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The use of divergent series in history

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THE USE OF DIVERGENT SERIES IN HISTORY

A Thesis

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in

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by

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ABSTRACT

This thesis seeks to present the history of non-convergent series. As we show in this thesis, in the past, divergent series have played an important role in mathematics. Euler, Cauchy, Abel, Fourier, Stirling and Poincare are just a few of the greatest mathematicians who used them. Today, non-convergent series play a marginal role in mathematics and are often not mentioned in the standard curriculum. Most students are not aware that they can be of any use, though such series are profitably employed in both physics and mathematics. When I discovered the text written by Bromwich, I thought it would be very interesting to learn more about non-convergent series. The study of divergent series may be divided into two parts: one concerning the asymptotic series and the other the theory of summability. In an asymptotic series the terms begin to decrease, and reach a minimum, afterwards increasing. If we take the sum to a stage at which the terms are sufficiently small, we may hope to obtain an approximation with a degree of accuracy represented by the last term retained; it can be proved that this is the case with many series which are convenient for numerical calculations, as we will see in Chapter Two. The theory of

summability is concerned with the question as to whether in any proper sense a "sum" may be assigned to the series, assumed divergent. One of the most important aspects of the theory of summability lies in its applications to Fourier series and other allied developments in mathematical physics. In this thesis we intend to study the asymptotic series.

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CHAPTER ONE

INTRODUCTION

The definitions of convergence and divergence are now commonplace in elementary analysis. The ideas were familiar to mathematicians before Newton and Leibniz (indeed to Archimedes)and all the great mathematicians of the seventeenth and eighteenth centuries, however recklessly they may seem to have manipulated series, knew well enough whether the series which they used were convergent. But it was not until the time of Cauchy that the definitions were formulated generally and explicitly. Newton and Leibniz, the first mathematicians to use infinite series systematically, had little temptation to use divergent series. The temptation became greater as analysis widened and it was soon found that they were useful and that operations performed on them uncritically often led to important results which could be verified independently. There is little written about divergent series before Euler. Mathematics after Euler moved slowly but steadily towards the orthodoxy ultimately imposed on it by Cauchy, Abel, and their successors, and divergent series were gradually banished from analysis, to reappear only in quite modern times. They had always had their opponents, such as d'Alembert, Laplace and Lagrange. After Cauchy, the opposition seemed definitely to have won. The analysts who used divergent series most, after Euler, were Fourier and Poisson (who was almost Cauchy's contemporary).

Our study will start with a few historical remarks on the use of non-convergent series. In Chapter Two we will give some general considerations on non-convergent series such as attaching a precise meaning to a non-convergent series, so that such series may be used for purposes of formal calculations, under proper restrictions. The Chapter continues with Euler's use of asymptotic series. We will show a few of his results. Then we are going to work with Bernoullian polynomials to find the remainder in Euler's formula. We end Chapter Two with an application of Euler's formula to Stirling's series. Various integrals of interest, both in Pure and Applied Mathematics, can be calculated most readily by means of asymptotic series. In Chapter Three we will use Integration by parts and Expansion of some function in the integral to obtain a suitable asymptotic series for a given integral. A few typical examples will be given: the error-function integral and the logarithmic integral. In this process we will be referring to the Gamma function. Next, we will work with asymptotic series for integrals containing sines and cosines: Fresnel's Integrals and the Sine- and Cosine-Integrals. In Chapter Four we will investigate Stirling's Series independent of Euler's Summation Formula as well as Stoke's Asymptotic Formula. Poincare's theory of asymptotic series along with a few of its applications will be studied in Chapter Five. Finally, in Chapter Six we will show some examples of the way in which asymptotic series present themselves in the solution of the differential equations.

CHAPTER TWO

SOME HISTORICAL REMARKS

Historical Remarks on the Use of Non-Convergent Series

Before the theory of convergence had been developed by Abel and Cauchy, mathematicians had no hesitation in using non-convergent series in both theoretical and numerical investigations.

In numerical work, however, they used only series which are now called asymptotic; in such series the terms begin to decrease, and reach a minimum, afterwards increasing. If we take the sum to a stage at which the terms are sufficiently small, we may hope to obtain an approximation of the function whose series turns out to be asymptotic with a degree of accuracy determined by the last term retained; and it can be proved that this is the case with many series which are convenient for numerical calculations.

An important class of such series consists of the series used by astronomers to calculate planetary positions: Poincare proved that, despite the fact that these series do not converge, the results of the calculations are confirmed by observation. The explanation of this fact could be inferred from Poincare's theory of asymptotic series (Chapter Five).

But mathematicians have often been led to use series of a different character, in which the terms never decrease, and may even increase to infinity. Typical examples of such series are:

(1) $$1-1+1-1+1-1+...;$$

(2) $$1-2+3-4+5-6+...;$$

(3) $$1-2+2^2-2^3+2^4-2^5+...;$$

Euler considered the "sum" of a non-convergent series as the finite numerical value of the arithmetical expression from the expansion of which the series was derived. Thus he defined the "sums" of the series (1)-(3) as follows:

$$(1)=\frac{1}{1+1}=\frac{1}{2}; \qquad (2)=\frac{1}{(1+1)^2}=\frac{1}{4}; \qquad (3)=\frac{1}{1+2}=\frac{1}{3};$$

Proof for (1):

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+...$$

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=1-x+x^2-x^3+...$$

Let $x=1$. Then $\frac{1}{1+1}=1-1+1-1+\ldots$

and the sum of (1) is $\frac{1}{2}$.

Proof for (2):

$$1-2+3-4+5-6+\ldots..$$
$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots=\sum_0^{\infty}(-1)^n x^n$$
$$\left(\frac{1}{1+x}\right)'=\frac{-1}{(1+x)^2}=-1+2x-3x^2+\ldots$$
$$\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+\ldots$$

Let $x=1$. Then $\frac{1}{(1+1)^2}=1-2+3-4+5-\ldots$

and the sum of (2) is $\frac{1}{(1+1)^2}=\frac{1}{4}$

Proof for (3):

$$1-2+2^2-2^3+2^4-2^5+\ldots$$
$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots$$

Let $x=2$. Then $\frac{1}{1+2}=1-2+2^2-2^3+\ldots$

and the sum of (3) is $\frac{1}{1+2}=\frac{1}{3}$.

Euler's definition depends on the inversion of two limits, which, taken in one order, give a definite value, and taken in the reverse order give a non-convergent series. Therefore, series (1) is:

$$\lim 1 - \lim x + \lim x^2 - \lim x^3 + \ldots$$

as x tends to 1; Euler's definition replaces this by

$$\lim(1 - x + x^2 - x^3 + \ldots)$$

So, generally, if $\sum f_n(c)$ is not convergent, Euler would define the "sum" as $\lim_{x\to c}\sum f_n(x)$, when this limit is definite.

Callet did not agree with Euler. Callet showed that the series:

$$1-1+1-1+1-1+\ldots$$

can also be obtained by writing $x=1$ in the series:

$$\begin{aligned}\frac{1+x}{1+x+x^2} &= \frac{(1+x)(1-x)}{(1+x+x^2)(1-x)} = \frac{1-x^2}{1-x^3} = (1-x^2)\frac{1}{(1-x^3)} = \\ &= \frac{1}{(1-x^3)} - x^2\frac{1}{(1-x^3)} = (1+x^3+x^6+\ldots)-(x^2+x^5+x^8+\ldots) = \\ &= 1-x^2+x^3-x^5+x^6-x^8+\ldots\end{aligned} \tag{5}$$

where by the left-hand side then becomes $\frac{2}{3}$ instead of $\frac{1}{2}$.

Lagrange, also, suggested that the series: $\frac{1+x}{1+x+x^2}$ should be written as: $1+0-x^2+x^3+0-x^5+x^6+0-x^8+\ldots$ and that then the derived series would be $1+0-1+1+0-1+1+0-1+\ldots$ The last series gives the sums to $1,2,3,4,5,6,\ldots$ terms as $1,1,0,1,1,0,\ldots$

$$s_0 = 1 \qquad s_3 = 1+0-1+1 = 1$$
$$s_1 = 1+0 = 1 \qquad s_4 = 1+0-1+1+0 = 1$$
$$s_2 = 1+0-1 = 0 \qquad s_5 = 1+0-1+1+0-1 = 0$$

Therefore the average sum is $\frac{2}{3}$, agreeing with Callet's result.

In fact, Frobenius pointed out that if $\sum_{n=0}^{\infty} a_n x^n$ is any power series having a radius of convergence equal to 1, then

$$\lim_{x\to 1}\sum_{n=0}^{\infty} a_n x^n = \lim_{n\to\infty}\frac{s_0+s_1+s_2+\ldots s_n}{n+1}$$

where $s_n = a_o + a_1 + \ldots + a_n$, putting Lagrange's remark on a more satisfactory basis.

So, the average sum is $\frac{1}{n+1}\sum_{j=1}^{n} s_j$.

$$\lim_{x\to 1}\sum_{n=0}^{\infty} a_n x^n = \lim_{n\to\infty}\frac{s_0+s_1+s_2+\ldots s_n}{n+1} = \lim_{n\to\infty}\frac{1+1+0+1+1+0+\ldots}{n+1}$$

Now we can notice that $s_0+s_1+s_2+\ldots s_n = (n+1)-k$, where k is the integral part of $\frac{1}{3}(n+1)$. Therefore the average sum is $\frac{2}{3}$, which is the value given by the left-hand side of (5).

In the original series (1), the sums are $1,0,1,0,1,0,\ldots$, of which the average is $1/2$, agreeing with Euler's sum.

Euler and other mathematicians made many discoveries by using series which do not converge. In fact, the older mathematicians had sufficient experimental evidence that the use of non-convergent series as if they were convergent led to correct results in the majority of cases when they presented themselves naturally.

An Example of the Use of a Non-convergent Series to Obtain a Correct Result

Let us find the Fourier series for the function:

$$f(x)=\frac{\pi\sinh x}{2\sinh\pi}$$

According to Fourier $f:[-\pi,\pi]\to R$

$$f(x)=\frac{a_0}{2}+\sum_1^{\infty}a_n\cos nx+\sum_1^{\infty}b_n\sin nx$$

where

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx$$
$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx$$

Fourier found that the coefficient b_n of $\sin nx$ is:

$$(-1)^{n-1}\left(\frac{1}{n}-\frac{1}{n^3}+\frac{1}{n^5}-\ldots\right)=(-1)^{n-1}\frac{n}{1+n^2}$$

which, for $n=1$ is a divergent series.

To see this, note that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\pi\sinh x}{2\sinh\pi}\sin nxdx = \int_{-\pi}^{\pi}\frac{\frac{1}{2}\left(e^x - e^{-x}\right)}{e^{\pi}-e^{-\pi}}\sin nxdx =$$

$$= \frac{1}{2\left(e^{\pi}-e^{-\pi}\right)}\int_{-\pi}^{\pi}\left(e^x - e^{-x}\right)\sin nxdx$$

Let

$$I = \int_{-\pi}^{\pi}\left(e^x - e^{-x}\right)\sin nxdx = -\left.\frac{\left(e^x - e^{-x}\right)\cos nx}{n}\right|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\left(e^x + e^{-x}\right)\cos nxdx$$

$$= \frac{\left(e^{-\pi}-e^{\pi}\right)\cos n\pi - \left(e^{\pi}-e^{-\pi}\right)\cos(-n)\pi}{n} + \frac{1}{n}\left.\left(\frac{\left(e^x + e^{-x}\right)\sin nx}{n}\right)\right|_{-\pi}^{\pi} -$$

$$-\frac{1}{n^2}\int_{-\pi}^{\pi}\left(e^x - e^{-x}\right)\sin nxdx$$

But $\left.\left(\frac{\left(e^x + e^{-x}\right)\sin nx}{n}\right)\right|_{-\pi}^{\pi} = 0$, therefore,

$$I = \frac{2\cos n\pi\left(e^{-\pi}-e^{\pi}\right)}{n} - \frac{1}{n^2}I$$

$$\frac{n^2+1}{n^2}I = \frac{2\cos n\pi\left(e^{-\pi}-e^{\pi}\right)}{n}$$

$$I = \frac{2n(\cos n\pi)\left(e^{-\pi}-e^{\pi}\right)}{n^2+1} = \begin{cases} \dfrac{2n\left(e^{\pi}-e^{-\pi}\right)}{n^2+1}, & \text{n=odd} \\ \dfrac{2n\left(e^{-\pi}-e^{\pi}\right)}{n^2+1}, & \text{n=even} \end{cases}$$

Therefore the coefficient of $\sin nx$ is

$$b_n = (-1)^{n-1}\frac{n}{n^2+1} = (-1)^{n-1}\frac{1}{n}\left(\frac{1}{1+1/n^2}\right)$$

which, by the geometric series expansion, becomes

$$b_n = (-1)^{n-1}\frac{n}{1+n^2} = (-1)^{n-1}\frac{1}{n}\left(1-\frac{1}{n^2}+\frac{1}{n^4}-...\right) = \left(-1^{n-1}\right)\left(\frac{1}{n}-\frac{1}{n^3}+\frac{1}{n^5}-...\right)$$

When $n=1$ we find b_1 which is the coefficient of $\sin x$:

$$b_1 = \frac{1}{1+1} = \frac{1}{2} = 1-1+1-1+1-1+....$$

So, we find that the sum of

(1) $\quad 1-1+1-1+1-1+....$

is $\frac{1}{2}$.

As a matter of fact, this is correct, since:

$$\int \sinh x \sin x dx = \frac{1}{2}(\cosh x \sin x - \sinh x \cos x)$$

so that:

$$\frac{2}{\pi}\int_0^{\pi} f(x)\sin x dx = \frac{1}{2}$$

Abel and Cauchy did not use non-convergent series in their work. They said that the use of non-convergent series had sometimes led to gross errors. However,the banishing of non-convergent series from their work was done with some hesitation.

Cauchy formulated the asymptotic property of Stirling's series by means of a method which can be applied to a large class of power-series. But the possibility of obtaining other useful asymptotic series was overlooked by later analysts; and, after Cauchy, mathematicians abandoned all attempts at utilizing non-convergent series. In England, however, Stokes published three remarkable papers (dated 1850, 1857, 1868), in which Cauchy's method for dealing with Stirling's series was applied to a number of other problems, such as the calculation of Bessel's functions for large values of the variable.

But no general theory of non-convergent series was forthcoming until 1886, when papers discussing the subject were written by Stieltjes and Poincare'. Since that time many researches have been published on the theory.

In the following articles we will work with the most important examples of asymptotic series, which have been found of importance in calculations.

General Considerations on Non-Convergent Series

In general, the "sum" of a series (convergent or divergent)was taken to be the number most naturally associated with it from the standpoint of mathematical operations. This concept, however, naturally led to inconsistency.

The notion of sum as thus loosely conceived was eventually replaced by the exact definition of Abel and Cauchy according to which the sum of any series

$$a_o + a_1 + a_2 + \ldots$$

is taken to mean the limit

$$s = \lim_{n\to\infty}\left(a_0 + a_1 + a_2 + \ldots + a_n\right).$$

Series for which this limit exists were termed convergent, all others divergent.

In view of the results obtained in the past by the use of non-convergent series, it seems probable that we can attach a perfectly precise meaning to a non-convergent series, so that such series may be used for purposes of formal calculation, under proper restrictions.

Of course it is evident that the "sum" associated with a non-convergent series is not to be confounded with the sum of a convergent series; but it will avoid confusion if the definition is such that the same operation, when applied to a convergent series, yields the sum in the ordinary sense.

Euler was perfectly aware of the distinction between his "sum" of a non-convergent series and the sum of a convergent series. Thus he says that the series:

$$1-2+2^2-2^3+2^4-\ldots=\frac{1}{1+2}=\frac{1}{3}$$

obviously cannot have the sum $\frac{1}{3}$ in the ordinary sense, since the sum of n terms differs more and more from $\frac{1}{3}$ as n becomes larger.

$$S_n=1-2+2^2-2^3+\ldots+(-1)^{n-1}2^{n-1}$$

$$S_n=\sum_{k=0}^{n-1}(-1)^k 2^k=\sum_{k=0}^{n-1}(-2)^k=\frac{1-(-2)^n}{1-(-2)}=\frac{1}{3}\left(1-(-2)^n\right)$$

And he adds that contradictions can be avoided by attributing a somewhat different meaning to the word sum. He defines the sum of any infinite series as the finite expression, by the expansion of which the series is generated. In this sense the sum of the infinite series

$$1-x+x^2-x^3+\ldots$$

will be $\frac{1}{1+x}$,because the series arises from the expansion of the fraction, whatever number is put in place of x. If this is agreed, the new definition of the word sum coincides with the ordinary meaning when a series converges; and since divergent series have no sum, in the proper sense of the word, no inconvenience can arise from this new terminology.

In practice, Euler used his definition almost exclusively in the form

$$\sum_{n=0}^{\infty} a_n = \lim_{x\to 1}\sum_{n=0}^{\infty} a_n x^n$$

and if restricted to this case, Euler's statement is correct.

The legitimate use of non-convergent series is always symbolic; the operations being merely convenient, though justifiable abbreviations of more complicated transformations in the background.

Even though we might just as well write the work in full, experience shows that the use of the asymptotic series often suggests useful transformations which otherwise might never be thought of.

An example of this may be taken from Euler's correspondence with Nicholas Bernoulli; Euler wanted to show how to attach a definite meaning to the series:

$$1-2!+3!-4!+5!-\ldots$$

He proves first that the series

$$x-1!x^2+2!x^3-3!x^4+\ldots$$

satisfies formally the differential equation

$$x^2\frac{dy}{dx}+y=x$$

Let

$$y = x - 1!x^2 + 2!x^3 - 3!x^4 + \ldots$$

$$\frac{dy}{dx} = 1 - 2!x + 3!x^2 - 4!x^3 + \ldots$$

$$x^2\frac{dy}{dx} = x^2 - 2!x^3 + 3!x^4 - 4!x^5 + \ldots$$

$$y = x - 1!x^2 + 2!x^3 - 3!x^4 + \ldots$$

$$x^2\frac{dy}{dx} + y = x \quad \text{and} \quad \frac{dy}{dx} + \frac{1}{x^2}y = \frac{1}{x}$$

$$\text{where } P = \frac{1}{x^2}, \quad Q = \frac{1}{x}$$

The integrating factor is $e^{\int Pdx} = e^{\int x^{-2}dx} = e^{-\frac{1}{x}}$ and the solution is $ye^{\int Pdx} = \int Qe^{\int Pdx}dx$, i.e.

$$ye^{-\frac{1}{x}} = \int \frac{1}{x}e^{-\frac{1}{x}}dx = e^{\frac{1}{x}}\int \frac{1}{x}e^{-\frac{1}{x}}dx = e^{\frac{1}{x}}\int \frac{1}{\xi}e^{-\frac{1}{\xi}}d\xi = \int_0^x \frac{1}{\xi}e^{\frac{1}{x}-\frac{1}{\xi}}d\xi$$

Let $\frac{1}{\xi} - \frac{1}{x} = t$.

So, the solution is

$$y = -\int_0^\infty e^{-t}\frac{-x}{1+xt}dt = \int_0^\infty \frac{xe^{-t}}{1+xt}dt$$

$$y = \int_0^\infty \frac{xe^{-t}}{1+xt}dt$$

Therefore, $\quad x - (1!)x^2 + (2!)x^3 - (3!)x^4 + \ldots = \int_0^\infty \frac{xe^{-t}}{1+xt}dt$

in agreement with the result found in Chapter Three below, showing that Euler was right that he had never been led into error by using his definition of "sum".

Numerical Evaluation of Non-convergent Series

A very natural method for the numerical evaluation of non-convergent series is given by Euler's transformation of slowly convergent series (Appendix A); as an illustration we take the series used by Euler:

(4) $$\log_{10} 2 - \log_{10} 3 + \log_{10} 4 - \ldots$$

Starting at $\log_{10} 10$, the differences are given in the table below:

$a_{10} = \log_{10} 10 = 1$		
$a_{11} = \log_{10} 11 = 1.0413927$	$a_{10} - a_{11} = -0.0413927$	
$a_{12} = \log_{10} 12 = 1.0791812$	$a_{11} - a_{12} = -0.0377885$	$(a_{10} - a_{11}) - (a_{11} - a_{12}) = -0.0036042$
$a_{13} = \log_{10} 13 = 1.1139434$	$a_{12} - a_{13} = -0.0347622$	$(a_{11} - a_{12}) - (a_{12} - a_{13}) = -0.0030263$
$a_{14} = \log_{10} 14 = 1.1461280$	$a_{13} - a_{14} = -0.0321846$	$(a_{12} - a_{13}) - (a_{13} - a_{14}) = -0.0025776$
$a_{15} = \log_{10} 15 = 1.1760913$	$a_{14} - a_{15} = -0.0299633$	$(a_{13} - a_{14}) - (a_{14} - a_{15}) = -0.0022213$
$a_{16} = \log_{10} 16 = 1.2041200$	$a_{15} - a_{16} = -0.0280287$	$(a_{14} - a_{15}) - (a_{15} - a_{16}) = -0.0019346$
$a_{17} = \log_{10} 17 = 1.2304489$	$a_{16} - a_{17} = -0.0263289$	
$a_{18} = \log_{10} 18 = 1.2552725$	$a_{17} - a_{18} = -0.0248236$	

$$(a_{10} - 2a_{11} + a_{12}) - (a_{11} - 2a_{12} + a_{13}) = -0.0005779$$
$$(a_{11} - 2a_{12} + a_{13}) - (a_{12} - 2a_{13} + a_{14}) = -0.0004487$$
$$(a_{12} - 2a_{13} + a_{14}) - (a_{13} - 2a_{14} + a_{15}) = -0.0003563$$
$$(a_{13} - 2a_{14} + a_{15}) - (a_{14} - 2a_{15} + a_{16}) = -0.0002867$$

$$(a_{10}-3a_{11}+3a_{12}-a_{13})-(a_{11}-3a_{12}+3a_{13}-a_{14})=-0.0001292$$
$$(a_{11}-3a_{12}+3a_{13}-a_{14})-(a_{12}-3a_{13}+3a_{14}-a_{15})=-0.0000924$$
$$(a_{12}-3a_{13}+3a_{14}-a_{15})-(a_{13}-3a_{14}+3a_{15}-a_{16})=-0.0000696$$

$$(a_{10}-4a_{11}+6a_{12}-4a_{13}+a_{14})-(a_{11}-4a_{12}+6a_{13}-4a_{14}+a_{15})=-0.0000368$$
$$(a_{11}-4a_{12}+6a_{13}-4a_{14}+a_{15})-(a_{12}-4a_{13}+6a_{14}-4a_{15}+a_{16})=-0.0000228$$

$$(a_{10}-5a_{11}+10a_{12}-10a_{13}+5a_{14}-a_{15})-(a_{11}-5a_{12}+10a_{13}-10a_{14}+5a_{15}-a_{16})=-0.0000140$$

From Appendix A we know that:

$$\sum_0^\infty(-1)^n v_n=\frac{1}{2}\left(v_0+\frac{1}{2}Dv_0+\frac{1}{2^2}D^2v_0+\frac{1}{2^3}D^3v_0+\ldots+\frac{1}{2^{p-1}}D^{p-1}v_0\right)+$$
$$+\frac{1}{2^p}\left(D^pv_0-D^pv_1+D^pv_2-\ldots\right)=$$
$$=\frac{1}{2}v_0+\frac{1}{4}Dv_0+\frac{1}{8}D^2v_0+\frac{1}{16}D^3v_0+\frac{1}{2^5}D^4v_0+\frac{1}{2^6}D^5v_0+\frac{1}{2^7}D^6v_0+\ldots$$

where $v_0=a_{10},\ v_1=a_{11},\ldots$ and $Dv_0=a_{10}-a_{11},\ D^2v_0=a_{10}-2a_{11}+a_{12},\ldots$

Therefore, the "sum" from $\log_{10}10$ onwards is approximated by:

$$.5000000-\left\{\frac{1}{4}(.0413927)+\frac{1}{8}(.0036042)+\frac{1}{16}(.0005779)+\frac{1}{32}.0001292+\right.$$
$$\left.+\frac{1}{64}(.0000368)+\frac{1}{128}(.0000140)\right\}=.5000000-(.0108396)$$

The sum of the first eight terms in the series is found to be (taking the terms in pairs)

$$\log_{10}2-\log_{10}3+\log_{10}4-\log_{10}5+\log_{10}6-\log_{10}7+\log_{10}8-\log_{10}9=$$
$$=-.1760913-.0969100-.0669467-.0511525=-.3911005$$

Combining these two results, the sum of the whole series appears to be: $.1088995-.0108396=.098060$,

which is exact up to six decimal places.

Euler's Use of Asymptotic Series

One of the earliest and most instructive examples of the application of non-convergent series was given by Euler in applying his formula of summation (Appendix E) to calculate certain finite sums.

In general, for any polynomial f and positive integer x,

$$f(1)+f(2)+...+f(x)=\int f(x)dx+\frac{1}{2}f(x)+\frac{1}{2!}B_1 f'(x)-\frac{1}{4!}B_2 f'''(x)+...$$

where B_i = Bernoulli's numbers.

We know from Appendix C that

$$B_r=\frac{(2r)!}{2^{2r-1}\pi^{2r}}\sum_{n=1}^{\infty}\frac{1}{n^{2r}}$$

It is obvious that Euler's summation formula converges for polynomials. For non-polynomials, however, we often get non-convergent (asymptotic) series on the right-hand side.

For example, taking $f(x)=\frac{1}{x}$, and $x=n$, we find

$$f(1)+f(2)+...+f(n)=\int f(x)dx+\frac{1}{2}f(x)+\frac{1}{2!}B_1 f^{(1)}(x)-\frac{1}{4!}B_2 f^{(3)}(x)+\frac{1}{6!}B_3 f^{(5)}(x)-...$$

$$1+\frac{1}{2}+...+\frac{1}{n}=\log n+\frac{1}{2n}-\frac{B_1}{2n^2}+\frac{B_2}{4n^4}-\frac{B_3}{6n^6}+...$$

Now this series does not converge.

We are going to prove that the series $\sum_r \frac{(-1)^r B_r}{2rn^{2r}}$ diverges using the ratio test.

For $a_r = \frac{(-1)^r B_r}{2rn^{2r}}$ calculate the ratio $\left|\frac{a_r}{a_{r-1}}\right|$:

$$\left|\frac{a_r}{a_{r-1}}\right| = \frac{B_r 2(r-1)n^{2(r-1)}}{2rn^{2r}B_{r-1}} = \frac{B_r}{B_{r-1}} \cdot \frac{r-1}{rn^2}$$

$$\frac{B_r}{B_{r-1}} = \frac{(2r)!2^{2r-3}\pi^{2r-2}}{(2r-2)!2^{2r-1}\pi^{2r}} \cdot \frac{\sum_n \frac{1}{n^{2r}}}{\sum_n \frac{1}{n^{2r-2}}} = \frac{(2r-1)2r}{4\pi^2} \cdot \frac{\sum_n \frac{1}{n^{2r}}}{\sum_n \frac{1}{n^{2r-2}}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{2r}} = 1 + \frac{1}{2^{2r}} + \frac{1}{3^{2r}} + \ldots + \frac{1}{n^{2r}} + \ldots > 1$$

If $r > 3$, $\sum_{n=1}^{\infty} \frac{1}{n^{2r-2}} < \sum_{n=1}^{\infty} \frac{1}{n^5} = \frac{1}{1 - \frac{1}{2^4}}$ (See Appendix I)

Therefore

$$\left|\frac{a_r}{a_{r-1}}\right| = \frac{B_r 2(r-1)n^{2(r-1)}}{2rn^{2r}B_{r-1}} = \frac{B_r}{B_{r-1}} \cdot \frac{r-1}{rn^2} > \frac{\left(1 - \frac{1}{2^4}\right)(r-1)2r(2r-1)}{4\pi^2 \cdot rn^2} =$$

$$= \frac{15(r-1)(2r-1)}{2\pi^2 n^2 \cdot 16} = \frac{15(r-1)\left(r - \frac{1}{2}\right)}{16\pi^2 n^2} > \frac{15(r-1)^2}{16\pi^2 n^2}$$

$\left|\frac{a_r}{a_{r-1}}\right| > \frac{15(r-1)^2}{16\pi^2 n^2}$, therefore we see that the terms in the series steadily increase in numerical value after a certain value of r (depending on n and roughly equal to the integer next greater than $1+n\pi$). We are not sure whether Euler realized that the series could never converge; but he was certainly aware of the fact that it does not converge for $n=1$. He used the series for $n=10$ to calculate the Euler constant

$$C = \lim_{n\to\infty}\left(1+\frac{1}{2}+...+\frac{1}{n}-\log n\right)$$

$$C = 0.5772156649015328(6060)...$$

which he regarded as the "sum" of the series $\frac{1}{2}+\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}-...$ for $n=1$.

