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SIMPLE GROUPS, PROGENITORS, AND RELATED TOPICS

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A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

————————————————

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

————————————————

by

Angelica Baccari

June 2018

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June 2018

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Abstract

The foundation of the work of this thesis is based around the involutory progenitor and the finite homomorphic images found therein. This process is developed by Robert T. Curtis and he defines it as $2^{*n} : N\{\pi w | \pi \in N, w\}$ where 2^{*n} denotes a free product of n copies of the cyclic group of order 2 generated by involutions. We repeat this process with different control groups and a different array of possible relations to discover interesting groups, such as sporadic, linear, or unitary groups, to name a few. Predominantly this work was produced from transitive groups in 6,10,12, and 18 letters. Which led to identify some appealing groups for this project, such as Janko group J_1, Symplectic groups $S(4,3)$ and $S(6,2)$, Mathieu group M_{12} and some linear groups such as $PGL_2(7)$ and $L_2(11)$. With this information, we performed double coset enumeration on some of our findings, M_{12} over $L_2(11)$ and $L_2(31)$ over D_{15}. We will also prove their isomorphism types with the help of the Jordan-Holder theorem, which aids us in defining the make up of the group. Some examples that we will encounter are the extensions of $L_2(31)^{\bullet}2$ and $A_5 : 2^2$.

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Chapter 1

Definitions, Theorems, and Lemmas

1.1 Preliminaries

1.1.1 Definitions

Definition 1.1. *A* **group** *G $(G, *)$ is a nonempty collection of elements with an associative operation $*$, such that:*

- *there exists an identity element, $e \in G$ such that $e * a = a * e$ for all $a \in G$;*

- *for every $a \in G$, there exists an element $b \in G$ such that $a * b = e = b * a$. [Rot95]*

Definition 1.2. *Let G be a set. A (binary)* **operation** *on G is a function that assigns each ordered pair of elements of G an element on G. [Rot95]*

Definition 1.3. *For group G, a* **subgroup** *S of G is a nonempty subset where $s \in G$ implies $s^{-1} \in G$ and $s, t \in G$ imply $st \in G$. We denote subgroup S of G as $S \leq G$. [Rot95]*

Definition 1.4. *Let H be a subgroup of group G. H is a* **proper** *subgroup of G if $H \neq G$. We denote this as $H < G$. [Rot95]*

Definition 1.5. *A* **symmetric group**, *S_X, is the group of all permutations of X, where $X \in \mathbb{N}$. S_X is a group under compositions. [Rot95]*

Definition 1.6. *If X is a nonempty set, a* **permutation** *of X is a bijection $\phi : X \longrightarrow X$. [Rot95]*

Definition 1.7. *A* **semigroup** *$(G, *)$ is a nonempty set G equipped with an associative operation. [Rot95]*

Definition 1.8. *If $x \in X$ and $\phi \in S_X$, then ϕ* **fixes** *x if $\phi(x) = x$ and ϕ* **moves** *x if $\phi(x) \neq x$. [Rot95]*

Definition 1.9. *For permutations $\alpha, \beta \in S_X$, α and β are* **disjoint** *if every element moved by one permutation is fixed by the other. Precisely,*

$$\text{if } \alpha(x) \neq x, \text{ then } \beta(a) = a \text{ and if } \alpha(y) = y, \text{ then } \beta(y) \neq y. \text{ [Rot95]}$$

Definition 1.10. *A permutation which interchanges a pair of elements is a* **transposition**. *[Rot95]*

Definition 1.11. *In group G, if $a,b \in G$, a and b* **commute** *if $a * b = b * a$. [Rot95]*

Definition 1.12. *A group G is* **abelian** *if every pair of elements in G commutes with one another. [Rot95]*

Definition 1.13. *Let X be a set and Δ by a family of words on X. A group G has* **generators** *X and* **relations** *Δ if $G \cong F/R$, where F is a free group with basis X and R is the normal subgroup of F generated by Δ. We say $< X|\Delta >$ is a* **presentation** *of G. [Rot95]*

Definition 1.14. *Let G be a group and $T = t_1, t_2, ..., t_n$ be a symmetric generating set for G with $|t_i| = m$. Then if $N = N_G(\bar{T})$, then we define the* **progenitor** *to be the semi direct product $m^{*n} : N$, where m^{*n} is the free product of n copies of the cyclic group C_n. [Cur07]*

Definition 1.15. *Let G be a group. If $H \leq G$, the* **normalizer** *of H in G is defined by $N_G(H) = \{a \in G | aHa^{-1} = H\}$. [Rot95]*

Definition 1.16. *Let G be a group. If $H \leq G$, the* **centralizer** *of H in G is:*

$$C_G(H) = \{x \in G : [x, h] = 1 \text{ for all } h \in H\}. \text{ [Rot95]}$$

Definition 1.17. *Let p be prime. If $G \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, then we say G is* **elementary abelian**. *[Rot95]*

Definition 1.18. *Let $(G, *)$ and $(H, \circ)$ be groups. The function $\phi : G \to H$ is a* **homomorphism** *if $\phi(a * b) = \phi(a) \circ \phi(b)$, for all $a, b \in G$. An* **isomorphism** *is a bijective homomorphism. We say G is isomorphic to H, $G \cong H$, if there is exists an isomorphism $f : G \to H$. [Rot95]*

Definition 1.19. *Let $f : G \to H$ be a homomorphism. The* **kernel of a homomorphism** *is the set $\{x \in G | f(x) = 1\}$, where 1 is the identity in H. We denote the kernel of f as* **ker f** *. [Rot95]*

Definition 1.20. *Let X be a nonempty subset of a group G. Let $w \in G$ where $w = x_1^{e_1} x_2^{e_2} \ldots x_n^{e_n}$, with $x_i \in X$ and $e_i = \pm 1$. We say that w is a* **word** *on X. [Rot95]*

Definition 1.21. *Let $a \in G$, where G is a group. The* **conjugacy class** *of a is given by $a^G = \{a^g | g \in G\} = \{g^{-1} a g | g \in G\}$. [Rot95]*

Definition 1.22. *The* **Dihedral Group** *D_n, n even and greater than 2, groups are formed by two elements, one of order $\frac{n}{2}$ and one of order 2. A presentation for a Dihedral Group is given by $< a, b | a^{\frac{n}{2}}, b^2, (ab)^2 >$. [Rot95]*

Definition 1.23. *A* **general linear group**, *$GL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$. [Rot95]*

Definition 1.24. *A* **special linear group**, *$SL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$. [Rot95]*

Definition 1.25. *A* **projective special linear group**, *$PSL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with determinant 1 over field $\mathbb{F}$ factored by its center:*

$$PSL(n, \mathbb{F}) = L_n(\mathbb{F}) = \frac{SL(n, \mathbb{F})}{Z(SL(n, \mathbb{F}))}. \ [Rot95]$$

Definition 1.26. *A* **projective general linear group**, *$PGL(n, \mathbb{F})$ is the set of all $n \times n$ matrices with nonzero determinant over field $\mathbb{F}$ factored by its center:*

$$PGL(n, \mathbb{F}) = \frac{GL(n, \mathbb{F})}{Z(GL(n, \mathbb{F}))}. \ [Rot95]$$

Definition 1.27. (**Monomial Character**) *Let G be a finite group and $H \leq G$. The character X of G is monomial if $X = \lambda^G$, where λ is a linear character of H. [Led87]*

Definition 1.28. (**Character**) *Let $A(x) = (A_{ij}(x))$ be a matrix representation of G of degree m. We consider the character polynomial of $A(x)$, namely*

$$det(\lambda I - A(x)) = \begin{bmatrix} \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \lambda - a_{11}(x) & -a_{12}(x) & \cdots & -a_{1m}(x) \\ \cdots & \cdots & \cdots & \cdots \\ \lambda - a_{m1}(x) & -a_{m2}(x) & \cdots & -a_{mm}(x) \end{bmatrix}$$

This is a polynomial of degree m in λ, and inspection shows that the coefficient of $-\lambda^{m-1}$ is equal to

$$\phi = a_{11}(x) + a_{22}(x) + ... + a_{mm}(x)$$

It is customary to call the right-hand side of this equation the trace of A(x), abbreviated to $tr A(x)$, so that

$$\phi(x) = tr A(x)$$

*We regard $\phi(x)$ as a function on G with values in K, and we call it the **character** of $A(x)$. [Led87]*

Definition 1.29. *The sun of squares of the degrees of the s=distinct irreducible characters of G is equal to $|G|$. The **degree of a character** χ is $\chi(1)$. Note that a character whose degree is 1 is called a linear character. [Led87]*

Definition 1.30. (**Lifting Process**) *Let N be a normal subgroup of G and suppose that $A_0(N_x)$ is a representation of degree m of the group G/N. Then $A(x) = A_0(N(x)$ defines a representation of G/N lifted from G/N. If $\phi_0(Nx)$ is a character of $A_0(Nx)$, then $\phi(x) = \phi_0(Nx)$ is the lifted character of $A(x)$. Also, if $u \in N$, then $A(u) = I_m, \phi(u) = m = \phi(1)$. Then lifting process preserves irreducibility. [Led87]*

Definition 1.31. (**Induced Character**) *Let $H \leq G$ and $\phi(u)$ be a character of H and defined $\phi(x) = 0$ if $x \in H$, then*

$$\phi^G(x) = \begin{cases} \phi(x) & x \in H \\ 0 & x \notin H \end{cases}$$

is an induced character of G. [Led87]

Definition 1.32. *Let G be a finite group and H be a subgroup such that $[G : H] = n$. Let C_α, $\alpha = 1, 2, ..., m$ be the conjugacy classes of G with $|C_\alpha| = h_\alpha, \alpha = 1, 2, 3, ..., m$. Let ϕ be a character of H and ϕ^G be the character of G induced from the character ϕ of H up to G. The values of ϕ^G on the m classes of G are given by:*

$$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w), \alpha = 1, 2, 3, ..., m. [Led87]$$

Definition 1.33. *Let G be a group. The **order** of G is the number of elements contained in G. We denote the order of G by $|G|$. [Rot95]*

Definition 1.34. *Let G be a group such that $K \leq G$. K is **normal** in G if $gKg^{-1} = K$, for every $g \in G$. We will use $K \triangleleft G$ to denote K as being normal in G. [Rot95]*

Definition 1.35. *Let G be a group and $S \subseteq G$. For $t \in G$, a **right coset** of S in G is the subset of G such that $St = \{st : s \in G\}$. We say t is a **representative** of the coset St. [Rot95]*

Definition 1.36. *Let G be a group. The **index** of $H \leq G$, denoted $[G : H]$, is the number of right cosets of H in G. [Rot95]*

Definition 1.37. *Let G be a group and H and K be subgroups of G. A **double coset** of H and K of the form $HgK = \{HgK | k \in K\}$ is determined by $g \in G$. [Rot95]*

Definition 1.38. *Let N be a group. The **point stabilizer** of w in N is given by:*

$$N^w = \{n \in N | w^n = w\}, \text{ where } w \text{ is a word in the } t_i\text{'s. [Rot95]}$$

Definition 1.39. *Let N be a group. The **coset stabilizer** of Nw in N is given by:*

$$N^{(w)} = \{n \in N | Nw^n = Nw\}, \text{ where } w \text{ is a word of the } t_i\text{'s. [Rot95]}$$

Definition 1.40. *Let G be a group. The **center** of G, $Z(G)$, is the set of all elements in G that commute with all elements of G. [Rot95]*

Definition 1.41. *A symmetric presentation of a group G is a definition of G of the form:*

$$G \cong \frac{2^{*n}:N}{\pi_1\omega_1, \pi_2\omega_2, \ldots}$$

where 2^{*n} denotes a free product of n copies of the cyclic group of order 2, N is transitive permutation group of degree n which permutes the n generators of the cyclic group by conjugation, thus defining semi-direct product, and the relators $\pi_1\omega_1, \pi_2\omega_2, \ldots$ have been factored out. [Led87]

Definition 1.42. *We defined*

$$\mathcal{N}^i = C_\mathcal{N}(t_i); \mathcal{N}^{ij} = C_\mathcal{N}(\langle t_i, t_j \rangle) \text{ etc,}$$

single point and two point stabilizer in $\mathcal{N}$ respectively. The coset stabilizing subgroup, $\mathcal{N}^{(w)}$, of $\mathcal{N}$ is given by

$$\mathcal{N}^{(w)} = \pi \in \mathcal{N} : \mathcal{N}w\pi = \mathcal{N}w,$$

for w a word in the symmetric generators. Clearly $\mathcal{N}^w \leq \mathcal{N}^{(w)}$, and the number of cosets in the double coset $[w] = \mathcal{N}w\mathcal{N}$ is given by $|\mathcal{N}|/|\mathcal{N}^{(w)}|$, since $\mathcal{N}w\pi_1 \neq \mathcal{N}w\pi_2$

$\Longleftrightarrow \mathcal{N}w\pi_1\pi_2^{-1} \neq \mathcal{N}w$

$\Longleftrightarrow \pi_1\pi_2 \notin \mathcal{N}^{(w)}$

$\Longleftrightarrow \mathcal{N}^{(w)}\pi_1\pi_2^{-1} \neq \mathcal{N}^{(w)}$

$\Longleftrightarrow \mathcal{N}^{(w)}\pi_1 \neq \mathcal{N}^{(w)}\pi_2.$

Double Coset Enumeration Arithmetic

In order to obtain the index of $\mathcal{N}$ in $\mathcal{G}$ we shall perform a manual double coset enumeration of $\mathcal{G}$ over $\mathcal{N}$; thus we must find all double cosets $[w]$ and work out how many single cosets each of them contains. We shall know that we have completed the double coset enumeration when the set of right cosets obtained is closed under right multiplication. Moreover, the completion test above is best performed by obtaining the orbits of $\mathcal{N}^{(w)}$ on the symmetric generators. We need only identify, for each $[w]$, the double coset to which $\mathcal{N}wt_i$ belongs for one symmetric generator t_i from each orbit. [Cur07]

Definition 1.43. First Isomorphism Theorem(F.I.T). *Let $\phi : G \to H$ is a homomorphism with $Ker\phi$. Then,*

$$\bullet Ker\phi \trianglelefteq G$$
$$\bullet G/Ker\phi \cong img\phi \ [Rot95]$$

1.1.2   Theorems

Theorem 1.44. *The number of irreducible character of G is equal to the number of conjugacy classes of G. [Cur07]*

Theorem 1.45. *Let $\phi : G \to H$ be a homomorphism with kernel K. Then K is a normal subgroup of G and $G/K \cong im\phi$. [Rot95]*

Theorem 1.46. *Let N and T be subgroups of G with N normal. Then $N \cap T$ is normal in T and $T/(N \cap T) \cong NT/N$. [Rot95]*

Theorem 1.47. *Every permutation $\alpha \in S_n$ is either a cycle or a product of disjoint cycles. [Rot95]*

Theorem 1.48. *Let $f : (G, *) \to (G', \circ)$ be a homomorphism. The following hold true:*

- *$f(e) = e'$, where e' is the identity in G',*

- *If $a \in G$, then $f(a^{-1}) = f(a)^{-1}$,*

- *If $a \in G$ and $n \in \mathbb{Z}$, then $f(a^n) = f(a)^n$. [Rot95]*

Theorem 1.49. *The intersection of any family of subgroups of a group G is again a subgroup of G. [Rot95]*

Theorem 1.50. *If $S \leq G$, then any two right (or left) cosets of S in G are either identical or disjoint. [Rot95]*

Theorem 1.51. *If G is a finite group and $H \leq G$, then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$. [Rot95]*

Theorem 1.52. *If S and T are subgroups of a finite group G, then*

$$|ST||S \cap T| = |S||T|. \ [Rot95]$$

Theorem 1.53. *If $N \triangleleft G$, then the cosets of N in G form a group, denoted by G/N, of order $[G : N]$. [Rot95]*

Theorem 1.54. *The commutator subgroup G' is a normal subgroup of G. Moreover, if $H \lhd G$, then G/H is abelian if and only if $G' \leq H$. [Rot95]*

Theorem 1.55. *Let G be a group with normal subgroups H and K. If $HK = G$ and $H \cap K = 1$, then $G \cong H \times K$. [Rot95]*

Theorem 1.56. *If $a \in G$, the number of conjugates of a is equal to the index of its centeralizer:*

$$|a^G| = [G : C_G(a)],$$

and this number is a divisor of $|G|$ when G is finite. [Rot95]

Theorem 1.57. *If $H \leq G$, then the number c of conjugates of H in G is equal to the index of its normalizer: $c = [G : N_G(H)]$, and c divides $|G|$ when G is finite. Moreover, $aHa^{-1} = bHb^{-1}$ if and only if $b^{-1}a \in N_G(H)$. [Rot95]*

Theorem 1.58. *Every group G can be imbedded as a subgroup of S_G. In particular, if $|G| = n$, then G can be imbedded in S_n. [Rot95]*

Theorem 1.59. *If $H \leq G$ and $[G : H] = n$, then there is a homomorphism $\rho : G \to S_n$ with $\ker\rho \leq H$. The homomorphism ρ is called the representation of G on the cosets of H. [Rot95]*

Theorem 1.60. *If X is a G-set with action α, then there is a homomorphism $\tilde{\alpha} : S_X$ given by $\tilde{\alpha} : x \mapsto gx = \alpha(g, x)$. Conversely, every homomorphism $\varphi : G \to S_X$ defines an action, namely, $gx = \varphi(g)x$, which makes X into a G-set. [Rot95]*

Theorem 1.61. *Every two composition series of a group G are equivalent.*
We will refer to this Theorem as the **Jordan-Hölder Theorem**. *[Rot95]*

Theorem 1.62. *Let X be a faithful primitive G-set of degree $n \geq 2$. If $H \lhd G$ and if $H \neq 1$, then X is a transitive H-set. Also, n divides $|H|$. [Rot95]*

1.1.3 Lemmas

Lemma 1.63. *Let X be a G-set, and let $xy \in X$.*

- *If $H \leq G$, then $Hx \cap Hy \neq \emptyset$ implies $Hx = Hy$.*

- *If $H \lhd G$, then the subsets Hx are blocks of X. [Rot95]*

Chapter 2

Building a Progenitor

2.1 Introduction

One of the major goals of this thesis project is to produce finite homomorphic images of finite non-abelian simple groups; to be able to construct such we follow the method of Robert T. Curtis' involutory progenitor. According to Curtis, a progenitor is a semi-direct product is defined as $P \cong 2^{*n} : N = \{\pi w | \pi \in N, w$ reduced word in the $t_i\}$[Cur07]; where N is our control group. Thus, a good starting point would be to construct a control group N .

2.2 Control Group

There are many ways to approach a control group. One of the possible starting points is to look into Transitive Groups in n letters. The following code produces all the subgroups that are found in the transitive group in 15 letters:

```
> TransitiveGroups(15);
[
    Permutation group acting on a set of cardinality 15
    Order = 15 = 3 * 5
        (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15),
    Permutation group acting on a set of cardinality 15
    Order = 30 = 2 * 3 * 5
        (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)
```

```
        (1, 14)(2, 13)(3, 12)(4, 11)(5, 10)(6, 9)(7, 8),
    Permutation group acting on a set of cardinality 15
    Order = 30 = 2 * 3 * 5
        (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)
        (1, 4)(2, 8)(3, 12)(6, 9)(7, 13)(11, 14),
    Permutation group acting on a set of cardinality 15
    Order = 30 = 2 * 3 * 5
        (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)
        (1, 11)(2, 7)(4, 14)(5, 10)(8, 13),
    Permutation group acting on a set of cardinality 15
    Order = 60 = 2^2 * 3 * 5
        (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)
        (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)... .
```

The list is cut short because the transitive group in 15 letter has a total of 104 subgroups, and listing all would take up too much space. We will demonstrate an example of building an involutory progenitor using the fifth listed subgroup of Transitive Group 15, which will be labeled as such:

```
> N:=TransitiveGroup(15,5);
> N;
Permutation group N acting on a set of cardinality 15
Order = 60 = 2^2 * 3 * 5
    (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)
    (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)
>
> S:=Sym(15);
> xx:=S!(1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8);
> yy:=S!(1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13);
> N:=sub<S|xx,yy>;
> #N;
60.
```

The permutations $x = (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)$ and $y = (1, 4, 10)(2, 5, 8)(37, 11)(6, 9, 15)(12, 14, 13)$ will generate the control group. The following code will help build a permutation representation of the control group N:

```
> FPGroup(N);
Finitely presented group on 2 generators
Relations
    $.1^5 = Id($)
    $.2^3 = Id($)
    ($.1 * $.2 * $.1)^2 = Id($)
```

this is the beginning of building the progenitor. From the code above, we will interpret .1 to be x and .2 to be y. Thus, the control group N is $N = < x, y | x^5, y^3, (xyx)^2 >$, where N is Alternating Group 5 and can be proved by the code below,

```
> NL:=NormalLattice(N);
> NL;

Normal subgroup lattice
-----------------------

[2]  Order 60  Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:

> IsIsomorphic(N,Alt(5));
true Homomorphism of GrpPerm: N, Degree 15,
Order 2^2 * 3 * 5 into GrpPerm: $,
Degree 5, Order 2^2 * 3 * 5 induced by
    (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)
    |--> (1, 2, 5, 3, 4)
    (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)
    |--> (1, 2, 4).
```

Since N is of order 60, our first instinct is to ask MAGMA if it is isomorphic to $Alt(5)$, another group that could have been tested that is also of order 60 could have been Dihedral Group 60 (or as MAGMA labeled Dihedral Group 30). However, the latter guess would have proven to be false.

The next approach is to find what stabilizes the t we choose to work with; in this case we will let t refer to t_1 out of the possible 15 t_i's in the control group. The permutations that stabilize t are given in letters (permutations), thus we will use the Schreier System to convert those permutations into words. The following code shows the process:

```
> N1:=Stabilizer(N,1);
> N1;
Permutation group N1 acting on a set of cardinality 15
Order = 4 = 2^2
    (2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15)
    (2, 5)(3, 13)(4, 10)(7, 14)(9, 15)(11, 12)
> NN<x,y>:=Group<x,y|x^5,y^3,(x*y*x)^2>;
```

```
> Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
> ArrayP:=[Id(N): i in [1..60]];
> for i in [2..60] do
for> P:=[Id(N): l in [1..#Sch[i]]];
for> for j in [1..#Sch[i]] do
for|for> if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
for|for> if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
for|for> if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
for|for> if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
for|for> end for;
for> PP:=Id(N);
for> for k in [1..#P] do
for|for> PP:=PP*P[k]; end for;
for> ArrayP[i]:=PP;
for> end for;
> for i in [1..60] do if ArrayP[i] eq N!(2, 11)(3, 4)(5, 12)
> (7, 9)(10, 13)(14,15) then print Sch[i]; end if; end for;
x^2 * y
> for i in [1..60] do if ArrayP[i] eq N!(2, 5)(3, 13)(4, 10)
> (7, 14)(9, 15)(11,12)then print Sch[i]; end if; end for;
x * y^-1 * x^-1 * y * x.
```

Lastly, we will make the following additions to complete the progenitor. Since we are working with an involutory progenitor, the order of ts is 2, we will make this addition to the progenitor. Also, we will make $t \sim t_1$ commute with the point stabilizers that were recently found. Therefore the progenitor is given as:

$$N \cong 2^{*15} : A_5 \cong < x, y, t | x^5, y^3, (xyx)^2, t^2, (t, x^2y), (t, xy^{-1}x^{-1}yx) > .$$

2.3   First Order Relations

To achieve the ultimate goal of producing a finite homomorphic image, relations must be added to the control group. This process is referred to as factoring by relations. Luckily, with the help of MAGMA, this can easily be done with the subsequent code:

```
> C:=Classes(N);
> Classes(N);
```

```
Conjugacy Classes of group N
----------------------------
[1]     Order 1        Length 1
        Rep Id(N)


[2]     Order 2        Length 15
        Rep (1, 7)(3, 8)(5, 14)(6, 11)(10, 12)(13, 15)


[3]     Order 3        Length 20
        Rep (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)


[4]     Order 5        Length 12
        Rep (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)


[5]     Order 5        Length 12
        Rep (1, 10, 14, 9, 3)(2, 7, 6, 15, 12)(4, 11, 8, 5, 13)


> #C;
5
> for i in [2..5] do
for> i,Orbits(Centralizer(N,C[i][3]));
for> end for;
2 [
    GSet{@ 2 @},
    GSet{@ 4 @},
    GSet{@ 9 @},
    GSet{@ 1, 7, 10, 12 @},
    GSet{@ 3, 8, 14, 5 @},
    GSet{@ 6, 11, 15, 13 @}
]
3 [
    GSet{@ 1, 4, 10 @},
    GSet{@ 2, 5, 8 @},
    GSet{@ 3, 7, 11 @},
    GSet{@ 6, 9, 15 @},
    GSet{@ 12, 14, 13 @}
]
4 [
    GSet{@ 1, 9, 10, 3, 14 @},
    GSet{@ 2, 15, 7, 12, 6 @},
    GSet{@ 4, 5, 11, 13, 8 @}
]
5 [
```

```
    GSet{@ 1, 10, 14, 9, 3 @},
    GSet{@ 2, 7, 6, 15, 12 @},
    GSet{@ 4, 11, 8, 5, 13 @}
]
> for j in [2..5] do
for> C[j][3];
for> for i in [1..60] do
for|for> if ArrayP[i] eq C[j][3]
for|for|if> then Sch[i]; end if;
for|for> end for;
for> end for;
(1, 7)(3, 8)(5, 14)(6, 11)(10, 12)(13, 15)
x * y * x
(1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)
y
(1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)
x
(1, 10, 14, 9, 3)(2, 7, 6, 15, 12)(4, 11, 8, 5, 13)
x^2.
```

Let us better analyze the information given within the following table, and proceed by multiplying each relation by a t of each orbit from each class.

Table 2.1: Conjugacy Classes of N

Conjugacy Classes of N=Alt(5)			
Class	Representative of the class	# of elements in class	Orbits
C1	Identity	1	$\{1\}\{2\}...\{15\}$
C2	xyx=(1,7)(3,8) (5,14)(6,11) (10,12)(13,15)	15	$\{2\}\{4\}\{9\}$ $\{1, 7, 10, 12\}$ $\{3, 8, 14, 5\}$ $\{6, 11, 15, 13\}$
C3	y=(1,4,10) (2,5,8)(3,7,11) (6,9,15)(12,14,13)	20	$\{1, 4, 10\}$ $\{2, 5, 8, \}$ $\{3, 7, 11\}$ $\{6, 9, 15\}$ $\{12,14,13\}$
C4	x=(1,9,10,3,14) (2,15,7,12,6) (4,5,11,13,8)	12	$\{1, 9, 10, 3, 14\}$ $\{2, 15, 7, 12, 6\}$ $\{4, 5, 11, 13, 8\}$
C5	x^2=(1,10,14,9,3) (2,7,6,15,12) (4,11,8,5,13)	12	$\{1, 10, 14, 9, 3\}$ $\{2, 7, 6, 15, 12\}$ $\{4, 11, 8, 5, 13\}$

Since we are working with t_1, in order to right multiply by t_i (not 1), we simply conjugate t_1. Therefore, all of the first order relations are as listed, where the variable r represents a power that the relation may be raised to.

