26)(6 24)(8 36)(10 34)(12 18)(13 30)(14 15)(16 29)(17 22)(19 35)(21 32)(23 27)(25 33)(28 31)

s_7 = (1 2)(3 8)(5 11)(6 15)(7 16)(12 22)(13 23)(14 24)(17 30)(18 27)(20 29)(21 31)(25 33)(26 28)(32 36)(34 35)

s_8 = (1 7)(2 16)(3 23)(4 18)(8 13)(9 32)(10 27)(11 15)(14 29)(17 33)(19 36)(20 24)(22 31)(25 28)(26 34)(30 35)

s_9 = (1 14)(2 24)(4 25)(5 15)(6 11)(7 29)(8 23)(9 33)(10 35)(16 20)(17 27)(18 26)(19 34)(22 31)(28 36)(30 32)

s_{10} = (1 10)(3 18)(4 23)(5 34)(7 27)(8 16)(9 31)(11 24)(12 26)(14 35)(15 19)(17 33)(20 28)(22 29)(25 36)(30 32)

s_{11} = (1 20)(2 29)(4 30)(7 24)(8 23)(9 26)(10 28)(11 15)(12 31)(14 16)(17 19)(18 34)(21 22)(25 36)(27 33)(32 35)

s_{12} = (1 19)(4 22)(5 35)(7 36)(8 9)(10 15)(11 24)(13 32)(14 34)(16 23)(17 20)(18 26)(21 30)(25 28)(27 33)(29 31)

s_{13} = (1 9)(4 11)(6 25)(7 32)(8 19)(10 31)(12 28)(13 36)(14 33)(15 24)(16 23)(17 27)(18 34)(20 26)(22 29)(30 35)

s_{14} = (1 4)(3 27)(6 33)(7 18)(8 16)(9 11)(10 23)(14 25)(15 24)(17 21)(19 22)(20 30)(26 34)(28 36)(29 31)(32 35).

Representation of Elements of $U_3(3)$:2

Now every element g of G can be represented as g = πw, where π is a permutation of $PGL_2(7)$, on 14 letters, and w is a product of at most two s_i's.

We give two examples to represent elements of $U_3(3)$:2 given as a permutations on 36 letters in the form πw, where π is a permutation of $PGL_2(7)$, on 14 letters, and w is a product of at most two s_i's.

Examples:

(1) Let α = (1 16)(2 7)(3 13)(4 18 10 27)(5 15 6 11)(8 23)(9 32 19 36)(12 31 21 22)(14 20)(17 35 26 25)(24 29)(28 34 30 33) $\in$ $U_3(3)$: 2

Then

α = πw, where π=(1 3)(4 5)(10 12)(13 14)

and w = $s_{11}s_9$

(see Appendix C for magma work).

(2) Let α = (1 29)(2 20)(3 8 13 23)(4 17 19 30)(5 11 6 15)(7 14)(9 28 10 26)(12 21)(16 24)(18 35 36 33)(22 31)(25 32 34 27) $\in U_3(3):2$

It turns out that

α = (1 2)(3 6)(8 9)(10 13)$s_5 s_4$

(see Appendix C).

CHAPTER SEVEN

SYMMETRIC REPRESENTATION FOR J_2:2 ELEMENTS

Introduction

In this chapter we will discuss how each element of the group J_2:2 can be represented as a permutation of $U_3(3)$:2 followed by a word of length at most two in terms of the symmetric generators s_i's.

A presentation for the progenitor $2^{*36}:(U_3(3):2)$ is given by:

$<x,y,t,s|x^7,\ y^2,\ (x*y)^3,\ (x,y)^4,\ t^2,\ (t^{(x\char`^3)},y),$

$(t^{(x\char`^4)},x*y)>\ (t^{(x\char`^4)},x*y),\ y*(t^x * t^{(x\char`^3)})^2,\ s^2,$

$(s,x),(s,y),(s,(t*x)^4 * x^2 * t)>,$

where the action of x,y,t on the symmetric generators is given by:

x ~(2 3 5 9 13 8 4)(6 11 18 27 21 12 7)
(10 16 20 29 23 14 17)(15 24 28 19 22 31 25)
(26 33 36 35 30 34 32)

y ~(4 5)(7 10)(8 13)(11 17)(12 20)(14 22)(18 19)
(23 28)(24 31)(27 29)(33 34)(35 36)

t ~ (1 2)(3 6)(4 7)(5 10)(8 14)(9 15)(11 19)
(12 20)(13 22)(16 26)(17 18)(21 30)(23 28)
(24 29)(25 32)(27 31)

In order to obtain a finite homomorphic image of the progenitor, we factor it by a relation determined as follows. By Curtis' Lemma,

$$C_N(<s_1,\ s_2>) = <t>.$$

This implies $t = s_1s_2s_1$. So we factor the progenitor by the relation $t = s_1s_2s_1$.

The index of G in the progenitor turns out to be 100.

Also $G \cong J_2{:}2$, where J_2 is the Hall-Janko group (see Abdul Jabbar [3]).

Double Coset Enumeration

Manual Double Coset enumeration of $J_2{:}2$:

The relator $s_1s_2s_1 = t$

$\Rightarrow s_1s_2s_1 = (1\ 2)(3\ 6)(4\ 7)(5\ 10)(8\ 14)(9\ 15)(11\ 19)$
$(12\ 20)(13\ 22)(16\ 26)(17\ 18)(21\ 30)(23\ 28)$
$(24\ 29)(25\ 32)(27\ 31)$.

Note $N\ s_1s_2 = N\ s_1$.

Now,

$s_1s_6 = s_1s_6s_2s_6s_6s_2$

$= s_1\ (1\ 3)(2\ 6)(4\ 24)(5\ 31)(7\ 18)(8\ 11)(9\ 16)(10\ 19)$
$(12\ 33)(13\ 17)(14\ 27)(15\ 26)(20\ 34)(22\ 29)(23\ 28)$
$(30\ 32)s_6s_1$

$\sim s_3s_6s_2$

In addition,

$s_1s_6 = s_1s_6s_9s_6s_6s_9$

$= s_1(1\ 26)(2\ 16)(3\ 15)(4\ 13)(5\ 8)(6\ 9)(7\ 14)(10\ 22)$

$(11\ 17)(12\ 34)(20\ 33)(21\ 30)(23\ 36)(24\ 31)(25\ 32)$

$(28\ 35)s_6s_9$

$\sim s_{26}\ s_6\ s_9$

$\sim s_{26}\ s_6\ s_{26}\ s_{26}\ s_9$

$= (1\ 9)(2\ 15)(3\ 16)(4\ 27)(5\ 29)(6\ 26)(7\ 17)(8\ 18)$

$(10\ 11)(13\ 19)(14\ 31)(22\ 24)(23\ 35)(28\ 36)(30\ 32)$

$(33\ 34)\ s_{26}\ s_9$

$\sim s_{26}\ s_9.$

Again,

$s_1s_6 = s_1s_6s_{26}s_6s_6s_{26}$

$= s_1\ (1\ 9)(2\ 15)(3\ 16)(4\ 27)(5\ 29)(6\ 26)(7\ 17)(8\ 18)$

$(10\ 11)(13\ 19)(14\ 31)(22\ 24)(23\ 35)(28\ 36)(30\ 32)$

$(33\ 34)s_6s_{26}$

$\sim s_9s_6s_{26}$

$= s_9s_6s_9s_9s_{26}$

$= (1\ 26)(2\ 16)(3\ 15)(4\ 13)(5\ 8)(6\ 9)(7\ 14)(10\ 22)$

$(11\ 17)(12\ 34)(20\ 33)(21\ 30)(23\ 36)(24\ 31)(25\ 32)$

$(28\ 35)s_9s_{26}$

$\sim s_9s_{26}.$

Moreover,

$$
\begin{aligned}
s_1 s_6 &= s_1 s_6 s_{35} s_6 s_6 s_{35} \\
&= s_1\ (1\ 36)(4\ 13)(5\ 18)(6\ 35)(7\ 34)(8\ 29)(9\ 28) \\
&\quad (10\ 21)(11\ 12)(14\ 25)(15\ 16)(17\ 32)(20\ 31)(22\ 33) \\
&\quad (23\ 26)(24\ 30)\ s_6 s_{35} \\
&\sim s_{36} s_6 s_{35} \\
&= s_{36} s_6 s_{36} s_{36} s_{35} \\
&= (1\ 35)(4\ 19)(5\ 8)(6\ 36)(7\ 21)(9\ 23)(10\ 33) \\
&\quad (11\ 32)(12\ 24)(13\ 27)(14\ 34)(15\ 16)(17\ 20)(22\ 25) \\
&\quad (26\ 28)(30\ 31)\ s_{36} s_{35} \\
&\sim s_{36} s_{35}.
\end{aligned}
$$

And

$$
\begin{aligned}
s_1 s_6 &= s_1 s_6 s_{36} s_6 s_6 s_{36} \\
&= s_1 (1\ 35)(4\ 19)(5\ 8)(6\ 36)(7\ 21)(9\ 23)(10\ 33) \\
&\quad (11\ 32)(12\ 24)(13\ 27)(14\ 34)(15\ 16)(17\ 20)(22\ 25) \\
&\quad (26\ 28)(30\ 31) s_6 s_{36} \\
&\sim s_{35} s_6 s_{36} \\
&= s_{35} s_6 s_{35} s_{35} s_{36} \\
&= (1\ 36)(4\ 13)(5\ 18)(6\ 35)(7\ 34)(8\ 29)(9\ 28) \\
&\quad (10\ 21)(11\ 12)(14\ 25)(15\ 16)(17\ 32)(20\ 31)(22\ 33) \\
&\quad (23\ 26)(24\ 30) s_{35} s_{36} \\
&\sim s_{35} s_{36}.
\end{aligned}
$$

Moreover,

$s_{15}s_{16} = s_{15}s_{16}s_2s_{16}s_{16}s_2$

$= s_{15}$ (1 26)(2 16)(3 15)(4 5)(6 9)(8 13)(11 31)

(12 34)(17 24)(18 27)(19 29)(20 33)(21 32)(23 35)

(25 30)(28 36) s_{16} s_2

$\sim s_3\ s_{16}\ s_2$

$= s_3\ s_{16}\ s_3\ s_3\ s_2$

= (1 9)(2 15)(3 16)(4 18)(5 19)(6 26)(7 31)(8 27)

(10 24)(11 22)(13 29)(14 17)(21 25)(23 36)(28 35)

(33 34) s_3 s_2

$\sim s_3\ s_2$.

Also,

$s_{23}s_{28} = s_{23}s_{28}s_9s_{28}s_{28}s_9$

$= s_{23}$ (1 36)(2 3)(4 13)(5 29)(6 35)(7 32)(8 18)

(9 28)(10 20)(11 25)(12 14)(17 34)(21 31)(22 30)

(23 26)(24 33) s_{28} s_9

$\sim s_{26}\ s_{28}\ s_9$

$= s_{26}\ s_{28}\ s_{26}\ s_{26}\ s_9$

= (1 35)(2 3)(4 19)(6 36)(7 20)(9 23)(10 30)(11 34)

(12 22)(13 27)(14 32)(17 21)(18 29)(24 25)(26 28)

(31 33) s_{26} s_9

$\sim s_{26}\ s_9$.

Thus we have 1 6 ~ 6 1 ~ 2 3 ~ 3 2 ~ 9 26 ~ 26 9 ~ 35 36 ~ 36 35 ~ 15 16 ~ 16 15 ~ 23 28 ~ 28 23.