The reason why this series can be used, although not convergent, is that the error in the value obtained by stopping at any particular stage in the series is less than the next term in the series. The truth of this statement follows from the general theorem proved in the next section.

To illustrate this point, consider the sums of the last series, and we find successively, with R_n as remainder and u_n as the next term after truncation, that

$$S_2 = \frac{1}{2} + \frac{B_1}{2} = .5833, \quad R_2 = C - S_2 = -.0061, \quad u_3 = -\frac{B_2}{4} = -.0083,$$
$$S_3 = \frac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} = .5750, \quad R_3 = C - S_3 = +.0022, \quad u_4 = \frac{B_3}{6} = +.0040,$$
$$S_4 = .5790, \quad R_4 = -.0018, \quad u_5 = -.0042,$$
$$S_5 = .5748, \quad R_2 = +.0024, \quad u_6 = +.0076,$$
$$S_6 = .5824, \quad R_6 = -.0052,$$

after which the terms steadily increase in numeral value. Thus, from this series we cannot obtain a closer approximation than S_4, which corresponds to stopping at the numerically least term u_4.

We quote a few of Euler's results for verification:

<u>Example 1</u>. Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} = 7.48547, \quad \text{if } n = 1000,$$
$$= 14.39273, \quad \text{if } n = 1000000.$$

Euler gives the values to 13 decimals.

Proof for (1):

$$C = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \lim_{n\to\infty} \ln n$$

For $n = 1000$, $\ln 1000 = 6.907755279$

$C = 0.5772156649$

Therefore

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}=7.48547$$

For $n=1000000$, $\ln 10000000=13.81551056$

$C=0.57721566$

Therefore $$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}=14.39273\,.$$

<u>Example 2</u>. Show that

i) $$1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}=\frac{1}{2}(C+\log n)+\log 2+\frac{B_1}{8n^2}-\frac{(2^3-1)B_2}{64n^4}+...,$$

and that

ii) $$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{2n-1}-\frac{1}{2n}=\log 2-\frac{1}{4n}+\frac{(2^2-1)B_1}{8n^2}-\frac{(2^4-1)B_2}{64n^4}+...,$$

Proof for (2i):

i) $$1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{2n-1}=\frac{1}{2}(C+\log n)+\log 2+\frac{B_1}{8n^2}-\frac{(2^3-1)B_2}{64n^4}+...,$$

We know that

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}=\ln n+\frac{1}{2^n}-\frac{B_1}{2n^2}+\frac{B_2}{4n^4}-\frac{B_3}{6n^6}+...$$

If n is replaced by 2n, then

$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2n}=C+\ln(2n)+\frac{1}{4n}-\frac{B_1}{2(2n)^2}+\frac{B_2}{4(2n)^4}-...=$$
$$=C+\ln 2+\ln n+\frac{1}{4n}-\frac{B_1}{8n^2}+\frac{B_2}{64n^4}-...$$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{2n}=\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{n}\right)=\frac{C}{2}+\frac{1}{2}\ln n+\frac{1}{4n}-\frac{B_1}{4n^2}+\frac{B_2}{8n^4}-...$$

$$1+\frac{1}{3}+...+\frac{1}{2n-1}=\frac{C}{2}+\ln 2+\frac{\ln n}{2}+\frac{B_1}{8n^2}+\frac{B_2}{64n^4}-\frac{B_2}{8n^4}-...$$

$$=\frac{C+\ln n}{2}+\frac{B_1}{8n^2}-\frac{2^3-1}{64n^4}B_2+...$$

Proof for (2ii):

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{2n-1}=\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{2n}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+...+\frac{1}{2n}\right)$$

$$=\left(C+\ln 2+\ln n+\frac{1}{4n}-\frac{B_1}{8n^2}+\frac{B_2}{64n^4}-...\right)-\left(C+\ln n+\frac{1}{2n}-\frac{B_1}{2n^2}+\frac{B_2}{4n^4}-...\right)$$

$$=\ln 2-\frac{1}{4n}+\frac{2^2-1}{8n^2}B_1-\frac{2^4-1}{64n^4}B_2+...$$

Example 3. Find a formula for

$$\frac{1}{a+b}+\frac{1}{2a+b}+\frac{1}{3a+b}+...+\frac{1}{na+b}$$

similar to Euler's formula.

Proof for (3):

We'll use Euler's formula of summation (See Appendix E)

$$f(1)+f(2)+...+f(x)=\int f(x)\,dx+\frac{1}{2}f(x)+\frac{1}{2!}B_1f'(x)-\frac{1}{4!}B_2f'''(x)+...$$

for $f(x)=\dfrac{1}{ax+b}=(ax+b)^{-1}$.

Then

$$\int f(x)\,dx=\frac{\ln(ax+b)}{a}+C$$

(note that C is not Euler's constant).

$$f'(x) = -(ax+b)^{-2}\,a = -a(ax+b)^{-2}$$

$$f''(x) = 2a^2(ax+b)^{-3}$$

$$f'''(x) = -2\cdot 3a^3(ax+b)^{-4}\text{, etc.}$$

$$\frac{1}{a+b}+\frac{1}{2a+b}+\ldots+\frac{1}{na+b}=$$
$$=\frac{1}{a}\ln(na+b)+\frac{1}{2(na+b)}-\frac{B_1 a}{2(na+b)^2}+\frac{\cancel{2\cdot 3}a^3B_2}{1\cdot\cancel{2\cdot 3}\cdot 4(na+b)^4}-\ldots$$
$$=\frac{1}{a}\ln(na+b)+\frac{1}{2(na+b)}-\frac{aB_1}{2(na+b)^2}+\frac{a^3B_2}{4(na+b)^4}-\frac{a^5B_3}{6(na+b)^6}+\ldots$$

<u>Example 4</u>. Taking $f(x)=\frac{1}{x^2}$, prove similarly that

$$\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}\ldots to\ \infty=\frac{1}{n}+\frac{1}{2n^2}+\frac{B_1}{n^3}-\frac{B_2}{n^5}+\frac{B_3}{n^7}-\ldots$$

Hence we find $\frac{1}{10^2}+\frac{1}{11^2}+\frac{1}{12^2}+\ldots to\ \infty=.1051663357$ and we deduce

that $\frac{\pi^2}{6}=1.6449340668$.

Proof for (4):

$$f(1)+f(2)+\ldots f(x)=\int f(x)\,dx+\frac{1}{2}f(x)+\frac{1}{2!}B_1f'(x)-\frac{1}{4!}B_2f'''(x)+\ldots$$

$$f(x) = \frac{1}{x^2} = x^{-2}$$

$$\int f(x) = -x^{-1}$$

$f'(x) = -2x^{-3}$, $f''(x) = 2\cdot 3x^{-4}$, $f'''(x) = -4!x^{-5}$, etc.

$$1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2} = -\frac{1}{n} + \frac{1}{2n^2} - \frac{B_1}{n^3} + \frac{B_2}{n^5} - \ldots$$

$$\sum_1^\infty \frac{1}{k^2} = \sum_1^n \frac{1}{k^2} + \sum_{n+1}^\infty \frac{1}{k^2}$$

$$\sum_{n+1}^\infty \frac{1}{k^2} = \sum_1^\infty \frac{1}{k^2} - \sum_1^n \frac{1}{k^2}$$

$$\sum_1^\infty \frac{1}{k^2} = \lim_{n\to\infty} \sum_1^n \frac{1}{k^2} = 0$$

$$\sum_{n+1}^\infty \frac{1}{k^2} = \frac{1}{n} - \frac{1}{2n^2} + \frac{B_1}{n^3} - \frac{B_2}{n^5} + \ldots$$

$$\frac{1}{n^2} + \sum_{n+1}^\infty \frac{1}{k^2} = \frac{1}{n} + \frac{1}{2n^2} + \frac{B_1}{n^3} - \frac{B_2}{n^5} + \ldots$$

If $n = 10$,

$$\frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2} + \ldots = \frac{1}{10} + \frac{1}{2\cdot 100} + \frac{B_1}{1000} = 0.1 + 0.005 + 0.000166 = 0.105166$$

We know $B_1 = \frac{1}{\pi^2}\sum_1^\infty \frac{1}{n^2}$, thus $\sum_1^\infty \frac{1}{k^2} = \pi^2 B_1 \simeq \frac{\pi^2}{6}$

Therefore,

$$\frac{\pi^2}{6} = \sum_1^\infty \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2} + \ldots = 1.6449340668$$

<u>Example 5</u>. Show similarly that

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} \ldots = 1.2020569032 .$$

Euler obtained in this manner the numerical values of $\sum \frac{1}{n^r}$ from $r = 2$ to 16, each calculated to 18 decimals. Stieltjes has carried on the calculations to 32 decimals from $r = 2$ to 70.

Proof for (5):

Take $f(x) = \frac{1}{x^3}$ in Euler's formula of summation:

$$f(1) + f(2) + \ldots f(x) = \int f(x)\,dx + \frac{1}{2} f(x) + \frac{1}{2!} B_1 f'(x) - \frac{1}{4!} B_2 f'''(x) + \ldots$$

$$f(x) = x^{-3}$$

$$\int f(x)\,dx = \frac{-1}{2x^2}$$

$f'(x) = -3x^{-4}$, $f''(x) = 3\cdot 4x^{-5}$, $f'''(x) = -3\cdot 4\cdot 5x^{-6}$, etc.

$$\sum_{1}^{\infty}\frac{1}{k^3} = \sum_{1}^{n}\frac{1}{k^3} + \sum_{n+1}^{\infty}\frac{1}{k^3}$$

$$\sum_{n+1}^{\infty}\frac{1}{k^3} = \sum_{1}^{\infty}\frac{1}{k^3} - \sum_{1}^{n}\frac{1}{k^3} = \lim_{m\to\infty}\sum_{1}^{m}\frac{1}{k^3} - \sum_{1}^{n}\frac{1}{k^3} = -\sum_{1}^{n}\frac{1}{k^3} -$$

$$-\left(-\frac{1}{2n^2} + \frac{1}{2n^3} - \frac{3B_1}{2!n^4} + \frac{5B_2}{2n^6} - \ldots\right) = \frac{1}{2n^2} - \frac{1}{2n^3} + \frac{3B_1}{2n^4} - \frac{5B_2}{2n^6} + \ldots$$

$$\frac{1}{n^3} + \sum_{n+1}^{\infty}\frac{1}{k^3} = \sum_{n}^{\infty}\frac{1}{k^3} = \frac{1}{2n^2} + \frac{1}{2n^3} + \frac{3B_1}{2n^4} - \frac{5B_2}{2n^6} + \ldots$$

If $n = 10$,

$$\frac{1}{10^3} + \frac{1}{11^3} + \ldots = \frac{1}{2\cdot 100} + \frac{1}{2\cdot 1000} + \frac{3B_1}{2\cdot 10000} = 0.005 + 0.0005 + 0.000025 = 0.005525$$

But $B_1 = \frac{1}{6}$.

Therefore,

$$\sum_{1}^{\infty}\frac{1}{k^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots + \frac{1}{9^9} + 0.005025 = 1.202056986\ .$$

Example 6. If $f(x)=\frac{1}{\left(l^2+x^2\right)}$, prove that

$$\frac{1}{l^2+1^2}+\frac{1}{l^2+2^2}+\ldots+\frac{1}{l^2+n^2}=\frac{1}{l}\left(\frac{\pi}{2}-\theta\right)-\frac{1}{2}\left(\frac{1}{l^2}-\frac{1}{l^2+n^2}\right)+$$

$$+\frac{\pi}{l\left(e^{2l\pi}-1\right)}-\frac{B_1}{2}\cdot\frac{\sin^2\theta\sin 2\theta}{l^3}+\frac{B_2}{4}\cdot\frac{\sin^4\theta\sin 4\theta}{l^5}-\ldots$$

where $\tan\theta=\frac{l}{n}$; the constant is determined by allowing n to tend to ∞ and using the series found in Appendix B.

Proof for (6):

We will use Euler's formula (Appendix E) for $f(x)=\frac{1}{l^2+x^2}$

$$\int\frac{1}{l^2+x^2}dx=\frac{1}{l}\arctan\frac{x}{l}+C$$

$$f'(x)=\frac{-2x}{\left(l^2+x^2\right)^2}=-2x\left(l^2+x^2\right)^{-2}$$

$$f''(x)=-\frac{2}{\left(l^2+x^2\right)^2}+\frac{2^3x^2}{\left(l^2+x^2\right)^3}=-2\left(l^2+x^2\right)^{-2}+2^3x^2\left(l^2+x^2\right)^{-3}$$

$$f'''(x)=\frac{2^3x}{\left(l^2+x^2\right)^3}+\frac{2^4x}{\left(l^2+x^2\right)^3}-\frac{2^4x^2\cdot 3x}{\left(l^2+x^2\right)^4}=\frac{2^3\cdot 3x}{\left(l^2+x^2\right)^3}-\frac{2^4\cdot 3x^3}{\left(l^2+x^2\right)^4}$$

$$=\frac{2^3\cdot 3x}{\left(l^2+x^2\right)^3}\left(1-\frac{2x^2}{l^2+x^2}\right)=\frac{2^3\cdot 3x\left(l^2-x^2\right)}{\left(l^2+x^2\right)^4}$$

$$\sum_{k=1}^{n}\frac{1}{l^2+k^2}=\frac{1}{l}\arctan\frac{n}{l}+C+\frac{1}{2\left(l^2+n^2\right)}-\frac{B_1 n}{\left(l^2+n^2\right)^2}-\frac{B_2 n\left(l^2-n^2\right)}{\left(l^2+n^2\right)^4}$$

$$\tan\theta=\frac{l}{n},\quad \arctan\frac{l}{n}=\theta$$

$$\arctan\frac{n}{l}=\frac{\pi}{2}-\theta \quad (\text{because}\quad \arctan x^2+\arctan\frac{1}{x^2}=\frac{\pi}{2})$$

$$\tan^2\theta=\frac{l^2}{n^2}\qquad l^2=n^2\tan^2\theta$$

$$l^2+n^2=n^2\tan^2\theta+n^2=n^2\left(1+\tan^2\theta\right)=n^2\left(1+\frac{\sin^2\theta}{\cos^2\theta}\right)=\frac{n^2}{\cos^2\theta}=l^2+n^2$$

$$\frac{B_1 n}{\left(l^2+n^2\right)^2}=\frac{B_1 n\cos^4\theta}{n^4}=\frac{B_1\cos^4\theta}{n\cdot n^2}=\frac{B_1\cos^4\theta\tan^2\theta}{n\cdot l^2}=$$

$$=\frac{B_1\cos^4\theta\dfrac{\sin^2\theta}{\cos^2\theta}}{n\cdot l^2}=\frac{B_1\sin^2\theta\cos^2\theta}{n\cdot l^2}=\frac{B_1\sin^2 2\theta}{4n\cdot l^2}=$$

$$=\frac{B_1\sin 2\theta\sin 2\theta}{4l^2\cdot n}=\frac{B_1\sin 2\theta}{4l^2}\cdot\frac{2\sin\theta\cos\theta\tan\theta}{l}$$

So, $$\frac{B_1 n}{\left(l^2+n^2\right)^2}=\frac{B_1\sin 2\theta\sin^2\theta}{2l^3}.$$

To find the constant C,

$$\sum_{k=1}^{\infty}\frac{1}{l^2+k^2}=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{l^2+k^2}=\frac{1}{l}\arctan\infty+C=\frac{\pi}{2l}+C$$

$$C=\sum_{k=1}^{\infty}\frac{1}{l^2+k^2}-\frac{\pi}{2l}$$

Use

$$\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2x}{x^2+4n^2\pi^2}$$ (See Appendix B)

$$\sum_{k=1}^{\infty}\frac{2x}{x^2+4k^2\pi^2}=2x\sum_{k=1}^{\infty}\frac{2x}{x^2+4k^2\pi^2}=\frac{1}{e^x-1}-\frac{1}{x}+\frac{1}{2}$$

Let $x=2l\pi$

$$\frac{1}{e^{2l\pi}-1}-\frac{1}{2l\pi}+\frac{1}{2}=4l\pi\sum_{k=1}^{\infty}\frac{1}{4l^2\pi^2+4k^2\pi^2}=4l\pi\sum_{k=1}^{\infty}\frac{1}{4\pi^2\left(l^2+k^2\right)}=\frac{l}{\pi}\sum_{1}^{\infty}\frac{1}{l^2+k^2}$$

$$\sum_{1}^{\infty}\frac{1}{l^2+k^2}=\frac{\pi}{l}\left(\frac{1}{e^{2l\pi}-1}-\frac{1}{2l\pi}+\frac{1}{2}\right)=\frac{\pi}{l\left(e^{2l\pi}-1\right)}-\frac{1}{2l^2}+\frac{\pi}{2l}$$

Therefore, $$C=\frac{\pi}{l\left(e^{2l\pi}-1\right)}-\frac{1}{2l^2}$$

$$\frac{B_2 n\left(n^2-l^2\right)}{\left(l^2+n^2\right)^4}=\frac{B_2 n\cos^8\theta}{n^8}=\frac{B_2 n\cos^8\theta}{n^8}\left(n^2-n^2\tan^2\theta\right)==\frac{B_2\cos^8\theta\left(\cos^2\theta-\sin^2\theta\right)}{n^5\cos^2\theta}=$$

$$=\frac{B_2\cos^6\theta\cos 2\theta}{n^5}=\frac{B_2\cos^6\theta\cos 2\theta\tan^5\theta}{l^5}=\frac{B_2\sin^5\theta\cos 2\theta\cos\theta}{l^5}=$$

$$=\frac{B_2\sin^4\theta\sin\theta\cos\theta\cos 2\theta}{l^5}=\frac{B_2\sin^4\theta\sin 2\theta\cos 2\theta}{2l^5}=$$

So, $$\frac{B_2 n\left(n^2-l^2\right)}{\left(l^2+n^2\right)^4}=\frac{B_2\sin^4\theta\sin 4\theta}{4l^5}.$$

Therefore,

$$\sum_{k=1}^{n}\frac{1}{l^2+k^2}=\frac{1}{l}\left(\frac{\pi}{2}-\theta\right)+\frac{\pi}{l\left(e^{2l\pi}-1\right)}-\frac{1}{2}\left(\frac{1}{l^2}-\frac{1}{l^2+n^2}\right)-\frac{B_1}{2}\cdot\frac{\sin^2\theta\sin 2\theta}{l^3}+$$
$$+\frac{B_2\sin^4\theta\sin 4\theta}{4l^5}-\ldots$$

Example 7. In particular, by writing $l=n$ (in Example 6) we find

$$\pi=4\mathrm{n}\left(\frac{1}{\mathrm{n}^2+1}+\frac{1}{\mathrm{n}^2+2^2}+\ldots+\frac{1}{\mathrm{n}^2+n^2}\right)+\frac{1}{n}-\frac{4\pi}{e^{2n\pi}-1}+$$
$$+\frac{B_1}{1\cdot n^2}-\frac{B_3}{3\cdot 2^2\cdot n^6}+\frac{B_5}{5\cdot 2^4\cdot n^{10}}-\frac{B_7}{7\cdot 2^6\cdot n^{14}}+\ldots$$

By writing $n=5$, Euler calculates the value of π to 15 decimals.

Proof for (7):

If $l=n$, $\tan\theta=1$, $\theta=\frac{\pi}{4}$

$$\sin 2\theta = \sin\frac{\pi}{2} = 1$$

$$\sin^2\theta = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}$$

$$\sin^4\theta = \frac{1}{2^2}$$

$$\sin 4\theta = \sin\pi = 0$$

It follows that the B_{2r} terms become 0.

$$\sin^6\theta = \frac{1}{2^3}$$

$$\sin 6\theta = \sin\frac{3\pi}{2} = -1$$

From Example(6) we have that

$$\sum_{k=1}^{n}\frac{1}{l^2+k^2} = \frac{1}{n}\cdot\frac{\pi}{4} + \frac{\pi}{n\left(e^{2n\pi}-1\right)} - \frac{1}{2}\cdot\frac{1}{2n^2} - \frac{B_1}{2}\cdot\frac{1}{2n^3} + \frac{B_3}{6}\cdot\frac{1}{2^3n^7} - \ldots =$$

$$= 4n\sum_{k=1}^{n}\frac{1}{l^2+k^2} = \pi + \frac{4\pi}{e^{2n\pi}-1} - \frac{1}{n} - \frac{B_1}{n^2} + \frac{B_3}{3\cdot 2^2\cdot n^6} - \ldots$$

$$\pi = 4n\sum_{k=1}^{n}\frac{1}{l^2+k^2} + \frac{1}{n} - \frac{4\pi}{e^{2n\pi}-1} + \frac{B_1}{n^2} - \frac{B_3}{3\cdot 2^2\cdot n^6} + \frac{B_5}{5\cdot 2^4\cdot n^{10}} - \ldots$$

The Remainder in Euler's Formula

We have seen (Appendix D) that the Bernoullian polynomials $\phi_n(t)$ satisfy the following relations:

$$\phi'_{2m}(x) = 2m\phi_{2m-1}(x), \qquad (m>1)$$

$$\phi'_{2m+1}(x) = (2m+1)\left(\phi_{2m}(x) + (-1)^{m-1} B_m\right), \qquad (m \geq 1)$$

and further, that $\phi_n(t)$ is zero both for $t=0$ and $t=1$.

If $t=1$, then $\phi_n(1)=0$ (because there is no $\frac{t^n}{n!}$ in the expansion of $\frac{t\left(e^{xt}-1\right)}{e^t-1}$). If $t=0$, then $\phi_n(0)=0$.

It follows that if F is any differentiable function,

$$\int_0^1 \phi_{2n}(t) F''(t)\,dt = F'(t)\phi_{2n}(t)\Big|_0^1 - \int_0^1 \phi'_{2n}(t) F'(t)\,dt = -\int_0^1 2n\phi_{2n-1}(t) F'(t)\,dt$$

Similarly

$$\int_0^1 \phi_{2n-1}(t) F'(t)\,dt = F(t)\phi_{2n-1}\Big|_0^1 - \int_0^1 \phi'_{2n-1}(t) F(t)\,dt =$$

$$= -\int_0^1 (2n-1)\left(\phi_{2n-2}(t) + (-1)^{n-2} B_{n-1}\right) F(t)\,dt$$

Combining these two results, we see that

$$\int_0^1 \phi_{2n}(t) F''(t)\,dt = 2n(2n-1)\int_0^1 \left(\phi_{2n-2}(t) + (-1)^{n-2} B_{n-1}\right) F(t)\,dt$$

or that

$$\int_0^1 \phi_{2n}(t) F''(t)\,dt - 2n(2n-1)\int_0^1 \phi_{2n-2}(t) F(t)\,dt = 2n(2n-1)B_{n-1}(-1)^n \int_0^1 F(t)\,dt\ .$$

Then, replacing $F(t)$ by $f^{2n-2}(x+t)$, we have

we have

$$\int_0^1 \phi_{2n}(t) f^{2n}(x+t)\,dt =$$

$$= 2n(2n-1)\int_0^1 \phi_{2n-2}(t) f^{2n-2}(x+t)\,dt + 2n(2n-1)B_{n-1}(-1)^{n-2}\int_0^1 f^{2n-2}(x+t)\,dt$$

Thus, if we write

$$X_n = \frac{1}{(2n)!}\int_0^1 \phi_{2n}(t) f^{2n}(x+t)\,dt\ ,$$

we find that

$$X_n - X_{n-1} =$$

$$= \frac{1}{(2n-2)!}\left(\int_0^1 \phi_{2n-2}(t) f^{2n-2}(x+t)\,dt + B_{n-1}(-1)^n \int_0^1 f^{2n-2}(x+t)\,dt - \int_0^1 \phi_{2n-2}(t) f^{2n-2}(x+t)\,dt\right)$$

$$= \frac{1}{(2n-2)!}B_{n-1}(-1)^n \int_0^1 f^{2n-2}(x+t)\,dt = \frac{B_{n-1}(-1)^n}{(2n-2)!} f^{2n-3}(x+t)\Big|_0^1$$

$$X_n - X_{n-1} = \frac{B_{n-1}(-1)^n}{(2n-2)!}\left(f^{2n-3}(x+1) - f^{2n-3}(x)\right).$$

This relation holds for values of $n > 1$; to complete the sequence, consider the integral

$$X_1 = \frac{1}{2}\int_0^1 \phi_2(t) f''(x+t)\,dt = \frac{1}{2}\int_0^1 (t^2 - t) f''(x+t)\,dt =$$

$$= \frac{1}{2}\left[(t^2 - t) f'(x+t)\Big|_0^1 - \int_0^1 (2t-1) f'(x+t)\,dt\right] =$$

$$= -\frac{1}{2}\int_0^1 2t f'(x+t)\,dt + \frac{1}{2}\int_0^1 f'(x+t)\,dt =$$

$$= -tf(x+t)\Big|_0^1 + \int_0^1 f(x+t)\,dt + \frac{1}{2} f(x+t)\Big|_0^1 =$$

$$= -f(x+1) + \int_0^1 f(x+t)\,dt + \frac{1}{2} f(x+1) - \frac{1}{2} f(x)$$

We find

$$X_1 = \int_0^1 f(x+t)\,dt - \frac{1}{2} f(x+1) - \frac{1}{2} f(x)$$

$$\frac{1}{2}\big(f(x+1) + f(x)\big) = \int_0^1 f(x+t)\,dt - X_1$$

and by the change of variables $x+t=\xi$, we have

$$\frac{1}{2}\big(f(x+1) + f(x)\big) = \int_x^{x+1} f(\xi)\,d\xi - X_1$$

Also, from the general formula

$$X_n - X_{n-1} = (-1)^n \frac{B_{n-1}}{(2n-2)!}\left(f^{2n-3}(x+1) - f^{2n-3}(x)\right)$$

we have

for $n=2$ $$-X_1 + X_2 = \frac{B_1}{2!}\left[f'(x+1) - f'(x)\right]$$

for $n=3$ $$-X_2 + X_3 = -\frac{B_2}{4!}\left[f'''(x+1) - f'''(x)\right],$$ etc.

That is, we have successively

$$\frac{1}{2}\left[f(x+1) + f(x)\right] = \int_x^{x+1} f(\xi)\,d\xi - X_1 =$$

$$= \int_x^{x+1} f(\xi)\,d\xi + \frac{B_1}{2!}\left[f'(x+1) - f'(x)\right] - X_2 =$$

$$= \int_x^{x+1} f(\xi)\,d\xi + \frac{B_1}{2!}\left[f'(x+1) - f'(x)\right] - \frac{B_2}{4!}\left[f'''(x+1) - f'''(x)\right] - X_3 \text{, etc.}$$

If we now write $x = a,\ a+1, \ldots, b-1$ where $b-a$ is any positive integer, and add the results, we obtain Euler's summation formula (as in Appendix E), but with a remainder term.