Table 2.2: conjugacy Class of N and Relations

Relations of N=Alt(5)		
Class	Representative of the class	Relations
C1	Identity	
C2	xyx	$(xyxt)^r$, $(xyxt^{y^x y})^r$ $(xyxt^y)^r$ $(xyxt^x)^r$ $(xyxt^{x^3})^r$ $(xyxt^{(y^x)^2})^r$
C3	y	$(yt)^r$, $(yt^{y^x y})^r$, $(yt^{x^3})^r$, $(yt^{(y^x)^2})^r$, $(yt^{(xy)^3})^r$
C4	x	$(xt)^r$, $(xt^{y^x y})^r$, $(xt^y)^r$
C5	x^2	$(x^2 t)^r$, $(x^2 t^{y^x y})^r$, $(x^2 t^y)^r$

From here, we will work with MAGMA, where we will research each power of r and check if the relations produced a finite homomorphic image. When working in this manner, it is ideal to not work with all of the relations, because it would take too long and also use up too much memory. In such situations, we may limit the relations we check in MAGMA to around twelve, though the individual ultimately decides the limit. In the table above all relations are raised to a generic power r, but when checking in the computer, we are bit more specific. Moving forward, we use a sub command of MAGMA (called NANO) where it test each power of the relation to see if there are finite groups, we run this in the background:

```
for r1, r2, r3, r4,r5, r6, r7, r8, r9, r10, r11, r12 in [0..10]
doG<x,y,t>:=Group<x,y,t|x^5, y^3, (x*y*x)^2, t^2, (t,x^2*y),
(t,x*y^-1*x^-1*y*x), (x*y*x*t)^r1, (y*t*t^(x))^r2,
(y*t*t^(x^2))^r3, (x*y*x*t^(y))^r4, (x*y*x*t^(x))^r5,
(y*t*t^(y))^r6, (y*t*t^(x^3))^r7,
(x*y*x*t^(x^3))^r8, (x^2*t)^r9, (y*t^(x^3))^r10,
(x*t)^r11, (y*t)^r12>;
if #G gt 60 then
r1, r2, r3, r4, r5, r6, r7, r8, r9, r10, r11, r12,
#G;
```

```
end if;
end for;
```

here, we may observe that each relations is being tested with numbers ranging from 1 to 10.

The table is listed in Chapter 6 (table 6.2), and it lists some of the finite homomorphic imaged found by factoring by first order relations.

2.4   Famous Lemma

In addition to the progenitor that was built in the previous section, we will continue by factoring the progenitor by additional relations, we will do so with the aid of the Famous Lemma.

Our goal throughout this project is to produce finite homomorphic images, but along with that, we also want our findings to be interesting. A task that can be bit overwhelming, how can one know what relations are worthy of trying out? A method that has proven to be resourceful is factoring by Famous Lemma relations, a process created by Robert T. Curtis. As stated in his book [Cur 07]:

Lemma 3.3

$N \cap < t_i, t_j > \leq C_N(N_{ij})$where N_{ij} denotes the stabilizer in N of the two points i and j.

Proof. If $\pi \in N$and $\pi = w(t_i, t_j)$, as word in the two symmetric generators t_i and t_j, and if $\sigma \in N_{ij}$, then

$$\pi^\sigma = w(t_i t_j)^\sigma = w(t_i^\sigma, t_j^\sigma) = w(t_{i\sigma}, t_{j\sigma}) = w(t_i, t_j) = \pi,$$

and so π commutes with every element on N_{ij} as required.

Such that for each two point stabilizer N_{ij}, find its centralizer in N and attempt to write elements of this centralizer as words in t_i and t_j.$\square$

As previously stated, the control group is :

$$N = < x, y, t | x^5, y^3, (xyx)^2, t^2, (t, x^2 y), (t, xy^{-1}x^{-1}yx) >.$$

Now to produce the relations detailed in the Famous Lemma, we recreate the process listed in the proof; meaning, we will stabilize the two points, t_1 and t_2, which will be our two point stabilizer of N, and follow by computing the centralizer of the two point stabilizer. Let us illustrate this with the control group N:

```
> N12:=Stabiliser(N,[1,2]);
```

```
> C12:=Centraliser(N,N12);
```

```
> C12;
Permutation group N acting on a set of cardinality 15
Order = 60 = 2^2 * 3 * 5
    (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)
    (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)
```

```
> for i in [1..60] do if ArrayP[i] eq N!
for|if> (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)
for|if> then print Sch[i]; end if; end for;
x
```

```
> for i in [1..60] do if ArrayP[i] eq N!
for|if> (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)
for|if> then print Sch[i]; end if; end for;
y
```

with the help of the Schreier System, we are able to translate the permutations that were given as the centralizer of the two point stabilizer of N, providing x and y. With the new Famous Lemma relations discovered above, x and y, there can be one of two possibilities:

$$= \begin{cases} (t_i t_j)^k = g & \text{where k is even and fixes 1 and 2} \\ (g t_i)^k = 1 & \text{where k is odd and g sends 1 to 2.} \end{cases} \quad \text{[Rot95]}$$

For our example, we will add the two relations (x, t) and (y, t). It just so happens that in this particular example, these Famous Lemma relations were also produced in the process of creating first order relations, and are already included in the end of our progenitor, and are listed in the end of the previos section.

Chapter 3

Double Coset Enumeration

3.1   Double Coset Enumeration of S_6

A group G is made up of a union of double cosets and when performing a double coset enumeration we are searching for all of the double cosets in G over a control group N. Where a double coset consists of single cosets and that will determine the size of the double coset. In this section, we will perform double coset enumeration on

$$G = \frac{2^{*5} : S_5}{(x^{-1}yxyt)^8, (x^2yx^{-1}t)^5, (x^2yx^2yxyt)^8} \cong S_6.$$

The control group is Symmetric Group 5 and the generators are $x = (1, 2, 3, 4, 5)$ and $y = (1, 2)$; we will refer to t as t_1 for this group. Let us compute the three listed relations:

$$[x^{-1}yxyt]^8 = e \text{ (let } \pi = x^{-1}yxy)$$

$$[\pi t]^8 = e$$

$$\pi^8 t^{\pi^7} t^{\pi^6} t^{\pi^5} t^{\pi^4} t^{\pi^3} t^{\pi^2} t^\pi t = e$$

$$(1, 3, 2)t_2 t_1 t_3 t_2 t_1 t_3 t_2 t_1 = e$$

$$(1, 3, 2)t_2 t_1 t_3 t_2 t_1 t_3 = t_1 t_2$$

$$[x^2yx^{-1}t]^5 = e \text{ (let } \pi = x^2yx^{-1})$$

$$[\pi t]^5 = e$$

$$\pi^5 t^{\pi^4} t^{\pi^3} t^{\pi^2} t^\pi t = e$$

$$(1,2,3,4)t_1t_4t_3t_2t_1 = e$$

$$(1,2,3,4)t_1t_4t_3 = t_1t_2$$

$$[x^2yx^2yxyt]^8 = e \text{ (let } \pi = x^2yx^2yxy)$$

$$[\pi t]^8 = e$$

$$\pi^8 t^{\pi^7} t^{\pi^6} t^{\pi^5} t^{\pi^4} t^{\pi^3} t^{\pi^2} t^\pi t = e$$

$$(1,3,2)t_2t_1t_3t_2t_1t_3t_2t_1 = e$$

$$(1,3,2)t_2t_1t_3t_2t_1t_3 = t_1t_2.$$

Now we may continue to perform the double coset enumerations of G over N.

In order to express G as a union of double cosets, we will interpret it as NgN, where g is an element of G; such that $G = NeN \cup Ng_1N \cup Ng_2N...$, where $g_i's$ are words in the $t_i's$. Our objective is to find the number of single cosets contained in each double coset $[w]$ defined as $[w] = \{Nw^n | n \in N\}$. We will know when we are done with our double coset enumeration when the set of right cosets is closed under right multiplication, meaning all of our potentially new double cosets are already recognized. We declare each double coset $[w]$ where Mwt_i will belong to one of the symmetric generators of t_i from each of the orbits of the coset stabiliser group; defined as $N^{(w)} = \{n \in N | Nw^n = Nw\}$, and where each w is a word of $t_i's$ on $X = \{1,2,3,4,5\}$.

N e N

We begin with the double coset $NeN = \{Ne^n | n \in N\}$, labeled [*], and contains one single coset, N. The coset stabilizer of N is the control group N and thus has a single orbit of $\{1,2,3,4,5\}$ of which we will perform right multiplication. Any

element of the orbit can be the representative, we chose the orbit representative of 1. This means that $Net_1 = Nt_1N$ is a new double coset [1] and since there is 5 single coset in the orbit of [*], 5 single cosets will move forward to the coset [1].

N t_1 N

For the double coset Nt_1N we need to find the point stabilizer, N^1, and the coset stablising group $N^{(1)}$ to discern how many single cosets are in [1],

$$N^{(1)} \leq < (2,3), (3,4,5), (3,5) >$$
$$\leq 24.$$

Thus, the order of the coset stabliser is 24. The number of single cosets is defined as $\frac{|N|}{|N^{(1)}|}$, so the number of single cosets in $N^{(1)} = \frac{|N|}{|N^{(1)}|} = \frac{120}{24} = 5$. And our index is represented by the sum of single cosets in each distinct double coset, so far we only have established the cosets [*] and [1], $1 + 5 = 6$. The total amount of single cosets is characterized as $\frac{G}{N}$, so the entirety of the double coset enumeration we should have a total of $\frac{720}{120} = 6$ single cosets. This will also be another clue to let us know that we have completed the double coset enumeration; notice that we are technically done because we have reached the limit. Carrying on, the orbits of [1] are $\{1\}, \{2,3,4,5\}$ and will pick a coset representative from each of the orbits to determine if any double cosets Mt_1t_i are new:

$$Mt_1t_1 = Me, \in [*] \ (1 \text{ will loop back to } [*])$$
$$Mt_1t_2 \in [1].$$

Consequently, there are not any new double cosets, since the potentially new double cosets, $[1,2]$ and $[1,1]$ either went back to [*] or looped back to [1]. There were 4 single cosets that looped back to [1]. We have completed the double cose enumeration of S_6.

Figure 3.1: Cayley Graph of S6

3.2 Double Coset Enumeration of $PGL_2(7)$

We factor the progenitor $2^{*7} : D_{14}$ by the relations $(at)^3$ and (bt^{a^3}). Let $G = \dfrac{2^{*7} : D_{14}}{(at)^3, (bt^{a^3})}$ be a symmetric presentation of G given by

$< a, b, t | a^7, b^2, (ab)^2, t^2, (t, ab), (at)^3, (bt^{a^3})^6 >$ where N$\cong D_{14} = < a, b >$ and $a = (1234567)$ and $b = (17)(26)(35)$ and $t = t_7$.

We will use manual double coset enumeration to find the index of N in G . By definition we may take G and express it as a union of double cosets NgN, where g is an element of G. So $G = NeN \cup Ng_1N \cup Ng_2N \cup ...$, where g_i's are words in the t_i's.

Our goal will be to complete a double coset enumeration of G over N in order to find the index of N in G. All distinct double cosets may be found by the following definition $[w]$, where $[w] = \{Nw^n | n \in N\}$ and from there on we may see how many single cosets are contained in each double coset. This procedure will be repeated until all potentially new double cosets have already been accounted for; and when the set of right cosets is closed under right-multiplication by t_i's. We symbolize, for each $[w]$, the double coset to which Nwt_i belongs for one symmetric generator t_i from each orbit of the coset stabilser $N^{(w)} = \{n \in N : Nw^n = Nw\}$, where w is a word of t_i's on $\{1, 2, 3, 4, 5, 6, 7\} = X$.

To begin with the double coset, we will start off with the identity; NeN, which we denote $[*]$. This double coset consists of the single coset N. The orbits of N on X are $\{1,2,3,4,5,6,7\}$. Next, we have chosen t_7 as our symmetric generator from the orbit $\{1,2,3,4,5,6,7\}$ and find where Nt_7 belongs. We find that Nt_7 belongs to Nt_7N and is

a new double coset. We denote Nt_7N by [7].

We will continue by finding how many single cosets [7] contains, but in order to do so we must find the number of elements in the coset stabilizing group of [7], denoted $N^{(7)}$. This is essetial because the number of single cosets in [7] is equal to $\frac{|N|}{|N^{(7)}|}$. We have:

$$N^{(7)} \geq |<e,(16)(25)(34)>|$$

$$\geq 2$$

Thus, the number of single cosets in $Nt_7N = \frac{|N|}{|N^{(7)}|} = \frac{14}{2} = 7$. Now our index is the number of distinct single cosets in each distinct double coset, such as [*] and [7]. As of now, we have $1+7 = 8$ single cosets, so our index is 8 as of now. We should also note that the orbits of [7] are $\{7\}$, $\{1,6\}$, $\{2,5\}$, and $\{3,4\}$.

We have computed the number of single cosets in this level of double cosets, so we continue to the next level by working with the orbits of [7]. The orbits of $N^{(7)}$ on X are $\{7\}$, $\{1,6\}$, $\{2,5\}$, and $\{3,4\}$, and we take t_7, t_1, t_2, and t_3 from each orbit respectively. From the orbit $\{7\}$ we get Nt_7t_7, which belongs to the double coset [*]. The representative t_1 from $\{1,6\}$ advances to the potentially new double coset Nt_7t_1 which we denote [71]. The representative t_2 from $\{2,5\}$ advances to the potentially new double coset Nt_7t_2 which we denote [72]. The representative t_3 from $\{3,4\}$ advances to the potentially new double coset Nt_7t_3 which we denote [73].

However, consider the following relation:

$t_7t_1 = (7362514)[t_7]^{(7246135)}$.

Hence in [7], the orbit representative from $\{7\}$ advances back to [*], the orbit representative from $\{1,6\}$ advances back to [7], the representative from $\{2,5\}$ goes to [72], and the representative from $\{3,4\}$ goes to [73]. So the only new, distinct double cosets are [72] and [73].

Consider the following relations:

$t_7t_2 = t_5t_3$. Hence $[t_7t_2]^{(14)(23)(57)} = t_5t_3 \Rightarrow [(14)(23)(57)] \in N^{(72)}$.

$t_7t_3 = t_2t_6$. Hence $[t_7t_3]^{(27)(36)(45)} = t_2t_6 \Rightarrow [(27)(36)(45)] \in N^{(73)}$.

Now, computing $N^{(72)}$ in N:

$$N^{(72)} \geq < N^{72}, (14)(23)(57) >$$
$$\geq < Id(G), (14)(23)(57) >$$
$$\geq 2$$

Computing $N^{(73)}$ in N:

$$N^{(73)} \geq < N^{73}, (27)(36)(45) >$$
$$\geq < Id(G), (27)(36)(45) >$$
$$\geq 2$$

The number of single cosets in $Nt_7t_2N = \frac{|N|}{|N^{(72)}|} = \frac{14}{2} = 7$. The number of single cosets in $Nt_7t_3N = \frac{|N|}{|N^{(73)}|} = \frac{14}{2} = 7$.

Hence our index is increased to $8 + 7 + 7 = 22$.

We now explore any potentially new double cosets coming from representatives from the orbits of $N^{(02)}$ on X and $N^{(03)}$ on X.

The orbits of $N^{(72)}$ on X are $\{7\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. The representative from the orbit $\{2\}$ advances to $[7]$. The other representatives will be the potentially new double cosets $[721]$, $[723]$, $[724]$, $[725]$, $[726]$, and $[727]$.

However, consider the following relations:

$t_7t_2t_7 = (7362514)[t_7t_2]^{(16)(25)(34)}$

$t_7t_2t_1 = (7246135)[t_7t_3]^{(76)(15)(24)}$

$t_7t_2t_3 = (7654321)[t_7]^{(7531642)}$

$t_7t_2t_4 = (7246135)[t_7t_3]^{(71)(62)(53)}$

$t_7t_2t_5 = (7246135)[t_7t_2]^{(73)(12)(62)}$

We now know that the only distict double coset is $[726]$ which only contains one element.

The orbits of $N^{(73)}$ on X are $\{7\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, and $\{6\}$. The representative from the orbit $\{3\}$ advances to $[7]$. The other representatives will be the potentially new double cosets $[731]$, $[732]$, $[734]$, $[735]$, $[736]$, and $[737]$.

However, we may consider the following relations:

$$t_7 t_3 t_7 = (7123456)[t_7 t_3]^{(7531642)}$$

$$t_7 t_3 t_2 = (7654321)[t_7 t_3]^{(7246135)}$$

$$t_7 t_3 t_4 = (7246135)[t_7 t_2]^{(71)(62)(53)}$$

$$t_7 t_3 t_5 = (7246135)[t_7 t_2]^{(76)(15)(24)}$$

$$t_7 t_3 t_6 = (7531642)[t_7]^{(7246135)}$$

Similarly, the only distict double coset is [731].

By considering the next relations, we may conclude that there will not be another level of double cosets. Thus finishing our double coset enumeration.

$$t_7 t_2 t_6 t_7 = (7654321)[t_7 t_2]^{(7123456)}$$

$$t_7 t_2 t_6 t_1 = (7531642)[t_7 t_2]^{(7246135)}$$

$$t_7 t_2 t_6 t_2 = (7362514)[t_7 t_2]^{(71)(62)(53)}$$

$$t_7 t_2 t_6 t_3 = (7246135)[t_7 t_2]^{(72)(63)(54)}$$

$$t_7 t_2 t_6 t_4 = (7123456)[t_7 t_2]^{(73)(12)(64)}$$

$$t_7 t_2 t_6 t_5 = (e)[t_7 t_2]^{(74)(13)(65)}$$

$$t_7 t_3 t_1 t_7 = (7123456)[t_7 t_3]^{(71)(62)(53)}$$

$$t_7 t_3 t_1 t_2 = (7246135)[t_7 t_3]^{(73)(12)(64)}$$

$$t_7 t_3 t_1 t_3 = (7123456)[t_7 t_3]^{(7246135)}$$

$$t_7 t_3 t_1 t_4 = (7531642)[t_7 t_3]^{(7362514)}$$

$$t_7 t_3 t_1 t_5 = (e)[t_7 t_3]^{(76)(15)(24)}$$

$$t_7 t_3 t_1 t_6 = (7654321)[t_7 t_3]^{(7531642)}$$

3.2.1 Proof $G \cong PGL_2(7)$

We factor the progenitor $2^{*7} : D_{14}$ by the relations $(xt)^3$ and (yt^{x^3}). Let $G \cong \frac{2^{*7}:D_{14}}{(xt)^3,(yt^{x^3})^6} \cong PGL_2(7)$ be a symmetric presentation of G given by

$< x, y, t | x^7, y^2, (xy)^2, t^2, (t, yb), (xt)^3, (yt^{x^3})^6 >$ where N$\cong D_{14} = < x, y >$ and $x = (1, 2, 3, 4, 5, 6, 7)$ and $y = (1, 7)(2, 6)(3, 5)$.

Definition Generators of PSL(2,q) (where p is a finite field of order p^2 where p is prime). Note, we will refer to this as $L_2(q)$

Figure 3.2: Cayley Graph of $PGL_2(7)$

In $L_2(7)$ our field is 7,

$F_7 = \mathbb{Z}_7 = 0, 1, 2, 3, 4, 5, 6$

$F_7 \cup \infty = 0, 1, 2, 3, 4, 5, 6 \cup \infty$.

The non-zero squares in F_7 are $1^2, 2^2, 3^2, 4^2, 5^2, 6^2$ in mod7 $= \{1, 2, 4\}$.

The group is generated by, $L(2, 7) = x \to \frac{ax+b}{cx+d}$ where $x \in F_7 \cup \infty, a, b, c, d \in F_7$ and $ad - bc = 1$ or equicantly a non-zero square.

Permutations of our presentation of $L_2(7)$ will be α, β, and γ, which are defined and listed below,

$\alpha : x \to x + 1 = (\infty)(0, 1, 2, 3, 4, 5, 6)$

$\beta : x \to 4x = (\infty)(0)(1, 4, 2)(3, 5, 6)$.

$\gamma : x \to \frac{-1}{x} = (0, \infty)(1, 6)(2, 3)(4, 5)$,

now since we are working with $PGL_2(7)$, which is $L_2(7) : 2$, we have an extra generator which is the automorphism,

$aut : x \to \frac{-6}{x} = (7, \infty)(2, 4)(5, 3)$. With the help of MAGMA we can find the homomorphic projection of t

```
> S:=Sym(7);
> xx:=S!(1,2,3,4,5,6,7);
> yy:=S!(1,7)(2,6)(3,5);
> N:=sub<S|xx,yy>;
> #N;
14
> G<x,y,t>:=Group<x,y,t|x^7,y^2,(x*y)^2,t^2,(t,x*y),(x*t)^3,
(y*t^(x^3))^6>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  A(1, 7)                        = L(2, 7)
    1
> S:=Sym(8);
> alpha:=S!(7,1,2,3,4,5,6);
> beta:=S!(1,4,2)(3,5,6);
> gamma:=S!(7,8)(1,6)(2,3)(4,5);
> aut:=S!(7,8)(2,4)(5,3);
> P:=sub<S|alpha, beta, gamma, aut>;
> IsIsomorphic(G1, P);
true Homomorphism of GrpPerm: G1, Degree 24,
 Order 2^4 * 3 * 7 into GrpPerm: P,
Degree 8, Order 2^4 * 3 * 7 induced by
 (2, 3, 5, 7, 10, 6, 4)(8, 9, 12, 18, 22, 20, 13)
 (11, 15, 19, 14, 16, 21, 17)
    |--> (2, 6, 4, 8, 5, 3, 7)

    (2, 3)(4, 5)(6, 7)(8, 12)(11, 16)(13, 18)(14, 15)(17, 21)
    (20, 22)
|--> (2, 4)(3, 5)(7, 8)

    (1, 2)(3, 4)(5, 8)(6, 9)(7, 11)(10, 14)(12, 19)(13, 15)
    (16, 17)(18, 20)(21,
        23)(22, 24)
|--> (1, 5)(2, 7)(3, 8)(4, 6).
```

This allows us know that homorphisms of $\phi(x)$, $\phi(y)$, and $\phi(t)$ are $(2,6,4,8,5,3,7)$, $(2,4)(3,5)(7,8)$ and $(1,5)(2,7)(3,8)(4,6)$ respectively. Now we will prove that G is isomorphic to $L_2(7)$ by constructing a homomorphism ϕ from the progenitor $2*^7 : D^{14}$ to $L_2(7)$. Let

$\phi(x) \equiv \frac{5x-1}{x-4} = (2, 6, 4, \infty, 5, 3, 0)$

$\phi(y) \equiv \frac{-6}{x-7} = (2, 4)(3, 5)(0, \infty)$

and we see that $|x| = 7, |y| = 2$, and $|xy| = 2$, thus $N =< x, y >\cong D_{14}$.

We now let $\phi(t_0) \equiv \frac{3x-6}{x-3} = (1, 5)(2, 0)(3, \infty)(4, 6)$.

This is how we resulted in the equations that makes $\phi(x)$ possible.

With the help of MAGMA, we will find out that $\phi(x) = (2, 6, 4, 8, 5, 3, 7)$(note that 8 is ∞ and 7 is 0); this means that that ∞ goes to 5. We will substitute these values into the formula $\frac{ax+b}{cx+d}$

$$\frac{a(\infty) + b}{c(\infty) + d} = 5$$
$$= \frac{a}{c} = 5$$

(3.1)

and 3 goes to 0

$$\frac{a(3) + b}{c(3) + d} = 0$$
$$= 3a + b = 0$$
$$-3a = b$$

(3.2)

4 goes to ∞

$$\frac{a(4) + b}{c(4) + d} = \infty$$
$$= 4c + d = 0$$
$$-4c = d$$

(3.3)

2 goes to 6

$$\frac{a(2) + b}{c(2) + d} = 6$$
$$2a + b = 12c + 6d$$

substitute the following findings, $a = 5c$ and $d = -4c$

(3.4)

$$2(5c) + b = 12c + 6(-4c)$$
$$b = -22c$$

finally to form $\phi(x)$

$$\begin{aligned}
\frac{a(x) + b}{c(4x) + d} &= \frac{(5c)x + (-22c)}{cx + (-4c)} \\
&= \frac{5x - 22}{x - 4} \\
&= \frac{5x - 1}{x - 4}
\end{aligned} \qquad (3.5)$$

Thus, $\phi(x) \equiv \frac{5x-1}{x-4} = (2, 6, 4, \infty, 5, 3, 0)$. We performed similar algebra to find $\phi(y)$ and $\phi(t)$.

We now need to show that $|t_0^N| = 7$

$\phi(t) = \phi(t_0) = (1, 5)(2, 0)(3, \infty)(4, 6)$

$\phi(t_0^x) = \phi(t_1) = (1, 3)(2, \infty))(4, 0)(5, 6)$

$\phi(t_0^{x^2}) = \phi(t_2) = (1, \infty))(2, 6)(3, 4)(5, 0)$

$\phi(t_0^{x^3}) = \phi(t_3) = (1, 2)(3, 5)(4, 6)(0, \infty))$

$\phi(t_0^{x^4}) = \phi(t_4) = (1, 6)(2, 5)(3, \infty))(4, 0)$

$\phi(t_0^{x^5}) = \phi(t_5) = (1, 4)(2, \infty))(3, 6)(5, 0)$

$\phi(t_0^{x^6}) = \phi(t_6) = (1, 0)(2, 6)(3, 5)(4, \infty))$

N permutes the six images of t_0, by conjugation $L_2(7)$ is given by,

$\phi(x) : (t_1, t_2, t_3, t_4, t_5, t_6, t_0)$ and $\phi(y) : (t_1, t_0)(t_2, t_6)(t_3, t_5)$,

thus $\phi(2*^7 : D_{14}) \cong PGL_2(7)$.

Now the relations given by $(xt)^3 = 1$ and $(yt^{x^3})^6 = 1$ where their order are 3 and 6 respectively match the order of the images. Thus, the order of $|\phi(x)\phi(t)| = 3$ and $|\phi(y)\phi(t)^{\phi(x)^3}| = 6$. Therefore, $PGL_2(7)$ is an image of G, so the $|G| \geq |PGL_2(7)|$ but $|G| \leq 336 = |PGL_2(7)|$ and so the equality holds and $G \cong PGL_2(7)$.