Now,

$s_1\ s_6\ s_4 \sim s_6\ s_1\ s_4$ (since $N\ s_1 s_6 = N s_6 s_1$)

$= s_6\ s_1\ s_4\ s_1\ s_4$

$= s_6$ (1 4)(2 7)(3 17)(5 25)(6 28)(8 12)(9 19)

(10 32)(11 14)(13 23)(15 20)(16 21)(18 22)

(24 29)(27 35)(31 34) s_4

$\sim s_{28}\ s_4$

$\Rightarrow\ s_1\ s_6\ s_5 \sim s_{23}\ s_5$

(Since $(s_1\ s_6\ s_4)^{\pi} \sim (s_{28}\ s_4)^{\pi} \Rightarrow s_1\ s_6\ s_5 \sim s_{23}\ s_5$

Where π = (4 5)(7 10)(8 13)(11 17)(12 20)(14 22)(18 19)

(23 28)(24 31)(27 29)(33 34)(35 36)

$\Rightarrow\ s_1\ s_6\ s_8 \sim s_{23}\ s_8$

(Since $(s_1\ s_6\ s_4)^{\pi} \sim (s_{28}\ s_4)^{\pi} \Rightarrow s_1\ s_6\ s_8 \sim s_{23}\ s_8$

Where π = (4 8)(5 13)(7 14)(10 22)(11 24)(17 31)(18 27)

(19 29)(21 25)(23 28)(30 32)(35 36))

$\Rightarrow\ s_1\ s_6\ s_{13} \sim s_{28}\ s_{13}$

(Since $(s_1\ s_6\ s_4)^{\pi} \sim (s_{28}\ s_4)^{\pi} \Rightarrow s_1\ s_6\ s_{13} \sim s_{28}\ s_{13}$

Where π = (4 13)(5 8)(7 22)(10 14)(11 31)(12 20)(17 24)

(18 29)(19 27)(21 25)(30 32)(33 34))

$\Rightarrow\ s_1\ s_6\ s_{30} \sim s_{15}\ s_{30}$

(Since $(s_1\ s_6\ s_4)^{\pi} \sim (s_{28}\ s_4)^{\pi} \Rightarrow s_1\ s_6\ s_{30} \sim s_{15}\ s_{30}$

Where π = (2 35)(3 36)(4 30)(5 33)(7 11)(8 34)(9 26)

(10 14)(12 18)(13 32)(15 28)(16 23)(19 25)(20 29)

(21 27)(22 31))

$\Rightarrow$ $s_1\ s_6\ s_{32} \sim s_{15}\ s_{32}$

(Since $(s_1\ s_6\ s_4)^{\pi} \sim (s_{28}\ s_4)^{\pi} \Rightarrow s_1\ s_6\ s_{32} \sim s_{15}\ s_{32}$

Where π = (2 35)(3 36)(4 32)(5 34)(7 31)(8 33)(9 26)

(11 22)(12 29)(13 30)(15 28)(16 23)(17 24)(18 20)

(19 21)(25 27))

$\Rightarrow$ $s_1\ s_6\ s_{33} \sim s_{16}\ s_{33}$

(Since $(s_1\ s_6\ s_4)^{\pi} \sim (s_{28}\ s_4)^{\pi} \Rightarrow s_1\ s_6\ s_{33} \sim s_{16}\ s_{33}$

Where π = (2 36)(3 35)(4 33)(5 32)(8 30)(9 26)(10 24)

(11 31)(12 19)(13 34)(14 17)(15 23)(16 28)(18 21)

(20 27)(25 29))

$\Rightarrow$ $s_1\ s_6\ s_{34} \sim s_{16}\ s_{34}$

(Since $(s_1\ s_6\ s_4)^{\pi} \sim (s_{28}\ s_4)^{\pi} \Rightarrow s_1\ s_6\ s_{34} \sim s_{16}\ s_{34}$

Where π = (2 36)(3 35)(4 34)(5 30)(7 22)(8 32)(9 26)

(10 17)(12 27)(13 33)(14 24)(15 23)(16 28)(18 25)

(19 20)(21 29))

Also,

$s_1\ s_6\ s_7 \sim s_3\ s_2\ s_7$

$= s_3\ s_2\ s_7\ s_2\ s_2$

$= s_3$ (1 4)(2 7)(3 23)(5 25)(6 18)(8 19)(9 12)

(10 32)(11 15)(13 17)(14 20)(22 28)(24 29)

$(26\ 30)(27\ 34)(31\ 35)\ s_2$

$\sim s_{23}\ s_2$

$\Rightarrow\ s_1\ s_6\ s_{10} \sim s_{28}\ s_2$

(Since $(s_1\ s_6\ s_7)^{\pi} \sim (s_{23}\ s_2)^{\pi} \Rightarrow s_1\ s_6\ s_{10} \sim s_{28}\ s_2$

Where $\pi = (4\ 5)(7\ 10)(8\ 13)(11\ 17)(12\ 20)(14\ 22)(18\ 19)$
$(23\ 28)(24\ 31)(27\ 29)(33\ 34)(35\ 36))$

$\Rightarrow\ s_1\ s_6\ s_{11} \sim s_{16}\ s_{35}$

(Since $(s_1\ s_6\ s_7)^{\pi} \sim (s_{23}\ s_2)^{\pi} \Rightarrow s_1\ s_6\ s_{11} \sim s_{16}\ s_{35}$

Where $\pi = (2\ 35)(3\ 36)(4\ 30)(5\ 33)(7\ 11)(8\ 34)(9\ 26)$
$(10\ 14)(12\ 18)(13\ 32)(15\ 28)(16\ 23)(19\ 25)(20\ 29)$
$(21\ 27)(22\ 31))$

$\Rightarrow\ s_1\ s_6\ s_{14} \sim s_{28}\ s_2$

(Since $(s_1\ s_6\ s_7)^{\pi} \sim (s_{23}\ s_2)^{\pi} \Rightarrow s_1\ s_6\ s_{14} \sim s_{28}\ s_2$

Where $\pi = (4\ 8)(5\ 13)(7\ 14)(10\ 22)(11\ 24)(17\ 31)(18\ 27)$
$(19\ 29)(21\ 25)(23\ 28)(30\ 32)(35\ 36))$

$\Rightarrow\ s_1\ s_6\ s_{17} \sim s_{23}\ s_3$

(Since $(s_1\ s_6\ s_7)^{\pi} \sim (s_{23}\ s_2)^{\pi} \Rightarrow s_1\ s_6\ s_{17} \sim s_{23}\ s_3$

Where $\pi = (2\ 3)(5\ 8)(7\ 17)(10\ 31)(11\ 14)(12\ 25)(15\ 16)$
$(18\ 29)(20\ 21)(22\ 24)(30\ 33)(32\ 34))$

$\Rightarrow\ s_1\ s_6\ s_{22} \sim s_{23}\ s_2$

(Since $(s_1\ s_6\ s_7)^{\pi} \sim (s_{23}\ s_2)^{\pi} \Rightarrow s_1\ s_6\ s_{22} \sim s_{23}\ s_2$

Where π = (4 13)(5 8)(7 22)(10 14)(11 31)(12 20)(17 24)(18 29)(19 27)(21 25)(30 32)(33 34))

$\Rightarrow$ $s_1\ s_6\ s_{24} \sim s_{32}\ s_3$

(Since $(s_1\ s_6\ s_7)^{\pi} \sim (s_{23}\ s_2)^{\pi} \Rightarrow s_1\ s_6\ s_{24} \sim s_{32}\ s_3$

Where π = (2 3)(4 13)(7 24)(10 11)(12 21)(14 31)(15 16)(17 22)(19 27)(20 25)(30 34)(32 33))

$\Rightarrow$ $s_1\ s_6\ s_{31} \sim s_{16}\ s_{35}$

(Since $(s_1\ s_6\ s_7)^{\pi} \sim (s_{23}\ s_2)^{\pi} \Rightarrow s_1\ s_6\ s_{31} \sim s_{16}\ s_{35}$

Where π = (2 35)(3 36)(4 32)(5 34)(7 31)(8 33)(9 26)(11 22)(12 29)(13 30)(15 28)(16 23)(17 24)(18 20)(19 21)(25 27))

Moreover,

$s_1\ s_6\ s_{12} = s_1\ s_6\ s_{12}\ s_6\ s_6$

$= s_1$ (1 15)(2 33)(4 25)(5 23)(6 12)(7 14)(8 21)(9 34)(10 29)(11 35)(13 28)(16 20)(17 32)(19 22)(24 36)(30 31) s_6

$\sim s_{15}\ s_6$

$\Rightarrow$ $s_1\ s_6\ s_{18} \sim s_{28}\ s_6$

(Since $(s_1\ s_6\ s_{12})^{\pi} \sim (s_{15}\ s_6)^{\pi} \Rightarrow s_1\ s_6\ s_{18} \sim s_{28}\ s_6$

Where π = (2 35)(3 36)(4 30)(5 33)(7 11)(8 34)(9 26)(10 14)(12 18)(13 32)(15 28)(16 23)(19 25)(20 29)(21 27)(22 31))

$\Rightarrow \quad s_1 \; s_6 \; s_{19} \sim s_{23} \; s_6$

(Since $(s_1 \; s_6 \; s_{12})^{\pi} \sim (s_{15} \; s_6)^{\pi} \Rightarrow s_1 \; s_6 \; s_{19} \sim s_{23} \; s_6$

Where π = (2 36)(3 35)(4 33)(5 32)(8 30)(9 26)(10 24)(11 31)(12 19)(13 34)(14 17)(15 23)(16 28)(18 21)(20 27)(25 29))

$\Rightarrow \quad s_1 \; s_6 \; s_{20} \sim s_{15} \; s_6$

(Since $(s_1 \; s_6 \; s_{12})^{\pi} \sim (s_{15} \; s_6)^{\pi} \Rightarrow s_1 \; s_6 \; s_{20} \sim s_{15} \; s_6$

Where π = (4 5)(7 10)(8 13)(11 17)(12 20)(14 22)(18 19)(23 28)(24 31)(27 29)(33 34)(35 36))

$\Rightarrow \quad s_1 \; s_6 \; s_{21} \sim s_{16} \; s_6$

(Since $(s_1 \; s_6 \; s_{12})^{\pi} \sim (s_{15} \; s_6)^{\pi} \Rightarrow s_1 \; s_6 \; s_{21} \sim s_{16} \; s_6$

Where π = (2 3)(4 13)(7 24)(10 11)(12 21)(14 31)(15 16)(17 22)(19 27)(20 25)(30 34)(32 33))

$\Rightarrow \quad s_1 \; s_6 \; s_{25} \sim s_{16} \; s_6$

(Since $(s_1 \; s_6 \; s_{12})^{\pi} \sim (s_{15} \; s_6)^{\pi} \Rightarrow s_1 \; s_6 \; s_{25} \sim s_{16} \; s_6$

Where π = (2 3)(5 8)(7 17)(10 31)(11 14)(12 25)(15 16)(18 29)(20 21)(22 24)(30 33)(32 34))

$\Rightarrow \quad s_1 \; s_6 \; s_{27} \sim s_{23} \; s_6$

(Since $(s_1 \; s_6 \; s_{12})^{\pi} \sim (s_{15} \; s_6)^{\pi} \Rightarrow s_1 \; s_6 \; s_{27} \sim s_{23} \; s_6$

Where π = (2 36)(3 35)(4 34)(5 30)(7 22)(8 32)(9 26)(10 17)(12 27)(13 33)(14 24)(15 23)(16 28)(18 25)(19 20)(21 29))

$\Rightarrow$ $s_1\ s_6\ s_{29} \sim s_{28}\ s_6$

(Since $(s_1\ s_6\ s_{12})^{\pi} \sim (s_{15}\ s_6)^{\pi} \Rightarrow s_1\ s_6\ s_{29} \sim s_{28}\ s_6$

Where π = (2 35)(3 36)(4 32)(5 34)(7 31)(8 33)(9 26)

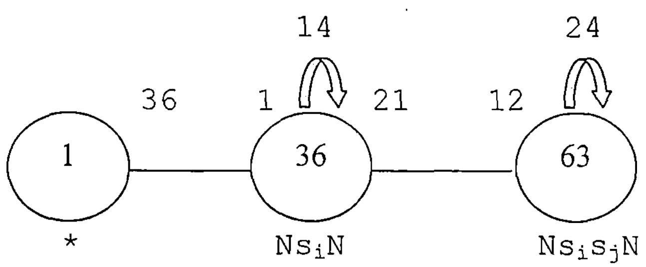
(11 22)(12 29)(13 30)(15 28)(16 23)(17 24)(18 20)

(19 21)(25 27))

The double cosets [w] = N w N in G.		
Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of Cosets
[*]	N	1
[1]	transitive on the remaining 35 points $N^{(1)} \sim PGL_2(7)$, of order 168 x 2 = 336 has orbits {1},{2, 3, 4, 5, 8, 9, 13, 26, 30, 32, 33, 34, 35, 36 }, and {6, 7, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 27, 28, 29, 31 }	36
[1 2]=[1]	and 1 6 ~ 6 1 ~ 2 3 ~ 3 2 ~ 9 26 ~ 26 9 ~ 35 36 ~ 36 35 ~ 15 16 ~ 16 15 ~ 23 28 ~ 28 23.	

[1 6]	Thus for i in	63
{1,2,3,6,9,15,16,23,26,28,35,36}		
[1 6 i] = [1]		
	and for j in	
	{4,5,7,8,10,11,12,13,14,17,18,19,20,21,22,	
	24,25,27,29,30,31,32,33,34}	
[1 6 j]=[1 6]		

The Cayley graph will be:



It is clear from the Cayley graph above of J_2:2 over $U_3(3)$:2 that every element of J_2:2 can be written as a permutation of $U_3(3)$:2, on 36 letters, followed by a word, in the symmetric generators s_i's, of length at most two.