Let $x = a$

$$\frac{1}{2}\left[f(a+1) + f(a)\right] = \int_a^{a+1} f(\xi)\,d\xi + \frac{B_1}{2!}\left[f'(a+1) - f'(a)\right] - \frac{B_2}{4!}\left[f'''(a+1) - f'''(a)\right] - \ldots$$

Let $x = a+1$

$$\frac{1}{2}\left[f(a+2) + f(a+1)\right] =$$

$$= \int_{a+1}^{a+2} f(\xi)\,d\xi + \frac{B_1}{2!}\left[f'(a+2) - f'(a+1)\right] - \frac{B_2}{4!}\left[f'''(a+2) - f'''(a+1)\right] - \ldots$$

Let $x = a + (b-a) - 1 = b-1$

$$\frac{1}{2}\left[f(b) + f(b-1)\right] =$$

$$= \int_{b-1}^{b} f(\xi)\,d\xi + \frac{B_1}{2!}\left[f'(b) - f'(b-1)\right] - \frac{B_2}{4!}\left[f'''(b) - f'''(b-1)\right] - \ldots$$

Adding, we find

$$f(a) + f(a+1) + \ldots + f(b) = \int_a^b f(\xi)\,d\xi + \frac{1}{2}\left[f(a) + f(b)\right] + \frac{B_1}{2!}\left[f'(b) - f'(a)\right] -$$

$$-\frac{B_2}{4!}\left[f'''(b)-f'''(a)\right]+\ldots+(-1)^n\frac{B_{n-1}}{(2n-2)!}\left(f^{2n-3}(b)-f^{2n-3}(a)\right)-R_n$$

where R_n is the remainder term and

$$R_n=\frac{1}{(2n)!}\int_0^1\phi_{2n}(t)\left[f^{2n}(a+t)+f^{2n}(a+1+t)+\ldots+f^{2n}(b-1+t)\right]dt$$

It is to be noticed also that

$$R_n-R_{n+1}=\frac{1}{(2n)!}\int_0^1\phi_{2n}(t)\left[f^{2n}(a+t)+f^{2n}(a+1+t)+\ldots+f^{2n}(b-1+t)\right]dt-$$

$$-\frac{1}{(2n+2)!}\int_0^1\phi_{2n+2}(t)\left[f^{2n+2}(a+t)+f^{2n+2}(a+1+t)+\ldots+f^{2n+2}(b-1+t)\right]dt=$$

$$=\frac{1}{(2n)!}\int_0^1\phi_{2n}(t)\left[f^{2n}(a+t)+f^{2n}(a+1+t)+\ldots+f^{2n}(b-1+t)\right]dt-$$

$$-\frac{1}{(2n+2)!}\left[(2n+2)(2n+1)\int_0^1\phi_{2n}(t)f^{2n}(a+t)\,dt+\right.$$

$$+(2n+2)(2n+1)B_n(-1)^{n+1}\int_0^1 f^{2n}(a+t)\,dt+$$

$$+(2n+2)(2n+1)\int_0^1\phi_{2n}(t)f^{2n}(a+1+t)\,dt+$$

$$\left.+(2n+2)(2n+1)B_n(-1)^{n+1}\int_0^1 f^{2n}(a+1+t)\,dt+\ldots\right]=$$

$$=\frac{1}{(2n)!}\int_0^1\phi_{2n}(t)\left[f^{2n}(a+t)+\ldots+f^{2n}(b-1+t)\right]dt-$$

$$-\frac{1}{2n!}\int_0^1\phi_{2n}(t)\left[f^{2n}(a+t)+\ldots+f^{2n}(b-1+t)\right]dt-$$

$$-\frac{1}{(2n)!}B_n(-1)^{n+1}\int_0^1\left[f^{2n}(a+t)+\ldots+f^{2n}(b-1+t)\right]dt=$$

$$=\frac{1}{(2n)!}B_n(-1)^n\left[f^{2n-1}(a+t)\Big|_0^1+\ldots+f^{2n}(b-1+t)\Big|_0^1\right]=$$

$$=\frac{(-1)^n B_n}{(2n)!}\left[f^{2n-1}(a+1)-f^{2n-1}(a)+f^{2n-1}(a+2)-f^{2n-1}(a+1)+\ldots+f^{2n-1}(b)-f^{2n-1}(b-1)\right]$$

So

$$R_n-R_{n+1}=\frac{(-1)^n B_n}{(2n)!}\left[f^{2n-1}(b)-f^{2n-1}(a)\right]$$

which gives the next term in Euler's summation formula.

Now it has been proved (Appendix D) that the Bernoullian polynomials $\phi_{2n}(t)$ and $\phi_{2n+2}(t)$ are both of constant sign, but their signs are opposite. ϕ_{2k} has the sign of $(-1)^k$. Thus, if we assume that the signs of $f^{2n}(x)$, $f^{2n+2}(x)$ are the same and that their common sign remains constant for all values of x from a to b, the integrals R_{n+1}, R_n have opposite signs.

Therefore, $|R_n|<|R_n-R_{n+1}|<\frac{B_n}{(2n)!}\left|f^{2n-1}(b)-f^{2n-1}(a)\right|$.

So, the error involved in omitting R_n from Euler's summation formula is numerically less than the next term, and has the same sign; that shows that, in fact, the series so obtained has the same property as a convergent series of decreasing terms, which have alternate signs.

Theoretically, however, the convergent series can be pushed to an arbitrary degree of approximation, while an asymptotic series cannot; but in practice it often happens that an asymptotic series gives a better approximation for numerical work than a convergent series, as in Examples 5 and 6 of the last article.

Application of Euler's Formula to Stirling's Series

Taking $f(x)=\log x$ in the general formula, we find

$\log(n!)$: $\quad f(1)=\log 1=0,\ f(2)=\log 2,\ f(3)=\log 3,\ \ldots\ f(n)=\log(n)$.

$f'(x)=x^{-1},\ f''(x)=-x^{-2},\ f'''(x)=2x^{-3},\ f^4(x)=-2\cdot 3\cdot x^{-4},\ f^5(x)=2\cdot 3\cdot 4x^{-5}$,etc.

$$f(1)+f(2)+f(3)+\ldots+f(n)=\int_1^n f(\xi)d\xi+\frac{1}{2}\left[f(1)+f(n)\right]+\frac{B_1}{2!}\left[f'(n)-f'(1)\right]-$$

$$-\frac{B_2}{4!}\left[f'''(n)-f'''(1)\right]+\frac{B_3}{6!}\left[f^5(n)-f^5(1)\right]-\ldots$$

$$\log 2+\log 3+\ldots+\log n=\int_1^n \log(x)\,dx+\frac{1}{2}\log n+\frac{B_1}{2}\cdot\frac{1}{n}-\frac{B_2}{4!}2n^{-3}+\frac{B_3}{6!}\left(2\cdot 3\cdot 4x^{-5}\right)-\ldots$$

Therefore, we find that

$$\log(n!)=\int_1^n \log x\,dx+\frac{1}{2}\log n+\frac{B_1}{2}\cdot\frac{1}{n}-\frac{B_2}{3\cdot 4}\cdot\frac{1}{n^3}+\frac{B_3}{5\cdot 6}\cdot\frac{1}{n^5}-\ldots+\text{constant}$$

where the error at each stage is numerically less than the next term, because $f^{2n}(x)$ is negative (for all positive values of x).

This gives, on integration,

$$\log(n!) = x\log x\Big|_1^n - \int_1^n dx + \frac{1}{2}\log n + \frac{B_1}{2}\cdot\frac{1}{n} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{n^3} + \frac{B_3}{5\cdot 6}\cdot\frac{1}{n^5} - \ldots + C$$

$$\log(n!) = \left(n+\frac{1}{2}\right)\log n - n + C_1 + \frac{B_1}{2}\cdot\frac{1}{n} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{n^3} + \frac{B_3}{5\cdot 6}\cdot\frac{1}{n^5} - \ldots$$

To find the constant C_1, we use Wallis's formula which gives

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdot \ldots to\ \infty = \lim_{n\to\infty}\frac{(2^n n!)^4}{(2n!)^2(2n+1)}$$

Thus

$$\log\left(\frac{1}{2}\pi\right) = \lim_{n\to\infty} 4\log(2)^n + 4\log n! - 2\log\{(2n)!\} - \log(2n+1) =$$

$$= \lim_{n\to\infty}\{4n\log 2 + 4\log(n!) - 2\log\{(2n)!\} - \log(2n+1)\}$$

Now our general formula gives

$$2\log(n!) - \log\{(2n)!\} = (2n+1)\log n - \cancel{2n} + 2C_1 - \left(2n+\frac{1}{2}\right)\log(2n) + \cancel{2n} - C_1 + \theta\left(\frac{1}{2}\right) =$$

$$= C_1 + \cancel{2n\log n} + \log n - 2n\log 2 - \cancel{2n\log n} - \frac{1}{2}\log 2 - \frac{1}{2}\log n + \theta\left(\frac{1}{2}\right) =$$

$$= C_1 + \frac{1}{2}\log n - \left(2n+\frac{1}{2}\right)\log 2 + \theta\left(\frac{1}{n}\right)$$

Hence

$$\log\left(\frac{1}{2}\pi\right)=\lim_{n\to\infty}\left\{4n\log 2+2C_1+\log n-(4n+1)\log 2-\log(2n+1)\right\}=$$

$$=\lim_{n\to\infty}\left\{2C_1+\log n-\log 2-\log(2n+1)\right\}=$$

$$=\lim_{n\to\infty}\left\{2C_1-\log 2+\log\frac{n}{2n+1}\right\}=2C_1-\log 2+\log\frac{1}{2}=$$

$$=2C_1-\log 2-\log 2=2C_1-2\log 2$$

So,

$$\log\left(\frac{1}{2}\pi\right)=2C_1-2\log 2$$

thus, $$2C_1=2\log 2+\log\left(\frac{1}{2}\pi\right)$$

$$2C_1=\log 2+\log\pi=\log 2\pi$$

thus $$C_1=\frac{1}{2}\log(2\pi)$$

Hence we have Stirling's series

$$\log(n!)=\left(n+\frac{1}{2}\right)\log n-n+\frac{1}{2}\log(2\pi)+\frac{B_1}{1\cdot 2}\cdot\frac{1}{n}-\frac{B_2}{3\cdot 4}\cdot\frac{1}{n^3}+\frac{B_3}{5\cdot 6}\cdot\frac{1}{n^5}-\dots$$

in which so far, n is a positive integer.

To obtain the series for $\log\{\Gamma(1+x)\}$ we use the product-formula of Appendix F.

Applying Euler's summation formula from x *to* $x+n$ for

$$f(x)=\log x\,,$$

we have

$$\log x+\log(x+1)+\ldots+\log(x+n)=$$

$$=\int_x^{x+n}\log\xi d\xi+\frac{1}{2}\left(\log x+\log(n+x)\right)+\frac{B_1}{2!}\left(\log'(n+x)-\log' x\right)-$$

$$-\frac{B_2}{4!}\left(\log'''(n+x)-\log''' x\right)+\ldots+O\left(\frac{1}{n}\right)=$$

$$=\int_x^{x+n}\log\xi d\xi+\frac{1}{2}\left(\log x+\log(n+x)\right)+\frac{B_1}{2!}\cdot\frac{1}{n+x}-\frac{B_1}{2}\cdot\frac{1}{x}-$$

$$-\frac{B_2}{4!}\cdot\frac{2}{(n+x)^3}+\frac{B2}{4!}\cdot\frac{2}{x^3}+\ldots+O\left(\frac{1}{n}\right)=$$

$$=\int_x^{x+n}\log\xi d\xi+\frac{1}{2}\left(\log x+\log(n+x)\right)-\frac{B_1}{1\cdot 2x+}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right)$$

Subtracting this from Stirling's formula for $\log(n!)$

$$\log(n!)=\left(n+\frac{1}{2}\right)\log n-n+\frac{1}{2}\log(2\pi)+\frac{B_1}{1\cdot 2n}-\frac{B_2}{3\cdot 4n^3}\ldots$$

we find

$$\log x+\log(x+1)+\ldots+\log(x+n)-\log 1-\log 2-\ldots-\log n=$$

$$=\int_x^{x+n}\log\xi d\xi+\frac{1}{2}\left\{\log x+\log(n+x)\right\}-\left(n+\frac{1}{2}\right)\log n+n-\frac{1}{2}\log(2\pi)-$$

$$-\frac{B_1}{1\cdot 2x}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right)$$

$$\log x+\log\left(1+\frac{x}{1}\right)+\log\left(1+\frac{x}{2}\right)+\ldots+\log\left(1+\frac{x}{n}\right)=$$

$$=\int_x^{x+n}\log\xi d\xi+\frac{1}{2}\log x+\frac{1}{2}\log(n+x)-n\log n+n-\frac{1}{2}\log n-\frac{1}{2}\log(2\pi)-$$

$$-\frac{B_1}{1\cdot 2x}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right)$$

$$\log\left\{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)\right\}=\int_x^{x+n}\log\xi\,d\xi-\frac{1}{2}\log x-\left(n\log n-n\right)-\frac{1}{2}\log(2\pi)+$$

$$+\frac{1}{2}\left\{\log(n+x)-\log n\right\}-\frac{B_1}{1\cdot 2x}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right)=$$

$$=\int_x^{x+n}\log\xi\,d\xi-\frac{1}{2}\log x-\int_0^n\log\xi\,d\xi-\frac{1}{2}\log(2\pi)+\frac{1}{2}\log\left(1+\frac{x}{n}\right)-\frac{B_1}{1\cdot 2x}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right)=$$

$$=\cancel{\int_x^n\log\xi\,d\xi}+\int_n^{n+x}\log\xi\,d\xi-\int_0^x\log\xi\,d\xi-\cancel{\int_x^n\log\xi\,d\xi}-\frac{1}{2}\log x-\frac{1}{2}\log(2\pi)+\frac{1}{2}\log\left(1+\frac{x}{n}\right)-$$

$$-\frac{B_1}{1\cdot 2x}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right)$$

$$=\int_n^{n+x}\log\xi\,d\xi-\int_0^x\log\xi\,d\xi-\frac{1}{2}\log x-\frac{1}{2}\log(2\pi)-\frac{B_1}{1\cdot 2x}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right)$$

Making a change of variable $\xi=n+\eta$, we find that the difference of the two integrals in the last formula is equal to

$$\int_n^{n+x}\log\xi\,d\xi-\int_0^x\log\xi\,d\xi=\int_0^x\log(n+\eta)d\eta-x\log x+x=$$

$$=\int_0^x\log n\left(1+\frac{\eta}{n}\right)d\eta-x\log x+x=\int_0^x\log\eta\,d\eta+\int_0^x\log\left(1+\frac{\eta}{n}\right)d\eta-x\log x+x=$$

$$=x\log n-x\log x+x+\int_0^x\log\left(1+\frac{\eta}{n}\right)d\eta=x\log n-x\log x+x+\ldots+O\left(\frac{1}{n}\right)$$

Thus, we find that

$$\log\left\{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)\right\}=$$

$$=x\log n-x\log x+x-\frac{1}{2}\log x-\frac{1}{2}\log(2\pi)+\frac{1}{2}\log\left(1+\frac{x}{\eta}\right)-\frac{B_1}{1\cdot 2x}+\frac{B_2}{3\cdot 4x^3}-\ldots+O\left(\frac{1}{n}\right).$$

Therefore,

$$x\log n-\log\left\{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)\right\}=$$

$$=\left(x+\frac{1}{2}\right)\log x-x+\frac{1}{2}\log(2\pi)+\frac{B_1}{1\cdot 2x}-\frac{B_2}{3\cdot 4x^3}+\ldots+O\left(\frac{1}{n}\right)+\frac{1}{2}\log\left(1+\frac{x}{n}\right)$$

Now, when $n\to\infty$, the left-hand side tends to $\log\{\Gamma(1+x)\}$ (See Appendix F) and so we have the result

$$\lim_{n\to\infty}\left\{x\log n-\log\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)\right\}=$$

$$\lim_{n\to\infty}\left\{\log\frac{n^x n!}{(1+x)(2+x)\ldots(n+x)}\right\}=\log\{\Gamma(1+x)\}$$

thus,

$$\log\{\Gamma(1+x)\}=\left(x+\frac{1}{2}\right)\log x-x+\frac{1}{2}\log(2\pi)+\frac{B_1}{2x}-\frac{B_2}{3\cdot 4x^3}+\ldots$$

which, as might perhaps have been anticipated, is of exactly the same form as the series originally found for $\log(n!)$. An independent discussion of this result will be found in Chapter Four.

Sometimes it is useful to have a slightly modified form of the result which can be used when x is of the form $x=m+p$, where m is large (not necessarily an integer), and p may also be large, but is small compared with m.

For this purpose we note that

$$\log(m+p)=\log m+\frac{p}{m}-\frac{p^2}{2m^2}+\frac{p^3}{3m^3}-\dots$$

Thus if we take p to be of order $\sqrt{m}$, at most, and reject terms of order $\frac{1}{m}$, we get the formula

$$\left(m+p+\frac{1}{2}\right)\log(m+p)=\left(m+p+\frac{1}{2}\right)\log m+p+\frac{1}{2}\frac{p^2}{m}+\frac{1}{2}\frac{p}{m}-\frac{1}{6}\frac{p^3}{m^2}+O\left(\frac{1}{m}\right),$$

and

$$\log\{\Gamma(1+m+p)\}=\left(m+p+\frac{1}{2}\right)\log m-m+\frac{1}{2}\log(2\pi)+\frac{1}{2}\frac{p^2}{m}+\frac{1}{2}\frac{p}{m}-\frac{1}{6}\frac{p^3}{m^2}+O\left(\frac{1}{m}\right).$$

CHAPTER THREE

ASYMPTOTIC SERIES AND INTEGRATION

Calculation of Integrals by Means of Asymptotic Series

Many integrals of interest, both in Pure and Applied Mathematics, can be calculated by means of asymptotic series. A few typical examples will be given below.

There are three methods which are usually effective in obtaining a suitable asymptotic series from a given integral:

(i) Integration by parts.

(ii) Expansion of some function in the integral.

(iii) Use of symbolic operators.

We will consider examples of the first two methods; it is usually impossible to use (iii) unless an estimate can be made as to the magnitude of the remainder in the expansion, and we will not give any examples here.

The Error Function Integral

This integral is usually expressed by the abbreviation $\operatorname{erf} x$, and is defined by the equation

$$\operatorname{erf} x = \int_0^x e^{-t^2} dt$$

Method(i)-Integration by Parts. A series suitable for calculation when x is small is deduced by expanding the exponential and integrating term-by-term. But this series is very inconvenient for numerical work when x exceeds 2. Noting that

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2},$$

an asymptotic series for the integral

$$u = \int_x^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} - \mathrm{erfx}$$

is found by writing $t^2 = s$.

First we will prove that

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

By definition,

$$\Gamma(1+x) = \int_0^\infty e^{-t} t^x\,dt$$

If $x = -\frac{1}{2}$, we obtain

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$$

We know from Appendix G that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Using $v=t^2$, we obtain

$$\sqrt{\pi}=\int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv=\int_0^\infty \frac{e^{-t^2}2t}{t}dt=2\int_0^\infty e^{-t^2}dt$$

Therefore,

$$\int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{2}\ .$$

Then, writing $t^2=s$ we have

$$u=\int_x^\infty e^{-t^2}dt=\int_{x^2}^\infty \frac{e^{-s}}{2\sqrt{s}}ds=\frac{1}{2}\int_{x^2}^\infty s^{-1/2}e^{-s}ds\ .$$

To the last integral we apply the transformation of integration by parts, which gives

$$u=\frac{1}{2}\left(-e^{-s}\frac{1}{\sqrt{s}}\bigg|_{x^2}^\infty-\frac{1}{2}\int_{x^2}^\infty\frac{e^{-s}}{s^{3/2}}\right)=\frac{1}{2x}e^{-x^2}-\frac{1}{2^2}\int_{x^2}^\infty\frac{e^{-s}}{s^{3/2}}ds=$$

$$=\frac{1}{2x}e^{-x^2}-\frac{1}{2^2}\left(-e^{-s}\frac{1}{s^{3/2}}\bigg|_{x^2}^\infty-\frac{3}{2}\int_{x^2}^\infty e^{-s}s^{-5/2}ds\right)=$$

$$=e^{-x^2}\left(\frac{1}{2x}\right)-\frac{1}{2^2}e^{-x^2}\frac{1}{x^3}+\frac{3}{2^3}\int_{x^2}^\infty\frac{e^{-s}}{s^{5/2}}ds=$$

$$=e^{-x^2}\left(\frac{1}{2x}-\frac{1}{2^2x^3}\right)+\frac{1\cdot 3}{2^3}\int_{x^2}^\infty\frac{e^{-s}}{s^{5/2}}ds=$$

$$=e^{-x^2}\left(\frac{1}{2x}-\frac{1}{2^2x^3}\right)+\frac{1\cdot 3}{2^3}\left(-e^{-s}\frac{1}{s^{5/2}}\bigg|_{x^2}^\infty-\frac{5}{2}\int_{x^2}^\infty e^{-s}\frac{1}{s^{7/2}}ds\right)=$$

$$=e^{-x^2}\left(\frac{1}{2x}-\frac{1}{2^2x^3}\right)+\frac{1\cdot 3}{2^3}\left(\frac{e^{-x^2}}{x^5}\right)-\frac{1\cdot 3\cdot 5}{2^4}\int_{x^2}^\infty e^{-s}\frac{1}{s^{7/2}}ds=$$

$$= e^{-x^2}\left(\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1\cdot 3}{2^3 x^5}\right) - \frac{1\cdot 3\cdot 5}{2^4}\int_{x^2}^{\infty} e^{-s}\frac{1}{s^{7/2}}ds =$$

$$= e^{-x^2}\left(\frac{1}{2x} - \frac{1}{2^2 x^3} + \frac{1\cdot 3}{2^3 x^5} - \frac{1\cdot 3\cdot 5}{2^4 x^7}\right) + \frac{1\cdot 3\cdot 5\cdot 7}{2^5}\int_{x^2}^{\infty}\frac{e^{-s}}{s^{9/2}}ds$$

Clearly this process can be continued as far as we want.

Because $s > x^2$, $\frac{1}{s^{9/2}} < \frac{1}{x^9}$, the remainder integral in the last formula is clearly less than

$$\int_{x^2}^{\infty}\frac{e^{-s}}{s^{9/2}}ds < \frac{1}{x^9}\int_{x^2}^{\infty} e^{-s}ds = \frac{1}{x^9}e^{-x^2}$$

$$\frac{1\cdot 3\cdot 5\cdot 7}{2^5}\int_{x^2}^{\infty}\frac{e^{-s}}{s^{9/2}}ds < \frac{1\cdot 3\cdot 5\cdot 7}{2^5 x^9}e^{-x^2},$$

and this is the next term in the series, after those retained.

Therefore, we see that the error obtained when stopping at any stage in the asymptotic series for the integral u is less than the following term in the series.

Method(ii)-Method of Expansion. Here we write $s = x^2 + v$ and

$$u = \int_{x^2}^{\infty}\frac{e^{-s}}{2\sqrt{s}}ds = \int_{0}^{\infty} e^{-x^2}e^{-v}\frac{dv}{2\sqrt{x^2+v}} = e^{-x^2}\int_{0}^{\infty}\frac{e^{-v}}{2\sqrt{x^2+v}}dv$$

Then write

$$\frac{1}{\sqrt{x^2+v}}=\frac{1}{x\sqrt{1+\frac{v}{x^2}}}=\frac{\left(1+\frac{v}{x^2}\right)^{-\frac{1}{2}}}{x}.$$

By the binomial expansion,

$$\left(1+\frac{v}{x^2}\right)^{-\frac{1}{2}}=1-\frac{v}{2x^2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{v^2}{x^4}-\frac{1}{3!}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{v^3}{x^6}+\ldots$$

so that

$$\frac{1}{\sqrt{x^2+v}}=\frac{1}{x}-\frac{1}{2}\frac{v}{x^3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{v^2}{x^5}-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{v^3}{x^7}+\ldots$$

and the remainder at any stage being less than the following term.

Now

$$\int_0^\infty e^{-v}v^n dv=-v^n e^{-v}\Big|_0^\infty+n\int_0^\infty v^{n-1}e^{-v}dv=n(n-1)\int_0^\infty v^{n-2}e^{-v}dv=\ldots=n!$$

and so we obtain again the same results for u and its asymptotic series.

$$u=e^{-x^2}\int_0^\infty\frac{1}{2}\left(\frac{e^{-v}}{x}-\frac{e^{-v}v}{2x^3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{e^{-v}v^2}{x^5}-\ldots\right)dv=e^{-x^2}\left(\frac{1}{2x}-\frac{1}{2^2x^3}+\frac{1\cdot 3}{2^3x^5}-\ldots\right)$$

The Logarithmic Integral

The integral

$$U=\int_x^\infty\frac{e^{-t}}{t}dt=\int_{e^{-x}}^0\frac{-dv}{-\log v}=-\int_0^{e^{-x}}\frac{dv}{\log v}$$

has often been denoted by the symbol $-\mathrm{li}\left(e^{-x}\right)$.

To obtain an asymptotic series for U, we write

$$\int_x^\infty \frac{e^{-t}}{t}dt = \int_0^1 \frac{e^{-t}}{t}dt + \int_1^\infty \frac{e^{-t}}{t}dt - \int_0^x \frac{e^{-t}}{t}dt = \int_0^1 \frac{e^{-t}}{t}dt + \int_1^\infty \frac{e^{-t}}{t}dt - \int_0^x \frac{e^{-t}}{t}dt + \int_0^x \frac{dt}{t} - \int_0^x \frac{dt}{t} =$$

$$= \int_0^1 \frac{e^{-t}}{t}dt + \int_1^\infty \frac{e^{-t}}{t}dt - \int_0^x \frac{e^{-t}}{t}dt + \int_0^x \frac{dt}{t} - \int_0^1 \frac{dt}{t} - \int_1^x \frac{dt}{t} =$$

$$= \int_1^\infty \frac{e^{-t}}{t}dt - \int_0^1 \left(1-e^{-t}\right)\frac{dt}{t} - \int_1^x \frac{dt}{t} + \int_0^x \left(1-e^{-t}\right)\frac{dt}{t}$$

First, we will prove that $\int_0^1 \left(1-e^{-t}\right)\frac{dt}{t} - \int_1^\infty e^{-t}\frac{dt}{t} = C$, the Euler's constant.