3.3 Double Coset Enumeration of $L_2(31)$ Over D_{15}

Let $G = \dfrac{2^{*15} : D_{15}}{(zt)^4, (x^2 z^{-1} t)^{10}, (x^2 z t)^3} \cong 2^\bullet L_2(31)$, where the group $G = 2^{*15} : D_{15}$ and is factored by the relations $(zt)^4, (x^2 z^{-1} t)^{10}, (x^2 z t)^3$; our goal will be to perform a double coset enumeration on G factored by the relators along with the center $Z(G)$. In order to find the center we will refer to MAGMA to aid in the search, the following code is a good starting point:

```
> G<x,y,z,t>:=Group<x,y,z,t|y^2,z^3,(x^-1 * y)^2,(x, z),
> (y * z^-1)^2, x^-5,t^2, (t,y * x * z),(z*t)^4,
> (x^2 * z^-1*t)^10,(x^2 * z*t)^3>;
> Index(G,sub<G|x,y,z>);
992
> f,G1,k:=CosetAction(G,sub<G|x,y,z>);
> CompositionFactors(G1);
    G
    |  A(1, 31)                      = L(2, 31)
    *
    |  Cyclic(2)
    1
> Center(G1);
Permutation group acting on a set of cardinality 992
Order = 2
> C:=Center(G1);
> Order(C.1);
2
> W:=WordGroup(G1);
> rho:=InverseWordMap(G1);
> C1:=C.1;
> C1@rho;
function(W)
    w5 := W.2 * W.1; w6 := w5 * W.4; w7 := w6 * W.3; w8 := w7 * W.4;
    w3 := W.3^-1; w9 := w8 * w3; w10 := w9 * W.4; return w10;
end function
> A:=function(W)
function>  w5 := W.2 * W.1; w6 := w5 * W.4; w7 := w6 * W.3;
function>  w8 := w7 * W.4; w3 := W.3^-1; w9 := w8 * w3;
function>  w10 := w9 * W.4; return w10;
function> end function;
> A(G);
y * x * t * z * t * z^-1 * t
> G<x,y,z,t>:=Group<x,y,z,t|y^2,z^3,(x^-1 * y)^2,(x, z),
> (y * z^-1)^2, x^-5,t^2, (t,y * x * z),(z*t)^4,(x^2 * z^-1*t)^10,
> (x^2 * z*t)^3,y * x * t * z * t * z^-1 * t>;
>
> f,G1,k:=CosetAction(G,sub<G|x,y,z>);
> CompositionFactors(G1);
    G
    |  A(1, 31)                      = L(2, 31)
    1
> #DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
25.
```

In the code above, we are curious to see if $G1$ has a center. Luckily it does. We continue by translating the permutations into words, which produce the center $Z(G) = < yxtztz^{-1}t >$. Lastly the words were added to the end of the progenitor. Now we can continue working with $L_2(31)$ and conveniently enough the number of double cosets was reduced to 25.

We will advance with $G = \dfrac{2^{*15} : D_{15}}{(zt)^4, (x^2z^{-1}t)^{10}, (x^2zt)^3, yxtztz^{-1}t} \cong L_2(31)$ and perform the double coset enumeration. The control group is D_{15}

with $x = (1, 2, 6, 10, 4)(3, 7, 12, 14, 9)(5, 8, 13, 15, 11)$,

$y = (2, 4)(3, 5)(6, 10)(7, 11)(8, 9)(12, 15)(13, 14)$,

$z = (1, 3, 5)(2, 7, 8)(4, 9, 11)(6, 12, 13)(10, 14, 15)$, and $t \sim t_{15}$. Let us compute our three listed relations.

$$[zt]^4 = e$$
$$z^4 t^{z^3} t^{z^2} t^z t = e$$
$$(1, 3, 5)(2, 7, 8)(4, 9, 11)(6, 12, 13)(10, 14, 15)t_{15}t_{14}t_{10}t_{15} = e$$
$$(1, 3, 5)(2, 7, 8)(4, 9, 11)(6, 12, 13)(10, 14, 15)t_{15}t_{14} = t_{15}t_{10}.$$

For the relation $(x^2zt)^3$, to make our work easier,

let $\pi = x^2 z^{-1} = (1, 2, 6, 10, 4)(3, 7, 12, 14, 9)(5, 8, 13, 15, 11)$, such that:

$$[x^2 x^{-1} t]^1 0 = e$$
$$[\pi t]^1 0 = e$$
$$\pi^{10} t^{\pi^9} t^{\pi^8} t^{\pi^7} t^{\pi^6} t^{\pi^5} t^{\pi^4} t^{\pi^3} t^{\pi^2} t^\pi t = e$$
$$e t_8 t_4 t_{12} t_5 t_{10} t_7 t_{11} t_6 t_3 t_{15} = e$$
$$t_8 t_4 t_{12} t_5 t_{10} t_7 t_{11} t_6 = t_{15} t_3.$$

Same as above, for the relation $(x^2zt)^3$,

let $\pi = x^2z = (1, 12, 11, 2, 14, 5, 6, 9, 8, 10, 3, 13, 4, 7, 15)$:

$$(x^2zt)^3 = e$$
$$\pi^3 = e$$
$$\pi^3 t^{\pi^2} t^\pi t = e$$
$$(1, 2, 6, 10, 4)(3, 7, 12, 14, 9)(5, 8, 13, 15, 11)t_{12}t_1t_{15} = e$$
$$(1, 2, 6, 10, 4)(3, 7, 12, 14, 9)(5, 8, 13, 15, 11)t_{12} = t_{15}t_1$$

Now we may continue to perform the double coset enumerations of G over N.

In order to express G as a union of double cosets, we will interpret it as NgN, where g is an element of G; such that $G = NeN \cup Ng_1N \cup Ng_2N...$, where $g_i's$ are words in the $t_i's$. Our objective is to find the number of single cosets contained in each double coset $[w]$ defined as $[w] = \{Nw^n | n \in N\}$. We will know when we are done with our double coset enumeration when the set of right cosets is closed under right multiplication, meaning all of our potentially new double cosets are already recognized. We declare each double coset $[w]$ where Mwt_i will belong to one of the symmetric generators of t_i from each of the orbits of the coset stabiliser group; defined as $N^{(w)} = \{n \in N | Nw^n = Nw\}$, and where each w is a word of $t_i's$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

N e N

We begin with the double coset $NeN = \{Ne^n | n \in N\}$, labeled $[*]$, and contains one single coset, N. The coset stabilizer of N is the control group N and thus has a single orbit of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ of which we will perform right multiplication. Any element of the orbit can be the representative, we choose the orbit representative of 1. This means that $Net_{15} = Nt_{15}N$ is a new double coset $[0]$ and since there is 15 single cosets in the orbit of $[*]$, 15 single cosets will move forward to the coset $[0]$. The reason being, that in this particular group, I chose to stabilize t_{15}, when bulding my progenitor, as a result, we will let $t \sim t_{15}$.

Nt$_0$N

The second double coset will be represented as $[0]$. The coset stabilizing group will be generated by $N^{(0)} =< (1,7)(2,3)(4,12)(5,8)(6,9)(10,14)(11,13) >$ such that $\dfrac{|N|}{|N^{(0)}|} = \dfrac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0]$. The orbits of $N^{(0)}$ are $\{15\}, \{1,7\}, \{2,3\}, \{4,12\}, \{5,8\}, \{6,9\}, \{10,4\}$ and $\{11,13\}$. We will then pick a representative from each orbit and right multiply, Nt_0, consider the following relations:

$$t_0 t_0 \in [0] \text{ (goes back to } [*]),$$
$$t_0 t_1 = x[t_0]^y \text{ (loops back to } [0]),$$
$$t_0 t_2 \in [0,2],$$
$$t_0 t_4 \in [0,4],$$
$$t_0 t_5 = yx^{-1}z^{-1}[t_0 t_2]^y x,$$
$$t_0 t_6 \in [0,6],$$
$$t_0 t_1 0 = yxz^{-1}[t_0]^{yxz} \text{ (loops back to } [0]),$$
$$t_0 t_{11} \in [0,11].$$

Thus the new double cosets are $[0,2], [0,4], [0,6]$ and $[0,11]$, where four single cosets continue onto $[0,2]$, 2 single cosets continue onto $[0,4], [0,6]$ and $[0,11]$ each. Moving forward, we will continue by looking into the double coset $[0,2]$

Nt$_0$t$_2$N

The double coset $Nt_0 t_2$ is labeled $[0,2]$. The coset stabilizing group will be generated by $N^{(0,2)} =< (2,4)(3,5)(6,10)(7,11)(8,9)(12,15)(13,14) >$ such that $\dfrac{|N|}{|N^{(0,2)}|} = \dfrac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0,2]$. The orbits of $N^{(0,2)}$ are $\{1\}, \{2,4\}, \{3,5\}, \{6,10\}, \{7,11\}, \{8,9\}, \{12,15\}$ and $\{13,14\}$.

We will then pick a representative from each orbit and right multiply, Nt_0t_2, consider the following relations:

$$t_0t_2t_1 \in [0, 2, 1],$$
$$t_0t_2t_4 = x^2z^{-1}[t_0]^y \text{ (goes back to } [0]),$$
$$t_0t_2t_3 \in [0, 2, 3],$$
$$t_0t_2t_6 \in [0, 2, 6],$$
$$t_0t_2t_7 = yzx^{-2}[t_0]^{yx},$$
$$t_0t_2t_8 = x^{-1}z^{-1}[t_0t_2]^{yx^2z} \text{ (loops back to } [0,2]),$$
$$t_0t_2t_{12} = yx^{-1}[t_0t_6]^{x^{-2}},$$
$$t_0t_2t_{14} = x[t_0t_{11}]^x.$$

Thus the new double cosets are $[0, 2, 1], [0, 2, 3], [0, 2, 6]$ where 2 single cosets continued onto each of the new double cosets, and 2 single cosets continue onto $[0, 6]$ and $[0, 11]$, each. Moving forward, we will continue by looking into the double coset $[0, 4]$

$\mathbf{Nt_0t_4N}$

The double coset Nt_0t_4 is labeled $[0, 4]$. The coset stabilizing group will be generated by $N^{(0,4)} = < (1, 8)(2, 5)(3, 7)(4, 13)(6, 11)(9, 12)(10, 15) >$ such that $\dfrac{|N|}{|N^{(0,4)}|} = \dfrac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 4]$. The orbits of $N^{(0,4)}$ are $\{14\}, \{1, 8\}, \{2, 5\}, \{3, 7\}, \{4, 13\}, \{6, 11\}, \{9, 12\}$ and $\{10, 15\}$. We will then pick a representative from each orbit and right multiply it to Nt_0t_4 consider the following relations:

$$t_0 t_4 t_{14} = yx^{-2}[t_0 t_4 t_2]^x,$$

$$t_0 t_4 t_1 \in [0, 4, 1],$$

$$t_0 t_4 t_2 \in [0, 4, 2],$$

$$t_0 t_4 t_3 = x^2 [t_0 t_{11}]^{zx^{-1}},$$

$$t_0 t_4 t_4 = [0] \text{ (goes back to } [0]),$$

$$t_0 t_4 t_6 = yxz^{-1}[t_0 t_{11}]^{x^{-2}},$$

$$t_0 t_4 t_9 = yzx^{-2}[t_0 t_6]^{x^2},$$

$$t_0 t_4 t_{10} \in [0, 4, 10].$$

Thus the new double cosets are $[0, 4, 2], [0, 4, 1], [0, 4, 10]$ where 2 single cosets continued to $[0, 4, 1]$ and $[0, 4, 10]$, and 3 continue to $[0, 4, 2]$. Also, 4 single cosets move forward to $[0, 11]$ and 2 go back to $[0]$.

We will continue by looking into the double coset $[0, 6]$

$\mathbf{N t_0 t_6 N}$

The double coset $N t_0 t_6$ is labeled $[0, 6]$. The coset stabilizing group will be generated by $N^{(0,6)} = < Id(N) >$ such that $\dfrac{|N|}{|N^{(0,6)}|} = \dfrac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0, 6]$. The orbits of $N^{(0,6)}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\},$ $\{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\},$ and $\{15\}$. We will then pick a representative from each orbit and right multiply it to $N t_0 t_6$, consider the following relations:

$$t_0 t_6 t_1 = yxz[t_0 t_4 t_1]^{yx^{-2}z^{-1}},$$

$$t_0 t_6 t_2 \in [0, 6, 2],$$

$$t_0 t_6 t_3 = z^{-1}[t_0 t_4 t_1]^{yzx^{-2}},$$

$$t_0 t_6 t_4 = xz^{-1}[t_0 t_2 t_6]^{x^{-1}z^{-1}},$$

$$t_0 t_6 t_5 = x^{-1}[t_0 t_6]^{yz},$$

$$t_0 t_6 t_6 = t_0 \text{ (goes back to } [0]),$$

$$t_0 t_6 t_7 \in [0, 6, 7],$$

$$t_0 t_6 t_8 = yxz^{-1}[t_0 t_2 t_1]^{zx^{-2}},$$

$$t_0 t_6 t_9 = yx^{-2}[t_0 t_2]^{x^2},$$

$$t_0 t_6 t_{10} \in [0, 6, 10],$$

$$t_0 t_6 t_{11} = zx^{-1}[t_0 t_{11}]^{x^2},$$

$$t_0 t_6 t_{12} = yx^2 z[t_0 t_4]^{yx^{-1}z^{-1}},$$

$$t_0 t_6 t_{13} = yx^{-1}z^{-1}[t_0 t_{11}]^{zx^{-2}},$$

$$t_0 t_6 t_{14} = x^{-2}z^{-1}[t_0 t_2 t_1]^{x^2 z^{-1}},$$

$$t_0 t_6 t_{15} \in [0, 6, 0].$$

Thus the new double cosets are $[0, 6, 2], [0, 6, 7], [0, 6, 10]$ and $[0, 6, 0]$. Also, 1 single coset continue to $[0, 2, 6], [0, 6], [0, 2]$ and $[0, 4]$ each, and 2 continue to $[0, 4, 1], [0, 2, 1]$ and $[0, 11]$ each. And one single coset goes back to $[0]$. We will continue by looking into the double coset $[0, 11]$

$\mathbf{N t_0 t_{11} N}$

The double coset $N t_0 t_{11}$ is labeled $[0, 11]$. The coset stabilizing group will be generated by $N^{(0,11)} = <Id(N)>$ such that $\frac{|N|}{|N^{(0,11)}|} = \frac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0, 11]$. The orbits of $N^{(0,11)}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\},$ $\{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\},$ and $\{15\}$. We will then pick a represen-

tative from each orbit and right multiply it to Nt_0t_{11}, consider the following relations:

$$t_0t_{11}t_1 = z^{-1}[t_0t_2t_3]^{yx^{-2}},$$

$$t_0t_{11}t_2 = x^{-2}[t_0t_4]^{xz^{-1}},$$

$$t_0t_{11}t_3 = z^{-1}[t_0t_6t_2]^{x^2},$$

$$t_0t_{11}t_4 = yx^{-2}z^{-1}[t_0t_4]^{yx^{-2}z^{-1}},$$

$$t_0t_{11}t_5 \in [0, 11, 5],$$

$$t_0t_{11}t_6 = yx^{-2}z^{-1}[t_0t_2t_6]^z,$$

$$t_0t_{11}t_7 = yx^{-1}z^{-1}[t_0t_4t_2]^{yx^{-1}z^{-1}},$$

$$t_0t_{11}t_8 = yx^{-1}z^{-1}[t_0t_4t_1]^{yx},$$

$$t_0t_{11}t_9 = yx^{-2}[t_0t_6]^{x^2z^{-1}},$$

$$t_0t_{11}t_{10} \in [0, 11, 10],$$

$$t_0t_{11}t_{11}[t_0t_{11}t_{11}] \text{ (goes back to } [0]),$$

$$t_0t_{11}t_{12} = xz^-1[t_0t_2]^{yx^{-1}},$$

$$t_0t_{11}t_{13} = xz^{-1}[t_0t_6]^{x^{-2}},$$

$$t_0t_{11}t_{14} = x^{-1}[t_0t_4t_{10}]^{x^2z^{-1}},$$

$$t_0t_{11}t_{15} \in [0, 11, 15].$$

As we can see, one single coset continued onto the new double cosets, $[0, 11, 5], [0, 11, 10]$, and $[0, 11, 0]$, each. Also, 1 single coset continue to $[0, 2, 3]$, $[0, 6, 2], [0, 2, 6], [0, 4, 2]$, $[0, 4, 1], [0, 4, 10]$ and $[0, 2]$, and 2 continue to $[0, 4]$ and $[0, 6]$. And one single coset goes back to $[0]$. Moving forward, we will continue by looking into the double coset $[0, 2, 1]$

$Nt_0t_2t_1N$

The double coset $Nt_0t_2t_1$ is labeled $[0, 2, 1]$. The coset stabilizing group will be generated by $N^{(0,2,1)} =< (2, 4)(3, 5)(6, 10)(7, 11)(8, 9)(12, 15)(13, 14) >$ such that $\frac{|N|}{|N^{(0,2,1)}|} = \frac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 2, 1]$. The orbits of $N^{(0,2,1)}$ are $\{1\}, \{2, 4\}, \{3, 5\}, \{6, 10\}, \{7, 11\}, \{8, 9\}, \{12, 15\}$ and

$\{13, 14\}$.

We will then pick a representative from each orbit and right multiply it to $Nt_0t_2t_1$, consider the following relations:

$$t_0t_2t_1t_1 = [t_0t_2] \text{ (goes back to } [0,2]),$$

$$t_0t_2t_1t_2 = yz[t_0t_6t_0]^{yzx^{-1}},$$

$$t_0t_2t_1t_3 = yz[t_0t_2t_3]^{yz},$$

$$t_0t_2t_1t_6 = zx^{-1}[t_0t_6t_2]^{yxz^{-1}},$$

$$t_0t_2t_1t_7 = yx^2[t_0t_6t_{10}]^x,$$

$$t_0t_2t_1t_8 = x^2z[t_0t_6]^{z^{-2}},$$

$$t_0t_2t_1t_{12} = yz[t_0t_6]^{yx^{-2}},$$

$$t_0t_2t_1t_{13} = x^{-1}z^{-1}[t_0t_6t_7]^x.$$

As a result, one single coset goes back to $[0, 2]$. While two single cosets advance to each of the following double cosets: $[0, 6, 15], [0, 2, 3], [0, 6, 2], [0, 6, 10], [0, 6, 7]$, and $[0, 6]$.

$\mathbf{Nt_0t_2t_3N}$

The double coset $Nt_0t_2t_3$ is labeled $[0, 2, 3]$. The coset stabilizing group will be generated by $N^{(0,2,3)} = < Id(G) >$ such that $\frac{|N|}{|N^{(0,2,3)}|} = \frac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0, 2, 3]$. The orbits of $N^{(0,2,3)}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}$, and $\{15\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_2t_3$, consider the following relations:

$$t_0 t_2 t_3 t_1 = yx^{-2}z^{-1}[t_0 t_2 t_1]^z,$$

$$t_0 t_2 t_3 t_2 = z[t_0 t_6 t_0]^{yzx^{-2}},$$

$$t_0 t_2 t_3 t_3 = [t_0 t_2] \text{ (goes back to } [0,2]),$$

$$t_0 t_2 t_3 t_4 \in [0,2,3,4],$$

$$t_0 t_2 t_3 t_5 = yzx^{-2}[t_0 t_2 t_3]^e,$$

$$t_0 t_2 t_3 t_6 = zx^{-1}[t_0 t_{11} t_5]^{zx^{-1}},$$

$$t_0 t_2 t_3 t_7 \in [0,2,3,7],$$

$$t_0 t_2 t_3 t_8 = x^{-1}z^{-1}[t_0 t_{11} t_0]^{yx^2},$$

$$t_0 t_2 t_3 t_9 = yx^2 z[t_0 t_2 t_3 t_7]^{xz},$$

$$t_0 t_2 t_3 t_{10} = z^{-1}[t_0 t_{11}]^{yx^{-2}},$$

$$t_0 t_2 t_3 t_{11} = zx^{-2}[t_0 t_6 t_7]^{yxz},$$

$$t_0 t_2 t_3 t_{12} = yx^2 z^{-1}[t_0 t_6 t_7]^{yz},$$

$$t_0 t_2 t_3 t_{13} = x^{-2}z^{-1}[t_0 t_2 t_6]^{yx},$$

$$t_0 t_2 t_3 t_{14} = yxz^{-1}[t_0 t_{11} t_5]^{zx^{-2}},$$

$$t_0 t_2 t_3 t_{15} = yxz^{-1}[t_0 t_6 t_2]^{z^{-1}}.$$

Thus, one single coset goes back to $[0,2]$ and one loops back to $[0,2,3]$. While two single cosets advance to the double cosets $[0,6,15], [0,2,3], [0,6,2], [0,6,10]$, and one single coset advance to the double cosets $[0,2,1], [0,6,0], [0,11,0], [0,11], [0,2,6]$ and $[0,6,2]$. The new double cosets are $[0,2,3,4]$ and $[0,2,3,7]$, and one single coset continued onto each.

$Nt_0t_2t_6N$

The double coset $Nt_0 t_2 t_6$ is labeled $[0,2,6]$. The coset stabilizing group will be generated by $N^{(0,2,6)} = < (1,13)(2,8)(3,12)(4,15)(5,6)(9,14)(10,11) >$ such that $\frac{|N|}{|N^{(0,2,6)}|} = \frac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset

$[0, 2, 6]$. The orbits of $N^{(0,2,6)}$ are $\{7\}, \{1, 13\}, \{2, 8\}, \{3, 12\}, \{4, 15\}, \{5, 6\}, \{9, 14\}$ and $\{10, 11\}$. We will then pick a representative from each orbit and right multiply it to $N t_0 t_2 t_6$, consider the following relations:

$$t_0 t_2 t_6 t_1 = y x^2 z^{-1} [t_0 t_{11}]^{y x^2}$$

$$t_0 t_2 t_6 t_2 = e[t_0 t_2 t_3 t_4]^{y x z^{-1}}$$

$$t_0 t_2 t_6 t_3 = z x^{-1} [t_0 t_6]^{x z}$$

$$t_0 t_2 t_6 t_4 = y x^{-1} [t_0 t_{11} t_5]^{x^{-1}}$$

$$t_0 t_2 t_6 t_6 = [t_0 t_2] \text{ (goes back to [0,2]),}$$

$$t_0 t_2 t_6 t_7 \in [0, 2, 6, 7],$$

$$t_0 t_2 t_6 t_9 = x^{-2} z^{-1} [t_0 t_2 t_3]^{y x}$$

$$t_0 t_2 t_6 t_{10} = z x^{-2} [t_0 t_6 t_7]^{z}.$$

Resulting in two single cosets goes back to $[0, 2]$, there is one new double coset $[0, 2, 6, 7]$. While two single cosets advance to the double cosets $[0, 2, 3, 4], [0, 6], [0, 6, 2], [0, 11, 5], [0, 2, 3]$.

$\mathbf{N t_0 t_4 t_1 N}$

The double coset $N t_0 t_4 t_1$ is labeled $[0, 4, 1]$. The coset stabilizing group will be generated by $N^{(0,4,1)} = < Id(G) >$ such that $\dfrac{|N|}{|N^{(0,4,1)}|} = \dfrac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0, 4, 1]$. The orbits of $N^{(0,4,1)}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}$, and $\{15\}$. We will then pick a representative from each orbit and right multiply it to $N t_0 t_4 t_1$, consider the

following relations:

$$t_0t_4t_1t_1 = [t_0t_4] \text{ (goes back to } [0,4]),$$

$$t_0t_4t_1t_2 = yx^{-2}z^{-1}[t_0t_2t_6]^{yxz^{-1}},$$

$$t_0t_4t_1t_3 = yzx^{-2}[t_0t_{11}]^{yx},$$

$$t_0t_4t_1t_4 = yx[t_0t_4t_{10}]^{yxz},$$

$$t_0t_4t_1t_5 = yx^2z^{-1}[t_0t_4t_2]^{x^{-1}},$$

$$t_0t_4t_1t_6 = yxz[t_0t_4t_2t_3]^{zx^{-1}},$$

$$t_0t_4t_1t_7 = yx^{-1}[t_0t_6t_0]^{yx^2z^{-1}},$$

$$t_0t_4t_1t_8 = x^{-1}z^{-1}[t_0t_6t_{10}]^{yx^2z},$$

$$t_0t_4t_1t_9 \in [0,4,1,9],$$

$$t_0t_4t_1t_{10} = z^{-1}[t_0t_6]^{yzx^{-2}},$$

$$t_0t_4t_1t_{11} \in [0,4,1,11],$$

$$t_0t_4t_1t_{12} = x^{-1}[t_0t_4t_{10}]^{yx^2z^{-1}},$$

$$t_0t_4t_1t_{13} = yzx^{-1}[t_0t_{11}t_{10}]^{yx^2},$$

$$t_0t_4t_1t_{14} = xz^{-1}[t_0t_4t_1]^{yxz^{-1}},$$

$$t_0t_4t_1t_{15} = y[t_0t_6]^{yx^{-2}z^{-1}}.$$

Thus we have two new double cosets $[0,4,1,9]$ and $[0,4,1,11]$ each with one single coset continuing to it. There is also one single coset going back to the double coset $[0,4]$ while one loops back to $[0,4,1]$. Also, one single coset moves forward to the double cosets $[0,2,6,7]$, $[0,11]$, $[0,4,2]$, $[0,4,2,3]$, $[0,6,0]$, $[0,6,10]$, and $[0,11,10]$, while two single coset continue to $[0,6]$ and $[0,4,10]$. Next, we will look into the double coset $Nt_0t_4t_{10}$.

$Nt_0t_4t_{1}0N$

The double coset $Nt_0t_4t_{10}$ is labeled $[0,4,10]$. The coset stabilizing group will be generated by $N^{(0,4,10)} = < Id(G) >$ such that $\frac{|N|}{|N^{(0,4,10)}|} = \frac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0,4,10]$. The orbits of $N^{(0,4,10)}$ are

$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\},$ and $\{15\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_4t_{10}$, consider the following relations:

$$t_0t_4t_{10}t_1 = zx^{-1}[t_0t_{11}t_{10}]^{yz^{-1}},$$

$$t_0t_4t_{10}t_2 = yxz^{-1}[t_0t_{11}t_0]^{yzx^{-1}},$$

$$t_0t_4t_{10}t_3 = x^{-1}[t_0t_4t_1]^{yx^2z^{-1}},$$

$$t_0t_4t_{10}t_4 = yx^{-1}z^{-1}[t_0t_2t_3t_7]^{yx},$$

$$t_0t_4t_{10}t_5 = xz^{-1}[t_0t_4t_2t_3]^{yx^2},$$

$$t_0t_4t_{10}t_6 = x^{-2}z^{-1}[t_0t_4t_1t_9]^{x^{-2}},$$

$$t_0t_4t_{10}t_7 = yzx^{-2}[t_0t_{11}t_{10}]^{x^2z},$$

$$t_0t_4t_{10}t_8 = x[t_0t_{11}]^{zx^{-2}},$$

$$t_0t_4t_{10}t_9 = e[t_0t_{11}t_5]^{x^2z},$$

$$t_0t_4t_{10}t_{10} = [t_0t_4] \text{ (goes back to } [0{,}4]),$$

$$t_0t_4t_{10}t_{11} = x^{-2}[t_0t_6t_0]^{z^{-1}},$$

$$t_0t_4t_{10}t_{12} = yxz^{-1}[t_0t_4t_1]^{yxz},$$

$$t_0t_4t_{10}t_{13} = zx^{-2}[t_0t_4t_1t_{11}]^{x^{-2}z^{-1}},$$

$$t_0t_4t_{10}t_{14} = zx^{-2}[t_0t_4t_2]^{yz},$$

$$t_0t_4t_{10}t_{15} = yx^{-2}z^{-1}[t_0t_4t_{10}]^e.$$

Moving forward, we see that there is one single coset going back to the double coset $[0, 4]$ while one loops back to $[0, 4, 10]$. Also, one single coset moves forward to the double cosets $[0, 11, 0]$, $[0, 2, 3, 7], [0, 4, 2, 3], [0, 4, 1, 9], [0, 11], [0, 11, 5], [0, 6, 0], [0, 4, 1, 11]$ and $[0, 4, 2]$, while two single cosets continue to $[0, 11, 10]$. Next, we will look into the double coset $Nt_0t_6t_2$.