The action of the symmetric generators on the 100 cosets of J_2:2 over $U_3(3)$:2 is given below

(see also Appendix D):

S_1 = (1, 2)(7, 13)(8, 14)(11, 20)(12, 24)(15, 25)(17, 29)(18, 32)(19, 35)(21, 36)(22, 38)(23, 39)(26, 42)(27, 44)(28, 48)(30, 49)(31, 50)(33, 51)(37, 55)(40, 58)(41, 57)(43, 62)(45, 61)(47, 67)(54, 56)(59, 76)(60, 77)(63, 84)(64, 85)(68, 78)(69, 83)(71, 74)(72, 75)(73, 79)(80, 86)(81, 90)(82, 92)(87, 95)(88, 91)(89, 93)(94, 99)(96, 98)(97, 100)

s_2 = (1, 3)(4, 13)(5, 14)(6, 20)(9, 29)(10, 32)(12, 57)(15, 62)(16, 48)(21, 56)(22, 54)(23, 41)(24, 39)(25, 42)(26, 43)(30, 78)(31, 83)(34, 35)(36, 38)(37, 79)(40, 68)(44, 46)(45, 69)(47, 73)(49, 58)(50, 61)(51, 52)(55, 67)(59, 84)(60, 63)(64, 85)(71, 81)(72, 82)(74, 80)(75, 95)(76, 77)(86, 90)(87,92)(88, 94)(89, 93)(91, 98)(96, 99)(97, 100)

s_3 = (1, 4)(3, 13)(5, 36)(6, 24)(8, 84)(9, 67)(10, 35)(11, 74)(14, 61)(16, 50)(17,80)(20, 55)(22, 76)(23, 86)(25, 65)(27, 93)(28, 59)(29, 39)(30, 68)(32,34)(33, 89)(37, 71)(38, 48)(40, 78)(41, 81)(42, 53)(43, 62)(44,51)(45, 63)(49, 58)(54, 60)(56, 83)(57, 73)(64, 72)(69, 77)(75, 91)(79, 90)(82,97)(85, 88)(87, 95)(92, 99)(94, 100)(96, 98)

s_4 = (1, 5)(3, 14)(4, 36)(6, 51)(7, 77)(9, 25)(10, 39)(12, 72)(13, 58)(16, 49)(17,75)(18, 95)(19, 64)(20, 52)(22, 59)(24, 29)(26, 87)(27, 85)(28, 76)(31, 69)(32, 42)(35, 44)(38, 48)(40, 60)(41, 62)(43, 82)(45, 83)(50, 61)(54, 63)(55, 66)(56, 78)(57, 92)(65, 67)(68, 84)(71, 74)(73, 79)(80, 88)(81,96)(86, 89)(90, 100)(91, 93)(94, 99)(97, 98)

S_5 = (1, 6)(3, 20)(4, 24)(5, 51)(7, 90)(9, 58)(10, 38)(13, 49)(14, 52)(15, 96)(16,42)(17, 86)(18, 98)(19, 85)(21, 88)(23, 80)(25, 32)(27, 64)(28, 91)(29,39)(30, 81)(35, 44)(36, 48)(37, 73)(41, 71)(43, 54)(47, 79)(50,53)(55, 67)(56, 99)(57, 68)(59, 72)(60, 87)(61, 70)(62,94)(63, 84)(69, 83)(74,78)(75, 93)(76, 89)(77, 97)(82, 92)(95, 100)

S_6 = (1, 7)(2, 13)(5, 77)(6, 90)(8, 54)(9, 81)(11, 41)(12, 71)(14, 83)(16, 60)(17,79)(18, 35)(19, 32)(20, 73)(21, 63)(24, 80)(25, 42)(28, 69)(29, 57)(30,58)(31, 76)(36, 59)(38, 61)(39, 55)(40, 49)(43, 65)(44, 51)(46,89)(47, 86)(48, 56)(50, 84)(52, 93)(53, 62)(64, 96)(67,74)(68, 78)(72, 82)(75,91)(85, 87)(88, 94)(92, 99)(95,100)(97, 98)

S_7 = (1, 8)(2, 14)(4, 84)(7, 54)(9, 92)(10, 82)(11, 51)(13, 78)(15, 72)(16, 63)(17,43)(18, 57)(20, 33)(21, 60)(23, 87)(24, 42)(25, 95)(28, 68)(29, 41)(30,59)(31, 61)(32, 62)(34, 85)(35, 44)(36, 76)(38, 58)(39, 75)(45,50)(46, 64)(48, 56)(49, 77)(55, 67)(65, 79)(66, 73)(69, 83)(71, 93)(74, 97)(80,88)(81, 96)(86, 90)(89, 99)(91, 98)(94, 100)

S_8 = (1, 9)(3, 29)(4, 67)(5, 25)(6, 58)(7, 81)(8, 92)(10, 61)(11, 71)(12, 41)(13,49)(14, 50)(16, 44)(18, 87)(19, 97)(20, 55)(23, 57)(24, 39)(26, 95)(30,90)(31, 82)(32, 42)(33, 100)(35, 51)(36, 65)(37, 74)(38, 70)(43, 72)(46,48)(54, 56)(59, 76)(60, 85)(62, 69)(63, 93)(64, 96)(68, 73)(75, 83)(77,98)(78, 80)(79, 86)(84, 94)(88, 91)(89, 99)

S_9 = (1, 10)(3, 32)(4, 35)(5, 39)(6, 38)(8, 82)(9, 61)(11, 94)(12, 75)(13, 34)(14,50)(15, 43)(16, 55)(17, 72)(20, 67)(21, 91)(24, 29)(25, 42)(26, 62)(27,89)(28, 88)(31, 92)(33, 93)(36, 48)(41, 69)(44, 51)(47, 99)(49,66)(54, 79)(56, 96)(57, 87)(58, 70)(59, 64)(60, 77)(63,71)(68, 78)(73, 98)(74,97)(76, 86)(80, 85)(81, 90)(83, 95)(84, 100)

S_{10} = (1, 11)(2, 20)(4, 74)(7, 41)(8, 51)(9, 71)(10, 94)(12, 81)(13, 68)(14, 33)(16,99)(17, 78)(18, 56)(22,96)(24, 86)(25, 36)(26, 88)(28, 62)(29, 57)(32,43)(34, 64)(35, 44)(37, 67)(38, 91)(39, 49)(40, 80)(42, 98)(46, 85)(47,55)(48, 54)(50, 61)(53, 69)(58, 90)(59, 72)(60, 87)(63, 93)(70, 83)(73,79)(75, 95)(76, 77)(82, 97)(84, 100)(89, 92)

S_{11} = (1, 12)(2, 24)(3, 57)(5, 72)(7, 71)(9, 41)(10, 75)(11, 81)(13, 78)(14, 61)(17,39)(19, 59)(20, 79)(21, 93)(22, 64)(23, 29)(25, 32)(26, 50)(31, 42)(33, 91)(34,96)(35, 89)(36, 76)(38, 88)(40, 73)(43, 82)(44, 48)(47,68)(49, 55)(51, 85)(52, 63)(54, 60)(56, 99)(58, 90)(62,69)(65, 98)(67, 74)(70,84)(77, 97)(80, 86)(83, 95)(87, 92)(94, 100)

S_{12} = (1, 15)(2, 25)(3, 62)(6, 96)(8, 72)(10, 43)(13, 35)(14, 83)(16, 98)(17,82)(18, 42)(20, 55)(21, 67)(23, 69)(24, 50)(26, 32)(27, 81)(29, 57)(30,64)(33, 77)(36, 47)(38, 48)(39, 75)(40, 97)(41, 45)(44, 100)(46, 59)(49,60)(51, 85)(52, 74)(54, 79)(56, 99)(58, 90)(61, 92)(63, 93)(66, 71)(68,84)(70, 76)(73, 91)(78, 80)(86, 89)(87, 95)(88, 94)

S_{13} = (1, 16)(3, 48)(4, 50)(5, 49)(6, 42)(7, 60)(8, 63)(9, 44)(10, 55)(11, 99)(13,58)(14, 61)(15, 98)(18, 96)(19, 100)(20, 67)(21, 54)(22, 56)(24, 53)(25,32)(29, 46)(33, 97)(35, 51)(36, 38)(39, 66)(40, 77)(41, 57)(43, 79)(45,84)(47, 94)(59, 68)(62, 88)(64, 81)(69, 76)(71, 93)(72, 75)(73, 91)(74,82)(78, 83)(80, 86)(85, 87)(89, 92)(90, 95)

S_{14} = (1, 17)(2, 29)(4, 80)(5, 75)(6, 86)(7, 79)(8, 43)(10, 72)(11, 78)(12, 39)(13,68)(14, 69)(15, 82)(18, 83)(20, 73)(23, 24)(25, 87)(27, 48)(28, 44)(32,62)(34, 100)(35, 51)(36, 38)(40, 74)(41, 57)(42, 50)(45, 95)(47, 90)(49,55)(52, 97)(54, 65)(56, 70)(58, 81)(59, 64)(60, 63)(61, 92)(67, 71)(76,89)(77, 98)(84, 94)(85, 88)(91, 93)(96, 99)

S_{15} = (1 18)(2 32)(5 95)(6 98)(7 35)(8 57)(9 87)(11 56)(13 19)(14 69)(15 42)(16 96)(17 83)(20 79)(22 99)(23 92)(24 50)(25 26)(28 73)(29 41)(36 67)(37 91)(38 88)(39 72)(43 62)(44 51)(45 75)(46 93)(48 54)(49 58)(52 89)(55 94)(59 84)(60 85)(61 82)(63 71)(64 81)(66 68)(70 78)(74 80)(76 86)(77 97)(90 100)

S_{16} = (1 19)(2 35)(5 64)(6 85)(7 32)(9 97)(12 59)(13 18)(14 20)(16 100)(21 80)(22 72)(23 88)(24 86)(25 42)(27 51)(29 48)(31 74)(33 44)(36, 76)(37, 82)(38, 91)(39, 75)(41, 69)(43, 53)(45, 94)(47 84)(49 58)(50 63)(54 79)(55 99)(56 83)(57 73)(60 87)(61 92)(62 65)(66 78)(67 71)(68 70)(77 98)(81 96)(89 93)(90 95)

S_{17} = (1 21)(2 36)(3 56)(6 88)(7 63)(8 60)(10 91)(12 93)(13 68)(14 69)(15 67)(16 54)(19 80)(20 55)(22 48)(23 85)(24 86)(25 47)(28 38)(29 44)(30 83)(31 78)(32 42)(33 75)(34 87)(35 89)(39 72)(41 81)(43 79)(49 77)(50 84)(51 64)(52 71)(53 95)(57 92)(58 61)(59 76)(62 94)(66 74)(73 98)(82 97)(90 100)(96 99)

S_{18} = (1 22)(2 38)(3 54)(4 76)(5 59)(11 96)(12 64)(13 49)(14 50)(16 56)(18 99)(19 72)(20 73)(21 48)(23 89)(24 80)

(25 67)(26 79)(28 36)(29 44)(32 62)(33 86)(34 81)(35 93)(37 43)(39 75)(40 61)(41 71)(42 98)(45 58)(51 85)(52 92)(53 82)(55 94)(57 87)(60 63)(66 90)(68 84)(69 77)(74 97)(78 83)(88 91)(95 100)

S_{19} = (1 23)(2 39)(3 41)(4 86)(6 80)(8 87)(9 57)(12 29)(13 58)(14 83)(15 69)(17 24)(18 92)(19 88)(20 67)(21 85)(22 89)(25 95)(30 55)(32 43)(33 76)(34 60)(35 93)(36 59)(37 49)(38 91)(42 50)(44 48)(45 62)(51 64)(52 99)(54 63)(56 96)(61 82)(65 94)(68 73)(70 77)(71 81)(72 75)(74 78)(79 90)(84 100)(97 98)

S_{20} = (1 26)(2 42)(3 43)(5 87)(9 95)(10 62)(11 88)(12 50)(13 35)(14 61)(15 32)(18 25)(20 73)(22 79)(24 31)(27 60)(28 94)(29 39)(30 100)(33 90)(36 67)(37 54)(38 91)(40 85)(41 69)(44 97)(46 80)(48 56)(49 77)(51 64)(52 84)(55 99)(57 92)(58 81)(59 68)(63 70)(66 86)(71 93)(72 82)(74 78)(75 83)(76 89)(96 98)