For that, we know that

$$C = \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right).$$

But

$$1+\frac{1}{2}+\ldots+\frac{1}{n} = \int_0^1 \left(1+x+x^2+\ldots+x^{n-1}\right)dx = \int_0^1 \frac{1-x^n}{1-x}dx = \int_0^n \left(1-\left(1-\frac{t}{n}\right)^n\right)\frac{dt}{t}.$$

Therefore,

$$C = \lim_{n\to\infty}\left(\int_0^n \left(1-\left(1-\frac{t}{n}\right)^n\right)\frac{dt}{t} - \int_1^n \frac{dt}{t}\right) =$$

$$= \lim_{n\to\infty}\left(\int_0^1 \left(1-\left(1-\frac{t}{n}\right)^n\right)\frac{dt}{t} + \int_1^n \frac{dt}{t} - \int_1^n \left(1-\frac{t}{n}\right)^n\frac{dt}{t} - \int_1^n \frac{dt}{t}\right) =$$

$$=\lim_{n\to\infty}\left(\int_0^1\left(1-\left(1-\frac{t}{n}\right)^n\right)\frac{dt}{t}-\int_1^n\left(1-\frac{t}{n}\right)^n\frac{dt}{t}\right)=\int_0^1\left(1-e^{-t}\right)\frac{dt}{t}-\int_1^\infty e^{-t}\frac{dt}{t}.$$

Now,

$$\int_x^\infty\frac{e^{-t}}{t}dt=-C-\log t\Big|_1^x+\int_0^x\left(1-e^{-t}\right)\frac{dt}{t}=-C-\log|x|+\int_0^x\left(1-e^{-t}\right)\frac{dt}{t}$$

To calculate the last integral in the above formula, we use the Maclaurin expansion:

$$1-e^{-t}=\frac{t}{1!}-\frac{t^2}{2!}-\frac{t^3}{3!}\dots$$

$$\frac{1-e^{-t}}{t}=1-\frac{t}{2!}+\frac{t^2}{3!}-\dots$$

Then

$$\int_0^x\left(1-e^{-t}\right)\frac{dt}{t}=\int_0^x\left(1-\frac{t}{2!}+\frac{t^2}{3!}-\frac{t^3}{4!}+\dots\right)dt=x-\frac{1}{2!}\frac{t^2}{2}\Big|_0^x+\frac{1}{3!}\frac{t^3}{3}\Big|_0^x-\frac{1}{4!}\frac{t^4}{4}+\dots=$$

$$=x-\frac{x^2}{2\cdot 2!}+\frac{x^3}{3\cdot 3!}-\frac{x^4}{4\cdot 4!}+\dots$$

Therefore,

$$U=\int_x^\infty\frac{e^{-t}}{t}dt=-C-\log|x|+x-\frac{1}{2}\frac{x^2}{2!}+\frac{1}{3}\frac{x^3}{3!}-\frac{1}{4}\frac{x^4}{4!}+\dots$$

where C is Euler's constant.

When x is negative all the integrals in the asymptotic series for U are convergent except $\int_1^x\frac{dt}{t}$, of which we must take the principal value; that is

$$\lim_{\varepsilon\to 0}\left(\int_1^{\varepsilon}\frac{dt}{t}+\int_{-\varepsilon}^{x}\frac{dt}{t}\right)=\lim_{\varepsilon\to 0}\left[\log\varepsilon+\log\left(-\frac{x}{\varepsilon}\right)\right]=\log(-x)=\log|x| .$$

But this expansion, although convergent for all values of x, is unsuitable for calculation when $|x|$ is large, just as the exponential series is not convenient for calculating high powers of e.

To overcome this, we may apply methods similar to those used for calculating the error function integral. If x is positive, we can use the method of integration by parts easily.

<u>Method(i)-Integration by Parts</u>. If $x>0$,

$$U=\int_x^{\infty}e^{-t}\frac{dt}{t}=-\frac{e^{-t}}{t}\Bigg|_x^{\infty}-\int_x^{\infty}t^{-2}e^{-t}=\frac{e^{-x}}{x}-\left(-\frac{e^{-t}}{t^2}\Bigg|_x^{\infty}-2\int_x^{\infty}t^{-3}e^{-t}dt\right)=$$

$$=\frac{e^{-x}}{x}-\left(\frac{e^{-x}}{x^2}-2\int_x^{\infty}t^{-3}e^{-t}dt\right)=\frac{e^{-x}}{x}-\frac{e^{-x}}{x^2}+2\left(-t^{-3}e^{-t}\Big|_x^{\infty}-3\int_x^{\infty}t^{-4}e^{-t}dt\right)$$

$$=\frac{e^{-x}}{x}-\frac{e^{-x}}{x^2}+\frac{2e^{-x}}{x^3}-2\cdot 3\int_x^{\infty}t^{-4}e^{-t}dt=$$

$$=e^{-x}\left(\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}\right)-2\cdot 3\left(-e^{-t}t^{-4}\Big|_x^{\infty}-\int_x^{\infty}t^{-5}e^{-t}dt\right)=$$

$$=e^{-x}\left(\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}\right)-2\cdot 3\left(e^{-x}\frac{1}{x^4}\right)+2\cdot 3\int_x^{\infty}t^{-5}e^{-t}dt=$$

$$=e^{-x}\left(\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\frac{2\cdot 3}{x^4}\right)+2\cdot 3\int_x^{\infty}t^{-5}e^{-t}dt$$

But, since $t > x$ we have $t^{-5} < x^{-5}$, so that

$$2\cdot 3\int_x^\infty t^{-5}e^{-t}dt < 2\cdot 3\int_x^\infty x^{-5}e^{-t}dt = \frac{2\cdot 3}{x^5}e^{-x} .$$

<u>Method (ii)-Method of Expansion</u>. To apply the method of expansion we write $t = x+v$, and then

$$U = \int_x^\infty e^{-t}\frac{dt}{t} = \int_0^\infty e^{-x}e^{-v}\frac{dv}{x+v} = e^{-x}\int_0^\infty \frac{e^{-v}}{x+v}dv$$

To calculate the last integral, we use the geometric series:

$$\frac{1}{x+v} = \frac{1}{x\left(1+\frac{v}{x}\right)} = \frac{1}{x} - \frac{v}{x^2} + \frac{v^2}{x^3} - \ldots + (-1)^{n-1}\frac{v^{n-1}}{x^n} + (-1)^n\frac{v^n}{x^n(x+v)}$$

Therefore,

$$U = e^{-x}\int_0^\infty \left(\frac{e^{-v}}{x} - \frac{ve^{-v}}{x^2} + \frac{v^2e^{-v}}{x^3} - \ldots + \frac{(-1)^n v^n e^{-v}}{x^n(x+v)}\right)dv$$

But

$$\int_0^\infty e^{-v}v^n dv = n!$$

from which we deduce

$$U = e^{-x}\left(\frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \ldots + (-1)^{n-1}\frac{(n-1)!}{x^n}\right) + e^{-x}\int_0^\infty \frac{(-1)^n v^n e^{-v}}{x^n(x+v)}dv$$

If we let

$$|R_n| = \int_0^\infty \frac{e^{-v}v^n}{x^n(x+v)}dv = \frac{1}{x^n}\int_0^\infty \frac{v^n}{e^v(x+v)}dv = \frac{1}{x^n}\int_0^\infty \frac{v^n e^{-v}}{(x+v)}dv$$

$$\text{then } |R_n| < \frac{1}{x^{n+1}} \int_0^\infty x^n e^{-v} dv = n!x^{-(n+1)}$$

When x is large, the terms of this series at first decrease very rapidly. That's why ,up to a certain degree of accuracy, this series is very convenient for numerical work when x is large; but, because the terms finally increase beyond all limits, we cannot get beyond a certain approximation.

For example, with $x=10$, the estimated limits for R_9, R_{10} are equal and are less than any other remainder. And the ratio of their common value to the first term in the series is about $1:2500$. To get this degree of accuracy from the first series we should need 35 terms. Again, with $x=20$, the ratio of R_{10} to the first term is less than $1:10^6$; but 80 terms of the ascending series do not suffice to obtain this degree of approximation.

When x is negative, we write $x=-\xi \quad \xi>0$ and $t=x+v=v-\xi$ and we find

$$U = e^{-x} \int_0^\infty \frac{e^{-v}}{x+v} dv = e^{\xi} \int_0^\infty \frac{e^{-v}}{v-\xi} dv = -e^{\xi} \int_0^\infty \frac{e^{-v}}{\xi-v} dv$$

$$-U = e^{\xi} P \int_0^\infty \frac{e^{-v}}{\xi-v} dv = e^{\xi} \left[\int_0^\infty \left(\frac{1}{\xi} + \frac{v}{\xi^2} + \frac{v^2}{\xi^3} + \ldots + \frac{v^{n-1}}{\xi^n} \right) e^{-v} dv + P \int_0^\infty \frac{v^n e^{-v}}{\xi^n (\xi-v)} dv \right]$$

where P denotes the principal value of the integral.

Thus

$$li\left(e^{\xi}\right)=e^{\xi}\left(\frac{1}{\xi}+\frac{1}{\xi^{2}}+\frac{2!}{\xi^{3}}+\ldots+\frac{(n-1)!}{\xi^{n}}+R_{n}\right)$$

where

$$R_{n}=P\int_{0}^{\infty}\frac{v^{n}e^{-v}}{\xi^{n}(\xi-v)}dv$$

Stieltjes has proven by an elaborate discussion that in this case also we get the best approximation by taking n equal to the integral part of ξ, and that the value of R_n is then of the order $e^{-\xi}\left(\frac{2\pi}{\xi}\right)^{\frac{1}{2}}$.

The two expansions of U can be used to find the "summation" of $1!-2!+3!-4!+\ldots$.

If we write $x=1$ and equate the series of ascending powers to the series of descending powers, we find that

$$U=e^{-1}(1-1+2!-3!+4!-\ldots)=-C+1-\frac{1}{2\cdot 2!}+\frac{1}{3\cdot 3!}-\frac{1}{4\cdot 4!}+\ldots$$

Therefore, we find that

$$1!-2!+3!-4!+\ldots=1+e\left(C-\left(1-\frac{1}{2\cdot 2!}+\frac{1}{3\cdot 3!}-\frac{1}{4\cdot 4!}+\ldots\right)\right).$$

Lacroix gives the value of $1!-2!+3!-4!+\ldots=0.4036526$, which agrees with Euler's result.

Asymptotic Series for Integrals
Containing Sines and Cosines

Fresnel's Integrals

Consider the two integrals

$$U=\int_x^\infty \frac{\cos t}{\sqrt{t}}dt, \qquad V=\int_x^\infty \frac{\sin t}{\sqrt{t}}dt, \qquad (x>0)$$

which are met in the theory of Physical Optics, and also in the theory of deep-water waves.

We have

$$U+iV=\int_x^\infty \frac{\cos t+i\sin t}{\sqrt{t}}dt=\int_x^\infty t^{-1/2}e^{it}dt=-ie^{it}t^{-1/2}\Big|_x^\infty-\frac{i}{2}\int_x^\infty t^{-3/2}e^{it}dt=$$

$$=ie^{ix}x^{-1/2}+\frac{1}{2i}\int_x^\infty t^{-3/2}e^{it}dt=\frac{-e^{ix}}{i\sqrt{x}}+\frac{1}{2i}\int_x^\infty t^{-3/2}e^{it}dt+\frac{1}{2i}\left(\frac{e^{it}}{it^{3/2}}\Big|_x^\infty+\frac{3}{2i}\int_x^\infty t^{-5/2}e^{it}dt\right)=$$

$$=\frac{-e^{ix}}{i\sqrt{x}}-\frac{e^{ix}}{2i^2x^{3/2}}+\frac{1\cdot 3}{2^2i^2}\int_x^\infty t^{-5/2}e^{it}dt=$$

$$=\frac{-e^{ix}}{ix^{1/2}}-\frac{e^{ix}}{2i^2x^{3/2}}+\frac{1\cdot 3}{2^2i^2}\left(\frac{t^{-5/2}e^{it}}{i}\Big|_x^\infty+\frac{5}{2}\int_x^\infty t^{-7/2}e^{it}dt\right)=$$

$$=\frac{-e^{ix}}{ix^{1/2}}-\frac{e^{ix}}{2i^2x^{3/2}}-\frac{1\cdot 3e^{ix}}{2^2i^3x^{5/2}}+\frac{1\cdot 3\cdot 5}{2^3i^2}\int_x^\infty t^{-7/2}e^{it}dt$$

Therefore,

$$U+iV=-e^{ix}\left(\frac{1}{i\sqrt{x}}+\frac{1}{2i^2x^{3/2}}+\frac{1\cdot 3}{2^2i^3x^{5/2}}+\dots\right)=\frac{ie^{ix}}{\sqrt{x}}\left(1+\frac{1}{2ix}-\frac{1\cdot 3}{(2ix)^2}+\frac{1\cdot 3\cdot 5}{(2ix)^3}+\dots\right)$$

Let us write

$$U+iV=\frac{ie^{ix}}{\sqrt{x}}(X-iY)$$

Then we have

$$U=\frac{1}{\sqrt{x}}(-X\sin x+Y\cos x),\ V=\frac{1}{\sqrt{x}}(X\cos x+Y\sin x)$$

where:

$$X=1-\frac{1\cdot 3}{(2x)^2}+\frac{1\cdot 3\cdot 5\cdot 7}{(2x)^4}-...,\qquad Y=\frac{1}{2x}-\frac{1\cdot 3\cdot 5}{(2x)^3}+\frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{(2x)^5}-...$$

The remainder in the series $U+iV$ after the four terms written above is

$$\frac{1\cdot 3\cdot 5\cdot 7}{(2i)^4}\int_x^\infty\frac{e^{it}}{t^{9/2}}dt\ .$$

Now we can employ Dirichlet's test: "An infinite integral whose integrand oscillates finitely becomes convergent after the insertion of a monotonic factor which tends to zero as a limit", $\left|\int_\xi^{\xi'}f(x)\varphi(x)dx\right|<Hf(\xi)$, where $\int\varphi(x)$ oscillates finitely and it remains less than *C*, C fixed for $\forall\xi$, $H\le 2C$, $f(x)$ is monotonic and $\lim_{x\to\infty}f(x)\to 0$.

Here we have:

$$\left|\int_x^\infty\frac{e^{it}}{t^{9/2}}dt\right|\le\frac{2}{x^{9/2}}$$

So, the remainder in the series $U+iV$ after the four terms written above is bounded as

$$\left|\frac{1\cdot 3\cdot 5\cdot 7}{(2i)^4}\int_x^\infty \frac{e^{it}}{t^{9/2}}dt\right| \le \frac{1\cdot 3\cdot 5\cdot 7}{2^4}\frac{2}{\sqrt{x^9}} = \frac{2}{\sqrt{x}}\frac{1\cdot 3\cdot 5\cdot 7}{(2x)^4}$$

which is twice the modulus of the following term of the series.

We now wish to find differential equations that are satisfied by X and Y. To this end is convenient to re-express X and Y in terms of the Gamma function:

$$\Gamma(1+x) = \int_0^\infty e^{-t}t^x dx \ .$$

Hence

$$\Gamma\left(\frac{1}{2}\right) = \Gamma\left(1-\frac{1}{2}\right) = \int_0^\infty e^{-v}v^{-\frac{1}{2}}dv = \int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv = \sqrt{\pi} \qquad \text{(See Appendix G)} .$$

On the other hand, for any n, by repeated integration by parts we see that

$$\int_0^\infty e^{-v}v^n \frac{dv}{\sqrt{v}} = \int_0^\infty e^{-v}v^{n-1/2}dv = \left(n-\frac{1}{2}\right)\int_0^\infty e^{-v}v^{n-3/2}dv = \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\int_0^\infty e^{-v}v^{n-5/2}dv =$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\ldots\left[n-\left(n-\frac{3}{2}\right)\right]\frac{1}{2}\int_0^\infty e^{-v}v^{-1/2}dv$$

Therefore,

$$\frac{1}{2}\cdot\frac{3}{2}\cdot\ldots\cdot\frac{2n-3}{2}\cdot\frac{2n-1}{2}\sqrt{\pi} = \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2^n}\sqrt{\pi} = \int_0^\infty e^{-v}v^n\frac{dv}{\sqrt{v}}$$

and so

$$X - iY = 1 + \frac{1}{2ix} - \frac{1\cdot 3}{(2ix)^2} + \frac{1\cdot 3\cdot 5}{(2ix)^3} + \ldots$$

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv = 1$$

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}\frac{v}{ix}dv = \frac{1}{\sqrt{\pi}}\cdot\frac{1}{ix}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}vdv = \frac{1}{2ix}, \text{ etc}$$

Therefore,

$$X - iY = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv\left\{1 + \frac{v}{ix} + \frac{v^2}{(ix)^2} + \ldots\right\} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv\left(\frac{x}{x+iv}\right) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv\frac{x^2 - xvi}{x^2+v^2}$$

the remainder at any stage in the expanded form of the integral being numerically less than the following term. Hence we obtain the formulae

$$X = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-v}}{\sqrt{v}}dv\left(\frac{x^2}{x^2+v^2}\right), \qquad Y = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-v}\sqrt{v}dv\left(\frac{x}{x^2+v^2}\right).$$

It is easy to prove that these expressions are equal to the original integrals by differentiating with respect to x. We have in fact

$$U + iV = \int_x^\infty \frac{e^{it}}{\sqrt{t}}dt$$

Then

$$\frac{d}{dx}(U+iV) = -\frac{e^{ix}}{\sqrt{x}}$$

$$U+iV=\frac{ie^{ix}}{\sqrt{x}}(X-iY)$$

$$\frac{d}{dx}\left\{\frac{ie^{ix}}{\sqrt{x}}(X-iY)\right\}=-\frac{e^{ix}}{\sqrt{x}}$$

Therefore,

$$\frac{i^2e^{ix}\sqrt{x}-ie^{ix}\dfrac{1}{2\sqrt{x}}}{x}(X-iY)+\frac{ie^{ix}}{\sqrt{x}}\frac{d}{dx}(X-iY)=-\frac{e^{ix}}{\sqrt{x}}$$

$$\frac{-2e^{ix}x-ie^{ix}}{2x\sqrt{x}}(X-iY)+\frac{ie^{ix}}{\sqrt{x}}\frac{d}{dx}(X-iY)=-\frac{e^{ix}}{\sqrt{x}}$$

$$\frac{-2e^{ix}x-ie^{ix}}{2x}(X-iY)+ie^{ix}\frac{d}{dx}(X-iY)=-e^{ix}$$

$$\left(\frac{2e^{ix}x}{2xe^{ix}}+\frac{ie^{ix}}{2xe^{ix}}\right)(X-iY)-i\frac{d}{dx}(X-iY)=1$$

$$\left(1+\frac{i}{2x}\right)(X-iY)-i\frac{d}{dx}(X-iY)=1$$

$$\left(i-\frac{1}{2x}\right)(X-iY)+\frac{d}{dx}(X-iY)=i$$

or

$$\frac{dX}{dx}-i\frac{dY}{dx}+iX+Y-\frac{1}{2x}X+\frac{i}{2x}Y=i$$

$$\frac{dY}{dx}-\frac{Y}{2x}-X=-1 \quad \text{and} \quad \frac{dX}{dx}+Y-\frac{X}{2x}=0$$

It is easy to verify that these equations are satisfied by the last pair of integrals for *X*, *Y*, and that these integrals tend to 1, 0 respectively as $x \to \infty$; thus we may infer that *U*, *V* and *X*, *Y* are actually related in the manner suggested by the foregoing work. The integrals *X*, *Y* seem to be due to Cauchy, and the asymptotic expansion to Poisson.

It is perhaps worth while to make the additional remark that the relation between *X*, *Y and U*, *V* are most naturally suggested by the use of asymptotic expansion.

<u>The Sine- and Cosine-Integrals</u>

Here we will find asymptotic formulae for the two integrals

$$P = \int_x^\infty \frac{\cos t}{t} dt, \quad Q = \int_x^\infty \frac{\sin t}{t} dt .$$

Then

$$P + iQ = \int_x^\infty \frac{\cos t + i \sin t}{t} dt = \int_x^\infty \frac{e^{it}}{t} dt .$$

The asymptotic formula is obtained on lines similar to those used in (1) above.

$$P + iQ = \int_x^\infty \frac{e^{it}}{t} dt = \frac{e^{it}}{it}\Bigg|_x^\infty + \frac{1}{i}\int_x^\infty t^{-2} e^{it} dt = -\frac{e^{ix}}{ix} + \frac{1}{i}\left(\frac{e^{it}}{it^2}\Bigg|_x^\infty + \frac{2}{i}\int_x^\infty t^{-3} e^{it} dt \right) =$$

$$= \frac{ie^{ix}}{x} + \frac{-e^{ix}}{i^2 x^2} + \frac{2}{i^2}\left(\frac{t^{-3} e^{it}}{i}\Bigg|_x^\infty + \frac{3}{i}\int_x^\infty t^{-4} e^{it} dt \right) = \frac{ie^{ix}}{x} + \frac{e^{ix}}{x^2} + \frac{2e^{ix}}{ix^3} + \frac{2 \cdot 3}{i^3}\int_x^\infty t^{-4} e^{it} dt =$$

$$=\frac{ie^{ix}}{x}+\frac{e^{ix}}{x^2}+\frac{2e^{ix}}{ix^3}+\frac{2\cdot 3}{i^3}\left(\frac{t^{-4}e^{it}}{i}\bigg|_x^\infty+\frac{4}{i}\int_x^\infty t^{-5}e^{it}dt\right)=$$

$$=\frac{ie^{ix}}{x}+\frac{ie^{ix}}{ix^2}+\frac{2ie^{ix}}{i^2x^3}-\frac{2\cdot 3}{i^3 i}\cdot\frac{e^{ix}}{x^4}+\frac{2\cdot 3\cdot 4}{i^4}\int_x^\infty t^{-5}e^{it}dt=$$

$$=\frac{ie^{ix}}{x}\left(1+\frac{1}{ix}+\frac{1\cdot 2}{(ix)^2}+\frac{1\cdot 2\cdot 3}{(ix)^3}+\frac{1\cdot 2\cdot 3\cdot 4}{(ix)^4}+\dots\right)$$

So, $$P+iQ=\frac{ie^{ix}}{x}\left(1+\frac{1}{ix}+\frac{2!}{(ix)^2}+\frac{3!}{(ix)^3}+\frac{4!}{(ix)^4}+\dots\right)$$

It follows that

$$\frac{P+iQ}{e^{ix}}=\frac{i}{x}\left(1+\frac{1}{ix}+\frac{2!}{(ix)^2}+\frac{3!}{(ix)^3}+\frac{4!}{(ix)^4}+\dots\right)$$

$$\frac{P+iQ}{\cos x+i\sin x}=\frac{(P+iQ)(\cos x-i\sin x)}{\cos^2 x+\sin^2 x}=P\cos x+Q\sin x-iP\sin x+iQ\cos x=$$

$$=\frac{1}{x}\left(i+\frac{1}{x}+\frac{2!}{ix^2}+\frac{3!}{i^2x^3}+\frac{4!}{i^3x^4}+\frac{5!}{i^4x^5}\dots\right)$$

Therefore, taking real and imaginary parts, we find

$$P\cos x+Q\sin x=\frac{1}{x}\left(\frac{1}{x}-\frac{3!}{x^3}+\frac{5!}{x^5}-\frac{7!}{x^7}+\dots\right)$$

$$-P\sin x+Q\cos x=\frac{1}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\frac{6!}{x^6}+\frac{8!}{x^8}\dots\right)$$

On the other hand

$$P\cos x+Q\sin x=\int_x^\infty\frac{\cos t\cos x+\sin t\sin x}{t}dt=\int_x^\infty\frac{\cos(t-x)}{t}dt$$

$$-P\sin x+Q\cos x=\int_x^{\infty}\frac{\sin t\cos x-\cos t\sin x}{t}dt=\int_x^{\infty}\frac{\sin(t-x)}{t}dt$$

Therefore, we find the most instructive formulae

$$\int_x^{\infty}\frac{\cos(t-x)}{t}dt=P\cos x+Q\sin x=\frac{1}{x}\left(\frac{1}{x}-\frac{3!}{x^3}+\frac{5!}{x^5}-\ldots\right)$$

$$\int_x^{\infty}\frac{\sin(t-x)}{t}dt=-P\sin x+Q\cos x=\frac{1}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\ldots\right)$$

We know that

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$$

Therefore, we see that the cosine-integral is represented by a series of reciprocal of the ordinary sine-series, and vice-versa.

The second formula leads to an easy method for calculating the maxima and minima of the sine-integral, which correspond to the values $x=n\pi$; thus we find

$$-P\sin n\pi+Q\cos n\pi=(-1)^n Q=\frac{1}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\ldots\right)$$

$$Q=I_n=\int_{n\pi}^{\infty}\frac{\sin t}{t}dt=\frac{(-1)^n}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\ldots\right),\qquad x=n\pi$$

For values of n greater than 2, it is found that the calculations can be easily carried out to four decimal places; thus

$$I_3=\frac{(-1)^3}{3\pi}\left(1-\frac{2!}{(3\pi)^2}+\frac{4!}{(3\pi)^4}-\dots\right)=\frac{-1}{3\pi}+\frac{2!}{3^3\pi^3}-\frac{4!}{3\cdot 3^4\pi^4}+\frac{6!}{3^7\pi^6}-\dots=-0.1040$$

$$I_4=\frac{(-1)^4}{4\pi}\left(1-\frac{2!}{(4\pi)^2}+\frac{4!}{(4\pi)^4}-\dots\right)=\frac{1}{4\pi}-\frac{2!}{4^3\pi^3}+\frac{4!}{4^5\pi^5}-\dots=0.0786$$

$I_5=-0.0631 \qquad I_6=0.0528$

If $x=n\pi+\frac{\pi}{2}=\left(n+\frac{1}{2}\right)\pi$ in the sine-series formula we can find the corresponding formula for the maxima and minima of the cosine-integral:

$$-P\sin\left(n\pi+\frac{\pi}{2}\right)+Q\cos\left(n\pi+\frac{\pi}{2}\right)=-P\left(\sin n\pi\cos\frac{\pi}{2}+\sin\frac{\pi}{2}\cos n\pi\right)=$$

$$=-P(-1)^n=(-1)^{n-1}P=(-1)^{n-1}\int_{\left(n+\frac{1}{2}\right)\pi}^{\infty}\frac{\cos t}{t}\,dt$$

Therefore,

$$\int_{\left(n+\frac{1}{2}\right)\pi}^{\infty}\frac{\cos t}{t}\,dt=\frac{(-1)^{n-1}}{x}\left(1-\frac{2!}{x^2}+\frac{4!}{x^4}-\dots\right), \qquad x=\left(n+\frac{1}{2}\right)\pi$$

We see that the remainder in the series $P+iQ$ (after the 4 terms) is

$$\frac{1\cdot 2\cdot 3\cdot 4}{i^4}\int_x^{\infty}t^{-5}e^{it}\,dt$$

By applying Dirichlet's test (as we did in (1) before) for $|\varphi(t)| = |e^{it}| \le 2$, $f(t) = \frac{1}{t^5}$ monotonic, $\lim_{t\to\infty} f(t) = 0$, we have that

$$\left|\int_x^\infty \frac{e^{it}}{t^5}\,dt\right| \le \frac{2}{x^5} < \frac{5}{x^5}$$

and

$$\left|\frac{1\cdot 2\cdot 3\cdot 4}{i^4}\int_x^\infty \frac{e^{it}}{t^5}\,dt\right| \le \left|\frac{1\cdot 2\cdot 3\cdot 4}{i^4}\cdot\frac{2}{x^5}\right| < \left|\frac{2i}{(ix)^5}\right| < 2\frac{5!}{(ix)^5} .$$

Therefore the remainder is less than twice the following term in each series.

CHAPTER FOUR

STIRLING'S SERIES

Introduction to Stirling's Series

In this section we will investigate Stirling's series without using Euler's summation formula.