$\mathbf{Nt_0t_6t_2N}$

The double coset $Nt_0t_6t_2$ is labeled $[0, 6, 2]$. The coset stabilizing group will

be generated by $N^{(0,6,2)} =< (1,3)(2,9)(4,7)(6,14)(8,11)(10,12)(13,15) >$ such that $\frac{|N|}{|N^{(0,6,2)}|} = \frac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0,6,2]$. The orbits of $N^{(0,6,2)}$ are $\{5\}, \{1,3\}, \{2,9\}, \{4,7\}, \{6,14\}, \{8,11\}, \{13,15\}$ and $\{10,12\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_6t_2$, consider the following relations:

$$t_0t_6t_2t_1 = yx^{-2}[t_0t_2t_3t_7]^{yx^{-1}},$$

$$t_0t_6t_2t_2 = [t_0t_6] \ (\text{ goes back to } [0,6]),$$

$$t_0t_6t_2t_4 = yx^{-2}[t_0t_6t_7]^{x^{-2}},$$

$$t_0t_6t_2t_5 = xz[t_0t_{11}t_{10}]^{x^{-1}},$$

$$t_0t_6t_2t_6 = e[t_0t_{11}]^{yx^2z},$$

$$t_0t_6t_2t_8 = x^{-2}z^{-1}[t_0t_2t_1]^{yx^{-1}z^{-1}},$$

$$t_0t_6t_2t_{10} = yxz[t_0t_2t_3]^z,$$

$$t_0t_6t_2t_{13} = yx^{-1}z^{-1}[t_0t_{11}t_5]^{x^2z}.$$

Hence, there is two single cosets going back to the double coset $[0,6]$ Also, one single coset moves forward to the double coset $[0,11,10]$, while two single cosets continue to $[0,11,5], [0,2,3,7], [0,6,7], [0,11], [0,2,1]$, and $[0,2,3]$. Next, we will look into the double coset $Nt_0t_6t_7$.

$Nt_0t_6t_7N$

The double coset $Nt_0t_6t_7$ is labeled $[0,6,7]$. The coset stabilizing group will be generated by $N^{(0,6,7)} =< Id(G) >$ such that $\frac{|N|}{|N^{(0,6,7)}|} = \frac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0,6,7]$. The orbits of $N^{(0,6,7)}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}$, and $\{15\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_6t_7$, consider the

following relations:

$$t_0 t_6 t_7 t_1 = x^2 z [t_0 t_2 t_3 t_4]^{yx^{-2}z^{-1}},$$

$$t_0 t_6 t_7 t_2 = yx^2 z [t_0 t_6 t_2]^{yx^2 z},$$

$$t_0 t_6 t_7 t_3 = yxz^{-1} [t_0 t_6 t_7]^e,$$

$$t_0 t_6 t_7 t_4 = x^{-1} [t_0 t_6 t_0]^{x^{-1}},$$

$$t_0 t_6 t_7 t_5 = yx^{-1} z^{-1} [t_0 t_4 t_1 t_1 1]^y,$$

$$t_0 t_6 t_7 t_6 = x^2 z^{-1} [t_0 t_2 t_3 t_7]^{x^{-1}z^{-1}},$$

$$t_0 t_6 t_7 t_7 = [t_0 t_6] \text{ (goes back to } [0,6]\text{)},$$

$$t_0 t_6 t_7 t_8 = x^{-2} [t_0 t_2 t_1]^{yx^{-1}},$$

$$t_0 t_6 t_7 t_9 = yx^{-1} z^{-1} [t_0 t_6 t_0]^{x^{-1}z^{-1}},$$

$$t_0 t_6 t_7 t_{10} = yx^{-2} [t_0 t_2 t_3]^{yz},$$

$$t_0 t_6 t_7 t_{11} = zx^{-2} [t_0 t_6 t_7]^{yzx^{-2}},$$

$$t_0 t_6 t_7 t_{12} = yxz^{-1} [t_0 t_2 t_3 t_4]^{yx^{-1}z^{-1}},$$

$$t_0 t_6 t_7 t_{13} = zx^{-2} [t_0 t_2 t_3]^{yxz},$$

$$t_0 t_6 t_7 t_{14} = x^2 z^{-1} [t_0 t_{11} t_5]^z,$$

$$t_0 t_6 t_7 t_{15} = x^2 z^{-1} [t_0 t_2 t_6]^{z^{-1}}.$$

Thus, there is one single coset going back to the double coset $[0,6]$ Also, one single coset moves forward to each of the double cosets $[0,6,2], [0,4,1,11], [0,2,3,7], [0,2,1], [0,11,5]$ and $[0,2,6]$ while two single cosets continue to $[0,2,3,4], [0,6,7], [0,6,0]$, and $[0,2,3]$. Next, we will look into the double coset $Nt_0 t_6 t_{15}$.

$Nt_0 t_6 t_0 N$

The double coset $Nt_0 t_6 t_{15}$ is labeled $[0,6,0]$. The coset stabilizing group will be generated by $N^{(0,6,0)} = < Id(G) >$ such that $\frac{|N|}{|N^{(0,6,0)}|} = \frac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0,6,0]$. The orbits of $N^{(0,6,0)}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}$, and $\{15\}$. We will

then pick a representative from each orbit and right multiply it to $Nt_0t_6t_{15}$, consider the following relations:

$$t_0t_6t_0t_1 = x[t_0t_6t_7]^x \quad t_0t_6t_0t_2 \qquad\qquad = x^{-2}[t_0t_6t_{10}]^{zx^{-1}},$$

$$t_0t_6t_0t_3 = z^{-1}[t_0t_6t_0]^{yz^{-1}},$$

$$t_0t_6t_0t_4 = x^2[t_0t_4t_{10}]^z,$$

$$t_0t_6t_0t_5 = yxz[t_0t_6t_7]^{xz},$$

$$t_0t_6t_0t_6 = yx^{-1}[t_0t_2t_3t_7]^x,$$

$$t_0t_6t_0t_7 = yz[t_0t_4t_1]^{yx^2z^{-1}},$$

$$t_0t_6t_0t_8 = yz^{-1}[t_0t_2t_6t_7]^{yx},$$

$$t_0t_6t_0t_9 = x[t_0t_2t_3t_7]^{x^2},$$

$$t_0t_6t_0t_{10} = yx^{-2}[t_0t_6t_{10}]^{x^{-2}},$$

$$t_0t_6t_0t_{11} = yxz^{-1}[t_0t_{11}t_5]^{yx},$$

$$t_0t_6t_0t_{12} = z[t_0t_2t_3]^{yzx^{-2}},$$

$$t_0t_6t_0t_{13} = yzx^{-2}[t_0t_{11}t_0]^{yx^{-2}z^{-1}},$$

$$t_0t_6t_0t_{14} = yzx^{-2}[t_0t_2t_1]^{yzx^{-1}},$$

$$t_0t_6t_0t_{15} = [t_0t_6] \text{ (goes back to [0,6]).}$$

In summary, there is one single coset going back to the double coset $[0,6]$ and also one single coset moves forward to each of the double cosets $[0,6,0]$, $[0,4,1]$, $[0,4,10]$, $[0,2,6,7]$, $[0,11,5]$, $[0,2,3]$, $[0,11,0]$ and $[0,2,1]$. While two single cosets continue to $[0,2,3,7]$, $[0,6,7]$, and $[0,6,10]$. Next, we will look into the double coset $Nt_0t_6t_{10}$.

$Nt_0t_6t_{10}N$

The double coset $Nt_0t_6t_{10}$ is labeled $[0,6,10]$. The coset stabilizing group will be generated by $N^{(0,6,10)} = <(1,15)(2,13)(3,14)(4,11)(5,10)(6,8)(7,12)>$ such that $\frac{|N|}{|N^{(0,6,10)}|} = \frac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0,6,10]$. The orbits of $N^{(0,6,10)}$ are $\{9\}, \{1,15\}, \{2,13\}, \{3,14\}, \{4,11\}, \{5,10\}, \{6,8\}$

and $\{7, 12\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_6t_{10}$, consider the following relations:

$$t_0t_6t_{10}t_9 = x^{-2}z^{-1}[t_0t_4t_1t_9]^{yxz},$$

$$t_0t_6t_{10}t_1 = yx^2[t_0t_6t_0]^{x^2},$$

$$t_0t_6t_{10}t_2 = xz^{-1}[t_0t_6t_0]^{yx^2},$$

$$t_0t_6t_{10}t_3 = yx^2z^{-1}[t_0t_2t_1]^{yx^{-1}},$$

$$t_0t_6t_{10}t_4 = yx^2[t_0t_{11}t_5]^{yx^{-2}z^{-1}},$$

$$t_0t_6t_{10}t_{10} = [t_0t_6] \text{ (goes back to [0,6]),}$$

$$t_0t_6t_{10}t_6 = yx^2[t_0t_{11}t_5]^{yx^{-2}z^{-1}},$$

$$t_0t_6t_{10}t_7 = x^{-1}z^{-1}[t_0t_2t_6t_7]^{yxz}$$

.

Thus there is two single cosets going back to the double coset $[0, 6]$ and also two single cosets moves forward to each of the double cosets $[0, 6, 7], [0, 2, 1], [0, 11, 5], [0, 4, 1]$, and four progressed to $[0, 6, 0]$. While one single coset continue to $[0, 4, 1, 9]$. Next, we will look into the double coset $Nt_0t_{11}t_{15}$.

$\mathbf{Nt_0t_{11}t_0N}$

The double coset $Nt_0t_{11}t_{15}$ is labeled $[0, 11, 0]$. The coset stabilizing group will be generated by $N^{(0,11,0)} =< (1,5)(2,11)(4,8)(6,15)(7,9)(10,13)(12,14) >$ such that $\dfrac{|N|}{|N^{(0,11,0)}|} = \dfrac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 11, 0]$. The orbits of $N^{(0,11,0)}$ are $\{3\}, \{1, 5\}, \{2, 11\}, \{4, 8\}, \{6, 15\}, \{7, 9\}, \{10, 13\}$ and $\{12, 14\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_{11}t_{15}$, consider the following relations:

$$t_0t_{11}t_0t_3 = z^{-1}[t_0t_4t_2t_3]^{z^{-1}},$$

$$t_0t_{11}t_0t_1 = yx^2z[t_0t_4t_2]^x,$$

$$t_0t_{11}t_0t_2 = yx^{-2}[t_0t_6t_0]^{yx^{-2}z^{-1}},$$

$$t_0t_{11}t_0t_4 = yx^2[t_0t_2t_3t_7]^{yz^{-1}},$$

$$t_0t_{11}t_0t_7 = x^{-1}z^{-1}[t_0t_2t_3]^{yx^2}$$

$$t_0t_{11}t_0t_{10} = yx^{-2}[t_0t_{11}t_{10}]^{zx^{-1}},$$

$$t_0t_{11}t_0t_{12} = yzx^{-2}[t_0t_4t_{10}]^{xz}$$

$$t_0t_{11}t_{15}t_{15} = [t_0t_{11}] \text{ (goes back to } [0,11]).$$

We see that there are two single cosets going back to the double coset $[0, 11]$. Also two single cosets moves forward to each of the double cosets $[0, 4, 2], [0, 6, 0], [0, 2, 3, 7], [0, 2, 3],$ $[0, 11, 10], [0, 4, 0]$, and one progressed to $[0, 4, 2, 3]$. Next, we will investigate the double coset $Nt_0t_{11}t_{10}$.

$Nt_0t_{11}t_{10}N$

The double coset $Nt_0t_{11}t_{10}$ is labeled $[0, 11, 10]$. The coset stabilizing group will be generated by $N^{(0,11,10)} =< (1, 12)(2, 7)(3, 6)(4, 14)(5, 13)(9, 10)(11, 15) >$ such that $\dfrac{|N|}{|N^{(0,11,10)}|} = \dfrac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 11, 0]$. The orbits of $N^{(0,11,10)}$ are $\{8\}, \{1, 12\}, \{2, 7\}, \{3, 6\}, \{4, 14\}, \{5, 13\}, \{9, 10\}$ and $\{11, 15\}$. We will then pick a representative from each orbit and right multiply it

to $Nt_0t_{11}t_{10}$, consider the following relations:

$$t_0t_{11}t_{10}t_8 = x^{-1}z^{-1}[t_0t_6t_2]^x,$$

$$t_0t_{11}t_{10}t_1 = yx^2z[t_0t_4t_2t_3]^{x^2z^{-1}},$$

$$t_0t_{11}t_{10}t_2 = xz[t_0t_2t_3t_7]^{x^2z^{-1}},$$

$$t_0t_{11}t_{10}t_3 = yz^{-1}[t_0t_4t_1]^{yx^2},$$

$$t_0t_{11}t_{10}t_4 = yx^{-1}z^{-1}[t_0t_4t_{10}]x^{-2}z^{-1},$$

$$t_0t_{11}t_{10}t_5 = zx^{-1}[t_0t_4t_{10}]^{yz^{-1}},$$

$$t_0t_{11}t_{10}t_{10} = [t_0t_11] \text{ (goes back to } [0,11]),$$

$$t_0t_{11}t_{10}t_{11} = yz[t_0t_{11}t_0]^{yxz}.$$

Thus, there are two single cosets going back to the double coset $[0, 11]$. Also two single cosets moves forward to each of the double cosets $[0, 4, 2, 3], [0, 2, 3, 7], [0, 4, 1], [0, 11, 0]$, and four progressed to $[0, 4, 10]$; while one progressed to $[0, 6, 2]$. Next, we will investigate the double coset $Nt_0t_{11}t_5$.

$\mathbf{Nt_0t_{11}t_5N}$

The double coset $Nt_0t_{11}t_5$ is labeled $[0, 11, 5]$. The coset stabilizing group will be generated by $N^{(0,11,5)} = < Id(G) >$ such that $\frac{|N|}{|N^{(0,11,5)}|} = \frac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0, 11, 5]$. The orbits of $N^{(0,11,5)}$ are $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\},$ and $\{15\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_{11}t_5$, consider

the following relations:

$$t_0 t_{11} t_5 t_1 = y[t_0 t_2 t_3]^{x^2 z^{-1}},$$

$$t_0 t_{11} t_5 t_2 = yxz^{-1}[t_0 t_2 t_6 t_7]^{zx^{-2}},$$

$$t_0 t_{11} t_5 t_3 = yz[t_0 t_6 t]2^{yx^{-2}},$$

$$t_0 t_{11} t_5 t_4 = yx^2 z[t_0 t_{11} t_5]^e,$$

$$t_0 t_{11} t_5 t_5 = [t_0 t_{11}] \text{ (goes back to } [0,11]),$$

$$t_0 t_{11} t_5 t_6 = e[t_0 t_4 t_{10}]x^{-2} z^{-1},$$

$$t_0 t_{11} t_5 t_7 = yx^2[t_0 t_2 t_3 t_7]^x,$$

$$t_0 t_{11} t_5 t_8 = yx^{-2}[t_0 t_4 t_1 t_{11}]^{x^{-1} z^{-1}},$$

$$t_0 t_{11} t_5 t_9 = x^2 z[t_0 t_4 t_1 t_9]^{yxz^{-1}},$$

$$t_0 t_{11} t_5 t_{10} = zx^{-2}[t_0 t_6 t_7]^{z^{-1}},$$

$$t_0 t_{11} t_5 t_{11} = yzx^{-1}[t_0 t_6 t_{10}]^{yx^{-2} z^{-1}},$$

$$t_0 t_{11} t_5 t_{12} = yxz[t_0 t_6 t_0]^{yx},$$

$$t_0 t_{11} t_5 t_{13} = x^{-1} z^{-1}[t_0 t_2 t_3 t_7]^e,$$

$$t_0 t_{11} t_5 t_{14} = yx^2 z^{-1}[t_0 t_2 t_6]^{zx^{-1}},$$

$$t_0 t_{11} t_5 t_{14} = yx^2 z^{-1}[t_0 t_2 t_6]^{zx^{-1}}.$$

Resulting in one single coset continued onto the double cosets, $[0, 6, 2], [0, 11, 5], [0, 4, 10],$ $[0, 4, 1, 11], [0, 4, 1, 9], [0, 6, 7], [0, 6, 10], [0, 6, 0],$ and $[0, 2, 6, 7]$, each. And one single coset goes back to $[0, 11]$. Next, we will investigate the double coset $N t_0 t_2 t_3 t_4$.

$\mathbf{N t_0 t_2 t_3 t_4 N}$

The double coset $N t_0 t_2 t_3 t_4$ is labeled $[0, 2, 3, 4]$. The coset stabilizing group will be generated by $N^{(0,2,3,4)} =< (1,2)(3,8)(4,6)(5,7)(9,13)(11,12)(14,15) >$ such that $\dfrac{|N|}{|N^{(0,2,3,4)}|} = \dfrac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 2, 3, 4]$. The orbits of $N^{(0,2,3,4)}$ are $\{10\}, \{1, 2\}, \{3, 8\}, \{4, 6\}, \{5, 7\}, \{9, 13\}, \{11, 12\}$ and $\{14, 15\}$. We will then pick a representative from each orbit and right multiply it

to $Nt_0t_2t_3t_4$, consider the following relations:

$$t_0t_2t_3t_4t_{10} = x^{-1}z^{-1}[t_0t_4t_1t_{11}]^{x^{-2}},$$

$$t_0t_2t_3t_4t_1 = yzx^{-1}[t_0t_2t_3t_4]^{yx},$$

$$t_0t_2t_3t_4t_3 = yx^{-1}z^{-1}[t_0t_2t_6t_7]^{yx},$$

$$t_0t_2t_3t_4t_4 = [t_0t_2t_3] \text{ (goes back to } [0,2,3]),$$

$$t_0t_2t_3t_4t_5 = xz[t_0t_2t_6]^{x^{-1}},$$

$$t_0t_2t_3t_4t_9 = x[t_0t_2t_3t_7]^{yx^2},$$

$$t_0t_2t_3t_4t_{11} = yz[t_0t_6t_7]^{x^2z},$$

$$t_0t_2t_3t_4t_{14} = x^{-2}z^{-1}[t_0t_6t_7]^{zx^{-2}}.$$

We can interpret that there are two single cosets going back to the double coset $[0,2,3]$ and two loops back to $[0,2,3,4]$. Also, two single cosets moves forward to each of the double cosets $[0,2,6,7], [0,2,6], [0,2,3,7], [0,6,7]$, and four progress to $[0,6,7]$. Next, we will investigate the double coset $Nt_0t_2t_3t_7$.

$Nt_0t_2t_3t_7N$

The double coset $Nt_0t_2t_3t_7$ is labeled $[0,2,3,7]$. The coset stabilizing group will be generated by $N^{(0,2,3,7)} =< Id(G) >$ such that $\frac{|N|}{|N^{(0,2,3,7)}|} = \frac{30}{1} = 30$, meaning that there are 30 single cosets in the double coset $[0,2,3,7]$. The orbits of $N^{(0,2,3,7)}$ $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}$, and $\{15\}$. We will then pick a representative from each orbit and right multiply it to $Nt_0t_2t_3t_7$, consider

the following relations:

$$t_0t_2t_3t_7t_1 = yxz^{-1}[t_0t_2t_6t_7]^{yx^{-2}}$$

$$t_0t_2t_3t_7t_2 = yx^2[t_0t_6t_0]^{x^{-1}}$$

$$t_0t_2t_3t_7t_3 = y[t_0t_{11}t_0t^{x^{-1}}$$

$$t_0t_2t_3t_7t_4 = yz^{-1}[t_0t_6t_2]^{x^{-1}z^{-1}}$$

$$t_0t_2t_3t_7t_5 = yx^2z[t_0t_4t_1t_{11}]^{x^2z}$$

$$t_0t_2t_3t_7t_6 = yzx^{-2}[t_0t_4t_{10}]^{yx}$$

$$t_0t_2t_3t_7t_7 = [t_0t_2t_3] \text{ (goes back to } [0,2,3]),$$

$$t_0t_2t_3t_7t_8 = yzx^{-2}[t_0t_11t_0]^{yz^{-1}}$$

$$t_0t_2t_3t_7t_9 = x^2z[t_0t_{11}t_{10}]^{yz^{-1}}$$

$$t_0t_2t_3t_7t_{11} = yx^2[t_0t_2t_3t_7]^e$$

$$t_0t_2t_3t_7t_{12} = x^{-1}[t_0t_6t_0]^{x^{-2}}$$

$$t_0t_2t_3t_7t_{13} = xz[t_0t_{11}t_5]^e$$

$$t_0t_2t_3t_7t_{14} = zx^{-2}[t_0t_6t_7]^{xz}$$

$$t_0t_2t_3t_7t_0 = x[t_0t_2t_3t_4]^{yx^2}$$

As a result there are two single cosets going back to the double coset $[0,2,3]$ and one loops back to $[0,2,3,7]$. Also two single cosets move forward to each of the double cosets $[0,6,0], [0,11,5]$, and one progress to $[0,2,6,7], [0,6,2], [0,4,1,11], [0,4,10], [0,11,0], [0,6,7]$, and $[0,2,3,4]$. Next, we will investigate the double coset $Nt_0t_2t_6t_7$.

$Nt_0t_2t_6t_7N$

The double coset $Nt_0t_2t_6t_7$ is labeled $[0,2,6,7]$. The coset stabilizing group will be generated by $N^{(0,2,6,7)} =< (1,13)(2,8)(3,12)(4,15)(5,6)(9,14)(10,11) >$ such that $\frac{|N|}{|N^{(0,2,6,7)}|} = \frac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0,2,6,7]$. The orbits of $N^{(0,2,6,7)}$ are $\{7\}, \{1,13\}, \{2,8\}, \{3,12\}, \{4,15\}, \{5,6\}, \{9,14\}$ and $\{10,11\}$. We will then pick a representative from each orbit and right multiply it

to $Nt_0t_2t_6t_7$, consider the following relations:

$$t_0t_2t_6t_7t_1 = x^{-1}z^{-1}[t_0t_6t_{10}]^{yxz}$$

$$t_0t_2t_6t_7t_2 = yx^2z[t_0t_2t_3t_4]^{yx^2z^{-1}}$$

$$t_0t_2t_6t_7t_3 = yx^2z[t_0t_6t_0]^{yx}$$

$$t_0t_2t_6t_7t_4 = yz^{-1}[t_0t_{11}t_5]^y$$

$$t_0t_2t_6t_7t_5 = yz^{-1}[t_0t_4t_1]^{yxz^{-1}}$$

$$t_0t_2t_6t_7t_7 = [t_0t_2t_6] \text{ (goes back to } [0,2,6]),$$

$$t_0t_2t_6t_7t_9 = yx^{-2}z^{-1}[t_0t_4t_1t_{11}]^{yx^2}$$

$$t_0t_2t_6t_7t_{10} = yz[t_0t_2t_3t_7]^{yx^{-2}}$$

We see that there is one single coset going back to the double coset $[0, 2, 6]$. Also two sin-gle cosets move forward to each of the double cosets $[0, 6, 10], [0, 2, 3, 4], [0, 6, 0], [0, 11, 5]$, $[0, 4, 1], [0, 4, 1, 11]$, and $[0, 2, 3, 7]$. Next, we will investigate the double coset $Nt_0t_2t_6t_7$.

$Nt_0t_4t_2t_3N$

The double coset $Nt_0t_4t_2t_3$ is labeled $[0, 4, 2, 3]$. The coset stabilizing group will be generated by $N^{(0,4,2,3)} =< (1,3)(2,9)(4,7)(6,14)(8,11)(10,12)(13,15) >$ such that $\frac{|N|}{|N^{(0,4,2,3)}|} = \frac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 4, 2, 3]$. The orbits of $N^{(0,4,2,3)}$ $\{5\}, \{1,3\}, \{2,9\}, \{4,7\}, \{6,14\}, \{8,11\}, \{10,12\}$ and $\{13, 15\}$. We will then pick a representative from each orbit and right multiply it to

$Nt_0t_4t_2t_3$, consider the following relations:

$$t_0t_4t_2t_3t_5 = z[t_0t_{11}t_0]^y$$

$$, t_0t_4t_2t_3t_2 = xz^{-1}[t_0t_4t_2t_3]^{x^2z^{-1}},$$

$$t_0t_4t_2t_3t_3 = [t_0t_4t_2] \text{ (goes back to } [0,4,2]),$$

$$t_0t_4t_2t_3t_4 = yzx^{-2}[t_0t_4t_1t_9^{yz^{-1}}],$$

$$t_0t_4t_2t_3t_6 = yzx^{-1}[t_0t_{11}t_{10}]^{z^{-1}},$$

$$t_0t_4t_2t_3t_8 = zx^{-1}[t_0t_4t_1t_9]^{yzx^{-1}},$$

$$t_0t_4t_2t_3t_{10} = x^{-1}[t_0t_4t_{10}]^{zx^{-2}},$$

$$t_0t_4t_2t_3t_{13} = yx^2z[t_0t_4t_1]^{yx-1z^{-1}}.$$

Thus, there are two single cosets going back to the double coset $[0, 4, 2]$, and two loop back to $[0, 4, 2, 3]$. Also two single cosets move forward to each of the double cosets $[0, 11, 10], [0, 4, 10]$, and $[0, 4, 1]$; while one continues to $[0, 11, 0]$ and four advance to $[0, 4, 1, 9]$. Next, we will investigate the double coset $Nt_0t_4t_1t_9$.

$Nt_0t_4t_1t_9N$

The double coset $Nt_0t_4t_1t_9$ is labeled $[0, 4, 1, 9]$. The coset stabilizing group will be generated by $N^{(0,4,1,9)} =< (1, 4)(2, 10)(3, 11)(5, 9)(7, 15)(8, 14)(12, 13) >$ such that $\frac{|N|}{|N^{(0,4,1,9)}|} = \frac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 4, 1, 9]$. The orbits of $N^{(0,4,1,9)}$ $\{6\}, \{1, 4\}, \{2, 10\}, \{3, 11\}, \{5, 9\}, \{7, 15\}, \{8, 14\}$ and

$\{12, 13\}$.

$$t_0 t_4 t_1 t_9 t_6 = x^{-2} z^{-1} [t_0 t_6 t_{10}]^{yxz},$$

$$t_0 t_4 t_1 t_9 t_1 = xz [t_0 t_4 t_{10}]^{yx^2},$$

$$t_0 t_4 t_1 t_9 t_2 = x^{-1} z^{-1} [t_0 t_4 t_1 t_9]^{yx^2},$$

$$t_0 t_4 t_1 t_9 t_3 = yx^2 z [t_0 t_4 t_1 t_{11}]^x,$$

$$t_0 t_4 t_1 t_9 t_7 = zx^{-1} [t_0 t_4 t_2 t_3]^{z^{-1}},$$

$$t_0 t_4 t_1 t_9 t_8 = yx^2 z [t_0 t_4 t_2 t_3]^z,$$

$$t_0 t_4 t_1 t_9 t_9 = [t_0 t_4 t_1] \text{ (goes back to [0,4,1]),}$$

$$t_0 t_4 t_1 t_9 t_{12} = x^2 z [t_0 t_{11} t_5]^{yxz^{-1}}.$$

In summary there are two single cosets going back to the double coset $[0, 4, 1]$, and two loop back to $[0, 4, 1, 9]$. Also two single cosets move forward to each of the double cosets $[0, 4, 10], [0, 4, 1, 11]$, and $[0, 11, 5]$; while one continues to $[0, 6, 10]$ and four advance to $[0, 4, 2, 3]$. Next, we will investigate the double coset $Nt_0 t_4 t_1 t_{11}$.