S_{21} = (1 27)(2 44)(4 93)(5 85)(6 64)(10 89)(13 32)(14 20)(15 81)(17 48)(19 51)(24 39)(25 95)(26 60)(28 29)(30 96)(31 99)(33 35)(36 38)(37 63)(40 87)(41 66)(42 98)(43 79)(45 71)(47 92)(49 77)(50 84)(53 57)(54 70)(55 94)(56 65)(58 90)(59 72)(61 82)(62 69)(67 74)(68 73)(75 91)(76 86)(78 83)(80 88)(97 100)

S_{22} = (1 28)(2 48)(4 59)(5 76)(6 91)(7 69)(8 68)(10 88)(11 62)(13 78)(14 83)(17 44)(18 73)(20 79)(21 38)(22 36)(24 39)(25 67)(26 94)(27 29)(30 84)(31 77)(32 43)(34 97)(35 51)(37 98)(41 53)(42 96)(49 60)(50 63)(52 100)(54 56)(55 99)(57 66)(58 61)(64 72)(71 81)(74 82)(75 93)(80 85)(86 89)(87 92)(90 95)

S_{23} = (1 30)(2 49)(3 78)(4 68)(6 81)(7 58)(8 59)(9 90)(13 40)(14 69)(15 64)(20 67)(21 83)(23 55)(24 29)(25 87)(26 100)(27 96)(28 84)(31 56)(32, 35)(33, 95)(36, 76)(37, 39)(38, 61)(41, 71)(42, 98)(43 82)(44 97)(46 72)(48 54)(50 63)(51 85)(52 94)(53 75)(57 73)(60 77)(62 88)(65 99)(74 80)(79 86)(89 92)(91 93)

S_{24} = (1 31)(2 50)(3 83)(5 69)(7 76)(8 61)(9 82)(10 92)(12 42)(13 68)(14 45)(19 74)(20 51)(21 78)(24 26)(25 32)(27 99)(28 77)(29 39)(30 56)(34 98)(35 93)(36 59)(37 97)(38 58)(41 62)(43 72)(44 100)(46 86)(47 89)(48 54)(49 60)(55 94)(57 87)(63 84)(64 81)(65 96)(66 80)(67

71)(73 91)(75 95)(79 90)(85 88)

S_{25} = (1 33)(2 51)(4 89)(8 20)(9 100)(10 93)(11 14)(12 91)(13 32)(15 77)(16 97)(19 44)(21 75)(22 86)(23 76)(24 80)(25 87)(26 90)(27 35)(29 48)(30 95)(36 59)(38 88)(39 72)(40 98)(41 62)(42 96)(43 54)(49 60)(50 61)(53 83)(55 67)(56 78)(57 68)(58 81)(63 71)(64 85)(65 73)(66 79)(69 70)(74 82)(84 94)(92 99)

S_{26} = (1 34)(3 35)(4 32)(8 85)(10 13)(11 64)(12 96)(14 20)(17 100)(21 87)(22 81)(23 60)(24 50)(25 53)(28 97)(29 48)(31 98)(36 67)(37 77)(38 61)(39 55)(41 71)(42 65)(43 62)(44 52)(45 90)(46 51)(47 95)(49 70)(54 63)(56 99)(57 92)(58 66)(59 72)(68 78)(69 76)(73 91)(74 82)(75 83)(79 86)(80 88)(84 94)(89 93)

S_{27} = (1 37)(2 55)(3 79)(4 71)(6 73)(9 74)(11 67)(13 58)(14 51)(18 91)(19 82)(20 47)(22 43)(23 49)(24 29)(25 36)(26 54)(27 63)(28 98)(30 39)(31 97)(32 62)(34 77)(35 89)(38 88)(41 81)(42 96)(44 100)(45 93)(46 75)(48 56)(50 84)(53 72)(57 68)(59 64)(60 70)(61 92)(69 76)(78 80)(83 95)(85 87)(86 90)(94 99)

S_{28} = (1 40)(2 58)(3 68)(4 78)(5 60)(7 49)(11 80)(12 73)(13 30)(14 50)(15 97)(16 77)(17 74)(20 79)(22 61)(24 86)(25 95)(26 85)(27 87)(29 41)(32 35)(33 98)(36 48)(38 45)(39 55)(42 96)(43 72)(44 100)(46 88)(47 57)(51 64)(52 82)(53 92)(54 63)(56 83)(59 84)(62 94)(65 91)(67 71)(69 76)(75 93)(81 90)(89 99)

S_{29} = (1 45)(2 61)(3 69)(4 63)(5 83)(8 50)(13 49)(14 31)(15 41)(16 84)(17 95)(18 75)(19 94)(20 51)(22 58)(23 62)(24 42)(25 87)(27 71)(29 57)(32 43)(34 90)(35 89)(36 48)(37 93)(38 40)(39 72)(44 97)(46 91)(47 100)(54 60)(55 99)(56 78)(59 68)(64 96)(65 88)(66 81)(67 74)(73 98)(76 77)(79 86)(80 85)(82 92)

S_{30} = (1 46)(3 44)(7 89)(8 64)(9 48)(11 85)(13 32)(14 20)(15 59)(16 29)(18 93)(24 66)(25 67)(26 80)(30 72)(31 86)(34 51)(35 52)(36 70)(37 75)(38 65)(39 53)(40 88)(41 57)(42 50)(43 82)(45 91)(47 76)(49 55)(54 56)(58 61)(60 87)(62 94)(63 71)(68 84)(69 77)(73 98)(74 78)(79 90)(81 96)(83 95)(92 99)(97 100)

S_{31} = (1 47)(2 67)(3 73)(6 79)(7 86)(10 99)(11 55)(12 68)(13

78)(14 51)(15 36)(16 94)(17 90)(19 84)(20 37)(21 25)(24 80)(27 92)(29 41)(31 89)(32 42)(34 95)(35 93)(38 48)(39 49)(40 57)(43 54)(44 97)(45 100)(46 76)(50 63)(53 87)(56 96)(58 81)(59 70)(60 85)(61 82)(62 88)(64 72)(69 77)(71 74)(75 83)(91 98)

S_{32} = (1 52)(3 51)(5 20)(6 14)(7 93)(12 63)(13 32)(15 74)(17 97)(18 89)(21 71)(22 92)(23 99)(24 42)(25 36)(26 84)(28 100)(29 48)(30 94)(34 44)(35 46)(38 58)(39 49)(40 82)(41 81)(43 72)(50 70)(53 61)(54 60)(55 65)(56 96)(57 87)(59 68)(62 88)(64 85)(66 67)(69 83)(73 79)(75 91)(76 86)(77 98)(78 80)(90 95)

S_{33} = (1 53)(4 42)(6 50)(7 62)(11 69)(13 35)(14 70)(16 24)(19 43)(20 51)(21 95)(22 82)(25 34)(27 57)(28 41)(29 66)(30 75)(32 65)(33 83)(36 67)(37 72)(38 58)(39 46)(40 92)(44 48)(47 87)(49 55)(52 61)(54 79)(56 78)(59 64)(60 85)(63 84)(68 73)(71 81)(74 97)(76 77)(80 86)(88 94)(89 99)(90 100)(91 93)(96 98)

S_{34} = (1 65)(4 25)(5 67)(7 43)(8 79)(9 36)(12 98)(13 35)(14 51)(17 54)(19 62)(20 66)(23 94)(24 50)(27 56)(29 44)(30 99)(31 96)(32 53)(33 73)(34 42)(38 46)(39 49)(40 91)(41 69)(45 88)(48 70)(52 55)(57 68)(58 61)(59 76)(60 63)(64 81)(71 74)(72 82)(75 93)(77 97)(78 83)(80 85)(84 100)(86 90)(87 95)(89 92)

S_{35} = (1 66)(5 55)(8 73)(10 49)(13 70)(14 51)(15 71)(16 39)(18 68)(19 78)(20 65)(21 74)(22 90)(24 46)(25 36)(26 86)(27 41)(28 57)(29 53)(31 80)(32 35)(33 79)(34 58)(38 61)(42 50)(43 54)(44 48)(45 81)(52 67)(56 83)(59 84)(60 77)(62 69)(63 93)(64 96)(72 75)(76 89)(82 97)(85 88)(87 92)(91 98)(94 99)(95 100)

S_{36} = (1 70)(6 61)(9 38)(10 58)(11 83)(12 84)(13 66)(14 53)(15 76)(17 56)(18 78)(19 68)(20 51)(23 77)(24 42)(25 67)(26 63)(27 54)(29 44)(32 35)(33 69)(34 49)(36 46)(37 60)(39 55)(41 62)(43 79)(47 59)(48 65)(50 52)(57 73)(64 72)(71 93)(74 80)(75 95)(81 90)(82 92)(85 87)(86 89)(88 91)(94 100)(96 99)(97 98)

Representation of Elements of $J_2:2$

As mentioned above, every element of G can be written as πw, where π is a permutation of $U_3(3):2$ and w is a product of at most two of the symmetric generators s_i's, which we refer to as symmetric representation of elements. Given a permutation representation, on 100 letters, of an element of $J_2:2$, we use the following algorithm to find a symmetric representation of the element. We give an example to explain the process.

Let $\alpha \in J_2:2$,

where, α = (1 39 24 9 20 4 58 49)(3 13 55 29)(5 46 48 61 52 50 32 34)(6 67)(7 30 23 17 57 74 71 68)(8 93 96 87 33 59 88 97)(10 16 53 44 25 36 35 70)(11 80 79 81)(12 41 90 47 86 78 40 37)(14 38 65 42)(15 64 76 22 99 31 82 98)(18 77 75 28 43 84 92 54)(19 60 72 27 83 89 100 69)(21 91 63 45 94 26 95 85)(56 62).

Now $N^{\alpha} = N\alpha$

$\Rightarrow 1^{\alpha} = 39$ (since N or * is labeled as 1)

$\Rightarrow N\alpha = Ns_1\, s_{19}$ (since t_2 is labeled as 6)

$\Rightarrow N\, \alpha\, s_{19}\, s_1 = N$

$\Rightarrow \alpha\, s_{19}\, s_1 = n$

$\Rightarrow \alpha$ = (1 19 11 2 28 23)(3 6 27 14 8 31)(4 30 21 7 16 26)(5 10)(9 13 33 22 15 36)(12 25 17 18 32 20)(24 29 34) $s_1\, s_{19}$,

where (1 19 11 2 28 23)(3 6 27 14 8 31)(4 30 21 7 16 26)(5

10)(9 13 33 22 15 36)(12 25 17 18 32 20)(24 29 34)

$\in U_3(3)$ (see Appendix D).

CHAPTER EIGHT

THE GROUP $G_2(4){:}2$

Introduction

In this chapter we will perform a manual double coset enumeration of the group $G_2(4){:}2$, Chevalley Group, over the Janko group $J_2{:}2$.