It can be proved that

$$\log\Gamma(1+x)=F(x)+2\int_0^\infty \frac{\arctan\frac{v}{x}}{e^{2\pi v}-1}\,dv \qquad \text{(See Appendix L)}$$

where

$$F(x)=\left(x+\frac{1}{2}\right)\log x-x+\frac{1}{2}\log(2\pi)\,.$$

We have

$$\arctan\frac{v}{x}=\frac{v}{x}-\frac{1}{3}\left(\frac{v}{x}\right)^3+\frac{1}{5}\left(\frac{v}{x}\right)^5-\ldots+(-1)^{n-1}\frac{1}{2n-1}\left(\frac{v}{x}\right)^{2n-1}+R_n$$

where $|R_n|<\dfrac{1}{2n+1}\left(\dfrac{v}{x}\right)^{2n+1}$

$$\int_0^\infty \frac{\arctan\frac{v}{x}}{e^{2\pi v}-1}\,dv=\int_0^\infty \frac{1}{e^{2\pi v}-1}\left(\frac{v}{x}-\frac{1}{3}\left(\frac{v}{x}\right)^3+\frac{1}{5}\left(\frac{v}{x}\right)^5-\ldots+(-1)^{n-1}\frac{1}{2n-1}\left(\frac{v}{x}\right)^{2n-1}+R_n\right)dv$$

From Appendix K we have

$$\int_0^\infty \frac{x^{2r-1}}{e^{2\pi x}-1}\,dx=\frac{B_r}{4r}$$

Hence we have

$$\int_0^\infty \frac{\arctan\frac{\nu}{x}}{e^{2\pi\nu}-1}d\nu = \frac{1}{x}\int_0^\infty \frac{\nu}{e^{2\pi\nu}-1}d\nu - \frac{1}{3x^3}\int_0^\infty \frac{\nu^3}{e^{2\pi\nu}-1}d\nu + \frac{1}{5x^5}\int_0^\infty \frac{\nu^5}{e^{2\pi\nu}-1}d\nu - \ldots +$$

$$+(-1)^{n-1}\frac{1}{(2n-1)x^{2n-1}}\int_0^\infty \frac{\nu^{2n-1}}{e^{2\pi\nu}-1}d\nu + \int_0^\infty \frac{R_n}{e^{2\pi\nu}-1}d\nu =$$

$$= \frac{1}{x}\frac{B_1}{4} - \frac{1}{3x^3}\frac{B_2}{4\cdot 2} + \frac{1}{5x^5}\frac{B_3}{4\cdot 3} - \ldots + (-1)^{n-1}\frac{B_n}{(2n-1)x^{2n-1}\cdot 4n} + R_n'$$

where R_n' is numerically less than the first term omitted from the series.

If we take the quotient of two consecutive terms and remark that $\frac{B_{n+1}}{B_n} = \frac{(2n+1)(2n+2)Q}{4\pi^2}$, where Q is a factor slightly less than 1, we see that the least value for the remainder is given by taking n equal to the integral part of πx; but the first two terms give a degree of accuracy which is ample for ordinary calculations.

Stoke's Asymptotic Expression

We are going to study Stoke's asymptotic expression for the series

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+a_1+1)\ldots\Gamma(n+a_r+1)}{\Gamma(n+b_1+1)\ldots\Gamma(n+b_s+1)}x^n = \sum X_n$$

where x is real and $s > r$.

$$X_n = \frac{\Gamma(n+a_1+1)\ldots\Gamma(n+a_r+1)}{\Gamma(n+b_1+1)\ldots\Gamma(n+b_s+1)}x^n$$

Write $s-r=\mu$, $\sum b-\sum a=\lambda$, and consider the term X_{t+p}, where t is large, and p is not of higher order than $\sqrt{t}$.

Neglecting terms of order $\frac{1}{\sqrt{t}}$, we find from Stirling's series from Chapter Two that

$$\log\Gamma(1+m+p)=\left(m+p+\frac{1}{2}\right)\log m-m+\frac{1}{2}\log(2\pi)+\frac{1}{2}\frac{p^2}{m}+\frac{1}{2}\frac{p}{m}-\frac{1}{6}\frac{p^3}{m^2}+O\left(\frac{1}{m}\right)$$

$$\log X_{t+p}=(t+p)\log x+\sum_{i=1}^{r}\log\Gamma(t+p+a_i+1)-\sum_{j=1}^{s}\log\Gamma(t+p+b_j+1)=$$

$$=(t+p)\log x+\sum_{i=1}^{r}\left[\left(t+p+a_i+\frac{1}{2}\right)\log t-t+\frac{1}{2}\log(2\pi)\right]-$$

$$-\sum_{j=1}^{s}\left[\left(t+p+b_j+\frac{1}{2}\right)\log t-t+\frac{1}{2}\log(2\pi)\right]+$$

$$+\sum_{i=1}^{r}\left[\frac{1}{2t}(p+a_i)^2+\frac{1}{2t}(p+a_i)-\frac{1}{6m^2}(p+a_i)^3+O\left(\frac{1}{t}\right)\right]-rt-$$

$$-\sum_{j=1}^{s}\left[\frac{1}{2t}(p+b_j)^2+\frac{1}{2t}(p+b_j)-\frac{1}{6m^2}(p+b_j)^3+O\left(\frac{1}{t}\right)\right]+st=$$

$$=(t+p)\log x+rt\log t+rp\log t+\left(\sum_{i=1}^{r}a_i\right)\log t+\frac{1}{2}r\log t++\frac{1}{2}r\log(2\pi)-$$

$$-sp\log t-st\log t-\left(\sum_{j=1}^{s}b_j\right)\log t-\frac{1}{2}s\log t-\frac{1}{2}s\log(2\pi)+\frac{1}{2t}rp^2+\frac{p}{t}\sum_{i=1}^{r}a_i+$$

$$+\frac{1}{2t}\sum_{i=1}^{r}a_i^2+\frac{rp}{2t}+\frac{1}{2t}\sum_{i=1}^{r}a_i-\frac{1}{2t}sp^2-\frac{p}{t}\sum_{j=1}^{s}b_j-\frac{1}{2t}\sum_{j=1}^{s}b_j^2-\frac{sp}{2t}-\frac{1}{2t}\sum_{j=1}^{s}b_j=$$

$$=(t+p)\log x-\mu p\log t-\lambda\log t-\frac{1}{2}\mu\log t-\frac{1}{2}\mu\log(2\pi)-\frac{1}{2t}\mu p^2-$$

$$-\frac{p\lambda}{t}-\frac{\mu p}{2t}-\frac{\lambda}{2t}+\frac{1}{2t}\sum a_i^2+b_j^2=$$

$$=(t+p)\log x-\mu t\log t-\mu p\log t-\lambda\log t-\frac{\mu}{2}\log t-\frac{\mu}{2}\log(2\pi)+\mu t-\frac{\mu p^2}{2t}$$

Therefore

$$\log X_{t+p}=(t+p)\log x-\mu\left\{\left(t+\frac{1}{2}\right)\log t+\frac{1}{2}\log(2\pi)-t\right\}-(p\mu+\lambda)\log t-\frac{1}{2}\frac{\mu p^2}{t}$$

It is convenient to suppose that x is of the form t^{μ}, where t is an integer (a restriction which can be removed by using elaborate methods); and then X_t is the greatest term because $\log x=\mu\log t$, so that the terms of the first degree in p cancel. We deduce that

$$\log X_{t+p}=(t+p)\mu\log t-\mu t\log t-\mu p\log t-\lambda\log t-\frac{\mu}{2}\log t-\frac{\mu}{2}\log(2\pi)+\mu t-\frac{\mu p^2}{2t}=$$

$$=\mu t-\frac{1}{2}\mu\log(2\pi t)-\lambda\log t-\frac{1}{2}\frac{\mu p^2}{t}$$

Therefore

$$X_{t+p}=e^{\mu t}e^{-\frac{1}{2}\mu\log(2\pi t)}e^{-\lambda\log t}e^{-\frac{\mu p^2}{2t}}=e^{\mu t}e^{-\log(2\pi t)^{\frac{\mu}{2}}}e^{\log t^{-\lambda}}e^{-\frac{\mu p^2}{2t}}$$

or $$X_{t+p} = \frac{e^{\mu t} t^{-\lambda}}{(2\pi t)^{\frac{\mu}{2}}} e^{-\frac{\mu p^2}{2t}}$$

Let $e^{-\frac{\mu}{2t}} = q$, $\lim_{t \to \infty} q = 1$

Then

$$X_{t+p} = \frac{e^{\mu t} t^{-\lambda}}{(2\pi t)^{\frac{\mu}{2}}} q^{p^2}$$

$$X_{t+p} + X_{t-p} = 2\frac{e^{\mu t} t^{-\lambda}}{(2\pi t)^{\frac{\mu}{2}}} q^{p^2}$$

$$\sum X_n = \frac{e^{t\mu} t^{-\lambda}}{(2\pi t)^{\frac{\mu}{2}}}\left(1 + 2q + 2q^4 + 2q^9 + \ldots\right)$$

Using Cesaro's theorem of divergent series, it can be proven that

$$\lim_{q \to 1}(1-q)^{\frac{1}{2}}\left(q + q^4 + q^9 + \ldots\right) = \frac{1}{2}\pi^{\frac{1}{2}}.$$

Therefore, the series in brackets is represented approximately by $\pi^{\frac{1}{2}}(1-q)^{-\frac{1}{2}}$, or by $\left(\frac{2\pi t}{\mu}\right)^{\frac{1}{2}}$

So, $$1 + q + q^4 + q^9 + \ldots \approx \pi^{\frac{1}{2}}(1-q)^{-\frac{1}{2}}.$$

Thus the asymptotic expression is

$$-\frac{e^{t\mu} t^{-\lambda}}{\mu^{\frac{1}{2}}(2\pi t)^{\frac{\mu-1}{2}}}, \quad \text{where} \quad t = x^{\frac{1}{\mu}}.$$

CHAPTER FIVE

POINCARE'S THEORY OF ASYMPTOTIC SERIES

Introduction to Poincare's Theory

Consider a function $J(x)$ expanded in inverse powers of x, $a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\dots$. The partial sums do not necessarily have to converge; but we suppose that taking any initial partial sum provides an "asymptotic" formula for f. We want the sum of the first $(n+1)$ terms to give an approximation to $J(x)$ which differs from $J(x)$ by less than $\frac{K_n}{x^{n+1}}$, where K_n depends only on n and not on x. Let S_n be the partial sum of the first $(n+1)$ terms. Poincare says that the series is *asymptotic* (or *semiconvergent*) to the function, if, for all n,

$$\lim_{x\to\infty} x^n\left(J-S_n\right)=0$$

This relation may be denoted by the symbol

$$J(x)\sim a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\dots .$$

In other words, the first $(n+1)$ terms of the series are of order $\frac{1}{x^{n+1}}$. Therefore, for a given value of n, the first $(n+1)$ terms of the series may be made as close as desired to the function $J(x)$ by making x sufficiently large. For each value of x and n there is an error of order $\frac{1}{x^{n+1}}$. Since the series actually diverges, there is an optimum number of terms in the series used to represent $J(x)$ for a given value of x. Associated with this is an unavoidable error. As x increases, the optimal number of terms increases and the error decreases. We note that if the original function is oscillatory (near infinity) then it cannot have an asymptotic expansion.

Let's consider an example. The logarithmic integral is defined as

$$U = \int_x^\infty \frac{e^{-t}}{t} dt .$$

As we proved in Chapter Three, the asymptotic series for this function can be generated via a series of partial integration, obtaining

$$U = \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \ldots + (-1)^n \frac{(n)!}{x^n}\right) + (-1)^{n+1} \int_x^\infty \frac{e^{-t}}{t^{n+2}} dt.$$

The infinite series obtained by taking the limit when $n \to \infty$ diverges, since the Cauchy convergence test yields

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{x}\right| = \infty .$$

Note that two successive terms in the series become equal in magnitude for n equal to the greatest integer less than or equal to x, indicating that the optimum number of terms for a given x is roughly the integer nearest x. As we proved in Chapter Three, the error involved using the first n terms is less than $\frac{(n+1)!e^{-x}}{x^{n+2}}$, which is exactly the next term in the series. We can see that as n increases, this estimate of the error first decreases and then increases without limit.

Note that the asymptotic series are fundamentally different to conventional power law expansions, such as

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots .$$

This series representation of $\sin x$ converges absolutely for all finite values of x. Thus, at any x the error associated with the series can be made as small as is desired by including a sufficiently large number of terms. In other words, unlike an asymptotic series, there is no intrinsic, or unavoidable, error associated with a convergent series. It follows that a convergent power law series representation of a function is unique inside the domain of convergence of the series.

On the other hand, an asymptotic series representation of a function is not unique. It is possible to have two different asymptotic series representations of the same function, as long as the difference between the two series is less than the error associated with each series (an example is the expansion of the confluent hypergeometric function).

It is to be noticed, however, that the same series may be asymptotic to more than one function; for example, since $\lim_{x\to\infty}\left(x^n e^{-x}\right)=0$ the same series will represent $J(x)$ and $J(x)+e^{-x}$.

Theorem

1) Asymptotic series can be added and subtracted as if they were convergent.
2) Asymptotic series can be multiplied together as if they were convergent. In particular, we can obtain any power of an asymptotic series.
3) If the first term of the asymptotic series is less than the radius of convergence, then we can substitute

and rearrange an asymptotic series in a power-series, and the result is an asymptotic series.

4) An asymptotic series can be integrated term by term to get another asymptotic series for the integral of the original function.

5) Consider a function $J(x)$ that has an asymptotic expansion. If its derivative has an asymptotic expansion, then the expansion of $J'(x)$ is the term-by - term differentiation of the expansion of $J(x)$.

Proof for (1):

It follows immediately from the definition of an asymptotic expansion.

Proof for (2):

Consider two asymptotic series:

$$J(x) \sim a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots, \quad K(x) \sim b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \ldots.$$

Then the formal product is

$$\Pi(x) = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \ldots$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0$.

We will show that the product $J(x) \cdot K(x)$ is represented asymptotically by $\Pi(x)$.

Let

$$S_n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots + \frac{a_n}{x^n}$$

$$T_n = b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots + \frac{b_n}{x^n}$$

$$\sum\nolimits_n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \dots + \frac{c_n}{x^n}$$

denote the sums of the first $(n+1)$ terms in these three series. Then we have:

$$J(x) = S_n + \frac{\rho}{x^n}, \qquad K(x) = T_n + \frac{\sigma}{x^n}$$

where ρ, σ are functions of x which tend to zero as $x \to \infty$.

Now, by definition Σ_n coincides with the product $S_n T_n$ up to and including the terms in $\frac{1}{x^n}$.

$$S_n T_n = \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \dots + \frac{a_n}{x^n}\right)\left(b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \frac{b_3}{x^3} + \dots + \frac{b_n}{x^n}\right) =$$

$$= a_0 b_0 + \frac{a_0 b_1 + b_0 a_1}{x} + \dots + \frac{a_0 b_n + a_1 b_{n-1} + \dots + a_0 b_n}{x^n} + O\left(\frac{1}{x^{n+1}}\right) + O\left(\frac{1}{x^{n+2}}\right) + \dots O\left(\frac{1}{x^{2n}}\right)$$

$$S_n T_n = c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{c_3}{x^3} + \dots + \frac{c_n}{x^n} + O\left(\frac{1}{x^{n+1}}, \dots \frac{1}{x^{2n}}\right)$$

Thus $S_n T_n - \sum_n$ contains terms from $\frac{1}{x^{n+1}}$ to $\frac{1}{x^{2n}}$.

We can write $S_n T_n = \sum_n + \frac{P_n}{x^{2n}}$, where P_n is a polynomial in x of degree $(n-1)$.

Therefore,

$$S_n T_n = \left(J(x)-\frac{\rho}{x^n}\right)\left(K(x)-\frac{\sigma}{x^n}\right)=\sum\nolimits_n+\frac{P_n}{x^{2n}}$$

$$\frac{x^n J(x)-\rho}{x^n}\cdot\frac{x^n K(x)-\sigma}{x^n}=\frac{x^{2n}\sum_n+P_n}{x^{2n}}$$

$$x^{2n}J(x)K(x)-x^n J(x)\sigma-x^n K(x)\rho+\rho\sigma=x^{2n}\sum\nolimits_n+P_n$$

$$x^{2n}\left\{J(x)K(x)-\sum\nolimits_n\right\}=\rho x^n K(x)+\sigma x^n J(x)+P_n-\rho\sigma$$

or $$x^n\left\{J(x)K(x)-\sum\nolimits_n\right\}=\rho K(x)+\sigma J(x)+\frac{P_n-\rho\sigma}{x^n}$$

As $x\to\infty$, $J(x)\to a_0$, $K(x)\to b_0$, and $\rho\to 0$, $\sigma\to 0$,

$$\lim_{x\to\infty}\left(x^n J(x)K(x)-\sum\nolimits_n\right)=\lim_{x\to\infty}\frac{P_n}{x^n}=0.$$

Therefore, the product $J(x)\cdot K(x)$ is represented asymptotically by $\Pi(x)$.

Proof for (3):

Let's consider the possibility of substituting an asymptotic series in a power-series. We have:

$$J(x)=a_0+J_1(x)$$

$$J(x)\sim a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\ldots$$

We know that a power-series $\sum a_n x^n$ represents a continuous function of x, say $f(x)$ within its radius of convergence $|x|=R$. Let us substitute a_0+J_1 for J in the series

$$f(J)=c_0+c_1J+c_2J^2+c_3J^3+\dots$$

and rearrange in powers of J_1, provided that $|a_0|$ is less than the radius of convergence, R; because $\lim_{x\to\infty} J_1(x)=0$, we can take x large enough so $|a_0|+|J_1|<R$.

Now, let's substitute the asymptotic series $\frac{a_1}{x}+\frac{a_2}{x^2}+\dots$ for J_1 in the series

$$F(J_1)=C_0+C_1J_1+C_2J_1^2+C_3J_1^3+\dots$$

Making a formal substitution, as if the series for J_1 were convergent, we obtain:

$$F(J_1)=C_0+C_1\left(\frac{a_1}{x}+\frac{a_2}{x^2}+\dots\right)+C_2\left(\frac{a_1}{x}+\frac{a_2}{x^2}+\dots\right)^2+\dots$$

Then we obtain some new series $\sum=D_0+\frac{D_1}{x}+\frac{D_2}{x^2}+\dots$ where

$D_0=C_0$, $D_1=C_1a_1$, $D_2=C_1a_2+C_2a_1^2$, $D_3=C_1a_3+2C_2a_1a_2+C_3a_1^3$, etc.

We will prove that the series $\sum$ represents $F(J_1)$ asymptotically.

Let S_n be the sum of the terms up to $\frac{1}{x^n}$ in J_1; S_n

represents J_1 asymptotically.

Let $\sum_n$ the sum of the terms up to $\frac{1}{x^n}$ in $\sum$.

$\sum_n = D_0 + \frac{D_1}{x} + \ldots \frac{D_n}{x^n}$, and $S_n = \frac{a_1}{x} + \frac{a_2}{x^2} \ldots \frac{a_n}{x^n}$

Now, if

$$\sum_n' = C_0 + C_1 S_n + C_2 S_n^2 + \ldots + C_n S_n^n,$$

because

$$\sum_n = C_0 + \frac{C_1 a_1}{x} + \frac{C_1 a_2 + C_2 a_1^2}{x^2} + \ldots$$

we can write

$$\sum_n' = C_0 + \frac{C_1 a_1}{x} + \frac{C_1 a_2}{x^2} + \frac{C_2 a_1^2}{x^2} + \ldots .$$

We see that $\sum_n'$ and $\sum_n$ agree up to terms in $\frac{1}{x^n}$, and consequently $\sum_n' - \sum_n$ is a polynomial in $\frac{1}{x}$, ranging from terms in $\left(\frac{1}{x}\right)^{n+1}$ to $\left(\frac{1}{x}\right)^{n^2}$; thus

$$(1) \qquad \lim_{x\to\infty} x^n \left(\sum_n' - \sum_n\right) = 0 .$$

Next, if

$T_n = C_0 + C_1 J_1 + C_2 J_1^2 + \ldots C_n J_1^n$, we have, since S_n represents J_1 asymptotically, $\lim_{x\to\infty} x^n \left(J_1^r - S_n^r\right) = 0$ for $r = 1, 2, \ldots$ and therefore

$$(2) \qquad \lim_{x\to\infty} x^n \left(T_n - \sum_n'\right) = 0 .$$

Finally,

$$F - T_n = C_0 + C_1 J_1 + C_2 J_1^2 + \ldots - \left(C_0 + C_1 J_1 + C_2 J_1^2 \ldots + C_n J_1^n\right) = C_{n+1} J_1^{n+1} + C_{n+2} J_1^{n+2} \ldots$$

thus, since $F(J_1)$ is convergent, $|F - T_n| < M J_1^{n+1}$, where M is a constant.

We find that

$$\lim_{x\to\infty} x^n (F - T_n) < \lim_{x\to\infty} x^n M J_1^{n+1} = 0$$

$$(3) \qquad \lim_{x\to\infty} x^n (F - T_n) = 0 ,$$

because $\lim_{x\to\infty}\left(M x^n J_1^{n+1}\right) = \lim_{x\to\infty}\left(M \frac{a_1^{n+1}}{x}\right) = 0$.

By combining (1), (2), and (3) we see now that

$$\lim_{x\to\infty} x^n \left(F - \sum_n\right) = \lim_{x\to\infty} x^n \left(F - T_n + T_n - \sum_n' + \sum_n' - \sum_n\right) = 0$$

Therefore,

$$\lim_{x\to\infty} x^n \left(F - \sum_n\right) = 0 .$$

Thus the series $\sum$ represents $F(J_1)$ asymptotically; therefore, an asymptotic series may be substituted in a power-series and rearranged (just as if convergent), provided that its first term is numerically less than the radius of convergence.

Note that we use the convergence of the series $f(J)$ in two places only, first in order to rearrange in powers of J_1, and secondly to establish the inequality $|F-T_n| < MJ_1^{n+1}$. Now this inequality is satisfied if the series $C_1J_1 + C_2J_1^2 + C_3J_1^3 + \ldots$ is asymptotic to $F(J_1)$; and then we must suppose that a_0 is zero in order to get any result at all, so that $J = J_1$ and we can entirely avoid the restriction that $f(J)$ is convergent. Thus, an asymptotic series, whose first term is zero may be substituted in another asymptotic series, and the result may be rearranged just as if both series were convergent.

Proof for (4):

Let us consider the integration of an asymptotic series in which $a_0 = 0,\ a_1 = 0$.

If

$$J(x) \sim \frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \ldots$$

$\lim\limits_{x\to\infty} x^n (J - S_n) = 0$ so, for any $\varepsilon > 0$ there is n_0, such that

$|x^n (J - S_n)| < \varepsilon$ or $|J - S_n| < \dfrac{\varepsilon}{x^n}$ for any $n \geq n_0$.

So,

$$|J - S_n| < \frac{\varepsilon}{x^n} \quad \text{if} \quad x \geq x_0$$

$$\left|\int_x^\infty (J - S_n)\right| \leq \int_x^\infty |J - S_n| < \int_x^\infty \frac{\varepsilon}{x^n}$$

$$\left|\int_x^\infty J dx - \int_x^\infty S_n dx\right| < \frac{\varepsilon}{(n-1)x^{n-1}}, \quad \text{if} \quad x > x_0$$

so that $\int_x^\infty J dx$ is represented asymptotically by

$$\int_x^\infty \left(\frac{a_2}{x^2} + \frac{a_3}{x^3} + \frac{a_4}{x^4} + \ldots\right) dx = -a_2 x^{-1}\Big|_x^\infty - \frac{a_3}{2} x^{-2}\Big|_x^\infty - \ldots = \frac{a_2}{x} + \frac{a_3}{2x^2} + \frac{a_4}{3x^3} + \ldots$$

We remark that an asymptotic series cannot, in general, be differentiated: the existence of an asymptotic series for $J(x)$ does not imply the existence of one for $J'(x)$. For example, $e^{-x}\sin(e^x)$ has an asymptotic series $0 + \frac{0}{x} + \frac{0}{x^2} + \ldots$.

But its derivative is $-e^{-x}\sin(e^x) + \cos(e^x)$, which oscillates as x tends to ∞; and consequently the derivative has no asymptotic expansion.

Proof for (5):

This follows by applying the theorem of integration to $J'(x)$.

Another proof can be made using the following Lemma:

If $\phi(x)$ has a definite finite limit as x tends to ∞, then $\phi'(x)$ either oscillates or tends to zero as a limit.

Proof for Lemma:

If $\phi(x)$ tends to a definite limit we can find x_0 so that $|\phi(x)-\phi(x_0)|<\varepsilon$ if $x>x_0$.

Thus, since $\phi'(\xi)=\dfrac{\phi(x)-\phi(x_0)}{x-x_0}$, where $x>\xi>x_0$, we find $|\phi'(\xi)|<\dfrac{\varepsilon}{x-x_0}$. So, $\phi'(x)$ cannot approach any definite limit other than zero; but the last inequality does not exclude oscillation, since ξ may not take all values greater than x_0 as x tends to ∞. $\phi'(x)$, if it has a definite limit, it must be zero.

Now, to prove (4), consider

$$J(x)\sim a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots .$$

Then we have

$$\lim_{x\to\infty} x^{n+1}\left\{J(x)-S_{n+1}(x)\right\}=0$$

and

$$\lim_{x\to\infty} x^{n+1}\left\{J(x)-S_n(x)-\frac{a_{n+1}}{x^{n+1}}\right\}=0 .$$

Therefore,

$$\lim_{x\to\infty} x^{n+1}\{J(x)-S_n(x)\}=a_{n+1} .$$

Thus the differential coefficient

$$x^{n+1}\{J'(x)-S_n'(x)\}+(n+1)x^n\{J(x)-S_n(x)\}$$

if it has a definite limit, must tend to 0.

But $x^n\{J(x)-S_n(x)\}\to 0$ so that, $\lim x^{n+1}\{J'(x)-S_n'(x)\}=0$ if it exists. That is, if $J'(x)$ has an asymptotic series, it is

$$-\frac{a_1}{x^2}-\frac{2a_2}{x^3}-\frac{3a_3}{x^4}-\dots .$$

Corollary

We can divide any asymptotic series by another asymptotic series (assuming that the first term a_0 is not zero) as if they were convergent.

Proof:

Let

$$J(x)\sim a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots .$$

Then

$$J(x)\sim a_0\left(1+\frac{a_1}{a_0x}+\frac{a_2}{a_0x^2}+\dots\right),$$

and we can write

$J(x)=a_0(1+K)$, where $K\sim\frac{a_1}{a_0x}+\frac{a_2}{a_0x^2}+\dots .$

Then $\left(J(x)\right)^{-1} = a_0^{-1}\dfrac{1}{1+K} = a_0^{-1}\sum_{n=0}^{\infty}(-K)^n$

$$\left(J(x)\right)^{-1} = a_0^{-1}\left(1-K+K^2-K^3+\ldots\right)$$

and we can construct an asymptotic series for $\{J(x)\}^{-1}$ by exactly the same rule as if the series for $J(x)$ were convergent.

Bromwich summarizes the situation thus:

> It is instructive to contrast the rules for transforming and combining asymptotic series with those previously established for convergent series. Thus, any two asymptotic series can be multiplied together: on the other hand, the product of two convergent series is not necessarily a convergent series. Similarly any asymptotic series may be integrated term-by-term, although not every convergent series can be integrated.
>
> On the other hand, as we have just explained, we cannot differentiate any asymptotic series unless we know from independent reasoning that the corresponding derivate has an asymptotic expansion; although, in dealing with a convergent series, we can apply the test for uniform convergence directly to the differentiated series, and so infer that the derived function has an expansion.
>
> These contrasts, however, are not surprising. In a convergent series, the parameter with respect to which we differentiate or integrate is strictly an auxiliary variable, and in no way enters into the definition of convergence of the series; but in an asymptotic series, the definition depends on the parameter x. (*An introduction to the Theory of Infinite Series*, 346)

It is sometimes convenient to extend our definition and say that J is represented asymptotically by the series

$$\Phi+\left(a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots\right)\Psi$$

when $\frac{J-\Phi}{\Psi}$ is represented by $a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots$, where Φ and Ψ are two suitable chosen functions of x.