$\mathbf{Nt_0 t_4 t_1 t_{11} N}$

The double coset $Nt_0 t_4 t_1 t_{11}$ is labeled $[0, 4, 1, 11]$. The coset stabilizing group will be generated by $N^{(0,4,1,11)} = < (2, 4)(3, 5)(6, 10)(7, 11)(8, 9)(12, 15)(13, 14) >$ such that $\dfrac{|N|}{|N^{(0,4,1,11)}|} = \dfrac{30}{2} = 15$, meaning that there are 15 single cosets in the double coset $[0, 4, 1, 11]$. The orbits of $N^{(0,4,1,11)}$ $\{1\}, \{2, 4\}, \{3, 5\}, \{6, 10\}, \{7, 11\}, \{8, 9\}, \{12, 15\}$ and

$\{7, 11\}$.

$$t_0 t_4 t_1 t_{11} t_1 = xz[t_0 t_2 t_3 t_4]^{yx^{-2}},$$

$$t_0 t_4 t_1 t_{11} t_2 = e[t_0 t_4 t_{10}]^{yx^{-2}z^{-1}},$$

$$t_0 t_4 t_1 t_{11} t_3 = yxz[t_0 t_6 t_7]^{y},$$

$$t_0 t_4 t_1 t_{11} t_6 = yz^{-1}[t_0 t_{11} t_5]^{xz},$$

$$t_0 t_4 t_1 t_{11} t_8 = yx^{-2}[t_0 t_4 t_1 t_9]^{yx},$$

$$t_0 t_4 t_1 t_{11} t_{12} = yxz^{-1}[t_0 t_2 t_6 t_7]^{yz^{-1}},$$

$$t_0 t_4 t_1 t_{11} t_{11} = [t_0 t_4 t_1] \ (\text{ goes back to } [0,4,1]),$$

$$t_0 t_4 t_1 t_{11} t_{13} = yz^{-1}[t_0 t_2 t_3 t_7]^{yx^2 z}.$$

Thus, there are two single cosets going back to the double coset $[0, 4, 1]$. Also two single cosets move forward to each of the double cosets $[0, 4, 10], [0, 6, 7], [0, 4, 1, 9], [0, 2, 6, 7]$ and $[0, 11, 5]$; while one continues to $[0, 2, 3, 4]$.

In conclusion, since our group is closed under right multiplication of $t'_i s$, we are done with our double coset enumeration of G , refer to the Cayley graph below.

3.3.1 Proof $G \cong L_2(31)$

We factor the progenitor $2^{*15} : D_{15}$ by the relations $(z * t)^4, (x^2 * z^{-1} * t)^{10}$ and $(x^2 * z * t)^3$. Let $G \cong \frac{2*15:D_{14}}{(z*t)^4, (x^2*z^{-1}*t)^{10}, (x^2*z*t)^3} \cong L_2(31)$ be a symmetric presentation of G given by $< x, y, z, t | y^2, z^3, (x^-1y)^2, (x, z), (yz^{-1})^2, x^{-5}, t^2, (t, yxz), (zt)^4, (x^2 z^{-1} * t)^{10}, (x^2 zt)^3, yxtztz^{-1}t >$ where N$\cong D_{15}$ and

$x = (1, 2, 6, 10, 4)(3, 7, 12, 14, 9)(5, 8, 13, 15, 11),$

$y = (2, 4)(3, 5)(6, 10)(7, 11)(8, 9)(12, 15)(13, 14),$ and

$z = (1, 3, 5)(2, 7, 8)(4, 9, 11)(6, 12, 13)(10, 14, 15).$

Definition Generators of PSL(2,q) (where p is a finite field of order p^2 where p is prime). Note, we will refer to this as $L_2(q)$

In $L_2(31)$ our field is 31,

$F_31 = \mathbb{Z}_31 =$

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25,$

$26, 27, 28, 29, 30\}$

$F_31 \cup \infty =$

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25,$

$26, 27, 28, 29, 30\} \cup \infty.$

The non-zero squares in F_30 are $1^2, 2^2, 3^2, ..., 29^2, 30^2$

in $\mod 7 = \{1, 2, 4, 5, 7, 8, 9, 10, 14, 16, 18, 19, 20, 25, 28\}.$

The group is generated by, $L(2, 31) = x \to \frac{ax+b}{cx+d}$ where $x \in F_31 \cup \infty, a, b, c, d \in F_31$ and $ad - bc = 1$ or equicantly a non-zero square.

Permutations of our presentation of $L_2(31)$ will be α, β, and γ, which are defined and listed below,

$\alpha : x \to x + 1 =$

$(\infty)(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25,$

$26, 27, 28, 29, 30, 31)$

$\beta : x \to 4x =$

$(\infty)(0)(1, 7, 18, 2, 14, 5, 4, 28, 10, 8, 25, 20, 16, 19, 9)(3, 21, 23, 6, 11, 15, 12, 22, 30,$

$24, 13, 29, 17, 26, 27)$

$\gamma : x \to \frac{-1}{x} =$

$(0, \infty)(1, 30)(2, 15)(3, 10)(4, 23)(5, 6)(7, 22)(8, 27)(9, 24)(11, 14)(12, 18)$

$(13, 19)(16, 29)(17, 20)\ (21, 28)(25, 26),$

alpha, beta and gamma suffice to generate $L_2(31)$.

With the help of MAGMA we can find the homomorphic projection of t; this let's us know that homorphisms of $\phi(x)$, $\phi(y)$, $\phi(z)$ and $\phi(t)$ are $(1, 32, 21, 20, 2)(3, 11, 19, 13, 9)$ $(5, 16, 30, 31, 10)(6, 17, 12, 22, 23)(7, 28, 26, 14, 29)(8, 27, 25, 15, 24),\ (1, 8)(2, 27)(3, 5)$ $(4, 18)(6, 7)(9, 16)(10, 11)(12, 14)(13, 30)(15, 21)(17, 29)(19, 31)(20, 25)(22, 26)(23, 28)(24, 32)$ and $(1, 23, 5)(2, 22, 10)(3, 28, 8)(6, 16, 32)(7, 24, 9)(11, 26, 27)(12, 31, 20)(13, 29, 15)$ $(14, 25, 19)(17, 30, 21),\ (1, 18)(2, 6)(3, 21)(4, 11)(5, 27)(7, 14)(8, 19)(9, 32)(10, 30)(12, 16)$ $(13, 22)(15, 28)(17, 31)(20, 25)(23, 26)(24, 29)$ respectively. Now we will prove that G is isomorphic to $L(2, 31)$ by constructing a homomorphism ϕ from the progenitor to $L(2, 31)$. Let

$\phi(x) \equiv \frac{21x-10}{x-1} = (1,\infty,21,20,2)(3,11,19,13,9)(5,16,30,0,10)(6,17,12,22,23)$
$(7,28,26,14,29)(8,27,25,15,24)$

$\phi(y) \equiv \frac{24x-29}{x-24} = (1,8)(2,27)(3,5)(4,18)(6,7)(9,16)(10,11)(12,14)(13,30)(15,21)$
$(17,29)(19,0)(20,25)(22,26)(23,28)(24,\infty)$

$\phi(z) \equiv \frac{8x-9}{x-5} = (1,23,5)(2,22,10)(3,28,8)(6,16,32)(7,24,9)(11,26,27)(12,31,20)$
$(13,29,15)(14,25,19)(17,30,21).$

We now let $\phi(t_0) \equiv \frac{9x+6}{x+9} = (1,18)(2,6)(3,21)(4,11)(5,27)(7,14)(8,19)(9,32)(10,30)$
$(12,16)(13,22)(15,28)(17,0)(20,25)(23,26)(24,29).$

Note, we will use 32 as ∞ and 31 as 0, reason being that in magma we use numbers to denote infinity and to keep trend we will also do this with 0. And by conjugation, we would prove $|t_0^N| = 15$. So that N permutes the fifteen images of t_0, by conjugation $L_2(7)$ is given by,

$\phi(x) : (t_1, t_{2,6}, t_{10}, t_4)(t_3, t_7, t_{12}, t_{14}, t_9)(t_5, t_8, t_{13}, t_{15}t_{11})$ and

$\phi(y) : (t_2, t_4)(t_3, t_5)(t_6, t_{10})(t_7, t_{11})(t_8, t_9)(t_{12}, t_{15})(t_{13}, t_{14})$

$\phi(z) : (t_1, t_3, t_5)(t_2, t_7 t_8)(t_4, t_9 t_{11})(t_6, t_{12} t_{13})(t_{10}, t_{14} t_{15}),$

thus $\phi(2*^{15} : D_{15}) \cong L_2(31).$

Thus $L_2(31)$ is an image of G, so the $|G| \geq |L_2(31)|$ but $|G| \leq 496 = |L_2(31)|$ and so the equality holds and $G \cong L_2(31)$.

Now the relations given by $(x^2 z^{-1} t)^{10}, (x^2 zt)^3$ and $(zt)^4 = 1$ where their order are $10, 3$ and 4 respectively match the order of the images. Thus, the order of $|\phi(x^2)\phi(z^{-1})t| = 10, |\phi(x^2)\phi(z)t| = 3$ and $|\phi(z)t| = 4$. Therefore, $L_2(31)$ is an image of G, so the $|G| \geq |L_2(31)|$ but $|G| \leq 496 = |L_2(31)|$ and so the equality holds and $G \cong L_2(31)$.

3.4 Double Coset Enumeration of M_{12} Over $L_2(11)$

3.4.1 Introduction

Usually, when we perform a double coset enumeration we work with the control group N, as we have done on other examples. Yet, sometimes we come across new and exciting groups that are too large to perform a double coset enumeration on; on such instances it is of help to factor our group by a maximal subgroup. This allows us to

work with a smaller isomorphic image of our original group. First, we will go in detail as to how we find a maximal subgroup.

3.4.2   Factoring G by Maximal Subgroup

We have the progenitor $2^{*15} : A_5 \cong M_{12}$ where $x = (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)$ $(4, 5, 11, 13, 8)$ and $y = (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)$, and is factored by the relations $(x^2 t)^4, (yt^{x^3})^6, (xt)^5$, and $(yt)^6$. When looking into our group with the help of MAGMA, we see that the number of double cosets that this Mathieu Group 12, or M_{12} for short, has is a bit too excessive, with 47 doble cosets. Thus we will proceed by factoring our group G by a maximal subgroup.

```
> G<x,y,t>:=Group<x,y,t|x^5, y^3, (x*y*x)^2, t^2, (t,x^2*y),
> (t,x*y^-1*x^-1*y*x), (x^2*t)^4, (y*t^(x^3))^6,
> (x*t)^5, (y*t)^6>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
>  CompositionFactors(G1);
    G
    |  M12
    1
> #DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
47

> IN:=sub<G1|f(x),f(y)>;
> M:=MaximalSubgroups(G1);
> #M;
11
>
> for i in [1..#M] do #M[i] `subgroup; end for;
72
660
240
192
192
432
432
1440
1440
7920
7920
```

Listed above are the possible maximal subgroups that we could use, though not all will work. Either because the maximal subgroup is not a factor of our group G, or because the list of conjugates within each maximal subgroup does not contain generators that will produce our control group made up of the permutations x and y from our group G. Ideally, we would like to start off with one of the maximal subgroups of largest order, but in our case, they did not work; however, the second maximal subgroup did, we refer to it as $M[2]$.

```
> M[2];
rec<recformat<order, length, subgroup, presentation> |
    order := 660,
    length := 144,
    subgroup := Permutation group acting on a set of
    cardinality 1584 Order = 660 = 2^2 * 3 * 5 * 11>


> D:=M[2]`subgroup;
> DD:=Conjugates(G1,D);
> DD:=Setseq(DD);
> for i in [1..#DD] do if f(x) in DD[i] and f(y) in DD[i]
for> then i; end if; end for;
7
14
```

Magma is very usuful for this because $M[2]$ is of length 144, meaning we would have to check each one of those 144 subgroups to see which will gernerate the permutations x and y. The following loop checks to see which of the 144 conjugates of our maximal subgroup $M[2]$ will work with our permutations x and y. In response, MAGMA found that the seventh conjugate of $M[2]$ will work for this situation (note the response from MAGMA will change every time we log on).

```
> for g in DD[7] do if sub <G1|f(x),f(y),g> eq DD[7]
for|if> then A:=g;
for|if> break; end if; end for;
> Order(A);
11
> W:=WordGroup(G1);
> rho:=InverseWordMap(G1);
> A@rho;
function(W)
    w4 := W.2 * W.3; w5 := W.2 * w4; w6 := w4 * W.2; w7 :=
```

```
    w5* w6; w8 := w6 * W.2; w9 := w7 * w8; w10 := W.1^2;
    w11 := w9* w10; w12 := W.1 * w6; w13 :=w11 * w12; w14
    := w12 * w8; w15 := w13 * w14; w16 := w10 * w14; w17 :=
    w15 *w16; w18 := w16 * w10; w19 := w17 * w18; w20 :=
    w12 * w16; w21 := w19 * w20; w22 := w6 * w20; w23 :=
    w21 * w22; w3 := W.3^-1; w24 := w3 * W.1; w25 := w3 *
     W.2; w26:= w24 * w25; w1 := W.1^-1; w27 := w1 * w3;
    w28 := w3* w27; w29  := w24 * w28; w30 := w26 * w29;
    w31 := w23* w30; w33 := w31 * W.3; w34 := w33 * W.2;
    w35 := w34* W.3; w36 := w35 * w31; w37 := w36 * W.2;
    w38 := w37 * W.3; w39 := w38 * W.1; return w39;
end function


> for  g in DD[7]  do if sub <G1|f(x),f(y),g> eq DD[7]
for|if> then g@rho;
for|if> break; end if; end for;
function(W)
    w4 := W.2 * W.3; w5 := W.2 * w4; w6 := w4 * W.2; w7 := w5
  * w6; w8 := w6 * W.2; w9 := w7 * w8; w10 := W.1^2; w11 :=
    w9 * w10; w12 := W.1 * w6; w13 := w11 * w12; w14 := w12 *
    w8; w15 := w13 * w14; w16 := w10 * w14; w17 := w15 *w16;
    w18 := w16 * w10; w19 := w17 * w18; w20 := w12 * w16;
    w21 := w19 * w20; w22 := w6 * w20; w23 := w21 * w22;
    w3 := W.3^-1; w24 := w3 * W.1; w25 := w3 * W.2; w26
    := w24 * w25; w1 := W.1^-1; w27 := w1 * w3; w28 := w3 *
    w27; w29 := w24 * w28; w30 := w26 * w29; w31 := w23
    * w30; w33 := w31 * W.3; w34 := w33 * W.2; w35 := w34
    * W.3; w36 := w35 * w31; w37 := w36 * W.2; w38 := w37
    * W.3; w39 := w38 * W.1; return w39;
end function


>A:=function(W)
    w4 := W.2 * W.3; w5 := W.2 * w4; w6 := w4 *
    W.2; w7 := w5 * w6; w8 := w6 * W.2; w9 :=
    w7 * w8; w10 := W.1^2; w11 := w9 * w10; w12
    := W.1 * w6; w13 := w11 * w12; w14 := w12 * w8;
    w15 := w13 * w14; w16 := w10 * w14; w17 := w15
    *w16; w18 := w16 * w10; w19 := w17 * w18; w20
    := w12 * w16;  w21 := w19 * w20;w22 := w6 * w20;
    w23 := w21 * w22; w3 := W.3^-1; w24 := w3 * W.1;
    w25 := w3 * W.2; w26 := w24 * w25; w1 := W.1^-1;
    w27 := w1 * w3; w28 := w3 * w27; w29 := w24 *
```

```
        w28; w30 := w26 * w29; w31 := w23 * w30; w33
        := w31 * W.3; w34 :=w33 * W.2; w35 := w34 * W.3;
        w36 := w35 * w31; w37 := w36 * W.2; w38 := w37
        * W.3; w39 := w38 * W.1; return w39;
end function;


>AA:=A(G);
>AA;
y^2 * t * y * t * y^2 * t * y^2 * x^3 * y * t * y * x * y
 * t * y^2 * t * y^2 * x^3 * y * t * y^2 * t * y^2 * x^3
* y * t * y^2 * t * y^2 * x^3 * y * t * y * x^3 * y * t *
y^2 * t * y^3 * t * y * x * y * t * y * x^3 * y * t * y^2
* t * y^2 * t^-1 * x * t^-1 * y * t^-1 * x * t^-1 * x^-1
* y * t * y^2 * t * y * t * y^2 * t * y^2 * x^3 * y *t * y
* x * y * t * y^2 * t * y^2 * x^3 * y * t * y^2 * t * y^2
* x^3 * y * t * y^2 * t * y^2 * x^3 * y * t * y * x^3 * y
* t * y^2 * t * y^3 * t * y * x * y * t * y * x^3 * y * t *
y^2 * t * y^2 * t^-1 *x * t^-1 * y * t^-1 * x * t^-1 *
x^-1 * t^-1 * y * t * x
```

We will now check if it is at all possible to condense the permutations listed above by using the Schreier System.

```
> G<x,y,t>:=Group<x,y,t|x^5, y^3, (x*y*x)^2, t^2, (t,x^2*y),
> (t,x*y^-1*x^-1*y*x),(x^2*t)^4,(y*t^(x^3))^6,(x*t)^5,(y*t)^6>;
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> NN:=G;
> N:=G;
> N:=G1;
> #N;
95040

>Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
>ArrayP:=[Id(N): i in [1..95040]];
>for i in [2..95040] do
>P:=[Id(N): l in [1..#Sch[i]]];
>for j in [1..#Sch[i]] do
>if Eltseq(Sch[i])[j] eq 1 then P[j]:=f(x); end if;
>if Eltseq(Sch[i])[j] eq -1 then P[j]:=f(x^-1); end if;
>if Eltseq(Sch[i])[j] eq 2 then P[j]:=f(y); end if;
>if Eltseq(Sch[i])[j] eq -2 then P[j]:=f(y^-1); end if;
>if Eltseq(Sch[i])[j] eq 3 then P[j]:=f(t); end if;
```

```
>end for;
>PP:=Id(N);
>for k in [1..#P] do
>PP:=PP*P[k]; end for;
>ArrayP[i]:=PP;
>end for;

>W:=WordGroup(G1);
>rho:=InverseWordMap(G1);
>AA:=f(A(G));
>Order(AA);
>for i in [1..95040] do if ArrayP[i] eq N!AA then Sch[i];
> end if; end for;
x^-1 * y^-1 * t * y^-1 * t * y^-1 * x * t * x
```

Our goal is achieved! The maximal subgroup, $M[2]$, of the group G can be expressed as $x^{-1}y^{-1}ty^{-1}ty^{-1}xtx$.

```
>IM:=sub<G1|f(x),f(y),
> f(x^-1 * y^-1 * t * y^-1 * t * y^-1 * x * t * x)>;
>#IM;
660
>CompositionFactors(IM);
    G
    |  A(1, 11)                         = L(2, 11)
    1
> #DoubleCosets(G,sub<G|x,y,
> x^-1 * y^-1 * t * y^-1 * t * y^-1 * x * t * x>,
> sub<G|x,y>);
11
```

3.4.3 Double Coset Enumeration of $L_2(11) \cong M_{12}$

We have successfully factored G by a maximal subgroup and will now continue by performing double coset enumerations of $L_2(11)$, which only has an appealing 11 double cosets.

Again, we have the progenitor $2^{*15} : A_5 \cong M_{12}$ where $x = (1, 9, 10, 3, 14)$ $(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)$ and $y = (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)$, factored by relations $(x^2t)^4, (yt^{x^3})^6, (xt)^5$, and $(yt)^6$. By letting t be represented by t_1, let us compute our four listed relations. To make our work easier, let $\pi = x^2 = (1, 10, 14, 9, 3)$ $(2, 7, 6, 15, 12)(4, 11, 8, 5, 13)$:

$$[x^2 t]^4 = e$$

$$[\pi t]^4 = e$$

$$\pi^4 t^{\pi^3} t^{\pi^2} t^\pi t = e$$

$$\pi = t_9 t_{14} t_{10} t_1$$

$$(1,3,9,14,10)(2,12,15,6,7)(4,13,5,8,11) t_9 t_{14} t_{10} t_1 = e$$

$$(1,3,9,14,10)(2,12,15,6,7)(4,13,5,8,11) t_9 t_{14} = t_1 t_{10}.$$

And $y = (1,4,10)(2,5,8)(3,7,11)(6,9,15)(12,14,13)$:

$$[y t^{x^3}]^6 = e$$

$$[y t_3]^6 = e \qquad (since (t_1)^{x^3} = t_3)$$

$$y^6 t^{y^5} t^{y^4} t^{y^3} t^{y^2} t^y t = e$$

$$y^6 t_{11} t_7 t_3 t_{11} t_8 t_5 t_3 = e$$

$$e t_{11} t_7 t_3 t_{11} t_8 t_5 t_3 = e$$

$$t_{11} t_7 t_3 t_{11} t_8 = t_3 t_5.$$

Letting $x = (1,9,10,3,14)(2,15,7,12,6)(4,5,11,13,8)$:

$$[xt]^5 = e$$

$$x^5 t^{x^4} t^{x^3} t^{x^2} t^x t = e$$

$$x^5 t_{14} t_3 t_{10} t_9 t_1 = e$$

$$e t_{14} t_3 t_{10} t_9 t_1 = e$$

$$t_{14} t_3 t_{10} = t_1 t_9.$$

And $y = (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)$:

$$[yt]^6 = e$$
$$y^6 t^{y^5} t^{y^4} t^{y^3} t^{y^2} t^y t = e$$
$$y^6 t_{10} t_4 t_1 t_{10} t_4 t_1 = e$$
$$e t_{10} t_4 t_1 t_{10} t_4 t_1 = e$$
$$t_{10} t_4 t_1 t_{10} = t_1 t_4.$$

Now $A_5 \leq L_2(11) \leq_{max} M_{12}$. Thus we will perform a double coset enumeration of G over M where $M = L_2(11)$ and $N \leq M \leq M$. As we identified in the code above, $M = < f(x), f(y), f(x^-1*y^-1*t*y^-1*t*y^-1*x*t*x) >$ is isomorphic to $L_2(11)$.

We will perform manual double coset enumeration of G over M. In order to express G as a union of double cosets, we will interpret it as MgN, where g is an element of G; such that $G = MeN \cup Mg_1 N \cup Mg_2 N...$, where $g_i's$ are words in the $t_i's$. Our objective is to find the number of single cosets contained in each double coset $[w]$ defined as $[w] = \{Mw^n | n \in M\}$. We will know when we are done with our double coset enumeration when the set of right cosets is closed under right multiplication, meaning all of our potential new double cosets are already recognized. We declare each double coset $[w]$ where Mwt_i will belong to one of the symmetric generators of t_i from each of the orbits of the coset stabiliser group; defined as $M^{(w)} = \{n \in N | Mw^n = Mw\}$, and where each w is a word of $t_i's$ on $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

N e N

We begin with the double coset $MeN = \{Ne^n | n \in N\}$, labeled $[*]$, and contains one single coset, M. The coset stabilizer of M is the control group N and thus has a single orbit of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ of which we will perform right multiplication. Any element of the orbit can be the representative, we chose the orbit representative of 1. This means that $Met_1 = Mt_1 N$ is a new double coset $[1]$ and since there is 15 single coset in the orbit of $[*]$, 15 single cosets will move forward to the

coset [1].

N t_1 **N**

For the double coset Mt_1N we need to find the point stabilizer, N^1, and the coset stablising group $N^{(1)}$ to discern how many single cosets are in [1],

$$N^{(1)} \leq \; < (2,11)(3,4)(5,12)(7,9)(10,13)(14,15), (2,5)(3,13)(4,10)(7,14)(9,15)(11,12) >$$
$$\leq 4.$$

Thus, the order of the coset stabliser is 4. The number of single cosets is defined as $\frac{|N|}{|N^{(1)}|}$, so the number of single cosets in $N^{(1)} = \frac{|N|}{|N^{(1)}|} = \frac{30}{4} = 15$. And our index is represented by the sum of single cosets in each distinct double coset, so far we only have established the cosets [*] and [1], $1 + 15 = 16$. The total amount of single cosets is characterized as $\frac{G}{M}$, so the entirety of the double coset enumeration we should have a total of $\frac{95040}{660} = 144$ single cosets. This will also be another clue to let us know that we have completed the double coset enumeration. Carrying on, the orbits of [1] are $\{1\}, \{6\}, \{8\}, \{2,11,5,12\}, \{3,4,13,10\}, \{7,9,14,15\}$ and will pick a coset representative from each of the orbits to determine if any double cosets Mt_1t_i are new:

$$Mt_1t_1 = Me, \in [*] \; (1 \text{ will loop back to } [*])$$
$$Mt_1t_6 \in [1,6]$$
$$Mt_1t_8 \in [1,8]$$
$$Mt_1t_2 \in [1,2]$$
$$Mt_1t_3 \in [1,3]$$
$$Mt_1t_7 \in [1,7].$$

Consequently, the new double cosets are $Mt_1t_6, Mt_1t_8, Mt_1t_2, Mt_1t_7$ and will be respectively labeled as $[1,6], [1,8], [1,2]$ and $[1,7]$.

N t_1t_6 **N**

Similarly as above, we will need to find the point stabilizer, $N^{1,6}$, and the coset stabilizing group $N^{(1,6)}$ to calculate the single cosets that live in $[1, 6]$. The order of the coset stabilizing group is, $N^{(1,6)} \leq < (2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15), (2, 5)(3, 13)(4, 10)(7, 14)(9, 15)(11, 12), (1, 6, 8)(2, 9, 4)(3, 12, 15)(5, 7, 13)(10, 11, 14) > \leq 12$. So the order of the coset stabilizer is $N^{(1,6)} = \frac{|N|}{|N^{(1,6)}|} = \frac{60}{12} = 5$, thus the order the double coset $[1, 6]$ contains is 5 single cosets. Therefore, the index is now $1 + 15 + 5 = 21$. The orbits of the coset $[1, 6]$ are $\{1, 6, 8\}, \{2, 9, 4\}, \{3, 12, 15\}, \{5, 7, 13\}, \{10, 11, 14\}$ and we will pick a representative from each orbit and right multiply, $Mt_1t_6t_i$:

$Mt_1t_6t_6 = Mt_1, \in [1]$ (3 single cosets loop back to $[1]$)

$Mt_1t_6t_2 \in [1, 6, 2]$

$Mt_1t_6t_3 = y^2x^{-1}y^{-1}xyt_5t_3t_1t_8t_1t_{10}$

$\Rightarrow Mt_1t_6t_3 = t_8t_1t_{10} \in [1, 6, 2]$

$\qquad$ (Since $\{N(t_1t_6t_2)^n | n \in N\}$ and $y^2x^{-1}y^{-1}xyt_5t_3t_1 \in M$),

$Mt_1t_6t_5 = x^{-1}y^{-3}x^2y^{-1}t_7t_{12}t_6t_1t_6t_5$

$\Rightarrow Mt_1t_6t_5 = t_1t_6t_5 \in [1, 6, 2]$

$\qquad$ (Since $\{N(t_1t_6t_2)^n | n \in N\}$ and $x^{-1}y^{-3}x^2y^{-1}t_7t_{12}t_6 \in M$),

$Mt_1t_6t_10 = yxy^{-1}x^{-1}t_{12}t_4t_1t_8t_1t_3$

$\Rightarrow Mt_1t_6t_5 = t_8t_1t_3 \in [1, 6, 2]$

$\qquad$ (Since $\{N(t_1t_6t_2)^n | n \in N\}$ and $yxy^{-1}x^{-1}t_{12}t_4t_1 \in M$).