A symmetric presentation for the progenitor $2^{100}{:}(J_2{:}2)$ is given by :

$$<x,y,t,s,u \mid x^7,\ y^2,\ (x*y)^3, (x,y)^4, t^2,\ (t^{(x\wedge 3)},y),\ (t\wedge(x\wedge 4),x*y), y*(t^x * t^{(x\wedge 3)})^2, s^2,\ (s,x),\ (s,y),\ s,(t*x)^4*x^2*t),\ s*s^t*s=t,\ u^2,\ (u,x), (u,y),\ (u,t)>.$$

and the action of N on the symmetric generators is as follows:

x ~ (3 4 6 10 16 9 5)(7 12 22 37 27 15 8)(11 19 26 45 30 17 21)(13 24 38 55 44 25 14)(18 31 40 23 28 47 33)(20 35 42 61 49 29 36)(32 50 58 39 48 67 51)(34 53 70 66 46 65 52)(41 59 79 93 96 82 60)(43 63 81 72 54 71 64)(56 74 85 62 84 90 75)(57 76 73 89 98 92 77)(68 86 91 99 97 87 69)(78 80 88 94 100 95 83)

y ~ (5 6)(8 11)(9 16)(12 21)(14 20)(15 26)(17 28)

(22 23)(24 36)(25 42)(29 48)(30 40)(31 47)(37 45)(38 39)(41 54)(43 62)(49 58)(50 67)(53 65)(55 61)(56 57)(59 80)(60 81)(63 71)(64 85)(66 70)(68 78)(69 79)(72 88)(73 83)(74 84)(75 91)(76 86)(77 90)(82 94)(87 96)(92 99)(95 98)(97 100)

t ~ (2 3)(4 7)(5 8)(6 11)(9 17)(10 18)(12 23)(15 26)(16 28)(19 34)(21 22)(24 41)(25 43)(27 46)(30 40)(31 45)(33 52)(36 54)(37 47)(38 56)(39 57)(42 62)(49 68)(50 69)(55 73)(58 78)(59 60)(61 83)(63 76)(64 85)(67 79)(71 86)(72 87)(74 90)(75 92)(77 84)(80 81)(82 95)(88 96)(89 93)(91 99)(94 98)(97 100)

s ~ (1 2)(7 13)(8 14)(11 20)(12 24)(15 25)(17 29)(18 32)(19 35)(21 36)(22 38)(23 39)(26 42)(27 44)(28 48)(30 49)(31 50)(33 51)(37 55)(40 58)(41 57)(43 62)(45 61)(47 67)(54 56)(59 76)(60 77)(63 84)(64 85)(68 78)(69 83)(71 74)(72 75)(73 79)(80 86)(81 90)(82 92)(8 95)(88 91)(89 93)(94 99)(96 98)(97 100)

We factor the progenitor by the relation $s = u_1 u_2 u_1$ to obtain the homomorphic image

$$G \cong \frac{2^{*100} :(J_2 : 2)}{s = u_1 u_2 u_1}$$

The index of $J_2:2$ in G is 416 and $G \cong G_2(4):2$ (see Abdul Jabbar [3])

Double Coset Enumeration

Manual double coset enumeration of G over N:

$S = u_1u_2u_1$

(1 2)(7 13)(8 14)(11 20)(12 24)(15 25)(17 29)(18 32)(19 35)(21 36)(22 38)(23 39)(26 42)(27 44)(28 48)(30 49)(31 50)(33 51)(37 55)(40 58)(41 57)(43 62)(45 61)(47 67)(54 56)(59 76)(60 77)(63 84)(64 85)(68 78)(69 83)(71 74)(72 75)(73 79)(80 86)(81 90)(82 92)(8 95)(88 91)(89 93)(94 99)(96 98)(97 100)

$= u_1u_2u_1$.

Note that this means that $Nu_1u_2 = Nu_1$.

Then the double coset [1 2]= [1].

Now

$u_1\ u_{13} = u_1\ u_{13}\ u_2\ u_{13}\ u_{13}\ u_2$

$= u_1$ (1 7)(2 13)(5 60)(6 81)(8 69)(9 90)(11 79)
(12 86)(14 56)(15 26)(16 77)(17 41)(18 35)
(19 32)(20 57)(21 76)(22 45)(23 37)(24 74)
(27 33)(28 54)(29 73)(30 58)(31 63)(36 84)
(40 49)(43 53)(46 93)(47 71)(48 83)(50 59)
(52 89)(62 65)(64 95)(67 80)(68 78)(72 88)

(75 92)(82 94)(85 98)(87 97)(91 99)(96 100) u_{13} u_2

$\sim$ u_7 u_{13} u_2

= (1 2)(5 16)(6 9)(7 13)(8 48)(11 29)(12 67)

(14 28)(15 42)(17 20)(18 32)(19 35)(21 50)

(22 61)(23 55)(24 47)(25 26)(27 51)(30 49)

(31 36)(33 44)(37 39)(38 45)(40 58)(41 73)

(46 52)(53 65)(54 83)(56 69)(57 79)(59 63)

(64 97)(68 78)(71 80)(72 99)(74 86)(75 94)

(76 84)(82 91)(85 100)(87 96)(88 92)(95 98) u_7 u_2

$\sim$ u_7 u_2

u_1 u_{13} = u_1 u_{13} u_3 u_{13} u_{13} u_3

= u_1 (1 4)(3 13)(5 50)(6 67)(8 59)(9 24)(10 35)

(11 80)(12 47)(14 38)(15 26)(16 36)(17 74)

(20 39)(21 31)(22 63)(23 71)(25 53)(27 89)

(28 84)(29 55)(30 68)(32 34)(33 93)(37 86)

(40 78)(41 90)(42 65)(45 76)(46 52)(48 61)

(49 58)(54 77)(60 69)(64 94)(72 100)(75 92)

(79 81)(82 85)(87 96)(88 97)(91 99)(95 98) u_{13}

u_3

$\sim$ u_4 u_{13} u_3

= (1 3)(4 13)(5 48)(6 29)(8 28)(9 20)(10 32)

(11 17)(12 73)(14 16)(15 43)(21 83)(22 69)

(23 79)(24 55)(26 62)(27 33)(30 78)(31 56)

(34 35)(36 61)(37 41)(38 50)(39 67)(40 68)
(44 52)(45 54)(46 51)(47 57)(49 58)(53 65)
(60 76)(63 77)(64 97)(71 90)(72 88)(75 96)
(81 86)(82 94)(85 100)(87 91)(92 98)(95 99)$u_4\, u_3$

$\sim u_4\, u_3$

$u_1\, u_{13} = u_1\, u_{13}\, u_4\, u_{13}\, u_{13}\, u_4$

$= u_1$ (1 3)(4 13)(5 48)(6 29)(8 28)(9 20)(10 32)
(11 17)(12 73)(14 16)(15 43)(21 83)(22 69)
(23 79)(24 55)(26 62)(27 33)(30 78)(31 56)
(34 35)(36 61)(37 41)(38 50)(39 67)(40 68)
(44 52)(45 54)(46 51)(47 57)(49 58)(53 65)
(60 76)(63 77)(64 97)(71 90)(72 88)(75 96)
(81 86)(82 94)(85 100)(87 91)(92 98)(95 99)

$u_{13}\, u_4$

$\sim u_3\, u_{13}\, u_4$

= (1 4)(3 13)(5 50)(6 67)(8 59)(9 24)(10 35)
(11 80)(12 47)(14 38)(15 26)(16 36)(17 74)
(20 39)(21 31)(22 63)(23 71)(25 53)(27 89)
(28 84)(29 55)(30 68)(32 34)(33 93)(37 86)
(40 78)(41 90)(42 65)(45 76)(46 52)(48 61)
(49 58)(54 77)(60 69)(64 94)(72 100)(75 92)
(79 81)(82 85)(87 96)(88 97)(91 99)(95 98)$u_3\, u_4$

$\sim u_3\, u_4$.

Therefore $N u_3 u_4 = N u_4 u_3$ (since $Nu_1u_{13} = Nu_4u_3$

and $Nu_1u_{13} = Nu_3u_4$)

then $Nu_iu_j = Nu_ju_i$ for all the single cosets Nu_iu_j in the double coset $Nu_1u_{13} N$. (1)

$u_1 u_{13} = u_1 u_{13} u_{10} u_{13} u_{13} u_{10}$

$= u_1$ (1 34)(3 35)(4 32)(5 16)(6 9)(8 97)(10 13)

(11 100)(12 95)(14 29)(15 26)(17 64)(20 48)

(21 98)(22 90)(23 77)(24 36)(25 65)(27 33)

(28 85)(31 87)(37 60)(41 86)(42 53)(44 46)

(45 81)(47 96)(49 70)(50 67)(51 52)(54 76)

(56 75)(57 91)(58 66)(59 94)(63 69)(68 78)

(71 79)(72 84)(73 92)(74 88)(80 82)(83 99)u_{13}

u_{10}

$\sim u_{34} u_{13} u_{10}$

= (1 10)(3 32)(4 35)(5 55)(6 61)(8 88)(9 38)

(11 72)(12 99)(13 34)(14 36)(15 62)(16 39)

(17 94)(20 24)(21 92)(22 45)(23 37)(26 43)

(27 93)(28 82)(29 67)(31 91)(33 89)(41 54)

(46 52)(47 75)(48 50)(49 66)(53 65)(56 95)

(57 98)(58 70)(59 100)(63 86)(64 84)(68 78)

(69 79)(71 76)(73 87)(74 85)(80 97)(83 96)$u_{34} u_{10}$

$\sim u_{34} u_{10}$

$\sim u_{10} u_{34}$ (from (1))

$u_1\, u_{13} = u_1\, u_{13}\, u_{18}\, u_{13}\, u_{13}\, u_{18}$

$= u_1$ (1 19)(2 35)(5 100)(6 97)(7 32)(8 28)(9 85)

(11 17)(12 84)(13 18)(14 29)(15 26)(16 64)

(20 48)(21 74)(22 94)(23 82)(24 71)(27 44)

(31 80)(33 51)(36 63)(37 88)(38 92)(39 99)

(41 54)(43 65)(45 72)(46 52)(47 59)(49 58)

(50 76)(53 62)(55 75)(60 98)(61 91)(66 78)

(67 86)(68 70)(69 79)(77 87)(81 95)(90 96) $u_{13}\, u_{18}$

$\sim u_{19}\, u_{13}\, u_{18}$

= (1 18)(2 32)(5 96)(6 87)(7 35)(8 73)(9 98)(11 83)

(12 47)(13 19)(14 54)(15 25)(16 95)(17 56)(20 41)

(21 31)(22 75)(23 91)(24 36)(26 42)(27 33)(28 57)

(29 79)(37 92)(38 82)(39 94)(45 99)(46 89)(48 69)

(49 58)(50 67)(52 93)(53 65)(55 72)(60 97)(61 88)

(63 86)(64 90)(66 68)(70 78)(71 76)(77 85)(81 100) u_{19}

u_{18}

$\sim u_{19}\, u_{18}$

$\sim u_{18}\, u_{19}$

$u_1\, u_{13} = u_1\, u_{13}\, u_{30}\, u_{13}\, u_{13}\, u_{30}$

$= u_1$ (1 40)(2 58)(3 68)(4 78)(5 77)(6 9)(7 49)(8 28)

(11 74)(12 57)(13 30)(14 36)(15 85)(16 60)(17 80)

(20 41)(21 31)(22 38)(23 37)(24 71)(25 96)(26 97)

(27 98)(29 79)(32 35)(33 87)(42 95)(43 94)(44 64)

(45 61)(46 82)(47 73)(48 50)(51 100)(52 88)

(53 91)(54 76)(62 72)(63 69)(65 92)(67 86)(75 89)

(93 99) $u_{13} u_{30}$

$\sim u_{40} u_{13} u_{30}$

= (1 30)(2 49)(3 78)(4 68)(5 16)(6 90)(7 58)(8 84)

(9 81)(11 17)(12 47)(13 40)(14 54)(15 100)(20 24)

(21 56)(22 45)(23 39)(25 98)(26 64)(27 95)(28 59)

(29 67)(31 83)(32 35)(33 96)(36 63)(37 55)(41 86)

(42 87)(43 88)(44 85)(46 94)(48 69)(50 76)(51 97)

(52 72)(53 99)(62 82)(65 75)(71 79)(89 91)(92 93)

$u_{40} u_{30}$

$\sim u_{40} u_{30}$

$\sim u_{30} u_{40}$

$u_1 u_{13} = u_1 u_{13} u_{66} u_{13} u_{13} u_{66}$

$= u_1$ (1 70)(5 16)(6 38)(8 28)(9 61)(10 58)(11 56)

(12 59)(13 66)(14 65)(15 63)(17 83)(18 78)(19 68)

(20 44)(21 31)(22 45)(23 60)(24 25)(26 76)(27 69)

(29 51)(32 35)(33 54)(34 49)(36 52)(37 77)(41 43)

(42 67)(46 50)(47 84)(48 53)(62 79)(64 94)(71 89)

(72 100)(75 96)(82 91)(85 98)(86 93)(87 97)

(88 92)(95 99) $u_{13} u_{66}$

$\sim u_{70} u_{13} u_{66}$

= (1 66)(5 39)(6 9)(8 57)(10 49)(11 17)(12 47)

(13 70)(14 44)(15 86)(16 55)(18 68)(19 78)(20 53)

(21 80)(22 81)(23 37)(24 52)(25 50)(26 71)(27 79)

(28 73)(29 65)(31 74)(32 35)(33 41)(34 58)(36 42)

(43 69)(45 90)(46 67)(48 51)(54 62)(63 89)(64 95)

(72 99)(75 94)(76 93)(82 85)(87 91)(88 97)(92 98)

(96 100) $u_{70}\, u_{66}$

$\sim u_{70}\, u_{66}$

$\sim u_{66}\, u_{70}$.