As an example, recall the asymptotic formula deduced from Stirling's series in Chapter Four:

$$\log\Gamma(1+x)=F(x)+2\int_0^\infty\frac{\arctan\frac{v}{x}}{e^{2\pi v}-1}dv=$$

$$=\left(x+\frac{1}{2}\right)\log x-x+\frac{1}{2}\log(2\pi)+\frac{B_1}{2x}-\frac{B_2}{3\cdot 4x^3}+\frac{B_3}{5\cdot 6x^5}-\dots=$$

$$=\log\left\{x^x(2\pi x)^{\frac{1}{2}}\right\}-x+\frac{B_1}{2x}-\frac{B_2}{3\cdot 4x^3}+\frac{B_3}{5\cdot 6x^5}-\dots$$

Thus $\Gamma(1+x)=x^x(2\pi x)^{\frac{1}{2}}e^{-x}e^{\frac{B_1}{2x}-\frac{B_2}{3\cdot 4x^3}+\frac{B_3}{5\cdot 6x^5}-\dots}$. Since we can substitute an asymptotic series into the (convergent) exponential series $e^x=1+x+\frac{x^2}{2!}+\dots$,

$$\Gamma(1+x)\sim e^{-x}x^x(2\pi x)^{\frac{1}{2}}\left(1+\frac{B_1}{2x}+\frac{B_1^2}{4x^2 2!}+\dots\right)$$

which may be rewritten:

$$\Gamma(1+x) \sim e^{-x}x^{x}(2\pi x)^{\frac{1}{2}}\left(1+\frac{C_1}{x}+\frac{C_2}{x^2}+\frac{C_3}{x^3}-\ldots\right)$$

where

$$C_1 = \frac{B_1}{2} = \frac{1}{12}, \quad C_2 = -\frac{B_1^2}{8} = \frac{1}{288}, \text{ etc.}$$

Poincare's theory generalizes for x complex and tending to ∞ in any definite direction. But a non-convergent series cannot represent asymptotically the same single-valued analytic function J for all arguments x. In fact if we can determine constants M, R, such that $\left|J - a_0 - \frac{a_1}{x}\right| < \frac{M}{|x|^2}$, when $|x| > R$, it can be shown that $J(x)$ is a regular function of $\frac{1}{x}$. Thus the asymptotic series must actually be convergent.

For different domains for x, we may have different asymptotic representations of the same function. This is illustrated in Stokes' discussion of the Bessel functions in Chapter Six).

Applications of Poincare's Theory

A significant application of Poincare's theory is to the solution of differential equations. We first obtain a formal solution in a non-convergent series. Independently we show that a solution with an asymptotic representation exists. (Thus we may either deduce a definite integral from the series first calculated or we may find a solution as a definite integral directly, and then identify it with the series.) Finally, the region of validity of the asymptotic representation is determined.

Poincare showed that every linear differential equation which has polynomial coefficients may be solved by asymptotic series, as long as the independent variable tends to infinity along a fixed direction. Poincare did not determine the regions of validity. Horn, in a number of special cases, filled in the gaps.

Barnes and Hardy applied Poincare's theory to the asymptotic representation of functions given by power-series, using the theory of contour integration. The method of Stokes given in Chapter Four is also useful for some real series.

Bromwich remarks that the ordinary Taylor's (or Maclaurin's) series of the Differential calculus has essentially an asymptotic character ($\frac{1}{x}$ being changed to x), until the remainder has been investigated.

Even when the series

$$f(0)+xf'(0)+\frac{1}{2}x^2f''(0)+\dots$$

is convergent, its sum is not necessarily equal to $f(x)$; but we can always show that $\{f(x)-S_n(x)\}$ is of higher order than the last term in $S_n(x)$. Or, in more precise form, we can claim that

$$\lim_{x\to 0}\frac{f(x)-S_n(x)}{x^n}=0$$

which has the same character as the definition from the beginning of this Chapter.

CHAPTER SIX

DIFFERENTIAL EQUATIONS

Introduction to Differential Equations

We will give some examples of the way in which asymptotic series present themselves in the solution of differential equations.

Let us try to solve the differential equation

$\frac{dy}{dx}=\frac{a}{x}+by \quad (b>0)$ by means of an asymptotic series

$$y=A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\dots$$

$$\frac{dy}{dx}=-\frac{A_1}{x^2}-\frac{2A_2}{x^3}-\frac{3A_3}{x^4}-\dots$$

On substitution, we find

$$-\frac{A_1}{x^2}-\frac{2A_2}{x^3}-\frac{3A_3}{x^4}-\dots=\frac{a}{x}+b\left(A_0+\frac{A_1}{x}+\frac{A_2}{x^2}+\dots\right)$$

This gives $A_0=0,\ A_1=-\frac{a}{b},\quad A_2=-\frac{A_1}{b}=\frac{a}{b^2},\quad A_3=-\frac{2A_2}{b}=-\frac{1\cdot 2a}{b^3},$ etc.

Therefore, we find the formal solution

$$y=-\frac{a}{bx}\left\{1-\frac{1}{bx}+2!\left(\frac{1}{bx}\right)^2-3!\left(\frac{1}{bx}\right)^3+\dots\right\}$$

and, as we have seen in Chapter Three, this represents the integral

$$y=-a\int_0^\infty \frac{e^{-t}}{t+bx}dt\,,$$

and it is easy now to verify directly that this integral does satisfy the given equation.

The Modified Bessel's Equation

Following Stokes, we have the equation in the form

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{n^2}{x^2}\right)y=0\,,$$

and then attempt to find a solution in the form $y=e^{\lambda x}x^{-\frac{1}{2}}\eta$, where η proves to be an asymptotic series.

$$\frac{dy}{dx}=\lambda e^{\lambda x}x^{-\frac{1}{2}}\eta-\frac{1}{2}e^{\lambda x}x^{-\frac{3}{2}}\eta+e^{\lambda x}x^{-\frac{1}{2}}\frac{d\eta}{dx}$$

$$\frac{1}{x}\frac{dy}{dx}=\lambda e^{\lambda x}x^{-\frac{3}{2}}\eta-\frac{1}{2}e^{\lambda x}x^{-\frac{5}{2}}\eta+e^{\lambda x}x^{-\frac{3}{2}}\frac{d\eta}{dx}$$

$$\frac{d^2y}{dx^2}=\lambda^2 e^{\lambda x}x^{-\frac{1}{2}}\eta-\frac{1}{2}\lambda e^{\lambda x}x^{-\frac{3}{2}}\eta+\lambda e^{\lambda x}x^{-\frac{1}{2}}\frac{d\eta}{dx}-\lambda\frac{1}{2}e^{\lambda x}x^{-\frac{3}{2}}\eta+$$

$$+\frac{1}{2}\cdot\frac{3}{2}e^{\lambda x}x^{-\frac{5}{2}}\eta-\frac{1}{2}e^{\lambda x}x^{-\frac{3}{2}}\frac{d\eta}{dx}+\lambda e^{\lambda x}x^{-\frac{1}{2}}\frac{d\eta}{dx}-\frac{1}{2}e^{\lambda x}x^{-\frac{3}{2}}\frac{d\eta}{dx}+e^{\lambda x}x^{-\frac{1}{2}}\frac{d^2\eta}{dx^2}$$

Then

$$x^{-\frac{1}{2}}\frac{d^2\eta}{dx^2}+\left(\lambda x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}+\lambda x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}+x^{-\frac{3}{2}}\right)\frac{d\eta}{dx}+\lambda^2x^{-\frac{1}{2}}\eta-\frac{1}{2}\lambda x^{-\frac{3}{2}}\eta-$$

$$-\frac{\lambda}{2}e^{-\frac{3}{2}}\eta+\frac{3}{4}x^{-\frac{5}{2}}\eta+\lambda x^{-\frac{3}{2}}\eta-\frac{1}{2}x^{-\frac{5}{2}}\eta-x^{-\frac{1}{2}}\eta-\frac{n^2}{x^2}x^{-\frac{1}{2}}\eta=0$$

Then

$$\frac{d^2\eta}{dx^2}+\left(\lambda-\frac{1}{2}x^{-1}+\lambda-\frac{1}{2}x^{-1}+x^{-1}\right)\frac{d\eta}{dx}+\eta\left(\lambda^2+\frac{3}{4}x^{-2}-\frac{1}{2}x^{-2}-1-\frac{n^2}{x^2}\right)=0$$

Thus

$$\frac{d^2\eta}{dx^2}+2\lambda\frac{d\eta}{dx}+\left(\lambda^2-1+\frac{1}{4}x^{-2}-\frac{n^2}{x^2}\right)\eta=0$$

The equation for η is found to be

$$x^2\left(\frac{d^2\eta}{dx^2}+2\lambda\frac{d\eta}{dx}\right)+\left\{\left(\lambda^2-1\right)x^2+\left(\frac{1}{4}-n^2\right)\right\}\eta=0$$

If we take $\lambda^2=1$,

$$x^2\left(\frac{d^2\eta}{dx^2}+2\lambda\frac{d\eta}{dx}\right)+\left(\frac{1}{4}-n^2\right)\eta=0$$

and then, writing

$$\eta=1+\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\dots$$

$$\frac{d\eta}{dx}=-\frac{A_1}{x^2}-\frac{2A_2}{x^3}-\frac{3A_3}{x^4}-\dots$$

$$\frac{d^2\eta}{dx^2}=\frac{1\cdot 2A_1}{x^3}+\frac{1\cdot 2\cdot 3A_2}{x^4}+\frac{3\cdot 4A_3}{x^5}-\dots$$

$$x^2\left(\frac{2A_1}{x^3}+\frac{2\cdot 3A_2}{x^4}+\frac{3\cdot 4A_3}{x^5}+\dots\right)+2\lambda x^2\left(-\frac{A_1}{x^2}-\frac{2A_2}{x^3}-\frac{3A_3}{x^4}-\dots\right)+$$

$$+\left(\frac{1}{4}-n^2\right)\left(1+\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\dots\right)=0$$

we obtain

$$\left(\frac{2A_1}{x^3}+\frac{2\cdot 3A_2}{x^4}+\frac{3\cdot 4A_3}{x^5}+\dots\right)-2\lambda\left(A_1+\frac{2A_2}{x}-\frac{3A_3}{x^2}-\dots\right)+$$

$$+\left(\frac{1}{4}-n^2\right)\left(1+\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\dots\right)=0$$

or

$$2\lambda A_1=\frac{1}{4}-n^2,$$

$$2A_1-4\lambda A_2+\left(\frac{1}{4}-n^2\right)A_1=0, \quad \text{thus} \quad 4\lambda A_2=\left(\frac{9}{4}-n^2\right)A_1$$

$$6\lambda A_3=\left(\frac{25}{4}-n^2\right)A_2, \quad \text{etc.}$$

Thus we may take

$$A_1=\frac{1-4n^2}{8\lambda}, \qquad A_2=\frac{1}{1\cdot 2}\frac{\left(1-4n^2\right)\left(9-4n^2\right)}{(8\lambda)^2}, \quad \text{etc.}$$

leading to the solutions

$$y=e^{\lambda x}x^{-\frac{1}{2}}\eta=e^{\lambda x}x^{-\frac{1}{2}}\left(1+\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\dots\right)=$$

$$=\frac{e^{\lambda x}}{\sqrt{x}}\left(1+\frac{1-4n^2}{8\lambda}+\frac{1}{1\cdot 2}\frac{\left(1-4n^2\right)\left(9-4n^2\right)}{(8\lambda x)^2}+\dots\right), \quad \text{where } \lambda=\pm 1 .$$

It is easy to see that these series cannot converge for any value of x (unless $2n$ is an odd integer; and then the series terminate); they do not agree with any of the series considered up to the present, but we can write

$$\Gamma\left(n+r+\frac{1}{2}\right)=\int_0^\infty e^{-t}t^{n+r-\frac{1}{2}}dt \ .$$

We know that

$$\Gamma(1+n)=\int_0^\infty e^{-v}v^n dv$$

and

$$x(x+1)(x+2)...(x+n-1)=\frac{\Gamma(n+x)}{\Gamma(x)} \quad \text{(Appendix F)} \ .$$

Thus

$$\int_0^\infty e^{-t}t^{n+r-\frac{1}{2}}dt=\Gamma\left(n+r+\frac{1}{2}\right)=\Gamma\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)...\left(n+\frac{1}{2}+r-1\right)=$$

$$=\Gamma\left(n+\frac{1}{2}\right)\frac{1}{2^r}(1+2n)(3+2n)(5+2n)...(2r-1+2n)$$

and for $\left(1-\frac{t}{2\lambda x}\right)^{n-\frac{1}{2}}$ we apply the binomial theorem.

In general

$$(1+x)^v=1+vx+v(v-1)\frac{x^2}{2!}+v(v-1)(v-2)\frac{x^3}{3!}+...$$

Therefore,

$$\left(1-\frac{t}{2\lambda x}\right)^{n-\frac{1}{2}}=1+\frac{(1-2n)t}{4\lambda x}+\frac{(1-2n)(3-2n)t^2}{1\cdot 2(4\lambda x)^2}+...$$

Thus the series can be written in the form

$$\frac{1}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^\infty e^{-t}t^{n-\frac{1}{2}}\left(1-\frac{t}{2\lambda x}\right)^{n-\frac{1}{2}}dt\ .$$

When x is real and positive (n being assumed positive), this integral has a meaning only if $\lambda=-1$; and then the remainder in the binomial expansion is less than the following term (at any rate after a certain stage), and thus the same is true of the asymptotic series. Consequently, for $\lambda=-1$, the asymptotic series is asymptotic to the integral

$$\frac{1}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^\infty e^{-t}\left\{\left(\frac{t(t+2x)}{2x}\right)\right\}^{n-\frac{1}{2}}dt\ .$$

If we write $t+x=x\cosh\theta$, and then multiply by the factor $e^{-x}x^{-\frac{1}{2}}$, we obtain the solution

$$y=\frac{x^n}{2^{n-\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)}\int_0^\infty e^{-x\cosh\theta}\sinh^{2n}\theta d\theta\ ,$$

which can be proved to satisfy the original differential equation, by substituting and integrating by parts. It may be expected that the two original series both satisfy the differential equation; although we cannot obtain a complete proof without some assistance from the Theory of Functions.

APPENDIX A

TRANSFORMATION OF SLOWLY CONVERGENT ALTERNATING SERIES

TRANSFORMATION OF SLOWLY CONVERGENT ALTERNATING SERIES

Let $\sum_{n=1}^{\infty}(-1)^{n-1}v_n$ be an alternating series.

Let us write: $v_n - v_{n+1} = Dv_n$ and

$$v_n - 2v_{n+1} + v_{n+2} = Dv_n - Dv_{n+1} = D^2v_n,$$

$$v_n - 3v_{n+1} + 3v_{n+2} - v_{n+3} = (Dv_n - Dv_{n+1}) - (Dv_{n+1} - Dv_{n+2})$$
$$= Dv_n - 2Dv_{n+1} + Dv_{n+2} = D^2v_n - D^2v_{n+1} = D^3v_n$$

and so on.

Then, if $|x| \le 1$, we have

$$(1+x)(v_0 - v_1x + v_2x^2 - ...) = v_0 - v_1x + v_2x^2 - v_3x^3 + v_0x - v_1x^2 + v_2x^3 - ...$$
$$= v_0 - xDv_0 - x^2Dv_1 + x^3Dv_2 - ...$$

and consequently

$$\sum_{0}^{\infty}(-1)^n v_n x^n = v_0 - v_1x + v_2x^2 - v_3x^3 + ... = \frac{v_0 + xDv_0 - x^2Dv_1 + x^3Dv_3...}{1+x} =$$
$$= \frac{v_0}{1+x} + \frac{xDv_0 - x^2Dv_1 + x^3Dv_3...}{1+x} = \frac{v_0}{1+x} + \frac{x}{1+x}\{Dv_0 - xDv_1 + x^2Dv_3 - ...\}$$
$$= \frac{v_0}{1+x} + y\{Dv_0 - xDv_1 + x^2Dv_3...\}$$

where $y = \dfrac{x}{1+x}$.

Repeating this operation with v_n replaced by $Dv_n, D^2v_n, ...D^pv_n$ successively, we find

$$\sum_{0}^{\infty}(-1)^n v_n x^n = \frac{1}{1+x}\{v_0 + yDv_0 + y^2D^2v_0 + ... + y^{p-1}D^{p-1}v_0\} + y^p\{D^pv_0 - xD^pv_1 + ...\}$$

It can be proved that in all cases when the original series converges, the reminder term $y^p\{D^pv_0 - xD^pv_1 + ...\}$ tends to zero as p increases to infinity, at least when x is positive.

The case of chief interest arises when $x=1$, and then we have

$$\sum_{0}^{\infty}(-1)^n v_n = \frac{1}{2}\left(v_0 + \frac{1}{2}Dv_0 + \frac{1}{2^2}D^2v_0 + \frac{1}{2^3}D^3v_0 + ... + \frac{1}{2^{p-1}}D^{p-1}v_0\right) +$$
$$+ \frac{1}{2^p}\left(D^pv_0 - D^pv_1 + D^pv_2 - ...\right)$$

We can write down a simple expression for the remainder, if $v_n = f(n)$, where $f(x)$ is a function such that the p^{th} derivative $f^p(x)$ has a fixed sign for all positive values of x, and steadily decreases in numerical value as x increases.

For, $Dv_n = f(n) - f(n+1) = -\int_0^1 f'(x_1 + n)dx_1$

and thus $D^2 v_n = +\int_0^1 dx_1 \int_0^1 f''(x_1 + x_2 + n)dx_2$

and in general $D^p v_n = (-1)^p \int_0^1 dx_1 \int_0^1 dx_2 ... \int_0^1 f^p(x_1 + x_2 + ... + x_p + n)dx_p$

Thus the series $D^p v_0 - D^p v_1 + D^p v_2 - ...$ consists of a succession of decreasing terms, of alternate signs. Its sum is therefore less than $D^p v_0$ in numerical value and consequently

$$\sum_0^\infty (-1)^n v_n = \frac{1}{2} v_0 + \frac{1}{4} D v_0 + \frac{1}{8} D^2 v_0 + ... + \frac{1}{2^p} D^{(p-1)} v_0 + R_p,$$

where $|R_p| < \frac{1}{2^p} |D^p v_0|$.

APPENDIX B

THE SERIES OF FRACTIONS FOR

$\cot x$, $\tan x$, $\operatorname{cosec} x$

THE SERIES OF FRACTIONS FOR
$\cot x$, $\tan x$, $\operatorname{cosec} x$

We will show the series of fractions for $\cot x$, $\tan x$, $\operatorname{cosec} x$, as well as prove the formula:

$$\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+\sum_1^\infty \frac{2x}{x^2+4n^2\pi^2} .$$

Let

$$F_n(x)=\frac{1}{2i}\left\{\left(1+\frac{ix}{n}\right)^n-\left(1-\frac{ix}{n}\right)^n\right\} .$$

Then

$$\sin x=\lim_{n\to\infty} F_n(x) \quad \text{and} \quad \cos x=\lim_{n\to\infty}\frac{dF_n}{dx} .$$

So that

$$\cot x=\lim_{n\to\infty}\frac{dF_n/dx}{F_n(x)} .$$

Now we can show if n is odd, say $n=2m+1$,

$$F_n(x)=x\prod_{r=1}^{m}\left(1-\frac{x^2}{n^2\tan^2\dfrac{r\pi}{n}}\right)$$

so that

$$\frac{dF_n(x)}{dx}=\prod_{r=1}^{m}\left(1-\frac{x^2}{n^2\tan^2\dfrac{r\pi}{n}}\right)+x\frac{d}{dx}\left\{\left(1-\frac{x^2}{n^2\tan^2\dfrac{\pi}{n}}\right)\left(1-\frac{x^2}{n^2\tan^2\dfrac{2\pi}{n}}\right)\cdots\left(1-\frac{x^2}{n^2\tan^2\dfrac{m\pi}{n}}\right)\right\}$$

Thus

$$\frac{dF_n(x)/dx}{F_n(x)}=\frac{1}{x}+\frac{-2x}{n^2\tan^2\dfrac{\pi}{n}}\sum_{r=1}^{m}\frac{1}{1-x^2/\left(n^2\tan^2\dfrac{r\pi}{n}\right)}=\frac{1}{x}+\sum_{r=1}^{m}\left(\frac{2x}{x^2-n^2\tan^2\dfrac{r\pi}{n}}\right)$$

$$\cot x=\lim_{n\to\infty}\frac{dF_n(x)/dx}{F_n(x)}=\lim_{n\to\infty}\left(\frac{1}{x}+\sum_{r=1}^{m}\frac{2x}{x^2-n^2\tan^2\dfrac{r\pi}{n}}\right)$$

We apply Tannery's Theorem (i.e., the comparison test for convergence) by taking the comparison series

$M_r = \dfrac{2|x|}{r^2\pi^2 - |x|^2}$, for we have

$$\left| x^2 - n^2 \tan^2\left(\frac{r\pi}{n}\right) \right| > r^2\pi^2 - |x|^2$$

$$\frac{1}{\left| x^2 - n^2 \tan^2\left(\frac{r\pi}{n}\right) \right|} < \frac{1}{r^2\pi^2 - |x|^2}.$$

Thus for all values of x, real or complex (except multiples of π), we have

$$\cot x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2\pi^2},$$

where n is taken as the variable of summation, instead of r. Now we have the following identities:

$$\tan x = \cot x - 2\cot 2x$$

$$\operatorname{cosec} x = \cot\frac{1}{2}x - \cot x$$

Thus we find, on subtraction

$$\tan x = \cot x - 2\cot 2x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2\pi^2} - \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2\cdot 4x}{(2x)^2 - n^2\pi^2}$$

Thus, we have

$$\tan x = \sum_{0}^{\infty} \frac{2x}{\left(n + \frac{1}{2}\right)^2 \pi^2 - x^2}$$

Similarly,

$$\operatorname{cosec} x = \frac{1}{x} + \sum_{1}^{\infty} (-1)^n \frac{2x}{x^2 - n^2\pi^2}.$$

Changing from x to ix, we find that

$$\cot ix = \frac{\cos ix}{\sin ix} = \frac{-i\cosh x}{\sinh x} = -i\coth x = -\frac{i}{x} - \sum_{n=1}^{\infty} \frac{2ix}{x^2 + n^2\pi^2}$$

$$\tan ix = \frac{\sin ix}{\cos ix} = \frac{-\sinh x}{i\cosh x} = i\tanh x = \sum_{n=0}^{\infty} \frac{2ix}{\left(n + \frac{1}{2}\right)^2 \pi^2 + x^2}$$

and

$$\operatorname{cosec} ix = \frac{1}{\sin ix} = -\frac{i}{\sinh x} = -i\operatorname{cosech} x = -\frac{i}{x} - \sum_{n=1}^{\infty} (-1)^n \frac{2ix}{x^2 + n^2\pi^2}.$$

Therefore,

$$\coth x = \frac{1}{x} + \sum_{1}^{\infty} \frac{2x}{x^2 + n^2\pi^2},$$

$$\tanh x = \sum_{0}^{\infty} \frac{2x}{x^2 + \left(n + \frac{1}{2}\right)^2 \pi^2},$$

$$\operatorname{cosec} x = \frac{1}{x} + \sum_{1}^{\infty} (-1)^n \frac{2x}{x^2 + n^2\pi^2}$$

We note that

$$\coth \frac{x}{2} = \frac{e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}}{e^{\frac{1}{2}x} - e^{-\frac{1}{2}x}} = \frac{e^x + 1}{e^x - 1} = 1 + \frac{2}{e^x - 1},$$

and accordingly we have

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + \sum_{1}^{\infty} \frac{2x}{x^2 + 4n^2\pi^2}.$$

APPENDIX C

THE POWER SERIES FOR $x/(e^x - 1)$

AND BERNOULLI'S NUMBERS

THE POWER SERIES FOR $x/(e^x-1)$ AND BERNOULLI'S NUMBERS

We know that

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

$$\frac{\left(e^x-1\right)}{x}=1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\dots$$

and the reciprocal function $\frac{x}{\left(e^x-1\right)}$ can be expanded in powers of x if $|x|<\rho$, where $\frac{\rho}{2!}+\frac{\rho^2}{3!}+\frac{\rho^3}{4!}+\dots<1$ (by the Theorem on expansion of reciprocal series). This last condition is certainly satisfied if: $e^\rho\le 1+2\rho$. This is true for $\rho\le 1.2$. Thus, we can certainly write:

$$\frac{x}{\left(e^x-1\right)}=1-\frac{x}{2}+A_2x^2+A_3x^3+A_4x^4+\dots,\ \text{ if }\ |x|<1.2$$

$$\frac{x}{\left(e^x-1\right)}+\frac{x}{2}=1+A_2x^2+A_3x^3+A_4x^4+\dots$$

A simple computation shows that function $\frac{x}{\left(e^x-1\right)}+\frac{x}{2}$ is an even function of x, so that $A_3=0,\ A_5=0,\ A_7=0,\ \dots$
Consequently, we can write

$$\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\dots=1-\frac{x}{2}\sum_{k=1}^{\infty}B_k\frac{x^{2k}}{(2k)!}$$

where B_i are called Bernoulli's numbers.
It is easy to verify by direct division that

$$B_1=\frac{1}{6},\ B_2=\frac{1}{30},\ B_3=\frac{1}{42},\ B_4=\frac{1}{30},\ B_5=\frac{5}{66}$$

It is also known (Appendix B) that

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{2x^2}{x^2+4n^2\pi^2}$$

Now if $|x|<2\pi$, (2π is the radius of convergence; the roots of $e^x=1$ are given by $x=2n\pi i$, and the least distance of any one of these from the origin is 2π), each fraction can be expanded in powers of x, giving

$$\frac{2x^2}{x^2+4n^2\pi^2}=\frac{2x^2}{4n^2\pi^2\left(1+\frac{x^2}{4n^2\pi^2}\right)}=\frac{x^2}{2n^2\pi^2}\cdot\frac{1}{1+\frac{x^2}{4n^2\pi^2}}$$

But $\frac{1}{1+y}=1-y+y^2-y^3+\ldots$

therefore

$$\frac{1}{1+\frac{x^2}{4n^2\pi^2}}=1-\frac{x^2}{4n^2\pi^2}+\frac{x^4}{16n^4\pi^4}-\frac{x^6}{64n^6\pi^6}+\ldots$$

So,

$$\frac{2x^2}{x^2+4n^2\pi^2}=\frac{x^2}{2n^2\pi^2}\left(1-\frac{x^2}{4n^2\pi^2}+\frac{x^4}{16n^4\pi^4}-\frac{x^6}{64n^6\pi^6}+\ldots\right)$$

Further, the resulting double series is absolutely convergent, since the series of absolute values is obtained by expanding similarly the convergent series

$$\sum_{n=1}^{\infty}\frac{2|x|^2}{4n^2\pi^2-|x|^2}.$$

It is therefore permissible to arrange the double series in powers of x, and then we obtain

$$\frac{x}{\left(e^x-1\right)}=1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{2x^2}{x^2+4n^2\pi^2}=1-\frac{x}{2}+\sum_{n=1}^{\infty}\frac{x^2}{2n^2\pi^2}\left(1-\frac{x^2}{4n^2\pi^2}+\frac{x^4}{16n^4\pi^4}-\frac{x^6}{64n^6\pi^6}+\ldots\right)$$

$$=1-\frac{x}{2}+\frac{x^2}{2\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)-\frac{x^4}{8\pi^4}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)+\frac{x^6}{32\pi^6}\left(\sum_{n=1}^{\infty}\frac{1}{n^6}\right)-\ldots$$

which is now seen to be valid for $|x|<2\pi$.