As we can see, in this particular double coset, we had 12 of the orbits move forward to the coset $[1, 6, 2]$ with the exception of one orbit representative going back to the coset $[1]$, resulting in 3 single cosets going back to $[1]$.

N t_1t_8 N

Alike, we will need to find the point stabilizer, $N^{1,8}$, and the coset stabilizing group $N^{(1,8)}$ to calculate the single cosets that live in $[1, 8]$. The order of the coset stabilizing group is, $N^{(1,8)} \leq < (2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15), (2, 5)(3, 13)(4, 10)(7, 14)(9, 15)(11, 12), (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8) > \leq 60$. As a result the order of the coset stabilizer is $N^{(1,8)} = \frac{|N|}{|N^{(1,8)}|} = \frac{60}{60} = 1$, thus the double coset $[1, 8]$ has 1 single coset. Therefore, the index is now $1 + 15 + 5 + 1 = 22$. The orbits of the coset $[1, 8]$ are $\{3\}, \{7\}, \{11\}, \{1, 9\}, \{2, 6\}, \{4, 8\}, \{5, 13\}, \{10, 14\}, \{12, 15\}$ and we will pick a representative from each orbit and right multiply, $Mt_1t_8t_i$:

$$Mt_1t_8t_3 = yx^{-2}yxy^{-1}t_5t_6t_{10}t_1t_7$$

$$\Rightarrow Mt_1t_8t_3 = t_7 \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $yx^{-2}yxy^{-1}t_5t_6t_{10}t_1 \in M$),

$$Mt_1t_8t_7 = yx^{-2}yxy^{-1}t_5t_6t_{10}t_1t_{11}$$

$$\Rightarrow Mt_1t_8t_7 = t_{11} \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $yx^{-2}yxy^{-1}t_5t_6t_{10}t_1 \in M$),

$$Mt_1t_8t_{11} = yx^{-2}yxy^{-1}t_5t_6t_{10}t_1t_3$$

$$\Rightarrow Mt_1t_8t_{11} = t_3 \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $yx^{-2}yxy^{-1}t_5t_6t_{10}t_1 \in M$),

$$Mt_1t_8t_1 = et_6$$

$$\Rightarrow Mt_1t_8t_1 = t_6 \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $et_6 \in M$),

(questions about this one)

$$Mt_1t_8t_2 = x^2yxy^{-2}x^{-1}t_5t_9t_{14}t_4$$

$$\Rightarrow Mt_1t_8t_2 = t_4 \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $x^2yxy^{-2}x^{-1}t_5t_9t_{14}t_1 \in M$),

$$Mt_1t_8t_8 = Mt_1, \in [1] \ (2 \text{ single cosets loop back to } [1])$$

$$Mt_1t_8t_5 = xyxy^{-1}xt_3t_1t_4t_{10}$$

$$\Rightarrow Mt_1t_8t_5 = t_{10} \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $xyxy^{-1}xt_3t_1t_4 \in M$),

$$Mt_1t_8t_{10} = xyxy^{-1}xt_3t_1t_4t_{15}$$

$$\Rightarrow Mt_1t_8t_{10} = t_{15} \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $xyxy^{-1}xt_3t_1t_4 \in M$),

$$Mt_1t_8t_{12} = y^2xt_{15}t_8t_9t_{13}$$

$$\Rightarrow Mt_1t_8t_{12} = t_{13} \in [1]$$

(Since $\{N(t_1)^n | n \in N\}$ and $y^2xt_{15}t_8t_9 \in M$).

As we can see, all the potentially new single cosets in the double coset $[1, 8]$ loop back to the double coset $[1]$.

N t_1t_2 **N**

The order of the coset stabilizing group of the coset $[1, 2]$ is made up of,

$N^{(12)} \leq < (1,14)(2,7)(3,9)(4,11)(6,12)(8,13) > \leq 2$; as a result the order of the coset stabilizer is $N^{(1,2)} = \frac{|N|}{|N^{(1,2)}|} = \frac{60}{2} = 30$, so the order of the double coset $[1,2]$ is 30. Therefore, the index is now $1 + 15 + 5 + 1 + 30 = 52$. The orbits of the coset $[1,2]$ are $\{5\}, \{10\}, \{15\}, \{1,14\}, \{2,7\}, \{3,9\}, \{4,11\}, \{6,12\}, \{8,13\}$. We will pick a representative from each orbit and right multiply, $Mt_1t_2t_i$, however consider the following relations:

$Mt_1t_2t_5 = xyxy^{-1}xy^{-1}t_{11}t_{10}t_1t_4t_8$

$\Rightarrow Mt_1t_2t_5 = t_4t_8 \in [1,2]$

$\qquad$ (Since $\{N(t_1t_2)^n | n \in N\}$ and $xyxy^{-1}xy^{-1}t_{11}t_{10}t_1 \in M$),

$Mt_1t_2t_{15} \in [1,2,15]$, $Mt_1t_2t_1 = x^2y^{-1}t_6t_5t_9t_1t_{12}$

$\Rightarrow Mt_1t_2t_1 = t_1t_{12} \in [1,2]$

$\qquad$ (Since $\{N(t_1t_2)^n | n \in N\}$ and $x^2y^{-1}t_6t_5t_9 \in M$),

$Mt_1t_2t_2 = Mt_1, \in [1]$ (2 single cosets loop back to $[1]$),

$Mt_1t_2t_3 = x^{-1}x^{-1}y^3xt_1t_3t_5t_6t_7$

$\Rightarrow Mt_1t_2t_3 = t_6t_7 \in [1,2]$

$\qquad$ (Since $\{N(t_1t_2)^n | n \in N\}$ and $x^{-1}x^{-1}y^3xt_1t_3t_5 \in M$),

$Mt_1t_2t_4 = yx^3y^{-1}t_1t_{14}t_{11}t_6t_{12}t_9$

$\Rightarrow Mt_1t_2t_4 = t_{12}t_9 \in [1,7]$

$\qquad$ (Since $\{N(t_1t_7)^n | n \in N\}$ and $yx^3y^{-1}t_{14}t_{11}t_6 \in M$),

$Mt_1t_2t_6 = xy^{-1}y^{-1}x^{-1}y^{-1}t_5t_6t_{10}t_1t_{13}t_5$

$\Rightarrow Mt_1t_2t_6 = t_{13}t_5 \in [1,7]$

$\qquad$ (Since $\{N(t_1t_7)^n | n \in N\}$ and $xy^{-1}y^{-1}x^{-1}y^{-1}t_5t_6t_{10} \in M$),

$Mt_1t_2t_8 = x^{-1}y^3xt_1t_3t_5t_{15}t_5t_{12}$

$\Rightarrow Mt_1t_2t_8 = t_{15}t_5t_{12} \in [1,6,2]$

$\qquad$ (Since $\{N(t_1t_6t_2)^n | n \in N\}$ and $x^{-1}y^3xt_1t_3t_5 \in M$).

$\qquad$ We have discovered one new double coset $[1,2,15]$ of which only the coset representative $\{15\}$ will advance. From this two letter double coset $[1,2]$, we see that two single cosets from the orbit $\{8,13\}$ will move forward to $[1,6,2]$. While single cosets from the orbits $\{5\}, \{1,14\}, \{3,9\}$ looped back to $[1,2]$ and the two single cosets form $\{4,11\}$ continued to $[1,7]$.

$\qquad$ **N** t_1t_7 **N**

Reiteratively, the order of the coset stabilizing group of the coset $[1,7]$ is made up of, $N^{(17)} \leq (1,10)(3,14)(5,8)(6,15)(7,12)(11,13) >\leq 2$; therefore the order of the coset stabilizer is $N^{(1,7)} = \frac{|N|}{|N^{(1,7)}|} = \frac{60}{2} = 30$, as concluded, the order of the double coset $[1,7]$ is 30. Resulting in the index increasing 30 single cosets; $1+15+5+1+30+30 = 82$. The orbits of the coset $[1,7]$ are $\{2\}, \{4\}, \{9\}, \{1,10\}, \{3,14\}, \{5,8\}, \{6,15\}$, $\{7,12\}, \{11,13\}$, in the next step we will pick a representative from each orbit and right multiply, $Mt_1t_7t_{is}$, however consider the following relations:

$Mt_1t_7t_2 \in [1,7,2]$

$Mt_1t_7t_4 \in [1,7,4]$

$Mt_1t_7t_9 = y^2x^2yx^{-1}t_{10}t_{14}t_8t_{12}t_{10}$

$\Rightarrow Mt_1t_7t_9 = t_{12}t_{10} \in [1,7]$

$\qquad$ (Since $\{N(t_1t_7)^n | n \in N\}$ and $^2x^2yx^{-1}t_{10}t_{14}t_8 \in M$),

$Mt_1t_7t_1 = (y^{-1}x)^3t_1t_{13}t_2t_9t_2t_{15}$

$\Rightarrow Mt_1t_7t_1 = t_9t_2t_{15} \in [1,6,2]$

$\qquad$ (Since $\{N(t_1t_6t_2)^n | n \in N\}$ and $(y^{-1}x)^3t_1t_{13}t_2 \in M$),

$Mt_1t_7t_3 = x^{-1}y^3xt_1t_3t_5t_5t_{11}$

$\Rightarrow Mt_1t_7t_3 = t_5t_{11} \in [1,2]$

$\qquad$ (Since $\{N(t_1t_2)^n | n \in N\}$ and $x^{-1}y^3xt_1t_3t_5 \in M$),

$Mt_1t_7t_5 = xy^{-1}t_6t_5t_9t_{10}t_6$

$\Rightarrow Mt_1t_7t_5 = t_{10}t_6 \in [1,7]$

$\qquad$ (Since $\{N(t_1t_7)^n | n \in N\}$ and $xy^{-1}t_6t_5t_9 \in M$),

$Mt_1t_7t_6 = x^2y^{-1}t_{10}t_{14}t_8t_{15}t_9$

$\Rightarrow Mt_1t_7t_6 = t_{15}t_9 \in [1,2]$

$\qquad$ (Since $\{N(t_1t_2)^n | n \in N\}$ and $x^2y^{-1}t_{10}t_{14}t_8 \in M$),

$Mt_1t_7t_7 = Mt_1, \in [1]$ (2 single cosets loop back to $[1]$),

$Mt_1t_7t_{11}x = y^3xyxt_{12}t_4t_1t_6t_4$

$\Rightarrow Mt_1t_7t_{11} = t_6t_4 \in [1,7]$

$\qquad$ (Since $\{N(t_1t_7)^n | n \in N\}$ and $y^3xyxt_{12}t_4t_1 \in M$).

Consequently, there are now two new double cosets, $[1,7,2]$ and $[1,7,4]$, each with one orbit moving forward; also, two double cosets moving forward to $[1,6,2]$, and four go to $[1,2]$. While, five orbits loop back for $[1,7]$, two loop back to $[1]$.

N $t_1 t_6 t_2$ **N**

We have now been able to advance to double cosets consisting of three letters, in this case we are working with the new distinct double coset $[1, 6, 2]$ which branched out from $[1, 6]$. The order of the coset stabilizing group of $[1, 6, 2]$ is generated by, $N^{(1,6,2)} \leq (2,5)(3,13)(4,10)(7,14)(9,15)(11,12) > \leq 2$; therefore the order of the coset stabilizer is $N^{(1,7)} = \frac{|N|}{|N^{(1,6,2)}|} = \frac{60}{2} = 30$, thus, the order of the double coset $[1, 6, 2]$ is 30. So the number of single cosets, or the index, is increasing 30 single cosets; $1 + 15 + 5 + 1 + 30 + 30 + 30 = 112$. The orbits of the coset $[1, 6, 2]$ are $\{1\}, \{6\}, \{8\}, \{2, 5\}, \{3, 13\}, \{4, 10\}, \{7, 14\}, \{9, 15\}, \{11, 12\}$, let us continue by picking a representative from each orbit and right multiplying, $M t_1 t_6 t_2 t_{is}$, however consider the following relations:

$$M t_1 t_6 t_2 t_{11} = xyxyx^3 t_{13} t_3 t_9 t_{13} t_{14} t_9$$
$$\Rightarrow M t_1 t_6 t_2 t_{11} = t_{13} t_{14} t_9 \in [1, 6, 2]$$
$$\text{(Since } \{N(t_1 t_6 t_2)^n | n \in N\} \text{ and } xyxyx^3 t_{13} t_3 t_9 \in M),$$
$$M t_1 t_6 t_2 t_6 = xy^{-1} x^{-1} yx^{-1} yxy^{-1} t_9 t_5 t_6 t_1 t_6 t_{11}$$
$$\Rightarrow M t_1 t_6 t_2 t_6 = t_1 t_6 t_{11} \in [1, 6, 2]$$
$$\text{(Since } \{N(t_1 t_6 t_2)^n | n \in N\} \text{ and } xy^{-1} x^{-1} yx^{-1} yxy^{-1} t_9 t_5 t_6 \in M),$$
$$M t_1 t_6 t_2 t_8 = y^2 x^{-1} x^y t_5 t_3 t_1 t_1 t_6 t_{12}$$
$$\Rightarrow M t_1 t_6 t_2 t_8 = t_1 t_6 t_{12} \in [1, 6, 2]$$
$$\text{(Since } \{N(t_1 t_6 t_2)^n | n \in N\} \text{ and } y^2 x^{-1} x^y t_5 t_3 t_1 \in M),$$
$$M t_1 t_6 t_2 t_2 = M t_1 t_6, \in [1, 6] \text{ (2 single cosets loop back to } [1, 6]),$$
$$M t_1 t_6 t_2 t_3 = xy^{-1} x^{-1} yy^x yt_6 t_2 t_{15} t_7 t_{10}$$
$$\Rightarrow M t_1 t_6 t_2 t_3 = t_7 t_{10} \in [1, 2]$$
$$\text{(Since } \{N(t_1 t_2)^n | n \in N\} \text{ and } xy^{-1} x^{-1} yy^x yt_6 t_2 t_{15} \in M),$$
$$M t_1 t_6 t_2 t_4 = x^{-1} yxy^{-1} x^{-1} t_4 t_{12} t_9 t_{14} t_{10} t_6 t_{11}$$
$$\Rightarrow M t_1 t_6 t_2 t_4 = t_{10} t_6 t_{11} \in [1, 7, 4]$$
$$\text{(Since } \{N(t_1 t_7 t_4)^n | n \in N\} \text{ and } x^{-1} yxy^{-1} x^{-1} t_4 t_{12} t_9 t_{14} \in M),$$
$$M t_1 t_6 t_2 t_1 \in [1, 6, 2, 9] \ M t_1 t_6 t_2 t_1 1 = xyxyx^3 t_{13} t_3 t_9 t_{13} t_{14} t_9$$
$$\Rightarrow M t_1 t_6 t_2 t_{11} = t_{13} t_{14} t_9 \in [1, 6, 2]$$
$$\text{(Since } \{N(t_1 t_6 t_2)^n | n \in N\} \text{ and } xyxyx^3 t_{13} t_3 t_9 \in M).$$

Subsequently, two double cosets go back to $[1, 6]$ and two go back to $[1, 2]$ and $[1, 7]$

each. From the singleton orbits, each ended up going to the coset $[1, 6, 2]$ along with two more; and two moved to $[1, 7, 4]$. Also, we discovered a new distinct double coset $[1, 6, 2, 9]$.

N $t_1 t_2 t_{15}$ **N**

The order of the coset stabilizing group of the coset $[1, 2, 15]$ is made up of,
$N^{(1,2,15)} \leq< (1, 2, 7, 14, 5)(3, 13, 15, 8, 9)(4, 11, 12, 10, 6), (1, 7)(3, 8)(5, 14)(6, 11)$
$(10, 12)(13, 15) >\leq 10$; as a result the order of the coset stabilizer is
$N^{(1,2,15)} = \frac{|N|}{|N^{(1,2,15)}|} = \frac{60}{10} = 6$, so the order of the double coset $[1, 2, 15]$ is 6. Therefore, the index is now $1 + 15 + 5 + 1 + 30 + 30 + 30 + 6 = 118$. The orbits of the coset $[1, 2, 15]$ are $\{2\}, \{4\}, \{9\}, \{1, 7\}, \{3, 8\}, \{5, 14\}, \{6, 11\}, \{10, 12\}, \{13, 15\}$. We will pick a representative from each orbit and right multiply each $Mt_1 t_2 t_{15} t_i$, however consider the following relations:

$Mt_1 t_2 t_{15} t_2 = x^{-1} y^2 t_{12} t_4 t_1 t_1 t_6 t_{11} t_7$
$\Rightarrow Mt_1 t_2 t_{15} t_2 = t_1 t_6 t_{11} t_7 \in [1, 6, 2, 9]$
$\qquad$ (Since $\{N(t_1 t_6 t_2 t_9)^n | n \in N\}$ and $x^{-1} y^2 t_{12} t_4 t_1 \in M$),

$Mt_1 t_2 t_{15} t_4 = yx^{-1} y^2 t_{12} t_4 t_1 t_{10} t_8 t_9$
$\Rightarrow Mt_1 t_2 t_{15} t_4 = t_{10} t_8 t_9 \in [1, 2, 15]$
$\qquad$ (Since $\{N(t_1 t_2 t_{15})^n | n \in N\}$ and $yx^{-1} y^2 t_{12} t_4 t_1 \in M$),

$Mt_1 t_2 t_{15} t_9 = (x * y)^2 t_1 t_5$
$\Rightarrow Mt_1 t_2 t_{15} t_9 = t_1 t_5 \in [1, 2]$
$\qquad$ (Since $\{N(t_1 t_2)^n | n \in N\}$ and $(xy)^2 \in M$),

$Mt_1 t_2 t_{15} t_1 = y^{-1} xy^2 x t_{13} t_5 t_9 t_1 t_9 t_2 t_{15} t10$
$\Rightarrow Mt_1 t_2 t_{15} t_1 = t_9 t_2 t_{15} t_{10} \in [1, 6, 2, 9]$
$\qquad$ (Since $\{N(t_1 t_6 t_2 t_9)^n | n \in N\}$ and $y^{-1} xy^2 x t_{13} t_5 t_9 t_1 \in M$),

$Mt_1 t_2 t_{15} t_3 = (x^{-1}, y^{-1}) t_1 t_2$
$\Rightarrow Mt_1 t_2 t_{15} t_3 = t_{14} t_5 \in [1, 2]$
$\qquad$ (Since $\{N(t_1 t_2)^n | n \in N\}$ and $(x^{-1}, y^{-1}) \in M$),

$Mt_1 t_2 t_{15} t_5 = y^{-1} xyxy^{-2} x t_{13} t_3 t_9 t_{14} t_{12} t_6 t_1$
$\Rightarrow Mt_1 t_2 t_{15} t_5 = t_{14} t_{12} t_6 t_1 \in [1, 6, 2, 9]$
$\qquad$ (Since $\{N(t_1 t_6 t_2 t_9)^n | n \in N\}$ and $y^{-1} xyxy^{-2} x t_{13} t_3 t_9 \in M$),

$Mt_1 t_2 t_{15} t_6 = x^2 y^{-1} x^{-1} t_{12} t_{14} t_8 t_1 t_{12} t_{15} t_8$

$\Rightarrow M t_1 t_2 t_{15} t_6 = t_{12} t_{15} t_8 \in [1, 2, 15]$

(Since $\{N(t_1 t_2 t_{15})^n | n \in N\}$ and $x^2 y^{-1} x^{-1} t_{12} t_{14} t_8 t_1 \in M$),

$M t_1 t_2 t_{15} t_{10} = y^{-3} x^2 y t_3 t_{13} t_{15} t_8 t_{13} t_2$

$\Rightarrow M t_1 t_2 t_{15} t_{10} = t_8 t_{13} t_{15} \in [1, 2, 15]$

(Since $\{N(t_1 t_2 t_{15})^n | n \in N\}$ and $y^{-3} x^2 y t_3 t_{13} t_{15} \in M$),

$M t_1 t_2 t_{15} t_{15} = M t_1 t_2, \in [1, 2]$ (2 single cosets loop back to $[1, 2]$).

N $t_1 t_7 t_4$ N

We will continue by exploring the potential double cosets proposed from the orbits of the double coset $[1, 7, 4]$. We identify that the orbits are $\{1, 7, 4\}, \{2, 5, 8\}, \{3, 7, 11\}$, $\{6, 9, 15\}, \{12, 14, 13\}$, and follow by performing right multiplication on $M t_1 t_7 t_4 t_{i's}$. Doing so could produce potential double cosets $[1, 7, 4, 4], [1, 7, 4, 2], [1, 7, 4, 3], [1, 7, 4, 6]$, $[1, 7, 4, 12]$; however, we first must consider the following relations:

$M t_1 t_7 t_4 t_4 = M t_1 t_7, \in [1, 7]$ (3 single cosets loop back to $[1, 7]$),

$M t_1 t_7 t_4 t_2 = y x^2 y x y^{-1} t_{12} t_6 t_2 t_7 t_5 t_{12}$

$\Rightarrow M t_1 t_7 t_4 t_2 = t_7 t_5 t_{12} \in [1, 7, 4]$

(Since $\{N(t_1 t_7 t_4)^n | n \in N\}$ and $y x^2 y x y^{-1} t_{12} t_6 t_2 \in M$),

$M t_1 t_7 t_4 t_3 = x^2 y^{-1} x^{-1} y t_{14} t_{10} t_4 t_{13} t_{14} t_8$

$\Rightarrow M t_1 t_7 t_4 t_3 = t_{13} t_{14} t_8 \in [1, 6, 2]$

(Since $\{N(t_1 t_6 t_2)^n | n \in N\}$ and $x^2 y^{-1} x^{-1} y t_{14} t_{10} t_4 \in M$),

$M t_1 t_7 t_4 t_6 = y^{-3} x^{-1} y^{-1} x^{-1} t_4 t_{12} t_9 t_{14} t_{13} t_6 t_9$

$\Rightarrow M t_1 t_7 t_4 t_6 = t_{13} t_6 t_9 \in [1, 7, 2]$

(Since $\{N(t_1 t_7 t_2)^n | n \in N\}$ and $y^{-3} x^{-1} y^{-1} x^{-1} t_4 t_{12} t_9 t_{14} \in M$),

$M t_1 t_7 t_4 t_{12} = y x y^{-2} t_{11} t_{10} t_1 t_7 t_{11} t_2$

$\Rightarrow M t_1 t_7 t_4 t_{12} = t_7 t_{11} t_2 \in [1, 6, 2]$

(Since $\{N(t_1 t_6 t_2)^n | n \in N\}$ and $y x y^- 2 t_{11} t_{10} t_1 \in M$).

Notice that there are no new double cosets. The single cosets from the orbits $\{3, 7, 11\}$ and $\{12, 13, 14\}$ continue to the double coset $[1, 6, 2]$, and the single cosets form $\{2, 5, 8\}$ and $\{6, 9, 15\}$ move to the double cosets $[1, 7, 4]$ and $[1, 7, 2]$ respectively. While the orbit representative from the potentially new double coset $[1, 7, 4, 4]$ would obviously loop back to $[1, 7]$. The order of the coset stabilizing group of the coset $[1, 7, 4]$ is generated by $N^{(1,7,4)} \leq< (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)$,

$(2, 5)(3, 13)(4, 10)(7, 14)(9, 15)(11, 12)(10, 12)(13, 15) >\le 6$; thus the order of the coset stabilizer is $N^{(1,7,4)} = \frac{|N|}{|N^{(1,7,4)}|} = \frac{60}{6} = 10$. Therefore, the index is now $1 + 15 + 5 + 1 + 30 + 30 + 30 + 6 + 10 = 128$.

N $t_1 t_7 t_2$ **N**

We will continue by exploring the potential double cosets proposed from the orbits of the double coset $[1, 7, 2]$. We identify that the orbits are $\{1\}, \{6\}, \{8\}, \{2, 11\}, \{3, 4\}$, $\{5, 12\}, \{7, 9\}, \{10, 13\}, \{14, 15\}$, and follow by performing right multiplication on $Mt_1 t_7 t_2 t_{i's}$. Doing so could produce potential double cosets $[1, 7, 2, 1], [1, 7, 2, 6], [1, 7, 2, 8]$, $[1, 7, 2, 2], [1, 7, 2, 3] [1, 7, 2, 5], [1, 7, 2, 7] [1, 7, 2, 10]$, and $[1, 7, 2, 14]$; however, we first must consider the following relations:

$Mt_1 t_7 t_2 t_1 = x^{-1} yx^{-1} yt_9 t_3 t_{13} t_{11} t_{12}$

$\Rightarrow Mt_1 t_7 t_2 t_1 = t_{11} t_{12} \in [1, 7]$

$\qquad$ (Since $\{N(t_1 t_7)^n | n \in N\}$ and $x^{-1} yx^{-1} yt_9 t_3 t_{13} \in M$),

$Mt_1 t_7 t_2 t_6 = xyx^{-1} t_8 t_{14} t_1 t_{13} t_{12} t_8 t_3$

$\Rightarrow Mt_1 t_7 t_2 t_6 = t_{12} t_8 t_3 \in [1, 7, 2]$

$\qquad$ (Since $\{N(t_1 t_7 t_2)^n | n \in N\}$ and $xyx^{-1} t_8 t_{14} t_1 t_{13} \in M$),

$Mt_1 t_7 t_2 t_8 = yx^3 yxyt_9 t_{14} t_{11} t_{12} t_9 t_5$

$\Rightarrow Mt_1 t_7 t_2 t_8 = t_{12} t_9 t_5 \in [1, 7, 4]$

$\qquad$ (Since $\{N(t_1 t_7 t_4)^n | n \in N\}$ and $yx^3 yxyt_9 t_{14} t_{11} \in M$),

$Mt_1 t_7 t_2 t_2 = Mt_1 t_7, \in [1, 7]$ (2 single cosets loop back to $[1, 7]$),

$Mt_1 t_7 t_2 t_3 = y^{x^{-1}} yx^{-1} yt_2 t_{13} t_1 t_9 t_1 t_{11}$

$\Rightarrow Mt_1 t_7 t_2 t_3 = t_9 t_1 t_{11} \in [1, 7, 2]$

$\qquad$ (Since $\{N(t_1 t_7 t_2)^n | n \in N\}$ and $y^{x^{-1}} yx^{-1} yt_2 t_{13} t_1 \in M$),

$Mt_1 t_7 t_2 t_5 = xy^2 xyt_7 t_8 t_{14} t_{15} t_{14} t_8$

$\Rightarrow Mt_1 t_7 t_2 t_5 = t_1 t_7 t_4 \in [1, 7, 4]$

$\qquad$ (Since $\{N(t_1 t_7 t_4)^n | n \in N\}$ and $xy^2 xyt_7 t_8 t_{14} \in M$),

$Mt_1 t_7 t_2 t_{10} = xyx^3 yxt_6 t_{12} t_7 t_2 t_8$

$\Rightarrow Mt_1 t_7 t_2 t_{10} = t_2 t_8 \in [1, 7]$

$\qquad$ (Since $\{N(t_1 t_7)^n | n \in N\}$ and $xyx^3 yxt_6 t_{12} t_7 \in M$),

$Mt_1 t_7 t_2 t_{14} = xy^{-3} x^2 t_{11} t_{14} t_9 t_3 t_{10} t_8$

$\Rightarrow Mt_1 t_7 t_2 t_{14} = t_3 t_{10} t_8 \in [1, 7]$

$\qquad$ (Since $\{N(t_1 t_7 t_2)^n | n \in N\}$ and $xy^{-3} x^2 t_{11} t_{14} t_9 \in M$).