Also,

$u_{32}\, u_{35} = u_{32}\, u_{35}\, u_2\, u_{35}\, u_{35}\, u_2$

$= u_{32}$ (1 19)(2 35)(5 85)(6 64)(7 32)(8 11)(9 100)

(12 80)(13 18)(15 26)(16 97)(17 28)(21 59)

(22 88)(23 72)(24 76)(27 51)(30 40)(31 84)

(33 44)(36 86)(37 94)(38 75)(39 91)(41 79)

(43 65)(45 82)(47 74)(50 71)(53 62)(54 69)

(55 92)(56 73)(57 83)(60 96)(61 99)(63 67)

(66 68)(70 78)(77 95)(81 87)(89 93)(90 98) $u_{35}\, u_2$

$\sim u_7\, u_{35}\, u_2$

= (1 18)(2 32)(5 98)(6 95)(7 35)(8 56)(9 96)

(11 57)(12 21)(13 19)(14 79)(15 25)(16 87)

(17 73)(20 69)(22 92)(23 99)(24 67)(26 42)

(28 83)(29 54)(30 40)(31 47)(36 50)(37 75)

(38 72)(39 88)(41 48)(44 51)(45 91)(46 93)

$(52\ 89)(53\ 65)(55\ 82)(59\ 74)(60\ 64)(61\ 94)$

$(66\ 78)(68\ 70)(77\ 100)(80\ 84)(81\ 85)(90\ 97)u_7\,u_2$

$\sim u_7\,u_2$

and

$u_{49}u_{58} = u_{49}\,u_{58}\;u_2\,u_{58}\;u_{58}\,u_2$

$= u_{49}\;(1\ 40)(2\ 58)(3\ 78)(4\ 68)(5\ 77)(7\ 49)(8\ 31)(11\ 73)$

$(12\ 80)(13\ 30)(15\ 87)(16\ 60)(17\ 57)(18\ 19)$

$(20\ 86)(21\ 28)(22\ 61)(23\ 37)(24\ 79)(25\ 100)$

$(26\ 98)(27\ 97)(29\ 71)(33\ 85)(38\ 45)(41\ 67)$

$(42\ 64)(43\ 99)(44\ 95)(46\ 91)(47\ 74)(51\ 96)$

$(52\ 92)(53\ 82)(54\ 69)(56\ 84)(59\ 83)(62\ 75)$

$(63\ 76)(65\ 88)(72\ 89)(81\ 90)(93\ 94)\;u_{58}\,u_2$

$\sim u_7\,u_{58}\,u_2$

$= (1\ 30)(2\ 49)(3\ 68)(4\ 78)(5\ 16)(6\ 81)(7\ 58)(8\ 56)$

$(9\ 90)(11\ 12)(13\ 40)(14\ 63)(15\ 96)(17\ 47)(18\ 19)$

$(20\ 29)(21\ 84)(23\ 39)(24\ 67)(25\ 97)(26\ 95)$

$(27\ 64)(28\ 83)(31\ 59)(33\ 100)(36\ 54)(37\ 55)$

$(38\ 61)(42\ 85)(43\ 92)(44\ 87)(46\ 99)(48\ 76)$

$(50\ 69)(51\ 98)(52\ 75)(53\ 94)(57\ 80)(62\ 91)$

$(65\ 72)(73\ 74)(82\ 89)(88\ 93)u_7\,u_2$

$\sim u_7\,u_2,$

$u_{68}\,u_{78} = u_{68}\,u_{78}\;u_3\,u_{78}\;u_{78}\,u_3$

$= u_{68}\;(1\ 30)(2\ 58)(3\ 78)(4\ 68)(5\ 45)(6\ 55)(7\ 49)(8\ 84)$

(9 39)(10 34)(12 47)(13 40)(15 94)(16 22)(20 71)

(21 83)(23 81)(24 79)(25 91)(26 72)(27 99)(28 59)

(29 86)(31 56)(33 75)(36 50)(37 90)(38 77)(41 67)

(42 92)(43 97)(44 82)(46 100)(51 88)(52 64)

(53 95)(60 61)(62 85)(63 76)(65 96)(74 80)(87 93)

(89 98) $u_{78}\, u_3$

$\sim u_4\, u_{78}\, u_3$

= (1 40)(2 49)(3 68)(4 78)(5 38)(6 23)(7 58)(8 28)

(9 37)(10 34)(11 80)(12 57)(13 30)(14 76)(15 82)

(16 61)(17 74)(20 29)(22 77)(25 75)(26 88)

(27 91)(33 92)(36 54)(39 81)(41 79)(42 99)

(43 100)(44 72)(45 60)(46 85)(47 73)(48 63)

(50 69)(51 94)(52 97)(53 98)(55 90)(56 83)

(62 64)(65 87)(89 96)(93 95) $u_4\, u_3$

$\sim u_4\, u_3$.

We have

1 13 ~ 13 1 ~ 2 7 ~ 7 2 ~ 3 4 ~ 4 3 ~ 34 10 ~ 10 34 ~ 19 18 ~ 18 19 ~ 40 30 ~ 30 40 ~ 70 66 ~ 66 70 ~ 32 35 ~ 35 32 ~ 49 58 ~ 58 49 ~ 68 78 ~ 78 68

Therefore the double coset [1 13] contains 315 distinct single cosets, since every 20 have the same name.

The double cosets [w] = N w N in G.		
Label [w]	Coset stabilizing subgroup $N^{(w)}$	Number of Cosets
[*]	N is transitive implies	1
[1]	$N^{(1)} \sim U_3(3):2$ $N^1 = < x,y,t>$ has orbits {1},{ 2,3,4,5,6,7,8,9,10,11,12,15,16, 17,18,19,21,22,23,26,27,28,30,31,33,34,37,40,45, 46,47,52,53,65,66,70}, and {13,14,20,24,25,29,32, 35,36,38,39,41,42,43,44, 48,49,50,51,54,55,56,57,58,59,60,61,62,63,64,67, 68,69,71,72,73,74,75,76,77,78,79,80,81,82,83,84, 85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100 }	100
[1 2]= [1]	1 13 ~ 13 1 ~ 2 7 ~ 7 2 ~ 3 4 ~ 4 3 ~ 34 10 ~ 10 34 ~ 19 18 ~ 18 19 ~ 40 30 ~ 30 40 ~ 70 66 ~ 66 70 ~ 32 35 ~ 35 32 ~ 49 58 ~ 58 49 ~ 68 78 ~ 78 68	
[1 13]		315

$N^{(1\ 13)}$ has orbits

{1,2,3,4,7,10,13,18,19,30,32,34,35,40,49,58,66,68,70,78} and {5,6,8,9,11,12,14,15,16,17,20,21,22,23,24,25,26,27,28,29,31,33,36,37,38,39,41,42,43,44,45,46,47,48,50,51,52,53,54,55,56,57,59,60,61,62,63,64,65,67,69,71,72,73,74,75,76,77,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100}.

1 13 1 ~ 13 1 1 ~ 1 ⇒ for all i in {1,2,3,4,7,10,13,18,19,30,32,34,35,40,49,58,66,68 ,70,78}

[1 13 i]= [13]

1 13 5 ~ 13 1 5

= 13 (1 5)(3 14)(4 36)(6 51)(7 77)(9 25)(10 39)(12 72)(13 58)(16 49)(17 75)(18 95)(19 64)(20 52)(22 59)(24 29)(26 87)(27 85)(28 76)(31 69)(32 42)(35 44)(38 48)(40 60)(41 62)(43 82)(45 83)(50 61)(54 63)(55 66)(56 78)(57 92)(65 67)(68 84)(71 74)(73 79)(80 88)(81 96)(86 89)(90 100)(91

93)(94 99)(97 98) 1

~ 58 1

but 58 1 ∈[1 13] (since N $(1\ 13)^{\pi}$ = N 58 1,

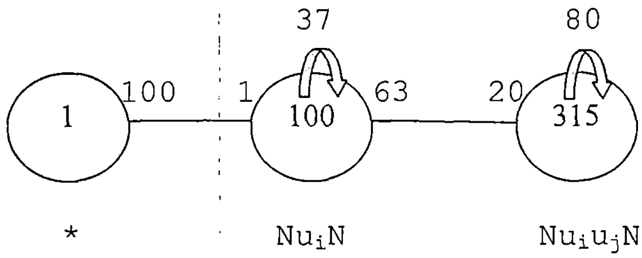
where

π = (1 58 38 53 100 80 3 30 40 2 45
82 29 83 19 66 70 10 9 50 62 17
68 13)(4 7 22 92 25 26 79 56 18
34 6 14 75 87 51 11 78 49 61 52
71 54 72 74)(5 88 67 69 99 97 20
84 37 36 93 15 86 76 43 47 60 65
95 98 42 33 64 73)(8 55 21 32 81
31 35 90)(12 77 46 41 59 23 48 94
57 16 91 44)(24 63 89 27 96
85)(28 39)∈N

Therefore, for all j in
{5,6,8,9,11,12,14,15,16,17,20,21,22,23,24,25,
26,27,28,29,31,33,36,37,38,39,41,42,43,44,45,46,47,48
,50,51,52,53,54,55,56,57,59,60,61,62,63,64,65,67,69,7
1,72,73,74,75,76,77,79,80,81,82,83,84,85,86,87,88,89,
90,91,92,93,94,95,96, 97, 98, 99, 100}

[1 13 j] = [1 13]

The Cayley graph will be as follows:



APPENDIX A

$PGL_2(11)$

Magma work for the Group $PGL_2(11)$

```
G<x,y,z,t>:=Group<x, y, z, t| x^5 , y^2, z^2,(x^-1 *
y)^2,(y * z)^2,(z * x)^3 ,y * x^3 * z * x^2 * z * x^-2 * z,
t^2, (t, z*x*z ),(t,y), (x*t)^4 >;
f, G1, k := CosetAction(G, sub< G | x, y, z >);
ts := [ Id(G1): i in [1 .. 6] ];
for i:=1 to 5 do ts[i] := f(t^(x^i)); end for;
ts[6]:=f(t^z);
S6:=SymmetricGroup(6);
x:=S6!(5, 1, 2, 3, 4);
y:=S6!(1,4)(2,3);
z:=S6!(5, 6)(1,4);
N:=sub<S6|x,y,z>;
print Order(N);
N5:=Stabiliser(N, 5);
N56:=Stabiliser(N5,6);
for g in N do if 5^g eq 1 and 6^g eq 3 then
N56:=sub<N|N56,g>; end if; end for;
for g in N do if 5^g eq 4 and 6^g eq 2 then
N56:=sub<N|N56,g>; end if; end for;
N56d:=Stabiliser(N5,6);
for g in N do if 6^g eq 5 and 5^g eq 6 then
N56d:=sub<N|N56d,g>; end if; end for;
for g in N do if 6^g eq 2 and 5^g eq 3 then
N56d:=sub<N|N56d,g>;
end if;end for;
for g in N do if 6^g eq 4 and 5^g eq 1 then
N56d:=sub<N|N56d,g>;
end if;end for;
for g in N do if 6^g eq 1 and 5^g eq 4 then
N56d:=sub<N|N56d,g>; end if; end for;
cst := [null : i in [1 .. 22]] where null is [Integers() |
];
prodim := function(pt, Q, I)
v := pt;
 for i in I do
  v := v^(Q[i]);
end for;
 return v;
end function;

for i := 1 to 6 do
cst[prodim(1, ts, [i])] := [i];
```

```
end for;
tr1:=Transversal(N,N56);
for i := 1 to 10 do
ss := [5,6]^tr1[i];
cst[prodim(1, ts, ss)] := ss;
end for;
tr2:=Transversal(N,N56d);
for i := 1 to 5 do
ss:=[5,6,5]^tr2[i];
        cst[prodim(1, ts, ss)] := ss;
      end for;
 print cst;
 print ts;
p:=ts[3]*ts[6]*ts[5];
print p;
print cst[1^p];
n:=p*ts[2];
zz := N![rep{j: j in [1..6] | (1^ts[i])^n eq 1^ts[j]}: i in
[1..6]];
print zz;
```

APPENDIX B

J_1

Magma work for the group J_1

(1)

```
N11:=Stabiliser(N,11);
N111:=Stabiliser(N11,1);
N1112:=Stabiliser(N111,2);
N11127:=Stabiliser(N1112,7);
for g in N do if 11^g eq 10 and 1^g eq 11 and 2^g eq 3 and
7^g eq 1 then N11127:=sub<N|N11127,g>;
for|if> for g in N do if 11^g eq 8 and 1^g eq 1 and 2^g eq
3 and 7^g eq 2 then N11127:=sub<N|N11127,g>;
for|if|for|if> end if; end for;
for|if> print Order(N11127);
for|if> end if; end for;
print Order(N11127);
55
IsTransitive(N11127);
true
print Orbits(N11127);
[
    GSet{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 }
]
```