We find $B_1=\frac{1}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$, $B_2=\frac{3}{\pi^4}\left(\sum_{n=1}^{\infty}\frac{1}{n^4}\right)$, $B_3=\frac{45}{2\pi^6}\left(\sum_{n=1}^{\infty}\frac{1}{n^6}\right)$

and, in general $B_r=\frac{(2r)!}{2^{2r-1}\pi^{2r}}\sum_{n=1}^{\infty}\frac{1}{n^{2r}}$

From the earlier computation of the Bernoullian numbers, we obtain asa corollary the results

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\quad \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90},\text{etc.}$$

APPENDIX D

BERNOULLIAN FUNCTIONS

BERNOULLIAN FUNCTIONS

The Bernoullian function of degree n, denoted by $\phi_n(x)$, is the coefficient of $\frac{t^n}{n!}$ in the expansion of $t\frac{e^{xt}-1}{e^t-1}$, which, by the foregoing, can be expanded in powers of t if $|t|<2\pi$. We know that

$$\frac{x}{e^x-1}=1-\frac{x}{2}+B_1\frac{x^2}{2!}-B_2\frac{x^4}{4!}+B_3\frac{x^6}{6!}-\ldots \qquad \text{(Appendix C)}$$

Since

$$e^{xt}-1=\frac{xt}{1!}+\frac{(xt)^2}{2!}+\frac{(xt)^3}{3!}+\ldots ,$$

$$t\frac{e^{xt}-1}{e^t-1}=\left(xt+\frac{x^2t^2}{2!}+\frac{x^3t^3}{3!}+\ldots\right)\left(1-\frac{t}{2}+B_1\frac{t^2}{2!}-B_2\frac{t^4}{4!}+\ldots\right)$$

Thus we have

$$\sum_{n=0}^{\infty}\phi_n(x)\frac{t^n}{n!}=\left(xt+\frac{x^2t^2}{2!}+\ldots\right)\left(1-\frac{t}{2}+B_1\frac{t^2}{2!}-B_2\frac{t^4}{4!}+\ldots\right)$$

So that

$$\phi_n(x)=x^n-\frac{n}{2}x^{n-1}+\frac{n(n-1)}{2!}B_1x^{n-2}-\frac{n(n-1)(n-2)(n-3)}{4!}B_2x^{n-4}+\ldots$$

where the polynomial terminates with either x or x^2. From this formula, or by direct multiplication, we find that the first six Bernoullian polynomials are:

$$\phi_1(x)=x,$$

$$\phi_2(x)=x^2-x=y,$$

$$\phi_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x=yz,$$

$$\phi_4(x)=x^4-2x^3+x^2=y^2,$$

$$\phi_5(x)=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x=yz\left(y-\frac{1}{3}\right),$$

$$\phi_6(x)=x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2=y^2\left(y-\frac{1}{2}\right),$$

where $y=x(x-1)$ and $z=x-\frac{1}{2}=\frac{1}{2}\frac{dy}{dx}$, with the last term x or x^2.

Since

$$\frac{t}{e^t-1}\left(e^{(x+1)t}-1\right)-\frac{t}{e^t-1}\left(e^{xt}-1\right)=\frac{t}{e^t-1}\left(e^{(x+1)t}-e^{xt}\right)=\frac{te^{xt}\left(e^t-1\right)}{e^t-1}=te^{xt},$$

$\phi_n(x+1)-\phi_n(x)$ is the coefficient of $\frac{t^n}{n!}$ in the expansion of $\frac{t}{e^t-1}\left\{e^{(x+1)t}-e^{xt}\right\}=te^{xt}$.

But

$$te^{xt}=t+xt^2+\frac{x^2}{2!}t^3+\frac{x^3}{3!}t^4+\ldots\frac{x^{n-1}}{(n-1)!}t^n+\ldots$$

so that

$$\phi_n(x+1)-\phi_n(x)=nx^{n-1}$$

thus,

$$\phi_2(x+1)=\phi_2(x)+2x=x^2-x+2x=x^2+x$$

$$\phi_3(x+1)=\phi_3(x)+3x^2=x^3-\frac{3}{2}x^2+\frac{1}{2}x+3x^2=x^3+\frac{3}{2}x^2+\frac{1}{2}x$$

and generally $\phi_n(x+1)$ differs from $\phi_n(x)$ only in the sign of the coefficient of x^{n-1}.

If we write $x=1,2,3,\ldots$ in the difference-equation and add the results, we see that, if x is any positive integer $(n>1)$

$$\begin{array}{ll}
x=1: & \phi_n(2)-\phi_n(1)=n \\
x=2: & \phi_n(3)-\phi_n(2)=n\cdot 2^{n-1} \\
x=3: & \phi_n(4)-\phi_n(3)=n\cdot 3^{n-1} \\
. & \\
. & \\
x: & \phi_n(x)-\phi_n(x-1)=n\cdot(x-1)^{n-1} \\
x+1: & \phi_n(x+1)-\phi_n(x)=n\cdot x^{n-1} \\
\hline
\multicolumn{2}{l}{n\left(1+2^{n-1}+3^{n-1}+\ldots+x^{n-1}\right)=\phi_n(x+1)-\phi_n(1)}
\end{array}$$

If $x=1$, then $t\frac{e^{xt}-1}{e^t-1}=t$ and $\phi_n(1)=0$ for all n.

Therefore

$$1+2^{n-1}+3^{n-1}+\ldots+x^{n-1}=\frac{1}{n}\phi_n(x+1)=\frac{1}{n}\left(nx^{n-1}+\phi_n(x)\right)=\frac{1}{n}\phi_n(x)+x^{n-1}$$

This gives the formula of Bernoulli for the summation of the $(n-1)^{th}$ powers of positive integers.

More generally, if $b-a$ is any integer,

$$\begin{array}{ll}
x=a & \phi_n(a+1)-\phi_n(a)=na^{n-1} \\
x=a+1 & \phi_n(a+2)-\phi_n(a+1)=n(a+1)^{n-1} \\
\ldots & \\
x=a+(b-a+1)=b-1 & \phi_n(b)-\phi_n(b-1)=n(b-1)^{n-1} \\
x=a+(b-a)=b & \phi_n(b+1)-\phi_n(b)=nb
\end{array}$$

Then, by adding the above, we obtain

$$n\left(a^{n-1}+(a+1)^{n-1}+\ldots+(b-1)^{n-1}\right)=\phi_n(b)-\phi_n(a)$$

$$a^{n-1}+(a+1)^{n-1}+(a+2)^{n-1}\ldots+(b-1)^{n-1}=\frac{1}{n}\left(\phi_n(b)-\phi_n(a)\right)$$

APPENDIX E

EULER'S SUMMATION FORMULA

EULER'S SUMMATION FORMULA

As we have seen in Appendix D, if x and n are positive integers,

$$1+2^{n-1}+3^{n-1}+\ldots+x^{n-1}=\frac{1}{n}\phi_n(x)+x^{n-1}=\frac{1}{n}\phi_n(x+1)=$$
$$=\frac{1}{n}x^n+\frac{1}{2}x^{n-1}+\frac{n-1}{2!}B_1x^{n-2}-\frac{(n-1)(n-2)(n-3)}{4!}B_2x^{n-4}+\ldots,$$

this polynomial containing $\frac{1}{2}(n+2)$ or $\frac{1}{2}(n+3)$ terms.

It is obvious that when $f(x)$ is a polynomial in x, we can obtain the value of the sum: $f(1)+f(2)+\ldots+f(x)$ by the addition of suitable multiples of the Bernoullian functions of proper degrees. But we can get a compact formula by using integration. We can write the foregoing polynomial in the form

$$\int x^{n-1}dx+\frac{1}{2}x^{n-1}+\frac{1}{2!}B_1\frac{d}{dx}\left(x^{n-1}\right)-\frac{1}{4!}B_2\frac{d^3}{dx^3}\left(x^{n-1}\right)+\ldots$$

Hence when $f(x)$ is a polynomial, we have Euler's summation formula

$$f(1)+f(2)+\ldots+f(x)=\int f(x)dx+\frac{1}{2}f(x)+\frac{1}{2!}B_1f'(x)-\frac{1}{4!}B_2f'''(x)+\ldots,$$

where there is no term on the right-hand side (in its final form) which is not divisible by x.

However, the most interesting applications of this formula arise when $f(x)$ is a rational algebraic fraction, or a transcendental function, and then of course the foregoing method of proof cannot be used; and the right-hand side becomes an infinite series which may not converge. We have considered a number of special examples of this kind in this thesis.

APPENDIX F

THE GAMMA-PRODUCT

Theorem. Let $P_n = (1+a_1)(1+a_2)\dots(1+a_n)$, where $a_1, a_2, a_3, \dots$ are numbers between 0 and 1. Then the convergence of the series $\sum a_n$ is necessary and sufficient for the convergence of the product $\prod_{n=1}^{\infty} P_n$.

Theorem. Suppose $u_1, u_2, u_3, \dots$ is a sequence such that $\sum u_n^2$ is convergent. Then the infinite product $(1+u_1)(1+u_2)\dots$ converges if $\sum u_n$ converges; diverges to infinity if $\sum u_n$ diverges to $+\infty$; diverges to *0* if $\sum u_n$ diverges to $-\infty$; oscillates if $\sum u_n$ oscillates.

It is evident that the product

$$P_n = \left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\left(1+\frac{x}{3}\right)\dots\left(1+\frac{x}{n}\right), \qquad x > -1$$

is divergent except for $x=0$ because

$$\sum_{n=1}^{\infty} \frac{x}{n} = x\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges except for } x=0.$$

We will first find the limit of $\frac{n^x}{P_n}$. Then we are going to show the following formulae:

$$x(x+1)(x+2)\dots(x+n-1) = \frac{\Gamma(n+x)}{\Gamma(x)}$$

and

$$\lim_{n\to\infty} \frac{n^x\Gamma(n)}{\Gamma(n+x)} = 1.$$

We have

$$\frac{x}{n} - \log\frac{P_n}{P_{n-1}} = \frac{x}{n} - \log\left(1+\frac{x}{n}\right) > 0$$

Because

$$0 < x - \log(1+x)$$

$$\frac{P_n}{P_{n-1}}=\frac{(1+x)\ldots\left(1+\frac{x}{n-1}\right)\left(1+\frac{x}{n}\right)}{(1+x)\ldots\left(1+\frac{x}{n-1}\right)}=1+\frac{x}{n}$$

Let

$$S_n=x\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\log P_n$$

Then

$$S_{n-1}=x\left(1+\frac{1}{2}+\ldots+\frac{1}{n-1}\right)-\log P_{n-1}$$

and

$$S_n-S_{n-1}=\frac{x}{n}-\left(\log P_n-\log P_{n-1}\right)=\frac{x}{n}-\log\left(1+\frac{x}{n}\right)>0$$

so the expression S_n increases with n.
In general,

$$0<u-\log(1+u<\frac{1}{2}u^2 \quad \text{for} \quad u>0$$

and

$$0<u-\log(1+u)<\frac{u^2}{2(1+u)} \quad \text{for} \quad -1<u<0$$

Let λ be the lower limit of the numbers $1,1+u_1,1+u_2\ldots,1+u_n\ldots$

$$0<u_{m+1}-\log\left(1+u_{m+1}\right)<\frac{u_{m+1}^2}{2\left(1+u_{m+1}\right)}$$

$$0<u_{m+2}-\log\left(1+u_{m+2}\right)<\frac{u_{m+2}^2}{2\left(1+u_{m+2}\right)}$$

$$\ldots$$

$$0<u_n-\log\left(1+u_n\right)<\frac{u_n^2}{2\left(1+u_n\right)}$$

By adding them, we obtain

$$0<\left(u_{m+1}+u_{m+2}+\ldots+u_n\right)-\log\left\{\left(1+u_{m+1}\right)\left(1+u_{m+2}\right)\ldots\left(1+u_n\right)\right\}<\frac{1}{2\lambda}\left(u^2{}_{m+1}+u^2{}_{m+2}+\ldots+u^2{}_n\right)$$

Therefore, if we consider

$$u_{m+1}=x,\ u_{m+2}=\frac{x}{2},\ \ldots u_n=\frac{x}{n}$$

we obtain

$$s_n=x\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)-\log\left\{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)\right\}<\frac{x^2}{2\lambda}\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right).$$

We know from Appendix 5 that $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<2$.

Therefore we have that

$$s_n < \frac{x^2}{2\lambda}\left(1+\frac{1}{2^2}+\ldots+\frac{1}{n^2}\right) < \frac{x^2}{\lambda}$$

$$P_n = \left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\left(1+\frac{x}{n}\right)$$

$$S_n = x\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) - \log P_n < \frac{x^2}{\lambda}$$

where λ is either 1, if x is positive, or $1+x$, if x is negative.

So, S_n increases and $S_n < \frac{x^2}{\lambda}$.

Hence, $S_n < x^2$ or $S_n < \frac{x^2}{1+x}$.

Therefore, we have an increasing and bounded sequence, so S_n approaches a definite limit S as n increases to ∞.

Further, from Appendix H

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right) = C,$$

where C is the Euler's constant.

Therefore,

$$\lim_{n\to\infty}\left(x\log n - \log P_n\right) = \lim\left\{S_n - x\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)\right\} = S - Cx$$

$$x\log n - \log P_n = \log n^x - \log P_n = \log\frac{n^x}{P_n}$$

$$\lim_{n\to\infty}\log\frac{n^x}{P_n} = S - Cx$$

$$\log\lim_{n\to\infty}\frac{n^x}{P_n} = S - Cx \in \mathbb{R}$$

So that $\frac{n^x}{P_n}$ has also a definite limit; this limit is denoted by $\Pi(x)$ in Gauss's notation.

Thus

$$\Pi(x)=\lim_{n\to\infty}\frac{n^x}{P_n}=\lim_{n\to\infty}\frac{n^x}{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)...\left(1+\frac{x}{n}\right)}$$

Thus

$$\Pi(x)=\lim_{n\to\infty}\frac{n^x\cdot n!}{(1+x)(2+x)...(n+x)}$$

which, again, can be written in Weierstrass's form,

$$S-Cx=\lim_{n\to\infty}\log\frac{n^x}{P_n}=\log\left(\lim_{n\to\infty}\frac{n^x}{P_n}\right)=\log\Pi(x)$$

$$Cx-S=\log\frac{1}{\Pi(x)}$$

Therefore,

$$\frac{1}{\Pi(x)}=e^{Cx-S}$$

$$e^{Cx}e^{-S}=e^{Cx}e^{-\lim S_n}=e^{Cx}\lim_{n\to\infty}e^{-S_n}=e^{Cx}\lim_{n\to\infty}e^{-x\left(1+\frac{1}{2}+...\frac{1}{n}\right)+\log P_n}=$$

$$=e^{Cx}\lim_{n\to\infty}\left(e^{\log P_n}e^{-x\left(1+\frac{1}{2}+...\frac{1}{n}\right)}\right)=e^{Cx}\lim_{n\to\infty}\left(e^{\log\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)...\left(1+\frac{x}{n}\right)}e^{-x}e^{-\frac{x}{2}}...e^{-\frac{x}{n}}\right)=$$

$$=e^{Cx}\lim_{n\to\infty}\left\{\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)...\left(1+\frac{x}{n}\right)e^{-x}e^{-\frac{x}{2}}...e^{-\frac{x}{n}}\right\}=$$

$$=e^{Cx}\lim_{n\to\infty}\prod_{r=1}^{n}\left(1+\frac{x}{r}\right)e^{-\frac{x}{r}}=e^{Cx}\prod_{r=1}^{\infty}\left(1+\frac{x}{r}\right)e^{-\frac{x}{r}}$$

Thus

$$\frac{1}{\Pi(x)}=e^{Cx}\prod_{r=1}^{\infty}\left(1+\frac{x}{r}\right)e^{-\frac{x}{r}}\quad\text{the Weierstrass's form}$$

When x is positive integer, Gauss's form gives $\Pi(x)=x!$ because

$$\frac{n^x n!}{(1+x)(2+x)...(n+x}=\frac{n^x x!}{(1+n)(2+n)...(x+n)}=\frac{x!}{\left(\frac{1+n}{n}\right)\left(\frac{2+n}{n}\right)...\left(\frac{x+n}{n}\right)}=$$

$$=\frac{x!}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{x}{n}\right)}$$

$$\Pi(x)=\lim_{n\to\infty}\frac{n^x n!}{(1+x)(2+x)\dots(n+x)}=\lim_{n\to\infty}\frac{x!}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\dots\left(1+\frac{x}{n}\right)}=x!$$

Although we have found it convenient to restrict $(1+x)$ to be positive, yet this is not necessary for convergence; and it is easy to see that the products for $\Pi(x)$ still converge if x has any negative value which is not an integer.

It is easy to verify by integration by parts that Euler's integral

$$\Gamma(1+x)=\int_0^\infty e^{-t}t^x dt=-t^x e^{-t}\Big|_{t=0}^{\infty}+\int_0^\infty xt^{x-1}e^{-t}dt=x\int_0^\infty t^{x-1}e^{-t}dt=x\Gamma(x)$$

Thus $\Gamma(1+x)=x\Gamma(x)$ or $\dfrac{\Gamma(1+x)}{\Gamma(x)}=x$

If x is an integer,

$$x!=1\cdot2\cdot3\cdot\ldots\cdot x=\frac{\Gamma(2)}{\Gamma(1)}\cdot\frac{\Gamma(3)}{\Gamma(2)}\cdot\frac{\Gamma(4)}{\Gamma(3)}\cdot\ldots\cdot\frac{\Gamma(1+x)}{\Gamma(x)}$$

Thus $\Gamma(1+x)$ has the property of being equal to $x!$ when x is an integer; and we may therefore anticipate the equation $\Gamma(1+x)=\Pi(x)=x!$ for x positive integer

If we change x to $x-1$ in the definition of $\Gamma(1+x)$ by the product P_n, we find that

$$\Gamma(x)=\Pi(x-1)=\lim_{n\to\infty}\frac{n^{x-1}n!}{x(1+x)(2+x)\dots(n+x-1)}$$

but

$$\Gamma(1+x)=\Pi(x)=\lim_{n\to\infty}\frac{n^x n!}{(1+x)(2+x)\dots(n+x)}$$

Therefore

$$\frac{\Gamma(1+x)}{\Gamma(x)}=\lim_{n\to\infty}\frac{n^x n!}{(1+x)(2+x)\dots(n+x)}\cdot\frac{x(1+x)(2+x)\dots(n+x-1)}{n^{x-1}n!}=x\lim_{n\to\infty}\frac{n}{n+x}=x$$

$$\Gamma(1+x)=x\Gamma(x)$$
$$\Gamma(2+x)=(x+1)\Gamma(x+1)$$
$$\Gamma(3+x)=(x+2)\Gamma(x+2)$$
$$\dots$$
$$\Gamma(n+x)=(x+n-1)\Gamma(x+n-1)$$

By multiplying, we obtain

$$\Gamma(n+x) = x(x+1)(x+2)...(x+n-1)\Gamma(x)$$

It follows that

$$x(x+1)(x+2)...(x+n-1) = \frac{\Gamma(n+x)}{\Gamma(x)}$$

and consequently the definition leads to the equation

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^{x-1}n!}{x(1+x)(2+x)...(n+x-1)} = \lim_{n\to\infty} \frac{n^{x-1}n!\Gamma(x)}{\Gamma(n+x)}$$

Because

$$n^{x-1}n! = n^{x-1}(n-1)!n = n^x(n-1)!$$

$$\Gamma(1+x) = x!, \quad \Gamma(n) = (n-1)!$$

$$n^{x-1}n! = n^x\Gamma(n)$$

we have that

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x\Gamma(n)\Gamma(x)}{\Gamma(n+x)} = \Gamma(x)\lim_{n\to\infty}\frac{n^x\Gamma(n)}{\Gamma(n+x)}$$

hence $$\lim_{n\to\infty}\frac{n^x\Gamma(n)}{\Gamma(n+x)} = 1 .$$

It is often convenient to write the last equation in the form $n^x\Gamma(n) \approx \Gamma(n+x)$. By reversing the foregoing argument we see that the function $\Gamma(x)$ is completely defined by the properties $\Gamma(1+x) = x\Gamma(x), \Gamma(n+x) \sim n^x\Gamma(x), \Gamma(1) = 1$.

APPENDIX G

EULER AND THE GAMMA FUNCTION

EULER AND THE GAMMA FUNCTION

We know that $\Gamma(x)=(x-1)!$ if x is a positive integer. To generalize the factorial function, initially defined only for positive integers and then extended to $0!$, Euler began by multiplying by an expression equal to 1.

$$n!=1\cdot2\cdot3\cdot4\cdot\ldots\cdot n=1\cdot2\cdot\ldots\cdot n\frac{(n+1)(n+2)\ldots(n+N)}{(n+1)(n+2)\ldots(n+N)}=$$

$$=1\cdot2\cdot3\cdot\ldots\cdot N\frac{(N+1)(N+2)\ldots(N+n)}{(N+1)(N+2)\ldots(N+n)}=$$

$$=\frac{1}{(n+1)}\cdot\frac{2}{(n+2)}\cdot\ldots\cdot\frac{N}{(n+N)}(N+1)\ldots(N+n)=$$

$$=\frac{1}{(n+1)}\cdot\frac{2}{(n+2)}\cdot\ldots\cdot\frac{N}{(n+N)}(N+1)^n\frac{(N+1)\ldots(N+n)}{(N+1)\ldots(N+n)}$$

To unsimplify this a bit more, Euler wrote

$$N+1=\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{N+1}{N}$$

so that the above expression for $n!$ becomes

$$n!=\frac{1}{(n+1)}\cdot\frac{2}{(n+2)}\cdot\ldots\cdot\frac{N}{(n+N)}\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\ldots\cdot\frac{N+1}{N}\right)^n\frac{(N+1)(N+2)\ldots(N+n)}{(N+1)(N+1)\ldots(N+1)}$$

This holds for any positive integers n and N, and Euler now wants to hold n fixed and let N tend to infinity. He first observes that each factor at the end, of form $\frac{(N+k)}{(N+1)}$, will tend to 1 as N grows, since k is fixed. Therefore he worries only about the first parts of the expression, up to the exponent. To evaluate that limit he first divides by n, then has to find the limit as N goes to infinity of the expression for $(n-1)!$ namely

$$\frac{n!}{n}=\frac{1}{n}\cdot\frac{1}{n+1}\cdot\frac{2}{n+2}\cdot\frac{3}{n+3}\cdots\frac{N}{n+N}\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{N+1}{N}\right)^n$$

This he rewrites as

$$(n-1)!=\frac{1}{n}\left[\frac{1}{n+1}\left(\frac{2}{1}\right)^n\right]\left[\frac{2}{n+2}\left(\frac{3}{2}\right)^n\right]\left[\frac{3}{n+3}\left(\frac{4}{3}\right)^n\right]\ldots$$

Euler remarked that each factor in this last expression is defined for any n except $0,-1,-2,-3,\ldots$ so he defined a new function, the gamma function, by

$$\Gamma(z)=\frac{1}{z}\left[\frac{1}{z+1}\left(\frac{2}{1}\right)^{z}\right]\left[\frac{2}{z+2}\left(\frac{3}{2}\right)^{z}\right]\left[\frac{3}{z+3}\left(\frac{4}{3}\right)^{z}\right]\ldots$$

where z is any complex number except $0,-1,-2,-3,\ldots$
To test whether this is at all sensible, he set out to compute $\Gamma\left(\frac{1}{2}\right)$ by the following, indirect but ingenious approach. By substituting and inverting, he expressed $\frac{1}{\Gamma(z)\Gamma(1-z)}$ as

$$\left\{z\left[\frac{z+1}{1}\right]\left(\frac{1}{2}\right)^{z}\left[\frac{z+2}{2}\right]\left(\frac{2}{3}\right)^{z}\left[\frac{z+3}{3}\right]\left(\frac{3}{4}\right)^{z}\ldots\right\}\left\{(1-z)\left[\frac{2-z}{1}\right]\left(\frac{1}{2}\right)^{1-z}\left[\frac{3-z}{2}\right]\left(\frac{2}{3}\right)^{1-z}\ldots\right\}$$

$$z(1-z)(1+z)\left(\frac{1}{2}\cdot\frac{2-z}{1}\cdot\frac{2+z}{2}\right)\left(\frac{2}{3}\cdot\frac{3-z}{2}\cdot\frac{3+z}{3}\right)\ldots=$$

$$=z(1-z)(1+z)\left(\frac{2-z}{2}\cdot\frac{2+z}{2}\right)\left(\frac{3-z}{3}\cdot\frac{3+z}{3}\right)\ldots$$

$$\frac{1}{\Gamma(z)\Gamma(1-z)}=z(1-z)(1+z)\left(\frac{2-z}{2}\cdot\frac{2+z}{2}\right)\left(\frac{3-z}{3}\cdot\frac{3+z}{3}\right)\ldots$$

Now Euler multiplied out pairs of factor to get

$$\frac{1}{\Gamma(z)\Gamma(1-z)}=z\left(1-z^{2}\right)\left(1-\frac{z^{2}}{4}\right)\left(1-\frac{z^{2}}{9}\right)\left(1-\frac{z^{2}}{16}\right)\ldots,$$

an expression which he recognized as one he found earlier for $\frac{\sin\pi z}{\pi}$. He put in $z=\frac{1}{2}$ and got

$$\frac{1}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}=\frac{\sin\frac{\pi}{2}}{\pi}=\frac{1}{\pi},$$

so that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

APPENDIX H

THE LOGARITHMIC SCALE AND APPLICATIONS

TO SPECIAL SERIES

THE LOGARITHMIC SCALE AND APPLICATIONS TO SPECIAL SERIES

Theorem. The series of positive terms $\sum a_n$ ($a_n \geq a_{n+1} > 0$) converges or diverges with the integral $\int_1^x f(x)\,dx$; if convergent, the sum of the series differs from the integral by less than a_1; if divergent, the limit of $(s_n - I_n)$ nevertheless exists and lies between *0* and a_1.

Proof:

Consider

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots, \text{ where } a_n = n^{-p}$$

If p is positive, $f(x)=\frac{1}{x^p}=x^{-p}$

$$\int_1^x f(x)dx = \int_1^x x^{-p}dx = \left.\frac{x^{-p+1}}{-p+1}\right|_1^x = \frac{\left(x^{1-p}-1\right)}{1-p}$$

The integral to ∞ is convergent only if $p>1$.

$$\lim_{x\to\infty}\int_1^x f(x)dx = \frac{-1}{1-p} = \frac{1}{p-1}$$

$$0 \leq \lim_{n\to\infty}\left(S_n - I_n\right) \leq a_1$$

and the sum is then contained between $1/(p-1)$ and $p/(p-1)$.

$$\frac{1}{p-1} \leq \sum n^{-p} \leq \frac{1}{p-1}+1$$

If $p=1$, the integral is equal to $\log x$, which shows that the harmonic series is divergent.

$$\int_1^x f(t)dt = \int_1^x t^{-1}dt = \ln t\Big|_1^x = \ln x$$

We infer that the limit:

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right)$$

exists and lies between 0 and 1.