There were no new double cosets. By right multiplying by a orbit representative, all potentially new double cosets either mapped back to $[1, 7]$, looped back onto $[1, 7, 2]$, or continued on to $[1, 7, 4]$. The order of the coset stabilizing group of the coset $[1, 7, 2]$ is generated by $N^{(1,7,2)} \leq < (1, 2, 10, 13, 11)(3, 6, 4, 15, 14)(5, 12, 7, 8, 9),$
$(2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15) > \leq 10$; thus the order of the coset stabilizer is
$N^{(1,7,2)} = \frac{|N|}{|N^{(1,7,2)}|} = \frac{60}{10} = 6$. Therefore, the index is now $1 + 15 + 5 + 1 + 30 + 30 + 30 + 6 + 10 + 6 = 134$.

N $t_1 t_6 t_2 t_9$ **N**

We have finally reached our last double coset, which is $[1, 6, 2, 9]$. Just as before, we will need to find the point stabilizer, $N^{1,6,2,9}$, and the coset stabilizing group $N^{(1,6,2,9)}$ to calculate the single cosets that live in $[1, 6, 2, 9]$. The order of the coset stabilizing group is, $N^{(1,6,2,9)} \leq < (1, 2)(3, 15)(4, 6)(5, 7)(8, 9)(10, 11), (1, 6, 8)(2, 9, 4)(3, 12, 15)$
$(5, 7, 13)(10, 11, 14)) > \leq 6$. So the order of the coset stabilizer is $N^{(1,6,2,9)} = \frac{|N|}{|N^{(1,6)}|} = \frac{60}{6} = 10$, thus the order the double coset $[1, 6, 2, 9]$ contains is 10 single cosets. The index is now $1 + 15 + 5 + 1 + 30 + 30 + 30 + 6 + 10 + 6 + 10 = 144$; and we have finally reached our maximum number for our index, recall this came from the order of G divided by M. The orbits of the coset $[1, 6, 2, 9]$ are $\{1, 6, 8\}, \{2, 9, 4\}, \{3, 12, 15\}, \{5, 7, 13\}, \{10, 11, 14\}$, we will pick a representative from each orbit and right multiply, $M t_1 t_6 t_2 t_9 t_i$:
$M t_1 t_6 t_2 t_9 t_1 = xy^{-1}xy^{-1}xyt_{12}t_7t_{15}t_9t_2t_{13}$
$\Rightarrow M t_1 t_6 t_2 t_9 t_1 = t_9 t_2 t_{13} \in [1, 6, 2]$
$\qquad$ (Since $\{N(t_1 t_6 t_2)^n | n \in N\}$ and $xy^{-1}xy^{-1}xyt_{12}t_7t_{15} \in M$),
$M t_1 t_6 t_2 t_9 t_9 = M t_1 t_6 t_2, \in [1, 6, 2]$ (3 single cosets loop back to $[1, 7]$),
$M t_1 t_6 t_2 t_9 t_3 = y^{-1}x^{-1}t_5t_4$
$\Rightarrow M t_1 t_6 t_2 t_9 t_3 = t_5 t_4 \in [1, 2]$
$\qquad$ (Since $\{N(t_1 t_2)^n | n \in N\}$ and $y^{-1}x^{-1} \in M$),
$M t_1 t_6 t_2 t_9 t_5 = xyx^{-1}y^2 t_{12}t_4t_1t_{13}t_{14}t_{11}t_7$
$\Rightarrow M t_1 t_6 t_2 t_9 t_5 = t_{13}t_{14}t_{11}t_7 \in [1, 6, 2, 9]$
$\qquad$ (Since $\{N(t_1 t_6 t_2 t_9)^n | n \in N\}$ and $xyx^{-1}y^2 t_{12}t_4t_1 \in M$),
$M t_1 t_6 t_2 t_9 t_{10} = x^{-1}yx^{-1}yxt_{15}t_8t_9t_3t_{12}t_6$
$\Rightarrow M t_1 t_6 t_2 t_9 t_{10} = t_3 t_{12} t_6 \in [1, 2, 15]$
$\qquad$ (Since $\{N(t_1 t_2 t_{15})^n | n \in N\}$ and $x^{-1}yx^{-1}yxt_{15}t_8t_9 \in M$).
Due to the fact that our group is closed under right multiplication of t_{is}, we have

completed our double coset enumeration of G over M, refer to the Cayley graph below.

Figure 3.3: Cayley Graph of $L_2(31)$

Figure 3.4: Cayley Graph of M_{12} Over $L_2(11)$

Chapter 4

Monomial Progenitors

4.1 Monomial Presentation of $5^{*6} :_m S_5$

$G = S_5$ is generated by xx and yy, where $xx = (1, 2, 3, 4, 5)$, and $yy = (1, 2)$.

The conjugacy classes of group G are

$C1 = Id(G),$

$C2 = (1, 2),$

$C3 = (1, 2)(3, 4),$

$C4 = (1, 2, 3).$

$C5 = (1, 2, 3, 4).$

$C6 = (1, 2, 3, 4, 5).$

$C7 = (1, 2, 3)(4, 5).$

Consider the subgroup H of G given below.

$H = Id(G), (2, 5, 3, 4), (2, 3)(4, 5), (1, 4, 3, 2, 5).$

The conjugacy classes of H are

$D1 = Id(G),$

$D2 = (1, 3)(2, 5),$

$D3 = (1, 2, 3, 5),$

$D4 = (1, 4, 3, 2, 5).$

Consider the irreducible characters ϕ (of H) and ϕ^G (of G) given below.

Class	$D1$	$D2$	$D3$	$D4$	$D5$
Size	1	5	5	5	4
Representative	$Id(G)$	$(1,3)(2,5)$	$(1,2,3,5)$	$(1,5,3,2)$	$(1,4,3,2,5)$
ϕ	1	-1	-I	I	1

Class	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$	$C7$
Size	1	10	15	20	30	24	20
Representative	$Id(G)$	$(1,2)$	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4)$	$(1,2,3,4,5)$	$(1,3,2)(4,5)$
ϕ^G	6	0	-2	0	0	1	0

Now we will induce the character $\phi = \chi.3$ of H up to $\chi.7$ of G to obtain the character ϕ^G of G

$$\phi \uparrow_H^G$$

$\phi_\alpha^G = \frac{n}{h_\alpha} \sum_{w \in H \cap C_\alpha} \phi(w)$, where $n = \frac{|G|}{|H|} = \frac{120}{20} = 6.$

$\phi_1^G = \frac{6}{1} \sum_{w \in H \cap C_1} \phi(w)$

So, $\phi_1^G = 6(\phi(1)) = 6(1) = 6.$

$\phi_2^G = \frac{6}{10} \sum_{w \in H \cap C_2} \phi(w)$

So, $\phi_2^G = \frac{3}{5}(\phi(0) = \frac{3}{5}(0) = 0.$

$\phi_3^G = \frac{6}{15} \sum_{w \in H \cap C_3} \phi(w)$

So, $\phi_3^G = \frac{2}{5}(5\phi(1,3)(2,5)) = \frac{2}{5}(5)(-1) = \frac{2}{5}(-5) = -2.$

$\phi_4^G = \frac{6}{20} \sum_{w \in H \cap C_4} \phi(w)$

So, $\phi_4^G = \frac{3}{10}(\phi(0)) = \frac{3}{4}(0) = 0.$

$\phi_5^G = \frac{6}{30} \sum_{w \in H \cap C_5} \phi(w)$

So, $\phi_5^G = \frac{1}{5}(\phi(0)) = \frac{1}{5}(0) = 0.$

$\phi_6^G = \frac{6}{24} \sum_{w \in H \cap C_6} \phi(w)$

So, $\phi_6^G = \frac{1}{4}(\phi(1,4,3,2,5)) = \frac{1}{4}(4)(1) = 1.$

$\phi_7^G = \frac{6}{20} \sum_{w \in H \cap C_7} \phi(w)$

So, $\phi_7^G = \frac{3}{10}(\phi(0)) = \frac{3}{4}(0) = 0.$

$\phi \uparrow_H^G = 6, 0, -2, 0, 0, 1, 0.$

Now to show the monomial representation has the generators

$$A(xx) = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{vmatrix} \text{ and } A(yy) = \begin{vmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}.$$

$G = He \cup H(1,2,3,4,5) \cup H(1,3,2,5) \cup H(1,3,5,2,4) \cup H(2,3,4,5) \cup H(1,5,2,3)$

Let $t_1 = e$, $t_2 = (1,2,3,4,5)$, and $t_3 = (1,3,2,5)$, $t_4 = (1,3,5,2,4)$, $t_5 = (2,3,4,5)$, $t_6 = (1,3,2,5).$

Then

$$
A(xx) = \begin{bmatrix}
\phi(t_1 x t_1^{-1}) & \phi(t_1 x t_2^{-1}) & \phi(t_1 x t_3^{-1}) & \phi(t_1 x t_4^{-1}) & \phi(t_1 x t_5^{-1}) & \phi(t_1 x t_6^{-1}) \\
\phi(t_2 x t_1^{-1}) & \phi(t_2 x t_2^{-1}) & \phi(t_2 x t_3^{-1}) & \phi(t_2 x t_4^{-1}) & \phi(t_2 x t_5^{-1}) & \phi(t_2 x t_6^{-1}) \\
\phi(t_3 x t_1^{-1}) & \phi(t_3 x t_2^{-1}) & \phi(t_3 x t_3^{-1}) & \phi(t_3 x t_4^{-1}) & \phi(t_3 x t_5^{-1}) & \phi(t_3 x t_6^{-1}) \\
\phi(t_4 x t_1^{-1}) & \phi(t_4 x t_2^{-1}) & \phi(t_4 x t_3^{-1}) & \phi(t_4 x t_4^{-1}) & \phi(t_4 x t_5^{-1}) & \phi(t_4 x t_6^{-1}) \\
\phi(t_5 x t_1^{-1}) & \phi(t_5 x t_2^{-1}) & \phi(t_5 x t_3^{-1}) & \phi(t_5 x t_4^{-1}) & \phi(t_5 x t_5^{-1}) & \phi(t_5 x t_6^{-1}) \\
\phi(t_6 x t_1^{-1}) & \phi(t_6 x t_2^{-1}) & \phi(t_6 x t_3^{-1}) & \phi(t_6 x t_4^{-1}) & \phi(t_6 x t_5^{-1}) & \phi(t_6 x t_6^{-1})
\end{bmatrix}
$$

Row 1

$\phi(t_1 x t_1^{-1}) = \phi(t_1(1,2,3,4,5)t_1^{-1}) = \phi(1,2,3,4,5) = 0$

$\phi(t_1 x t_2^{-1}) = \phi(t_1(1,2,3,4,5)t_2^{-1}) = \phi((1,2,3,4,5)(1,2,3,4,5)^{-1}) = 1$

$\phi(t_1 x t_3^{-1}) = \phi(t_1(1,2,3,4,5)t_3^{-1}) = \phi((1,2,3,4,5)(1,5,2,3)) = \phi(1,3,4,2) = 0$

$\phi(t_1 x t_4^{-1}) = \phi(t_1(1,2,3,4,5)t_4^{-1}) = \phi((1,2,3,4,5)(1,4,2,5,3)) = \phi(1,5,4,3,2) = 0$

$\phi(t_1 x t_5^{-1}) = \phi(t_1(1,2,3,4,5)t_5^{-1}) = \phi((1,2,3,4,5)(2,5,4,3)) = \phi(1,5) = 0$

$\phi(t_1 x t_6^{-1}) = \phi(t_1(1,2,3,4,5)t_6^{-1}) = \phi((1,2,3,4,5)(1,3,2,5)) = \phi(1,5,3,4) = 0$

Row 2

$\phi(t_2 x t_1^{-1}) = \phi(t_2(1,2,3,4,5)t_1^{-1}) = \phi((1,2,3,4,5)(1,2,3,4,5)) = \phi(1,3,5,2,4) = 0$

$\phi(t_2 x t_2^{-1}) = \phi(t_2(1,2,3,4,5)t_2^{-1}) = \phi((1,2,3,4,5)(1,2,3,4,5)(1,2,3,4,5)^{-1})$

$= \phi(1,2,3,4,5) = 0$

$\phi(t_2 x t_3^{-1}) = \phi(t_2(1,2,3,4,5)t_3^{-1}) = \phi((1,2,3,4,5)(1,2,3,4,5)(1,5,2,3))$

$= \phi(2,4,5,3) = 0$

$\phi(t_2 x t_4^{-1}) = \phi(t_2(1,2,3,4,5)t_4^{-1}) = \phi((1,3,5,2,4)(1,4,2,5,3))$

$= \phi((1)(2)(3)(4)(5)) = 1$

$\phi(t_2 x t_5^{-1}) = \phi(t_2(1,2,3,4,5)t_5^{-1}) = \phi((1,3,5,2,4)(2,5,4,3)) = \phi(1,2,3,4) = 0$

$\phi(t_2 x t_6^{-1}) = \phi(t_2(1,2,3,4,5)t_6^{-1}) = \phi((1,3,5,2,4)(1,3,2,5)) = \phi(1,2,4,3) = 0$

Row 3

$\phi(t_3xt_1^{-1}) = \phi(t_3(1,2,3,4,5)t_1^{-1} = \phi((1,3,2,5)(1,2,3,4,5)) = \phi(1,4,5,2) = 3$

$\phi(t_3xt_2^{-1}) = \phi(t_3(1,2,3,4,5)t_2^{-1} = \phi((1,3,2,5)(1,2,3,4,5)(1,5,4,3,2)$

$= \phi(1,3,2,5) = 0$

$\phi(t_3xt_3^{-1}) = \phi(t_3(1,2,3,4,5)t_3^{-1} = \phi((1,3,2,5)(1,2,3,4,5)(1,5,2,3))$

$= \phi(1,4,2,5,3) = 0$

$\phi(t_3xt_4^{-1}) = \phi(t_3(1,2,3,4,5)t_4^{-1} = \phi((1,3,2,5)(1,2,3,4,5)(1,4,2,5,3))$

$= \phi(1,2,4,3) = 0$

$\phi(t_3xt_5^{-1}) = \phi(t_3(1,2,3,4,5)t_5^{-1} = \phi((1,3,2,5)(1,2,3,4,5)(2,5,4,3))$

$= \phi(1,3,2) = 0$

$\phi(t_3xt_6^{-1}) = \phi(t_3(1,2,3,4,5)t_6^{-1} = \phi((1,3,2,5)(1,2,3,4,5)(1,3,2,5))$

$= \phi(1,2)(2,3)) = 0$

Row 4

$\phi(t_4xt_1^{-1}) = \phi(t_4(1,2,3,4,5)t_1^{-1} = \phi((1,3,5,2,4)(1,2,3,4,5))$

$= \phi(1,4,2,5,3) = 0$

$\phi(t_4xt_2^{-1}) = \phi(t_4(1,2,3,4,5)t_2^{-1} = \phi((1,3,5,2,4)(1,2,3,4,5)(1,5,4,3,2)$

$= \phi(1,3,5,2,4) = 0$

$\phi(t_4xt_3^{-1}) = \phi(t_4(1,2,3,4,5)t_3^{-1} = \phi((1,3,5,2,4)(1,2,3,4,5)(1,5,2,3))$

$= \phi(1,2,3,4,5) = 0$

$\phi(t_4xt_4^{-1}) = \phi(t_4(1,2,3,4,5)t_4^{-1} = \phi((1,3,5,2,4)(1,2,3,4,5)(1,4,2,5,3))$

$= \phi(1,2,4,3) = 0$

$\phi(t_4xt_5^{-1}) = \phi(t_4(1,2,3,4,5)t_5^{-1} = \phi((1,3,5,2,4)(1,2,3,4,5)(2,5,4,3))$

$= \phi(1,3)(2,4,5)) = 0$

$\phi(t_4xt_6^{-1}) = \phi(t_4(1,2,3,4,5)t_6^{-1} = \phi((1,3,5,2,4)(1,2,3,4,5)(1,3,2,5))$

$= \phi((1,4,5,2)(3)) = -2 = 3$

Row 5

$\phi(t_5xt_1^{-1}) = \phi(t_5(1,2,3,4,5)t_1^{-1} = \phi((2,3,4,5)(1,2,3,4,5)) = \phi((1,2,4)(3,5)) = 0$

$\phi(t_5xt_2^{-1}) = \phi(t_5(1,2,3,4,5)t_2^{-1} = \phi((2,3,4,5)(1,2,3,4,5)(1,5,4,3,2))$

$= \phi((1)(2,3,4,5)) = 0$

$\phi(t_5 x t_3^{-1}) = \phi(t_5(1,2,3,4,5)t_3^{-1} = \phi((2,3,4,5)(1,2,3,4,5)(1,5,2,3))$

$= \phi(1,3,2,4,5) = 0$

$\phi(t_5 x t_4^{-1}) = \phi(t_5(1,2,3,4,5)t_4^{-1} = \phi((2,3,4,5)(1,2,3,4,5)(1,4,2,5,3))$

$= \phi((1,5)(2)(3)(4)) = 0$

$\phi(t_5 x t_5^{-1}) = \phi(t_5(1,2,3,4,5)t_5^{-1} = \phi((2,3,4,5)(1,2,3,4,5)(2,5,4,3))$

$= \phi(1,5,2,3,4) = 1$

$\phi(t_5 x t_6^{-1}) = \phi(t_5(1,2,3,4,5)t_6^{-1} = \phi((2,3,4,5)(1,2,3,4,5)(1,3,2,5))$

$= \phi(1,5,2,4,3) = 0$

Row 6

$\phi(t_6 x t_1^{-1}) = \phi(t_6(1,2,3,4,5)t_1^{-1} = \phi((1,5,2,3)(1,2,3,4,5)) = \phi(2,4,5,3) = 0$

$\phi(t_6 x t_2^{-1}) = \phi(t_6(1,2,3,4,5)t_2^{-1} = \phi((1,5,2,3)(1,2,3,4,5)(1,5,4,3,2))$

$= \phi((4)(1,5,2,3)) = 0$

$\phi(t_6 x t_3^{-1}) = \phi(t_6(1,2,3,4,5)t_3^{-1} = \phi((1,5,2,3)(1,2,3,4,5)(1,5,2,3))$

$= \phi((1,5)(2,4)(3)) = -1 = 4$

$\phi(t_6 x t_4^{-1}) = \phi(t_6(1,2,3,4,5)t_4^{-1} = \phi((1,5,2,3)(1,2,3,4,5)(1,4,2,5,3))$

$= \phi((2)(1,4,3,5)) = 0$

$\phi(t_6 x t_5^{-1}) = \phi(t_6(1,2,3,4,5)t_5^{-1} = \phi((1,5,2,3)(1,2,3,4,5)(2,5,4,3))$

$= \phi(2,3,5) = 0$

$\phi(t_6 x t_6^{-1}) = \phi(t_6(1,2,3,4,5)t_6^{-1} = \phi((1,5,2,3)(1,2,3,4,5)(1,3,2,5))$

$= \phi(1,3,5,2,4) = 0$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix}$$

$$A(yy) = \begin{bmatrix} \phi(t_1yt_1^{-1}) & \phi(t_1yt_2^{-1}) & \phi(t_1yt_3^{-1}) & \phi(t_1yt_4^{-1}) & \phi(t_1yt_5^{-1}) & \phi(t_1yt_6^{-1}) \\ \phi(t_2yt_1^{-1}) & \phi(t_2yt_2^{-1}) & \phi(t_2yt_3^{-1} & \phi(t_2yt_4^{-1}) & \phi(t_2yt_5^{-1}) & \phi(t_2yt_6^{-1})) \\ \phi(t_3yt_1^{-1}) & \phi(t_3yt_2^{-1}) & \phi(t_3yt_3^{-1}) & \phi(t_3yt_4^{-1}) & \phi(t_3yt_5^{-1}) & \phi(t_3yt_6^{-1}) \\ \phi(t_4yt_1^{-1}) & \phi(t_4yt_2^{-1}) & \phi(t_4yt_3^{-1}) & \phi(t_4yt_4^{-1}) & \phi(t_4yt_5^{-1}) & \phi(t_4yt_6^{-1}) \\ \phi(t_5yt_1^{-1}) & \phi(t_5yt_2^{-1}) & \phi(t_5yt_3^{-1}) & \phi(t_5yt_4^{-1}) & \phi(t_5yt_5^{-1}) & \phi(t_5yt_6^{-1}) \\ \phi(t_6yt_1^{-1}) & \phi(t_6yt_2^{-1}) & \phi(t_6yt_3^{-1}) & \phi(t_6yt_4^{-1}) & \phi(t_6yt_5^{-1}) & \phi(t_6yt_6^{-1}) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now to find a permutation representation of $A(xx)$ and $A(yy)$ of the monomial representation.

$$A(xx) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \end{bmatrix}$$

thus $a_{12} = 1$, $a_{24} = 1$, $a_{31} = 3$, $a_{46} = 3$, $a_{55} = 1$, $a_{63} = 4$.

Therefore,

$t_1 \rightarrow t_2$,

$t_2 \to t_4,$

$t_3 \to t_1^3.$

$t_4 \to t_6^3,$

$t_5 \to t_5,$

$t_6 \to t_3^4.$

1	2	3	4	5	6	7	8	9	10	11	12
t_1	t_2	t_3	t_4	t_5	t_6	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2	t_4	t_1^3	t_6^3	t_5	t_3^4	t_2^2	t_4^2	t_1	t_6	t_5^2	t_3^3
2	4	13	18	5	21	8	10	1	6	11	15

(continued)

13	14	15	16	17	18	19	20	21	22	23	24
t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
t_2^3	t_4^3	t_1^4	t_6^4	t_5^3	t_3^2	t_2^4	t_4^4	t_1^2	t_6^2	t_5^4	t_3^1
14	16	19	24	17	9	20	22	7	12	23	3

Therefore, $A(xx) = (1, 2, 4, 18, 9)(3, 13, 14, 16, 24)(5)$

$(6, 21, 7, 8, 10)(11)(17)(23)(12, 15, 19, 20, 22).$

Similarly for $A(yy)$,

$$A(yy) = \begin{bmatrix} 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

thus $a_{13} = 4$, $a_{25} = 1$, $a_{31} = 4$, $a_{46} = 1$, $a_{52} = 1$, $a_{64} = 1$.

Therefore,

$t_1 \to t_3^4,$

$t_2 \to t_5,$

$t_3 \to t_1^4.$

$t_4 \to t_6,$

$t_5 \to t_2,$

$t_6 \to t_4.$

1	2	3	4	5	6	7	8	9	10	11	12
t_1	t_2	t_3	t_4	t_5	t_6	t_1^2	t_2^2	t_3^2	t_4^2	t_5^2	t_6^2
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_3^4	t_5	t_1^4	t_6	t_2	t_4	t_3^3	t_5^2	t_1^3	t_6^2	t_2^2	t_4^2
21	5	19	6	2	4	15	11	13	12	8	10

(continued)

13	14	15	16	17	18	19	20	21	22	23	24
t_1^3	t_2^3	t_3^3	t_4^3	t_5^3	t_6^3	t_1^4	t_2^4	t_3^4	t_4^4	t_5^4	t_6^4
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
t_3^2	t_5^3	t_1^2	t_6^3	t_2^3	t_4^3	t_3	t_5^4	t_1	t_6^4	t_2^4	t_4^4
9	17	7	18	14	16	3	23	1	24	20	22

Therefore, $A(yy) = (1, 21)(2, 5)(3, 19)(4, 6)(7, 15)(8, 11)$
$(12, 10)(13, 9)(14, 17)(16, 18)(20, 23)(24, 22).$

Now we will built presentation of the monomial progenitor $5^{*6} :_m S_5$; we know that the presentation for S_5 is $< x, y | x^5, y^2, (x^{-1}y)^4, (xyx^{-2}yx)^2 >$.

We need to find the Normaliser$\{t_1, t_1^2, t_1^3, t_1^4\}$.

And we chose for this set to map to $1, 7, 13, 19$ respectively in the table above.

Stabiliser$(N, \{1, 7, 13, 19\}) = (2, 22, 11, 12, 9)(3, 14, 10, 5, 6)(4, 17, 18, 15, 20)$
$(8, 16, 23, 24, 21),$

and $(1, 13, 19, 7)(2, 21, 4, 5)(3, 22, 23, 20)(6, 12, 24, 18)(8, 15, 10, 11)(9, 16, 17, 14).$

Therefore, t_1's commute with $(1, 13, 19, 7)(2, 21, 4, 5)(3, 22, 23, 20)$
$(6, 12, 24, 18)(8, 15, 10, 11)(9, 16, 17, 14)$.

Thus, our presentation is

$$< x, y, t | x^5, y^2, (x^- 1 * y)^4, (x * y * x^- 2 * y * x)^2, t^5, (t, x^2 * y * x * y), t^{(x^{-1}y)} = t^3 >.$$

Now we can add relations and find finite homomorphic images.