(2)

```
1 []
2 [ 11 ]
3 [ 1 ]
4 [ 10 ]
5 [ 2 ]
6 [ 1, 11 ]
7 [ 10, 11 ]
8 [ 9 ]
9 [ 3 ]
10 [ 2, 11 ]
11 [ 2, 1 ]
12 [ 11, 10 ]
13 [ 11, 1 ]
14 [ 9, 10 ]
15 [ 5 ]
16 [ 9, 11 ]
17 [ 8 ]
18 [ 4 ]
19 [ 3, 11 ]
20 [ 3, 1 ]
21 [ 1, 10 ]
22 [ 3, 2 ]
```

23 [10, 1]
24 [11, 5, 8]
25 [11, 2]
26 [10, 9]
27 [1, 2]
28 [5, 2]
29 [11, 8, 4]
30 [8, 9]
31 [6]
32 [5, 11]
33 [8, 10]
34 [8, 11]
35 [7]
36 [4, 11]
37 [4, 1]
38 [2, 10]
39 [4, 2]
40 [2, 4, 3]
41 [1, 10, 11]
42 [11, 9]
43 [4, 3]
44 [4, 10]
45 [2, 3]
46 [10, 1, 11]
47 [1, 11, 10]
48 [1, 3]
49 [2, 5]
50 [10, 9, 11]
51 [9, 8]
52 [1, 2, 11]
53 [6, 3]
54 [11, 8, 3]
55 [2, 7, 10]
56 [8, 5]
57 [8, 9, 11]
58 [7, 8]
59 [6, 11]
60 [6, 1]
61 [8, 2]
62 [10, 8]
63 [7, 9]
64 [9, 1]
65 [7, 10]
66 [7, 11]
67 [5, 1]

68 [3, 10]
69 [6, 2, 10]
70 [10, 2]
71 [8, 4, 1]
72 [1, 9]
73 [5, 3]
74 [9, 10, 4]
75 [3, 5, 4]
76 [1, 3, 2]
77 [2, 11, 1]
78 [11, 9, 10]
79 [11, 5]
80 [5, 4]
81 [3, 4]
82 [10, 4]
83 [3, 9]
84 [11, 2, 1]
85 [1, 5, 3]
86 [1, 11, 2]
87 [8, 7, 3]
88 [8, 5, 1]
89 [2, 4]
90 [1, 4]
91 [7, 6, 8]
92 [3, 6]
93 [2, 5, 11]
94 [11, 10, 1]
95 [9, 8, 10]
96 [5, 8]
97 [8, 7]
98 [9, 7, 11]
99 [7, 5, 9]
100 [7, 4]
101 [11, 4, 2]
102 [1, 9, 4]
103 [10, 7, 2]
104 [3, 8, 11]
105 [3, 10, 7]
106 [9, 6]
107 [8, 5, 11]
108 [6, 4, 11]
109 [4, 2, 9]
110 [6, 8]
111 [6, 5, 7]
112 [6, 7]

113 [7, 1]
114 [5, 10]
115 [7, 2]
116 [9, 8, 4]
117 [9, 3]
118 [2, 8]
119 [9, 7]
120 [6, 5]
121 [11, 1, 6]
122 [6, 2]
123 [10, 7]
124 [6, 9]
125 [8, 1]
126 [6, 10]
127 [1, 5]
128 [11, 7, 4]
129 [2, 9]
130 [7, 3, 11]
131 [5, 1, 9]
132 [11, 3]
133 [7, 4, 10]
134 [9, 5, 2]
135 [11, 8]
136 [6, 4]
137 [9, 4]
138 [3, 5]
139 [10, 11, 5]
140 [8, 9, 3]
141 [7, 11, 9]
142 [8, 11, 4]
143 [1, 4, 10]
144 [3, 1, 2]
145 [4, 2, 6]
146 [6, 7, 1]
147 [8, 4]
148 [10, 8, 9]
149 [1, 6]
150 [11, 10, 3]
151 [4, 5]
152 [4, 7, 2]
153 [2, 1, 3]
154 [2, 3, 8]
155 [7, 6, 2]
156 [8, 2, 5]
157 [10, 3]

158 [3, 10, 5]
159 [8, 7, 9]
160 [4, 7]
161 [3, 6, 11]
162 [3, 6, 1]
163 [5, 8, 2]
164 [8, 6]
165 [10, 5, 4]
166 [7, 6]
167 [2, 5, 10]
168 [5, 6, 11]
169 [4, 8]
170 [10, 11, 9]
171 [10, 3, 1]
172 [2, 10, 5]
173 [3, 6, 10]
174 [9, 6, 1]
175 [4, 9, 1]
176 [3, 8]
177 [2, 10, 9]
178 [2, 9, 6]
179 [5, 7]
180 [6, 7, 9]
181 [6, 7, 10]
182 [5, 6]
183 [5, 8, 3]
184 [9, 2]
185 [6, 9, 3]
186 [4, 9]
187 [8, 3]
188 [6, 3, 9]
189 [10, 9, 5]
190 [4, 5, 10]
191 [1, 7]
192 [1, 2, 7]
193 [7, 3]
194 [2, 6]
195 [7, 5]
196 [1, 8]
197 [5, 9]
198 [11, 4]
199 [1, 8, 5]
200 [8, 5, 4]
201 [10, 5]
202 [7, 10, 3]

203 [9, 1, 5]
204 [4, 6]
205 [7, 8, 2]
206 [9, 10, 8]
207 [6, 9, 4, 3]
208 [8, 1, 10]
209 [10, 2, 6]
210 [4, 2, 3]
211 [9, 5]
212 [9, 7, 8]
213 [2, 7]
214 [5, 8, 1]
215 [6, 5, 1]
216 [10, 9, 2]
217 [3, 2, 4]
218 [3, 4, 9]
219 [7, 1, 4]
220 [4, 11, 6]
221 [2, 9, 4]
222 [11, 1, 3, 10]
223 [4, 7, 11]
224 [4, 7, 1]
225 [1, 2, 4, 11]
226 [10, 1, 8]
227 [9, 8, 1]
228 [1, 11, 7]
229 [6, 11, 3]
230 [3, 7]
231 [4, 6, 2]
232 [2, 3, 5, 1]
233 [7, 5, 4]
234 [3, 11, 6]
235 [11, 3, 9, 8]
236 [2, 5, 9]
237 [4, 1, 7]
238 [5, 10, 2]
239 [5, 6, 8]
240 [5, 3, 4]
241 [4, 3, 5]
242 [1, 10, 6]
243 [5, 2, 8]
244 [11, 10, 6]
245 [9, 1, 7, 6]
246 [11, 6]
247 [11, 7]

```
248 [ 1, 4, 8 ]
249 [ 11, 4, 1, 7 ]
250 [ 11, 3, 7 ]
251 [ 6, 9, 2 ]
252 [ 11, 2, 9 ]
253 [ 8, 1, 9, 4 ]
254 [ 3, 4, 6, 2 ]
255 [ 10, 6 ]
256 [ 1, 3, 8 ]
257 [ 8, 6, 7 ]
258 [ 5, 7, 3 ]
259 [ 10, 11, 2, 9 ]
260 [ 1, 10, 3, 11 ]
261 [ 10, 6, 1 ]
262 [ 7, 5, 6 ]
263 [ 6, 4, 5 ]
264 [ 11, 2, 7 ]
265 [ 4, 3, 7, 10 ]
266 [ 7, 3, 9 ]
```

(3)

```
prodim := function(pt, Q, I)
v := pt;
for i in I do
v :=v^(Q[i]);  end for; return v;
end function;
J<x, y, t> := Group< x, y, t | x^11 = y^2= (x*y)^3=
(x^4*y*x^6*y)^2 = 1, t^2 = (t, y) = (t^x, y) = (t^(x^8), y)
=(y*(t^(x^3)))^5 = (x*t)^6 = 1>;
f,J1,k:=CosetAction(J,sub<J|x,y>);
ts := [ (t^(x^i)) @ f : i in [1 .. 11] ];
S11 := SymmetricGroup(11);
aa := S11 ! (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11);
bb := S11 ! (3, 4)(2, 10)(5, 9)(6, 7);
cc := S11 ! (1, 3, 9, 5, 4)(2, 6, 7, 10, 8); L11 := sub<
S11 | aa,
bb, cc >;
cst := [null : i in [1 .. 266]] where null is [Integers() |
];
for i := 1 to 11 do
cst[prodim(1, ts, [i])] := [i]; for>    end for;
for i := 1 to 11 do
    for j in {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} diff {i}
do
cst[prodim(1, ts, [i, j])] := [i, j];
```

```
end for;  end for;
tra1 := Transversal(L11, sub<L11 | cc>);
for i := 1 to 132 do
ss := [3, 6, 1]^tra1[i];
   cst[prodim(1, ts,ss)] := ss;
  end for;
 tra2 := Transversal(L11, sub<L11 | aa, cc>);
  for i := 1 to 12 do
  ss := [1, 10, 3, 11]^tra2[i];
  cst[prodim(1,ts, ss)] := ss;
 end for;
 xx :=L11 ! (1,2,3,4,5,6,7,8,9,10,11); uu := {1,2,3}; p :=
[1 : i in [1..266]];
for i := 1 to 11 do
p[prodim(1, ts, [i])] := prodim(1, ts, [i]^xx); end for;
for i := 1 to 11 do
for j in {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} diff {i}
do
 p[prodim(1, ts, [i, j])] := prodim(1, ts, [i,j]^xx);end
for; end for;
 for i := 1 to 132 do
t := [3, 6, 1]^tra1[i];
 p[prodim(1, ts, t)]:= prodim(1, ts, t^xx); end for;
  for i := 1 to 12 do
 t := [1, 10, 3, 11]^tra2[i];
p[prodim(1, ts,t)] := prodim(1, ts, t^xx);
end for;
p:=J1!p;
print p;
print p*ts[1]*ts[2]*ts[3];
```

APPENDIX C

$U_3(3):2$

Magma work for the Group $U_3(3):2$

```
G<x,y,t,s>:=Group<x,y,t,s|x^7, y^2, t^2, (x^-1 * t)^2 , (y
* x)^3 , t * x^-1
 * y * x  * t * y , x^2 * y * x^3 * y * x^-4 * y * x^-4 *
y*x , s^2, (s^(x^3),y),(s^(x
^4), x*y), t * s * s^t * s, y * (s * s^(t *
x^6))^2,y*(s^(x^3))*(s^(t*x^6))*(s
^x)*s>;
f, G1, k := CosetAction(G, sub< G | x, y,t >);
ts := [ Id(G1): i in [1 .. 14] ];
for i:=1 to 7 do ts[i] := f(s^(x^i)); end for;
ts[14]:=f(s^t); ts[13]:=f((s^t)^x); ts[12]:=f((s^t)^(x^2));
ts[11]:=f((s^t)^(x^3)); ts[10]:=f((s^t)^(x^4));
ts[9]:=f((s^t)^(x^5)); ts[8]:=f((s^t)^(x^6));
S14:=SymmetricGroup(14);
x:=S14!(1,2,3,4,5,6,7)(14,13,12,11,10,9,8);
y:=S14!(2,6)(4,5)(14,10)(13,12);
t:=S14!(7,14)(1,8)(2,9)(3,10)(4,11)(5,12)(6,13);
N:=sub<S14|x,y,t>;
N7:=Stabiliser(N, 7);
N71:=Stabiliser(N7,1);
for g in N do if 7^g eq 1 and 1^g eq 7 then
N71:=sub<N|N71,g>;
end if; end for;
for g in N do if 7^g eq 3 and 1^g eq 8 then
N71:=sub<N|N71,g>;
end if;end for;
for g in N do if 7^g eq 12 and 1^g eq 13 then
N71:=sub<N|N71,g>;
end if;end for;
cst := [null : i in [1 .. 36]] where null is [Integers() |
];
 prodim := function(pt, Q, I)
  v := pt;
 for i in I do
 v := v^(Q[i]);
 end for;
 return v;
 end function;
 for i := 1 to 14 do
   cst[prodim(1, ts, [i])] := [i];
   end for;
 N71c:=N71;
 tr:=Transversal(N,N71c);
```

```
 for i := 1 to 21 do
     ss := [7,1]^tr[i];
     cst[prodim(1, ts, ss)] := ss;
     end for;
 for i in [1..36] do print i, cst[i]; end for;
 print ts;

p:=ts[7]*ts[8]
print cst[1^p];
n:=p*ts[9]*ts[11];
zz := N![rep{j: j in [1..14] | (1^ts[i])^n eq 1^ts[j]}: i
in
[1..14]];
print zz;

p:=ts[11]*ts[7];
print cst[1^p];
n:=p*ts[4]*ts[5];
zz := N![rep{j: j in [1..14] | (1^ts[i])^n eq 1^ts[j]}: i
in
[1..14]];
 print zz;
```