This limit is Euler's constant.

$$0 \leq \lim_{n\to\infty}\left(S_n - I_n\right) \leq a_1$$

$$0 \leq \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n\right) \leq 1$$

The value of the constant is 0.57721… (see Chapter Two), and will be denoted usually by C.
It is often convenient to write

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\rightarrow\log n+C$$

<u>Application</u>: $1+\frac{1}{2^2}+\frac{1}{3^2}\ldots+\frac{1}{n^2}<2$

Proof:

Let $f(n)=\frac{1}{n^p}$

Let

$$I_n=\int_1^n f(x)\,dx=\int_1^n\frac{1}{x^p}\,dx=\int_1^n x^{-p}\,dx=\left.\frac{x^{1-p}}{1-p}\right|_1^n=\frac{n^{1-p}-1}{1-p}.$$

If $p>1$,

$$\lim_{n\to\infty}I_n=\frac{1}{p-1}$$

$$0\le\lim\left(S_n-I_n\right)\le a_1$$

$$\frac{1}{p-1}\le\lim S_n\le\frac{1}{p-1}+1=\frac{p}{p-1}$$

In our case $p=2$, therefore

$$1+\frac{1}{2^2}+\frac{1}{3^2}\ldots+\frac{1}{n^2}<2$$

APPENDIX I

APPLICATION OF ABSOLUTE CONVERGENCE

FOR THE SERIES $\sum_{n=1}^{\infty}\left((-1)^{n-1}/n^{p}\right)$

APPLICATION OF ABSOLUTE CONVERGENCE FOR THE SERIES $\sum_{n=1}^{\infty}\left((-1)^{n-1}/n^{p}\right)$

Let $\zeta(p)=\sum_{n=1}^{\infty}\frac{1}{n^{p}}$ be the zeta-function and $\zeta_{a}(p)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{p}}$ its corresponding alternating series. We can express the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{p}}$ in terms of the sum of the corresponding series of positive terms by the formula

$$\zeta_{a}(p)=\left(1-\frac{1}{2^{p-1}}\right)\zeta \quad \text{for} \quad p>1.$$

To prove the above formula we use the fact that if a series $\sum a_{n}$ is absolutely convergent, its sum is not altered by derangement.

Hence, because $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{p}}$ is absolutely convergent, its sum is independent of the order of its terms.
Therefore,

$$1-\frac{1}{2^{p}}+\frac{1}{3^{p}}-\frac{1}{4^{p}}+\frac{1}{5^{p}}-...=1+\frac{1}{3^{p}}+\frac{1}{5^{p}}+...-\left(\frac{1}{2^{p}}+\frac{1}{4^{p}}+\frac{1}{6^{p}}+...\right)=$$

$$=1+\frac{1}{2^{p}}+\frac{1}{3^{p}}+\frac{1}{4^{p}}+\frac{1}{5^{p}}+...-2\left(\frac{1}{2^{p}}+\frac{1}{4^{p}}+\frac{1}{6^{p}}+...\right)=$$

$$=1+\frac{1}{2^{p}}+\frac{1}{3^{p}}+\frac{1}{4^{p}}+\frac{1}{5^{p}}+...-\left(\frac{1}{2^{p-1}}+\frac{1}{2^{p-1}2^{p}}+\frac{1}{2^{p-1}3^{p}}+...\right)=$$

$$=1+\frac{1}{2^{p}}+\frac{1}{3^{p}}+\frac{1}{4^{p}}+\frac{1}{5^{p}}+...-\frac{1}{2^{p-1}}\left(1+\frac{1}{2^{p}}+\frac{1}{3^{p}}+...\right)=$$

$$=\left(1-\frac{1}{2^{p-1}}\right)\left(1+\frac{1}{2^{p}}+\frac{1}{3^{p}}+\frac{1}{4^{p}}+\frac{1}{5^{p}}+...\right)$$

Therefore $\zeta_{a}(p)=\left(1-\frac{1}{2^{p-1}}\right)\zeta(p)$.

But $1-\frac{1}{2^{p}}<\zeta_{a}(p)<1$, therefore $\zeta(p)<\dfrac{1}{1-\frac{1}{2^{p-1}}}$.

So, for $p=5$ we have that $\sum_{n=1}^{\infty}\frac{1}{n^{5}}<\frac{1}{1-2^{-4}}$.

APPENDIX J

APPLICATIONS OF UNIFORM CONVERGENCE

APPLICATIONS OF UNIFORM CONVERGENCE

An integral $\int_a^\infty F(x,y)\,dx$ which converges uniformly in an interval (α,β) has properties strictly analogous to those of uniformly convergent series.

Theorem 1. If $f(x,y)$ is a continuous function of y in the interval (α,β), the integral is also a continuous function of y, provided that it converges uniformly in the interval (α,β).

Theorem 2. Under the same conditions as in Theorem 1, we may integrate with respect to y under the sign of integration, provided that the range falls within the interval (α,β).

Theorem 3. The equation $\frac{d}{dy}\int_a^\infty f(x,y)\,dx = \int_a^\infty \frac{\delta f}{\delta y}\,dx$ is valid, provided that the integral on the right converges uniformly and that the integral on the left is convergent.

Theorem 4. If $\lim_{n\to\infty} f(x,n) = g(x)$, $\lim_{n\to\infty}\lambda_n = \infty$ then $\lim_{n\to\infty}\int_a^{\lambda_n} f(x,n)\,dx = \int_a^\infty g(x)\,dx$, provided that $f(x,n)$ tends to its limit $g(x)$ uniformly in any fixed interval, and that we can determine a positive function $M(x)$ to satisfy $|f(x,n)| \le M(x)$ for all values of n, while $\int_a^\infty M(x)dx$ converges.

Application. Consider the integral

$$J = \int_0^\infty e^{-xy}\left(e^{-ax} - e^{-bx}\right)\frac{dx}{x}$$

where a, b may be complex, provided that they have their real parts positive or zero. Then J is uniformly convergent for all positive or zero values of y.

Proof:

If we differentiate with respect to y, we obtain

$$\frac{d}{dy}\int_0^\infty e^{-xy}\left(e^{-ax} - e^{-bx}\right)\frac{dx}{x} = \int_0^\infty -xe^{-xy}\left(e^{-ax} - e^{-bx}\right)\frac{dx}{x} = -\int_0^\infty e^{-xy}\left(e^{-ax} - e^{-bx}\right)dx =$$

$$= -\int_0^\infty e^{-(xy+ax)}dx + \int_0^\infty e^{-(bx+xy)}dx = -\int_0^\infty e^{-x(a+y)}dx + \int_0^\infty e^{-x(b+y)}dx =$$

$$= \frac{e^{-x(a+y)}}{(a+y)}\bigg|_0^\infty + \frac{e^{-x(b+y)}}{-(b+y)}\bigg|_0^\infty = -\frac{1}{a+y} + \frac{1}{b+y}$$

and this integral converges uniformly so long as $y \geq l > 0$. Its value is therefore equal to $\frac{dJ}{dy}$, in virtue of Theorem 3 above.

So,
$$\frac{dJ}{dy} = -\frac{1}{a+y} + \frac{1}{b+y}.$$

By Theorem 1, $\lim\limits_{y\to\infty} J = 0$, so that

$$J = -\int_y^\infty \left(\frac{1}{a+y} - \frac{1}{b+y}\right)$$

$$\lim_{y\to 0} J = \int_0^\infty \lim_{y\to 0}\left(e^{-xy}\left(e^{-ax} - e^{-bx}\right)\frac{dx}{x}\right) = \int_0^\infty \left(e^{-ax} - e^{-bx}\right)\frac{dx}{x} = \int_0^\infty \left(\frac{1}{a+y} - \frac{1}{b+y}\right)dy =$$

$$= \ln(a+y)\big|_0^\infty - \ln(b+y)\big|_0^\infty = \ln\frac{a+y}{b+y}\bigg|_0^\infty = 0 - \ln\frac{a}{b} = \ln\frac{b}{a}$$

In particular, if we write $a = 1,\ b = i$, we have

$$\int_0^\infty \left(e^{-x} - e^{-ix}\right)\frac{dx}{x} = \int_0^\infty \left(\frac{1}{1+y} - \frac{1}{y+i}\right)dy = \ln(1+y) - \int_0^\infty \frac{y-i}{y^2+1}dy =$$

$$= \ln(1+y) - \int_0^\infty \frac{y}{y^2+1}dy + i\int_0^\infty \frac{1}{y^2+1}dy = \ln(1+y) - \frac{1}{2}\ln\left(y^2+1\right) + i\tan^{-1}y =$$

$$= \left(\ln\frac{1+y}{\sqrt{y^2+1}} + i\tan^{-1}y\right)\Bigg|_0^\infty = \frac{i\pi}{2}$$

So,

$$\int_0^\infty \left(e^{-x} - \cos x + i\sin x\right)\frac{dx}{x} = \frac{1}{2}\pi i$$

or

$$\int_0^\infty \left(e^{-x} - \cos x\right)\frac{dx}{x} = 0, \qquad \int_0^\infty \sin x\frac{dx}{x} = \frac{1}{2}\pi$$

APPENDIX K

INTEGRATION OF AN INFINITE SERIES OVER AN INFINITE INTERVAL AND THE INVERSION OF A REPEATED INFINITE INTEGRAL

INTEGRATION OF AN INFINITE SERIES OVER AN INFINITE INTERVAL AND THE INVERSION OF A REPEATED INFINITE INTEGRAL

Integration of an infinite series over an infinite interval. Many cases of practical importance are covered by the following test:

Theorem. If $\sum f_n(x)$ converges uniformly in any fixed interval $a \le x \le b$ where b is arbitrary, and if $\phi(x)$ is continuous for all finite values of x, then

$$\int_a^\infty \phi(x)\left[\sum f_n(x)\right]dx = \sum \int_a^\infty \phi(x) f_n(x)\,dx$$

provided that either the integral $\int_a^\infty |\phi(x)|\left\{\sum |f_n(x)|\right\}dx$ or the series $\sum \int_a^\infty |\phi(x)| \cdot |f_n(x)|$ is convergent.

Application. Show that

$$\int_0^\infty \frac{\sin bx}{e^{ax}-1}dx = \frac{b}{a^2+b^2} + \frac{b}{(2a)^2+b^2} + \ldots$$

where a is positive, and $b = p + iq$, where $|q| = s < a$.

Proof:

Since

$$|\sin(bx)| = \left[\sinh^2(qx) + \sin^2(px)\right]^{\frac{1}{2}} < \cosh sx < e^{8x}$$

and the integral

$$\int_0^\infty \left(\frac{e^{8x}}{e^{ax}-1}\right)dx$$

is convergent, it follows from the Theorem above that term-by-term integration is permissible, because the terms in the series

$$\frac{1}{e^{ax}-1} = e^{-ax} + e^{-2ax} + e^{-3ax} + \ldots$$

are all positive.

Thus we have

$$\sum_{r=1}^\infty e^{-rax} = \sum_{r=1}^\infty \left(e^{-ax}\right)^r = \frac{e^{-ax}}{1-e^{-ax}} = \frac{1}{e^{ax}-1}$$

$$\int_0^\infty \frac{\sin bx}{e^{ax}-1}dx = \int_0^\infty \sum_{r=1}^\infty (\sin bx) e^{-rax}dx = \sum_{r=1}^\infty \int_0^\infty e^{-rax} \sin bx\,dx$$

Let

$$I=\int_0^\infty e^{-rax}\sin bx dx=-\frac{1}{ra}e^{-rax}\sin bx\Big|_0^\infty+\frac{b}{ra}\int_0^\infty e^{-rax}\cos bx dx=$$

$$=\frac{b}{ra}\left(-\frac{1}{ra}e^{-rax}\cos bx\Big|_0^\infty-\frac{b}{ra}\int_0^\infty e^{-rax}\sin bx dx\right)==\frac{b}{ra}\left(\frac{1}{ra}-\frac{b}{ra}I\right)$$

Thus $$I=\frac{b}{(ar)^2+b^2}\qquad r=1,2,....$$

Therefore

$$\int_0^\infty\frac{\sin bx}{e^{ax}-1}dx=\frac{b}{a^2+b^2}+\frac{b}{(2a)^2+b^2}+...$$

In the case when $a=2\pi$, this expression is equal to

$$\int_0^\infty\frac{\sin bx}{e^{ax}-1}dx=\sum_{r=1}^\infty\frac{b}{(2r\pi)^2+b^2}$$

We know that

$$\frac{1}{e^x-1}=\frac{1}{x}-\frac{1}{2}+\sum_{n=1}^\infty\frac{2x}{x^2+4n^2\pi^2}$$

So,

$$\frac{1}{e^b-1}=\frac{1}{b}-\frac{1}{2}+\sum_{r=1}^\infty\frac{2b}{b^2+4r^2\pi^2}$$

Then

$$2\sum_{r=1}^\infty\frac{b}{b^2+4r^2\pi^2}=\frac{1}{e^b-1}-\frac{1}{b}+\frac{1}{2}$$

$$\sum_{r=1}^\infty\frac{b}{b^2+(2r\pi)^2}=\int_0^\infty\frac{\sin bx}{e^{2\pi x}-1}dx=\frac{1}{2}\left(\frac{1}{e^b-1}-\frac{1}{b}+\frac{1}{2}\right)$$

<u>The inversion of a repeated infinite integral</u>. We will prove that

$$\int_0^\infty e^{-y_0t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt=2\int_0^\infty\frac{\arctan\frac{x}{y_0}}{e^{2\pi x}-1}dx\ .$$

We know

$$\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}=2\int_0^\infty\frac{\sin xt}{e^{2\pi x}-1}dx$$

and

$$\int_0^\infty e^{-yt}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt=2\int_0^\infty e^{-yt}dt\int_0^\infty\frac{\sin xt}{e^{2\pi x}-1}dx$$

We will show that last integral is absolutely convergent. First, we will prove that

$$\int_0^\infty \frac{x^{2r-1}}{e^{2\pi x}-1}dx = \frac{B_r}{4r} \quad .$$

For that, we take

$$\int_0^\infty \frac{\sin bx}{e^{2\pi x}-1}dx = \int_0^\infty \frac{1}{e^{2\pi x}-1}\sum_{n=0}^{\infty}\frac{(-1)^n (bx)^{2n+1}}{(2n+1)!}dx = \sum_{n=0}^{\infty}\frac{(-1)^n b^{2n+1}}{(2n+1)!}\int_0^\infty \frac{x^{2n+1}}{e^{2\pi x}-1}dx \; .$$

On the other hand,

$$\int_0^\infty \frac{\sin bx}{e^{2\pi x}-1}dx = \frac{1}{2}\left(\frac{1}{e^b-1}-\frac{1}{b}+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{1}{b}-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{B_n}{(2n)!}b^{2n-1}-\frac{1}{b}+\frac{1}{2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n B_n}{(2n)!}b^{2n-1}$$

From the last two expression, we obtain that

$$\int_0^\infty \frac{x^{2r-1}}{e^{2\pi x}-1}dx = \frac{B_r}{4r} \; .$$

Taking $r=1, \quad B_1 = \frac{1}{6}$, we obtain

$$\int_0^\infty \frac{|\sin xt|}{e^{2\pi x}-1}dx < \int_0^\infty \frac{xt}{e^{2\pi x}-1}dx = \frac{t}{24}$$

We see that $\left|e^{-yt}\right| = e^{-\xi t} \quad if \;\; y = \xi + i\eta$.

Thus,

$$2\int_0^\infty \left|e^{-yt}\right|dt\int_0^\infty \frac{|\sin xt|}{e^{2\pi x}-1}dx < \int_0^\infty e^{-\xi t}\frac{t}{12}dt = \frac{1}{12}\int_0^\infty te^{-\xi t}dt =$$

$$= \frac{1}{12}\left(-\frac{t}{\xi}e^{-\xi t}\Big|_0^\infty + \frac{1}{\xi}\int_0^\infty e^{-\xi t}dt\right) = \frac{1}{12}\left(\frac{1}{\xi}\cdot\frac{e^{-\xi t}}{-\xi}\right) < \frac{1}{12\xi^2}$$

which proves the absolute converge; we can therefore invert the order of integration without altering the value of integral, and we then find

$$\int_0^\infty e^{-yt}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt = 2\int_0^\infty\int_0^\infty e^{-yt}\frac{\sin xt}{e^{2\pi x}-1}dxdt$$

We proved earlier that

$$I = \int_0^\infty e^{-rax}\sin bxdx = \frac{b}{(ar)^2+b^2}$$

If $r=1, \;\; a=y, \;\; b=x$, $\qquad \displaystyle\int_0^\infty e^{-yt}\sin xtdt = \frac{x}{y^2+x^2}$

Therefore

$$\int_0^\infty e^{-yt}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt = 2\int_0^\infty \frac{xdx}{\left(x^2+y^2\right)\left(e^{2\pi x}-1\right)}$$

Now, if we write $y=\xi+i\eta$ in the last equation, we can integrate with respect to ξ under the integral sign, between ξ_0 *and* ∞; for

$$\left|e^{-yt}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)\right| < \frac{t}{12}e^{-\xi t}$$

and so
$$\int_{\xi_0}^\infty d\xi \int_0^\infty \left|e^{-yt}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)\right| dt < \frac{1}{12\xi_0} \qquad (\xi_0>0)$$

so that this double integral is absolutely convergent. Similarly we find that the right-hand side is absolutely convergent, since $\left|x^2+y^2\right| \geq \xi^2$, so that

$$\int_{\xi_0}^\infty d\xi \int_0^\infty \frac{xdx}{\left|x^2+y^2\right|\left(e^{2\pi x}-1\right)} \leq \int_{\xi_0}^\infty \frac{d\xi}{\xi^2}\int_0^\infty \frac{xdx}{e^{2\pi x}-1} = \frac{1}{24\xi_0}$$

Thus, we find the further equation

$$\int_0^\infty \left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt \int_{\xi_0}^\infty e^{-yt}d\xi = 2\int_0^\infty \frac{dx}{e^{2\pi x}-1}\int_{\xi_0}^\infty \frac{xd\xi}{x^2+y^2}$$

which gives
$$\int_0^\infty \frac{e^{-y_0 t}}{t}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt = 2\int_0^\infty \frac{\arctan\frac{x}{y_0}}{e^{2\pi x}-1}dx$$
where $y_0=\xi_0+i\eta$.

APPENDIX L

INTEGRALS FOR $\log\Gamma(1+x)$

INTEGRALS FOR $\log\Gamma(1+x)$

We have proved in Appendix J above that if the real parts of a, b are positive,

$$\log\frac{b}{a}=\int_0^\infty\left(e^{-at}-e^{-bt}\right)\frac{dt}{t}$$

Hence, if the real part of $1+x$ is positive, writing $b=r,\ a=r+x$ we have

$$\log\frac{r}{r+x}=\int_0^\infty\left(e^{-(r+x)t}-e^{-rt}\right)\frac{dt}{t}=\int_0^\infty e^{-rt}\left(e^{-xt}-1\right)\frac{dt}{t},\qquad r=1,2,3,\ldots$$

If $b=n,\ a=1$ then

$$\log n=\int_0^\infty\left(e^{-t}-e^{-nt}\right)\frac{dt}{t}$$

We know from Appendix F that

$$\Gamma(1+x)=\lim_{n\to\infty}\frac{n^x n!}{(1+x)(2+x)\ldots(n+x)}$$

Therefore

$$\log\Gamma(1+x)=\lim_{n\to\infty}\log\frac{n^x n!}{(1+x)(2+x)\ldots(n+x)}=$$

$$=\lim_{n\to\infty}\left(\log n^x+\log\frac{1\cdot2\cdot3\cdot\ldots n}{(1+x)(2+x)\ldots(n+x)}\right)=$$

$$=\lim_{n\to\infty}\left(x\log n+\sum_{r=1}^{n}\log\frac{r}{r+x}\right)$$

Thus we are led to consider the function

$$S(x,n)=x\log n+\sum_{r=1}^{n}\log\frac{r}{r+x}=x\int_0^\infty\left(e^{-t}-e^{-nt}\right)\frac{dt}{t}-\sum_{r=1}^{n}\int_0^\infty\left(1-e^{-xt}\right)e^{-rt}\frac{dt}{t}$$

Now

$$\sum_{r=1}^{n}e^{-rt}=\frac{e^{-t}\left(1-e^{-nt}\right)}{1-e^{-t}}=\frac{1-e^{-nt}}{e^t-1}$$

so that

$$S(x,n)=x\int_0^\infty\left(e^{-t}-e^{-nt}\right)\frac{dt}{t}-\sum_{r=1}^{n}\int_0^\infty e^{-rt}\frac{dt}{t}+\sum_{r=1}^{n}\int_0^\infty e^{-xt}e^{-rt}\frac{dt}{t}=$$

$$=x\int_0^\infty\left(e^{-t}-e^{-nt}\right)\frac{dt}{t}-\int_0^\infty\frac{1-e^{-nt}}{e^t-1}\frac{dt}{t}+\int_0^\infty e^{-xt}\frac{1-e^{-nt}}{e^t-1}\frac{dt}{t}=$$

$$= \int_0^\infty \left(xe^{-t} - xe^{-nt} - \frac{1-e^{-nt}}{e^t-1} + \frac{e^{-xt} - e^{-xt}e^{-nt}}{e^t-1} \right) \frac{dt}{t} =$$

$$= \int_0^\infty \left(xe^{-t} - xe^{-nt} - \frac{1-e^{-xt}}{e^t-1} + \frac{e^{-nt}\left(1-e^{-xt}\right)}{e^t-1} \right) \frac{dt}{t}$$

So that,

$$S(x,n) = \int_0^\infty \left(xe^{-t} - \frac{1-e^{-xt}}{e^t-1} \right) \frac{dt}{t} - \int_0^\infty e^{-nt} \left(x - \frac{1-e^{-xt}}{e^t-1} \right) \frac{dt}{t}$$

Let

$$F(x) = \int_0^\infty \left(xe^{-t} - \frac{1-e^{-xt}}{e^t-1} \right) \frac{dt}{t}$$

$$G(x,n) = -\int_0^\infty e^{-nt} \left(x - \frac{1-e^{-xt}}{e^t-1} \right) \frac{dt}{t}$$

Then,

$$S(x,n) = F(x) + G(x,n)$$

It is to be observed that both in $F(x)$ and in $G(x)$ the integrands are finite at $t=0$.

From Appendix D ,

$$\frac{1}{e^t-1} = \frac{1}{t} - \frac{1}{2} + \sum_{n=1}^\infty \frac{2t}{t^2+4n^2\pi^2}$$

$$1-e^{-xt} = -\left(-xt + \frac{x^2+2}{2!} - \ldots \right) = xt - \frac{x^2+2}{2!} + \frac{x^3+3}{3!} - \ldots$$

$$\frac{1-e^{-xt}}{e^t-1} = x - \frac{1}{2}\left(x+x^2\right)t + t^2\left(\frac{x^3}{3!} - \frac{x^2}{4} \right) - \ldots$$

$$x - \frac{1-e^{-xt}}{e^t-1} = \frac{1}{2}\left(x+x^2\right)t + t^2\left(\frac{x^3}{3!} - \frac{x^2}{4} \right) - \ldots$$

$$\frac{1}{t}\left(x - \frac{1-e^{-xt}}{e^t-1} \right) = \frac{1}{2}\left(x+x^2\right)t + X_1 t + X_2 t^2 + \ldots$$

and similarly for the other integrand.

Thus, when $t<1$ $t^2, t^3, \ldots, t^n \to 0$ so $\frac{1}{t}\left| x - \frac{1-e^{-xt}}{e^t-1} \right|$ cannot exceed some fixed value, independent of t; but if $t>1$, this expression is less than $|x| + \frac{e+1}{e-1}$, because $\left|e^{-xt}\right| < e^t$ (since the real part of $1+x$ is positive).

$e^t > e^1$, $\quad e^t - 1 > e - 1$, $\quad \dfrac{1}{e^t-1} < \dfrac{1}{e-1}$, $\quad \left|\dfrac{1}{t}\left(x - \dfrac{1-e^{-xt}}{e^t-1}\right)\right| \le |x| + \dfrac{e+1}{e-1}$

Thus we can determine a value of X, independent of t, such that

$$\left|\frac{1}{t}\left(x - \frac{1-e^{-xt}}{e^t-1}\right)\right| < X$$

Then

$$|G(x,n)| < \int_0^\infty X e^{-nt}\,dt < \frac{X}{n}$$

so that $\quad \lim\limits_{n\to\infty} G(x,n) = 0$.

Hence

$$\log\Gamma(1+x) = \lim_{n\to\infty} S(x,n) = F(x) = \int_0^\infty \left(xe^{-t} - \frac{1-e^{-xt}}{e^t-1}\right)\frac{dt}{t}$$

This integral can be divided into two parts:

$$\phi(x) = \int_0^\infty \left[xe^{-t} - \frac{1}{e^t-1} + \left(\frac{1}{t} - \frac{1}{2}\right)e^{-xt}\right]\frac{dt}{t}$$

$$\psi(x) = \int_0^\infty e^{-xt}\left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right)\frac{dt}{t} = 2\int_0^\infty \frac{\arctan(y/x)}{e^{2\pi y}-1}\,dy\ ,$$

the last expression following from Appendix K.
So,

$$\log\Gamma(1+x) = \phi(x) + \psi(x)$$

$$\log\Gamma(1+x) = \phi(x) + 2\int_0^\infty \frac{\arctan(y/x)}{e^{2\pi x}-1}\,dy$$

When $\xi \to \infty$, where $\xi = \operatorname{Re} x$, then $\lim \psi(x) = 0$ because $|\psi(x)| < \dfrac{1}{12\xi}$.

Thus, when $\xi \to \infty$, we have $\lim \psi(x) = 0$.
It can also be proved using the Analogue of Abel's Lemma that $|\psi(x)| < \dfrac{1}{6|\eta|}$. Therefore $\lim \psi(x) = 0$, when $\eta \to \infty$, ξ being kept positive.
Therefore,

$$\phi(x) - \phi(1) = \int_0^\infty \left\{xe^{-t} - \frac{1}{e^t-1} + \left(\frac{1}{t} - \frac{1}{2}\right)e^{-xt}\right\}\frac{dt}{t} - \int_0^\infty \left\{e^{-t} - \frac{1}{e^t-1} + \left(\frac{1}{t} - \frac{1}{2}\right)e^{-t}\right\}\frac{dt}{t} =$$

$$= \int_0^\infty \left\{(x-1)e^{-t} + \left(\frac{1}{t} - \frac{1}{2}\right)\left(e^{-xt} - e^{-t}\right)\right\}\frac{dt}{t} =$$

$$= \int_0^\infty \frac{xe^{-t} - e^{-t}}{t} + \frac{e^{-xt} - e^{-t}}{t^2} dt + \frac{1}{2}\log x = \left(x + \frac{1}{2}\right)\log x - (x-1) .$$

We know that

$$\int_0^\infty \frac{xe^{-ax} - e^{-bx}}{x} dx = \ln\frac{b}{a}$$

Thus we see that, if $A = 1 + \phi(1)$,

$$\phi(x) = \left(x + \frac{1}{2}\right)\log x - x + A .$$

To determine A, we make use

$$\log\left(x + \frac{1}{2}\right) + \log\Gamma(x+1) + 2x\log 2 - \log\Gamma(2x+1) = \frac{1}{2}\log\pi .$$

Thus we have, since $\lim\psi(x) = 0$,

$$\lim\left[\phi\left(x - \frac{1}{2}\right) + \phi(x) + 2x\log 2 - \phi(2x)\right] = \frac{1}{2}\log\pi ,$$

which gives, on inserting the value of $\phi(x)$,

$$\lim\left[A + x\log\left(1 - \frac{1}{2x}\right) + \frac{1}{2} - \frac{1}{2}\log 2\right] = \frac{1}{2}\log\pi$$

or

$$A = \frac{1}{2}\log(2\pi) .$$

Thus we can write

$$\log\Gamma(1+x) = \left(x + \frac{1}{2}\right)\log x - x + \frac{1}{2}\log 2\pi + \psi(x)$$

where $\psi(x) = 2\int_0^\infty \frac{\arctan(y/x)}{e^{2\pi y} - 1} dy$ and $|\psi(x)| < \frac{1}{12\xi}$.

REFERENCES

[1] Bromwich, T.J., An Introduction to the Theory of Infinite Series, 2nd Edition, MacMillan and Company Limited, London, 1926.

[2] Ford, Walter B., Asymptotic Series-Divergent Series, Chelsea Publishing Company, New York, 1960.

[3] Hardy, G.H., Divergent Series, AMS Chelsea Publishing, Rhode Island, 1991.

[4] Marsden, Jerrold E., Basic Complex Analysis, W.H. Freeman and Company, San Francisco, 1973.