APPENDIX D

J_2:2

Magma work for the Group J_2:2

```
G<x,y,t,s>:=Group<x,y,t,s|x^7, y^2, (x*y)^3, (x,y)^4, t^2,
(t^(x^3),y), (t^(x^4),x*y),y*(t^x*t^(x^3))^2,
s^2,(s,x),(s,y),(s,(t*x)^4*x^2*t), s*s^t*s=t>;
print Index(G, sub<G|x,y,t>: CosetLimit:=5000000,
Hard:=true);
U3<x,y,t>:=Group<x,y,t|x^7, y^2, (x*y)^3, (x,y)^4, t^2,
(t^(x^3),y),
(t^(x^4),x*y), y*(t^x*t^(x^3))^2>;
H:=sub<U3|x,y,(t*x)^4*x^2*t>;
f,G1,k:=CosetAction(U3,H);
G<x,y,t,s>:=Group<x,y,t,s|x^7, y^2, (x*y)^3, (x,y)^4, t^2,
(t^(x^3),y), (t^(x^4),x*y),
y*(t^x*t^(x^3))^2, s^2,(s,x),(s,y),(s,(t*x)^4*x^2*t),
s*s^t*s=t>;
print Index(G, sub<G|x,y,t>: CosetLimit:=5000000,
Hard:=true);
f, G1, k := CosetAction(G, sub< G | x, y,t >);
ts := [ Id(G1): i in [1 .. 36] ];
ts[1]:=f(s);
ts[2]:=f(s^t);
ts[3]:=f(s^(t*x));
ts[4]:=f(s^(t*x^6));
ts[5]:=f(s^(t*x^2));
ts[6]:=f((s^(t*x))^t);
ts[7]:=f((s^(t*x^6))^t);
ts[8]:=f(s^(t*x^5));
ts[9]:=f(((s^(t*x^6))^t)^(t*x^4));
ts[10]:=f((s^(t*x^5))^(t*x^2));
ts[11]:= f((s^(t*x))^(t*x));
ts[12]:= f((s^(t*x^6))^(t*x^6));
ts[13]:=f(s^(t*x^4));
ts[14]:=f((s^(t*x^5))^t);
ts[15]:=f((((s^(t*x^6))^t)^(t*x^4))^t);
ts[16]:=f(((s^(t*x^6))^(t*x^6))^(t*x^6));
ts[17]:=f((s^(t*x^2))^(t*x^6));
ts[18]:=f((s^(t*x))^(t*x^2));
ts[19]:=f((((s^(t*x^6))^t)^(t*x^4))^(t*x^3));
ts[20]:=f((s^(t*x^2))^(t*x^2));
ts[21]:=f(((((s^(t*x^6))^t)^(t*x^4))^(t*x^3))^(t*x^3));
ts[22]:=f((((s^(t*x^6))^t)^(t*x^4))^(t*x^4));
ts[23]:=f(((s^(t*x^6))^(t*x^6))^(t*x^2));
ts[24]:=f((((s^(t*x^6))^(t*x^6))^(t*x^2))^(t*x^6));
ts[25]:=f(((s^(t*x))^(t*x))^(t*x^3));
```

```
ts[26]:=f((((((s^(t*x^6))^t)^(t*x^4))^(t*x^3))^(t*x^3))^(t*
x^3));
ts[27]:=f((s^(t*x))^(t*x^3));
ts[28]:=f(((s^(t*x))^(t*x^3))^(t*x^4));
ts[29]:=f((s^(t*x^5))^(t*x^5));
ts[30]:=f((((((s^(t*x^6))^t)^(t*x^4))^(t*x^3))^(t*x^3))^t);
ts[31]:=f((s^(t*x^4))^(t*x));
ts[32]:=f((((((s^(t*x^6))^t)^(t*x^4))^(t*x^3))^(t*x^3))^(t*
x^2));
ts[33]:=f((((s^(t*x))^(t*x))^(t*x^3))^(t*x^2));
ts[34]:=f((((((s^(t*x^6))^t)^(t*x^4))^(t*x^3))^(t*x^3))^(t*
x));
ts[35]:=f(((((s^(t*x))^(t*x))^(t*x^3))^(t*x^2))^(t*x^2));
ts[36]:=f( (((s^(t*x))^(t*x))^(t*x^3))^(t*x^3));
S36:=Sym(36);
x:=S36!(2, 3, 5, 9, 13, 8, 4)(6, 11, 18, 27, 21, 12, 7)(10,
16, 20, 29, 23,14, 17)(15,
24, 28, 19, 22, 31, 25)(26, 33, 36, 35, 30, 34, 32);
y:=S36!(4, 5)(7, 10)(8, 13)(11, 17)(12, 20)(14, 22)(18,
19)(23, 28)(24, 31)(27, 29)(33,34)(35, 36);
t:=S36!(1, 2)(3, 6)(4, 7)(5, 10)(8, 14)(9, 15)(11, 19)(12,
20)(13, 22)(16, 26)(17, 18)(21, 30)(23, 28)(24, 29)
(25, 32)(27, 31);
N:=sub<S36|x,y,t>;
N1:=Stabiliser(N, 1);
N16:=Stabiliser(N1,6);
for g in N do if 6^g eq 1 and 1^g eq 6 then
N16:=sub<N|N16,g>;
end if; end for;
for g in N do if 3^g eq 15 and 2^g eq 16 then
N16:=sub<N|N16,g>;
end if;end for;
for g in N do if 9^g eq 23 and 26^g eq 28 then
N16:=sub<N|N16,g>;
end if;end for;
cst := [null : i in [1 .. 100]] where null is [Integers() |
];
prodim := function(pt, Q, I)
  v := pt;
 for i in I do
 v := v^(Q[i]);
 end for;
 return v;
end function;
 for i := 1 to 36 do
```

```
 cst[prodim(1, ts, [i])] := [i];
 end for;
 N16c:=N16;
 tr:=Transversal(N,N16c);
 for i := 1 to 63 do
 ss := [1,6]^tr[i];
  cst[prodim(1, ts, ss)] := ss;
   end for;
p:=ts[3]*ts[5]*ts[8];
cst[1^p];
n:=p*ts[19]*ts[1];
zz := N![rep{j: j in [1..36] | (1^ts[i])^n eq 1^ts[j]}: i
in [1..36]];
print zz;

for i in [1..100] do print i, cst[i]; end for;
1 []
2 [ 1 ]
3 [ 2 ]
4 [ 3 ]
5 [ 4 ]
6 [ 5 ]
7 [ 6 ]
8 [ 7 ]
9 [ 8 ]
10 [ 9 ]
11 [ 10 ]
12 [ 11 ]
13 [ 1, 6 ]
14 [ 1, 7 ]
15 [ 12 ]
16 [ 13 ]
17 [ 14 ]
18 [ 15 ]
19 [ 16 ]
20 [ 1, 10 ]
21 [ 17 ]
22 [ 18 ]
23 [ 19 ]
24 [ 1, 11 ]
25 [ 1, 12 ]
26 [ 20 ]
27 [ 21 ]
28 [ 22 ]
29 [ 1, 14 ]
```

30 [23]
31 [24]
32 [1, 15]
33 [25]
34 [26]
35 [1, 16]
36 [1, 17]
37 [27]
38 [1, 18]
39 [1, 19]
40 [28]
41 [2, 19]
42 [1, 20]
43 [2, 20]
44 [1, 21]
45 [29]
46 [30]
47 [31]
48 [1, 22]
49 [1, 23]
50 [1, 24]
51 [1, 25]
52 [32]
53 [33]
54 [2, 18]
55 [1, 27]
56 [2, 17]
57 [2, 11]
58 [1, 28]
59 [3, 22]
60 [4, 28]
61 [1, 29]
62 [34, 16]
63 [36, 20]
64 [16, 4]
65 [34]
66 [35]
67 [1, 31]
68 [3, 23]
69 [36, 25]
70 [36]
71 [3, 27]
72 [9, 14]
73 [35, 7]
74 [3, 10]

75 [15, 29]
76 [3, 18]
77 [13, 28]
78 [3, 28]
79 [35, 25]
80 [5, 19]
81 [35, 29]
82 [28, 32]
83 [15, 14]
84 [3, 7]
85 [26, 7]
86 [3, 19]
87 [15, 8]
88 [9, 22]
89 [9, 21]
90 [18, 35]
91 [15, 27]
92 [9, 24]
93 [3, 21]
94 [13, 31]
95 [26, 31]
96 [15, 13]
97 [13, 25]
98 [15, 5]
99 [9, 31]
100 [23, 20]

APPENDIX E

$G_2(4):2$

Magma work for the Group $G_2(4){:}2$

```
S100:=Sym(100);
x:=S100! (3, 4, 6, 10, 16, 9, 5)(7, 12, 22, 37, 27, 15,
8)(11, 19, 26, 45, 30, 17, 21)(13, 24, 38, 55, 44, 25,
14)(18, 31, 40, 23, 28, 47, 33)(20, 35, 42, 61, 49, 29,
36)(32, 50, 58, 39, 48, 67, 51)(34, 53, 70, 66, 46, 65,
52)(41, 59, 79, 93, 96, 82, 60)(43, 63, 81, 72, 54, 71,
64)(56, 74, 85, 62, 84, 90, 75)(57, 76, 73, 89, 98, 92,
77)(68, 86, 91, 99, 97, 87, 69)(78, 80, 88, 94, 100, 95,
83);
y:=S100! (5, 6)(8, 11)(9, 16)(12, 21)(14, 20)(15, 26)(17,
28)(22, 23)(24, 36)(25, 42)(29, 48)(30, 40)(31, 47)(37,
45)(38, 39)(41, 54)(43, 62)(49, 58)(50, 67)(53, 65)(55,
61)(56, 57)(59, 80)(60, 81)(63, 71)(64, 85)(66, 70)(68,
78)(69, 79)(72, 88)(73, 83)(74, 84)(75, 91)(76, 86)(77,
90)(82, 94)(87, 96)(92, 99)(95, 98)(97, 100);
t:=S100! (2, 3)(4, 7)(5, 8)(6, 11)(9, 17)(10, 18)(12,
23)(15, 26)(16, 28)(19, 34)(21,22)(24, 41)(25, 43)(27,
46)(30, 40)(31, 45)(33, 52)(36, 54)(37, 47)(38, 56)(39,
57)(42, 62)(49, 68)(50, 69)(55, 73)(58, 78)(59, 60)(61,
83)(63,76)(64, 85)(67, 79)(71, 86)(72, 87)(74, 90)(75,
92)(77, 84)(80, 81)(82, 95)(88, 96)(89, 93)(91, 99)(94,
98)(97, 100);
s:=S100! (1, 2)(7, 13)(8, 14)(11, 20)(12, 24)(15, 25)(17,
29)(18, 32)(19, 35)(21, 36)(22, 38)(23, 39)(26, 42)(27,
44)(28, 48)(30, 49)(31, 50)(33, 51)(37, 55)(40, 58)(41,
57)(43, 62)(45, 61)(47, 67)(54, 56)(59, 76)(60, 77)(63,
84)(64, 85)(68, 78)(69, 83)(71, 74)(72, 75)(73, 79)(80,
86)(81, 90)(82, 92)(87, 95)(88, 91)(89, 93)(94, 99)(96,
98)(97, 100); N:=sub<S100|x,y,t,s>;
print Order(N);
G<x,y,t,s,u>:=Group<x,y,t,s,u|x^7, y^2, (x*y)^3, (x,y)^4,
t^2,
(t^(x^3),y), (t^(x^4),x*y),y*(t^x*t^(x^3))^2, s^2,(s,x),
(s,y), (s,(t*x)^4*x^2*t),
s*s^t*s=t,u^2,(u,x),(u,y),(u,t),s=u*u^s*u>;
print Index(G, sub<G|x,y,t,s>: CosetLimit:=5000000,
Hard:=true);
 N1:=Stabiliser(N,1);
N113:=Stabiliser(N1,13);
for g in N do if 1^g eq 13 and 13^g eq 1 then
N113:=sub<N|N113,g>;
end if; end for;
```

```
for g in N do if 2^g eq 30 and 7^g eq 40 then
N113:=sub<N|N113,g>;
end if;end for;
for g in N do if 58^g eq 68 and 49^g eq 78 then
N113:=sub<N|N113,g>;
end if;end for;
for g in N do if 66^g eq 32 and 70^g eq 35 then
N113:=sub<N|N113,g>;
end if;end for;
print Order(N113);
print Orbits(N113);
```

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