Chapter 5

Extension Problems

5.1 $G \cong L_2(31)^{\bullet}2$

We produced this group $G1$ from the progenitor written of Symmetric Group 15, we added relations and left it running in the background with the help of NANO. The powers of the relations are now the parameters of the group G. Thus we have $r1 := 0; r2 := 0; r3 := 0; r3 := 0; r4 := 0; r5 := 4; r6 := 0; r7 := 10; r8 := 3;$ $G < x, y, z, t >:= Group < x, y, z, t | y^2, z^3, (x^{-1}y)^2, (x, z), (yz^{-1})^2, x^{-5}, t^2, (t, yxz),$ $(yt)^{r1}, (xt)^{r2}, (xzt)^{r3}, (zx^{-1}t)^{r4}, (zt)^{r5}, (x^2t)^{r6}, (x^2z^{-1}t)^{r7}, (x^2zt)^{r8} >$. Let's us now disregard the parameters with are being set equal to zero; $G =< x, y, z, t | y^2, z^3, (x^{-1}y)^2,$ $(x, z), (yz^{-1})^2, x^{-5}, t^2, (t, yxz), (zt)^4, (x^2z^{-1}t)^{10}, (x^2zt)^3 >$, this will be the representation that we will prove in this isomorphism type. The finite representation of $G1$ is made up of the composition factors below,

```
>  CompositionFactors(G1);
    G
    |   A(1, 31)                         = L(2, 31)
    *
    |   Cyclic(2)
    1  *
```

and the composition series of G is

$$G = G_0 \supseteq G1 \supseteq G2 \text{ where } G2 = 1,$$

and the composition factors are,

$$G = (G_0/G_1)(G_1/G_2)$$
$$= (G_0/G_1)(G_1)$$
$$= L(2, 31)C_2.$$

Our goal is to solve the extension problem. Based on the compositions series it is going to come down to a type of association between the two subgroups of $G1$, which are Cyclic Group 2, which we will refer to as C_2, and Linear Group 2,31, denoted $PSL(2, 31)$ or $L(2, 31)$. To initiate, we usually start off by looking for a largest abelian group to factor by, however, lets us try a different (if at all possible) approach- lets start off by asking MAGMA if the group $G1$ has a center.

```
>Center(G1);
    Permutation group acting on a set of cardinality 992
    Order = 2*
```

Great, we see that this group $G1$ has a center that we can factor by. This starting point can be executed with any group, however, not every group will have a center, then our next attempt would have to be to look for a largest abelian group within $G1$. Secondly, let's consider the make up, or normal lattice of the group,

```
> NL:=NormalLattice(G1);
> NL;

Normal subgroup lattice
-----------------------
```

```
[4]  Order 29760  Length 1  Maximal Subgroups: 2 3
---
[3]  Order 14880  Length 1  Maximal Subgroups: 1
---
[2]  Order 2      Length 1  Maximal Subgroups: 1
---
[1]  Order 1      Length 1  Maximal Subgroups:
```

We see here that we have a factor or order 2 which MAGMA labeled as $NL[2]$, surely this is our center of order 2. Following this, we want to separate this subgroup of order 2 from the rest of $G1$, so we would factor $G1$ by $NL[2]$

```
> q,ff:=quo<G1|NL[2]>;
> q;
Permutation group q acting on a set of cardinality 496
Order = 14880 = 2^5 * 3 * 5 * 31
```

By looking at our composition factors of $G1$, we can already begin to assume that our new subgroup q is $L_2(31)$, nevertheless, we must prove that is is so,

```
> IsIsomorphic(q,PSL(2,31));
true Homomorphism of GrpPerm: q, Degree 496,
Order 2^5 * 3 * 5 * 31 into
GrpPerm: $, Degree 32, Order 2^5 * 3 * 5 * 31 induced by...
```

There are three generators of our quotient group q; however, the generators are very large to include in this document, but this can easily be done with the code $Generators(q)$. To write the presentation of the isomorphism type, we must produce a presentation of q, with the generators a,b,c,d.

```
>  FPGroup(q);
Finitely presented group on 4 generators
Relations
    $.1^5 = Id($)
    $.2^2 = Id($)
    $.3^3 = Id($)
    $.4^2 = Id($)
    ($.1^-1 * $.2)^2 = Id($)
    ($.1, $.3) = Id($)
    ($.2 * $.3^-1)^2 = Id($)
    $.4 * $.3^-1 * $.4 * $.1^-1 * $.2 * $.4 * $.3 = Id($)
```

```
$.1 * $.4 * $.3^-1 * $.2 * $.1 * $.4 * $.2 * $.3 = Id($)
($.1^-1 * $.4 * $.2 * $.4 * $.2)^2 = Id($)
$.1^-1 * $.4 * $.3^-1 * $.4 * $.3^-1 * $.1^2 * $.4 * $.1 *
$.2 * $.4 *
$.1^-1 = Id($)
$.1^-1 * $.4 * $.1^-1 * $.4 * $.1 * $.4 * $.3^-1 * $.1 *
$.4 * $.1 * $.4 *
$.1^-1 * $.4 * $.3 = Id($)

b^2,
c^3,
d^2,
(a^{-1} * b)^2,
(a, c),
(b * c^{-1)^2,
d * c^-1 * d * a^-1 * b * d * c,
a * d * c^-1 * b * a * d * b * c,
(a^-1 * d * b * d * b)^2,
a^-1 * d * c^-1 * d * c^-1 * a^2 * d * a * b * d *
a^-1,
a^-1 * d * a^-1 * d * a * d * c^-1 * a * d * a * d *
a^-1 * d * c.
```

We translated the code from MAGMA which was given using dollar signs and numbers to represent the words $a, b, c,$ and d. Reason being that we reserve the ending of the alphabet when building our progenitor for a group; therefore we use the beginning of the alphabet when proving the isomorphic type or our homomorphic finite image of our group $G1$. So that we have, $a^5, b^2, c^3, d^2, (a^-1b)^2, (a, c), (bc^{-1})^2,$

$dc^{-1}da^{-1}bdc, adc^{-1}badbc, (a^{-1}dbdb)^2, a^{-1}dc^{-1}dc^{-1}a^2dabda^{-1}, a^{-1}da^{-1}dadc^{-1}adada^{-1}dc$

as our presentation for q. Next, to find the generators of C_2, we will first identify the generators of $NL[2]$ and store it as e. We will name the prsentation of the isomorphism type H. Lastly, every generator of q must commute with e, and this addition is added to the end of presentation for q, thus we have:

```
>  H<a,b,c,d,e>:=Group<a,b,c,d,e|a^5,b^2,c^3,d^2,(a^-1 * b)^2,
> (a, c),(b*c^-1)^2,d*c^-1*d*a^-1*b*d*c,a*d*c^-1*b*a*d*b*c,
> (a^-1*d*b*d*b)^2,a^-1*d*c^-1*d*c^-1*a^2*d*a*b*d* a^-1,
> a^-1*d*a^-1*d*a*d*c^-1*a*d*a*d*a^-1*d*c,
> e^2,(a,e), (b,e), (c,e), (d,e)>;
> f,H1,k:=CosetAction(H,sub<H|Id(H)>);
```

```
> s,t:=IsIsomorphic(G1,H1);
> s;
true
> t;
Homomorphism of GrpPerm: G1, Degree 992, Order 2^6 * 3 * 5 * 31
into GrpPerm:
H1, Degree 29760, Order 2^6 * 3 * 5 * 31 induced by
```

As we can see above, we solved this extension problem by verifying that H is isomorphic to $G1$. Therefore we have proved $2^{\bullet}L(2, 31)$ is the isomorphism type of $G1$.

5.2 $\quad G \cong A_{(}5) : 2^4$

Consider the following finite homomorphic image,
$G =< x, y, t | x^5, y^3, (xy^{-1}x)^2, t^2, (y, t), ((x^{-1}, y^{-1}), t), (yx^{-1}t)^5, (xyx^{-1}y^{-1}xyt)^4 >$, of which we will prove its isomorphism type. The finite representation of $G1$ is made up of the composition factors below,

```
>  CompositionFactors(G1);
   G
   |  Alternating(5)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   1
```

and the composition series of G is

$$G = G_0 \supseteq G1 \supseteq G2 \supseteq G3 \supseteq G4 \supseteq G5 \text{ where } G5 = 1,$$

and the composition factors are,

$$G = (G_0/G_1)(G_1/G_2)(G_2/G_3)(G_3/G_4)(G_4/G_5)$$
$$= Alt(5)C_2 C_2 C_2 C_2.$$

Let us now review the normal lattice, and follow by factoring by the largest abelian group,

```
> NL:=NormalLattice(G1);
> NL;

Normal subgroup lattice
-----------------------

[3]  Order 960  Length 1  Maximal Subgroups: 2
---
[2]  Order 16   Length 1  Maximal Subgroups: 1
---
[1]  Order 1    Length 1  Maximal Subgroups:

> for i in [1..#NL] do if IsAbelian(NL[i]) then i;
> end if; end for;
1
2.
```

By observing the code above, we see that the largest abelian subgroup of $G1$ is $NL[2]$, we will thus proceed by factoring by the largest abelian group,

```
> q,ff:=quo<G1|NL[2]>;
> q;
Permutation group q acting on a set of cardinality 6
Order = 60 = 2^2 * 3 * 5
    (2, 3, 5, 4, 6)
    (1, 2, 4)(3, 5, 6)
    Id(q).
```

As a result, we have now divided the group $G1$ into two, where on one hand we have q and on the other $NL[2]$. Consider the subgroup $NL[2]$ of order 16, could this consist

of the string of cyclic groups that were shown in the composition factors? We would answer this questions with the help of MAGMA, examine the following code,

```
> X:=[2,2,2,2];
> IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
true Mapping from: GrpPerm: $, Degree 16, Order 2^4 to GrpPerm:
$, Degree 8, Order 2^4
Composition of Mapping from: GrpPerm: $, Degree 16, Order 2^4
to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: $, Degree 8, Order 2^4
```

thus we have proven that $NL[2]$ is four Cyclic Group (2), meaning $C_2{}^4$. The order of q is 60 similarly to Alt(5) which is also listed in the composition factors above, let us verify to see if indeed this is correct:

```
> IsIsomorphic(q, Alt(5));
true Homomorphism of GrpPerm: q, Degree 6, Order 2^2 * 3 * 5
into GrpPerm: $,
Degree 5, Order 2^2 * 3 * 5 induced by
    (2, 3, 5, 4, 6) |--> (1, 2, 4, 3, 5)
    (1, 2, 4)(3, 5, 6) |--> (1, 5, 2)
    Id(q) |--> Id($)
```

surely enough q is isomorphic to Alternating Group A_5. Finally, we will write a presentation uniting the two subgroups q and $NL[2]$ as a semi-direct product.

To continue we will need to work with transversals, so that we may relate elements of q that are not in $NL[2]$ to elements in $G1$, this is done in the code below:

```
> T:=Transversal(G1,NL[2]);
> NL[2];
Permutation group acting on a set of cardinality 16
Order = 16 = 2^4
    Id($)
    (1, 2)(3, 6)(4, 8)(5, 9)(7, 11)(10, 12)(13, 16)(14, 15)
    (1, 3)(2, 6)(4, 14)(5, 10)(7, 13)(8, 15)(9, 12)(11, 16)
    (1, 5)(2, 9)(3, 10)(4, 16)(6, 12)(7, 15)(8, 13)(11, 14)
    (1, 7)(2, 11)(3, 13)(4, 12)(5, 15)(6, 16)(8, 10)(9, 14)
> C:=G1!(1, 2)(3, 6)(4, 8)(5, 9)(7, 11)(10, 12)(13, 16)(14, 15);
> D:=G1!(1, 3)(2, 6)(4, 14)(5, 10)(7, 13)(8, 15)(9, 12)(11, 16);
> E:=G1!(1, 5)(2, 9)(3, 10)(4, 16)(6, 12)(7, 15)(8, 13)(11, 14);
```

```
> I:=G1!(1, 7)(2, 11)(3, 13)(4, 12)(5, 15)(6, 16)(8, 10)(9, 14);
> q.1;
(2, 3, 5, 4, 6)
> q.2;
(1, 2, 4)(3, 5, 6)
> q.3;
Id(q)
> ff(T[2]) eq q.1;
true
> ff(T[3]) eq q.2;
true
> A:=G1!T[2];
> B:=G1!T[3];
```

we have also found appropriate representations of the generators in q which we will label a and b, and the names of the generators of $NL[2]$ will be c, d, e and i. The quality that makes semi-direct products special in the presentation is that we will raise all the generators of $NL[2]$ to the power of generators of q, meaning we are left to answer the following: $c^a =?, d^a =?, e^a =?, i^a =?, c^b =?, d^b =?, e^b =?, i^b =?$. We turned to MAGMA for help consider the following code:

```
> C^A;
(1, 3)(2, 6)(4, 14)(5, 10)(7, 13)(8, 15)(9, 12)(11, 16)
> C^A eq D;
true
> D^A;
(1, 5)(2, 9)(3, 10)(4, 16)(6, 12)(7, 15)(8, 13)(11, 14)
> D^A eq E;
true
> E^A;
(1, 7)(2, 11)(3, 13)(4, 12)(5, 15)(6, 16)(8, 10)(9, 14)
> E^A eq I;
true
> I^A;
(1, 4)(2, 8)(3, 14)(5, 16)(6, 15)(7, 12)(9, 13)(10, 11)
> for i,j,k,l in [1..4] do if I^A eq C^i*D^j*E^k*I^l then
> i,j,k,l; end if; end\
 for;
1 1 1 1
1 1 1 3
1 1 3 1
1 1 3 3
1 3 1 1
```

```
1 3 1 3
1 3 3 1
1 3 3 3
3 1 1 1
3 1 1 3
3 1 3 1
3 1 3 3
3 3 1 1
3 3 1 3
3 3 3 1
3 3 3 3
> C^B;
(1, 2)(3, 6)(4, 8)(5, 9)(7, 11)(10, 12)(13, 16)(14, 15)
> C^B eq C;
true
> D^B;
(1, 3)(2, 6)(4, 14)(5, 10)(7, 13)(8, 15)(9, 12)(11, 16)
> D^B eq D;
true
> E^A;
(1, 7)(2, 11)(3, 13)(4, 12)(5, 15)(6, 16)(8, 10)(9, 14)
> for i,j,k,l in [1..4] do if E^B eq C^i*D^j*E^k*I^l then
> i,j,k,l; end if; end\
 for;
1 1 1 1
1 1 1 3
1 1 3 1
1 1 3 3
1 3 1 1
1 3 1 3
1 3 3 1
1 3 3 3
3 1 1 1
3 1 1 3
3 1 3 1
3 1 3 3
3 3 1 1
3 3 1 3
3 3 3 1
> I^B;
(1, 5)(2, 9)(3, 10)(4, 16)(6, 12)(7, 15)(8, 13)(11, 14)
> I^B eq E;
true
```

We now can write a presentation of H, and confirm that it is a semi-direct presentation of $G1$.

```
> H<a,b,c,d,e,i>:=Group<a,b,c,d,e,i|a^5, b^3, (a*b^-1*a)^2,
> c^2, d^2, e^2, i^2, (c *d)^2, (c *e)^2, (d *e)^2,
> (c *i)^2, (d *i)^2, (e *i)^2, c^a=d, d^a=e, e^a=i,
> i^a=c*d*e*i, c^b=c, d^b=d, e^b=c*d*e*i,i^b=e>;
> f1,H1,k2:=CosetAction(H,sub<H|Id(H)>);
> #H1;
960
> IsIsomorphic(G1,H1);
true Homomorphism of GrpPerm: G1, Degree 16,
Order 2^6 * 3 * 5 into GrpPerm: H1,
Degree 960, Order 2^6 * 3 * 5 induced by...
```

Therefore, we have proved the isomorphism type of $G \cong Alt(5) : 2^4$.

5.3 $G \cong 2^2 :^{\bullet} (2^4 : (3^2 : 2))$

The presentation of $G =< x, y, z, t | x^2, y^3, z^3, (y, z), x * z^- 1 * x * z, y^- 1 * x * y^- 1 * x * z, t^2, (t, z * y^- 1), (y * z * t^x)^3, (x * y * t)^6 >$

has the composition factors

```
>  CompositionFactors(G1);
   G
   |  Cyclic(2)
   *
   |  Cyclic(3)
   *
   |  Cyclic(3)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   *
   |  Cyclic(2)
   1
```

and the composition series of G is

$G = G_0 \supseteq G1 \supseteq G2 \supseteq G3 \supseteq G4 \supset G5 \supseteq G6 \supseteq G7 \supseteq G8 \supseteq G9$ where $G9 = 1$

Our goal is to solve the extension problem, first let us begin by reviewing our normal lattice of the group $G1$,

```
> NL:=NormalLattice(G1);
> NL;

Normal subgroup lattice
-----------------------

[8]  Order 1152  Length 1  Maximal Subgroups: 6 7
---
[7]  Order 576   Length 1  Maximal Subgroups: 4 5
[6]  Order 384   Length 1  Maximal Subgroups: 5
---
[5]  Order 192   Length 1  Maximal Subgroups: 3
[4]  Order 192   Length 1  Maximal Subgroups: 3
---
[3]  Order 64    Length 1  Maximal Subgroups: 2
---
[2]  Order 4     Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:.
```

With the help of MAGMA ,the following code will produce the largest abelian group,

```
> for i in [1..#NL] do if IsAbelian(NL[i]) then i; end if;
> end for;
1
2
```

this will tell us that the largest abelian subgroup from the normal lattice of group G1 is NL[2] which is of order 4. However, we are left to wonder what NL[2] is made up of, we know, thanks to our normal lattice, that it is of order 4 which leaves us to discover if this is Cyclic 4 or 2^2.

```
>X:=[2,2];
>IsIsomorphic(NL[2],AbelianGroup(GrpPerm,X));
```

```
true Mapping from: GrpPerm: $, Degree 64, Order 2^2 to
GrpPerm: $, Degree 4,
Order 2^2
```

We see that $NL[2]$ is 2^2. From here we will find the quotient group, meaning we will factor G1 by NL[2] resulting in the quotient group, which we will name q.

```
> q,ff:=quo<G1|NL[2]>;
> q;
Permutation group q acting on a set of cardinality 16
Order = 288 = 2^5 * 3^2
    (2, 3)(4, 5)(7, 8)(9, 10)(12, 14)(13, 15)
    (2, 4, 7)(6, 10, 14)(9, 11, 15)(12, 13, 16)
    (2, 4, 7)(3, 5, 8)(6, 11, 16)(9, 13, 14)(10, 15, 12)
    (1, 2)(3, 6)(4, 7)(5, 9)(8, 12)(10, 14)(11, 15)(13, 16)
> CompositionFactors(q);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
> nlq:=NormalLattice(q);
>  nlq;

Normal subgroup lattice
-----------------------

[7]  Order 288  Length 1  Maximal Subgroups: 5 6
---
[6]  Order 144  Length 1  Maximal Subgroups: 3 4
[5]  Order 96   Length 1  Maximal Subgroups: 4
---
[4]  Order 48   Length 1  Maximal Subgroups: 2
[3]  Order 48   Length 1  Maximal Subgroups: 2
```

```
---
[2]  Order 16    Length 1  Maximal Subgroups: 1
---
[1]  Order 1     Length 1  Maximal Subgroups:.
```

The quotient subgroup is order 288, and if we refer back to the normal lattice of $G1$, we notice that there isn't a subgroup of order 288, thus the relation between $NL[2]$ and q will be that of either semi-direct product or mixed extension, because there isn't a direct product relationship between them. Now we must find a form to relate q to $NL[2]$ to create an isomorphic representation of $G1$, and for this we will use Transversals.

Transversals are needed to relate elements of q that is not a subgroup of NL[2], to elements in $G1$, providinng some type of a connector between the two subgroups within $G1$.

```
> T:=Transversal(G1,NL[2]);
```

Now we need to see which of these transversals relates to q; we need to be careful here, somtimes MAGMA stores the transversals in a different order each time one runs the code,

```
> ff(T[2]) eq q.1;
true
> ff(T[3]) eq q.2;
true
> ff(T[4]) eq q.3;
true
ff(T[5]) eq q.4;.
```

Now we know which four transversals relate to q, and we also know that only four elements are neccesary to generate q (refer back to when we printed the subgroup q with the MAGMA command q;). Continue by labeling the generators of q. Where $T[2]$ is $q.1$ and will be labeled L, and similar operation $T[3]$, $T[4]$ and $T[5]$ will be labeled M, N, and O respectively .

The generatos of $NL[2]$ are,

```
> NL[2];
Permutation group acting on a set of cardinality 64
Order = 4 = 2^2
(1, 63)(2, 56)(3, 32)(4, 48)(5, 50)(6, 31)(7, 40)(8, 37)
    (9, 55)(10, 54)(11, 34)(12, 19)(13, 21)(14, 60)
```

```
      (15, 24)(16, 42)(17, 43)(18, 28)(20, 30)(22, 44)
      (23, 41)(25, 59)(26, 58)(27, 35)(29, 52)(33, 57)
      (36, 53)(38, 46)(39, 45)(47, 64)(49, 62)(51, 61)
(1, 49)(2, 29)(3, 20)(4, 64)(5, 61)(6, 53)(7, 10)(8, 57)
      (9, 41)(11, 42)(12, 38)(13, 58)(14, 44)(15, 39)
      (16, 34)(17, 27)(18, 25)(19, 46)(21, 26)(22, 60)
      (23, 55)(24, 45)(28, 59)(30, 32)(31,
```

we will label each J and K respectively. To write a representation of the semi-direct product form we must write each element of $NL[2]$ and raise it to the power of each element in q, such as,

```
j^l= j
j^m= k
j^n= j*k
j^o= j
k^l= k
k^m= j*k
k^n= j
k^o= k
```

To finish we need presentation of q

```
> FPGroup(q);
Finitely presented group on 4 generators
Relations
    $.1^2 = Id($)
    $.2^3 = Id($)
    $.3^3 = Id($)
    $.4^2 = Id($)
    ($.2, $.3) = Id($)
    $.1 * $.3^-1 * $.1 * $.3 = Id($)
    $.2^-1 * $.1 * $.2^-1 * $.1 * $.3 = Id($)
    $.2^-1 * $.4 * $.3^-1 * $.2 * $.4 * $.3 = Id($)
    ($.4 * $.2^-1)^3 = Id($)
    ($.1 * $.4)^4 = Id($)
```

which in words is, $H < j, k, l, m, n, o >:= Group < j, k, l, m, n, o | j^2, k^2, (j, k), l^2,$ $m^3, n^3, o^2, (m, n), l*n^{-1}*l*n, m^{-1}*l*m^{-1}*l*n, m^{-1}*o*n^{-1}*m*o*n, (o*m^-1)^3, (l*o)^4 >;$ to make this presentation semi-direct we will add in the equalites between q and $NL[2]$ that we have listed above, resulting in H:

$$H < j, k, l, m, n, o > := Group < j, k, l, m, n, o | j^2, k^2, (j, k), l^2, m^3,$$

$$n^3, o^2, (m, n), l*n^{-1}*l*n, m^{-1}*l*m^{-1}*l*n, m^{-1}*o*n^{-1}*m*o*n, (o*m^-1)^3, (l*o)^4,$$

$$j^l = j, j^m = k, j^n = j*k, j^o = j, k^l = k, k^m = j*k, k^n = j, k^o = k >;$$

We have finaly constructed a semi-direct presentation of $G1$. To ensure that our work is correct we will refer to MAGMA,

```
> H<j,k,l,m,n,o>:=Group<j,k,l,m,n,o|j^2,k^2,(j,k),
> l^2,m^3,n^3,o^2,(m,n),l *n^-1 * l *n,m^-1 * l *m^-1 * l *n ,
> m^-1 *o *n^-1 *m *o *n,(o *m^-1)^3 ,(l *o)^4 ,j^l=j ,j^m= k,
> j^n=j*k , j^o=j ,k^l=k ,k^m=j*k ,k^n= j,k^o=k>;
> f,G1,k1:=CosetAction(G,sub<G|x,y,z>);
>   #G1;
1152
> f1,H1,k2:=CosetAction(H,sub<H|Id(H)>);
>   #H1;
1152
> IsIsomorphic(G1,H1);
false.
```

We were on the right track, we see that both the order of $G1$ and of our presentation labeled H are of order 1152, however H is not a proper isomomorphic image of $G1$. This means that it is not a semi-direct isomorphism type, it is actually a mixted extension. To create a isomphic image that is a mixed extension, we have to check what MAGMA said the $FPGroup(q)$ is and make sure that each truly is equivalent to identity,

```
> (T[3],T[4]);
Id($)
>
> T[2] *T[4]^-1 * T[2] *T[4];
Id($)
>
> T[3]^-1*T[2]*T[3]^-1*T[2]*T[4];
Id($)
>
> T[3]^-1*T[5]*T[4]^-1*T[3]*T[5]*T[4];
Id($)
>
> (T[5]*T[3]^-1)^3;
Id($)
>
> (T[2]*T[5])^4;
(1, 49)(2, 29)(3, 20)(4, 64)(5, 61)(6, 53)(7, 10)(8, 57)
```

```
        (9, 41)(11, 42)(12, 38)(13, 58)(14, 44)(15, 39)
        (16, 34)(17, 27)(18, 25)(19, 46)(21, 26)(22, 60)
        (23, 55)(24, 45)(28, 59)(30, 32)(31, 36)(33, 37)
        (35, 43)(40, 54)(47, 48)(50, 51)(52, 56)(62, 63).
```

We see that there is a problem when $(l * o)^4 \neq Id$, but actually $(l * o)^4 = k$; so all that is left to do is make that small adjustment.

```
> H<j,k,l,m,n,o>:=Group<j,k,l,m,n,o|j^2,k^2,(j,k),
> l^2,m^3,n^3,o^2,(m,n),l *n^-1 * l *n,m^-1 * l *m^-1 * l *n ,
> m^-1 *o *n^-1 *m *o *n,(o *m^-1)^3 ,(l *o)^4=k ,j^l=j ,
> j^m= k, j^n=j*k , j^o=j ,k^l=k ,k^m=j*k ,k^n= j,k^o=k>;
> #G;
1152
> f1,G1,k2:=CosetAction(G,sub<G|Id(G)>);
> #G1;
1152
> f1,H1,k2:=CosetAction(H,sub<H|Id(H)>);
> #H1;
1152
> IsIsomorphic(G1,H1);
true Mapping from: GrpPerm: G1 to GrpPerm: H1
Composition of Mapping from: GrpPerm: G1 to GrpPC and
Mapping from: GrpPC to GrpPC and
Mapping from: GrpPC to GrpPerm: H1
```

Thus, our $G1$ is made up of $2 :^{\bullet} q$, but now we are left with the question of what can q be expressed by?

Within our quotient group q, we will factor q by its largest abelian subgroup $nlq[2]$; this new quotient group will be labeled $q1$;

```
> q,ff:=quo<G1|NL[2]>;
> q;
Permutation group q acting on a set of cardinality 16
Order = 288 = 2^5 * 3^2
    (2, 3)(4, 5)(7, 8)(9, 10)(12, 14)(13, 15)
    (2, 4, 7)(6, 10, 14)(9, 11, 15)(12, 13, 16)
    (2, 4, 7)(3, 5, 8)(6, 11, 16)(9, 13, 14)(10, 15, 12)
    (1, 2)(3, 6)(4, 7)(5, 9)(8, 12)(10, 14)(11, 15)(13, 16)
> for i in [1..#nlq] do if IsAbelian(nlq[i]) then i; end if;
    end for;
1
2
```

```
> q1,ff1:=quo<q|nlq[2]>;
> nlq1:=NormalLattice(q1);
> nlq1;

Normal subgroup lattice
-----------------------


[6]  Order 18  Length 1  Maximal Subgroups: 4 5
---
[5]  Order 9   Length 1  Maximal Subgroups: 2 3
[4]  Order 6   Length 1  Maximal Subgroups: 3
---
[3]  Order 3   Length 1  Maximal Subgroups: 1
[2]  Order 3   Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:

> q1;
Permutation group q1 acting on a set of cardinality 6
Order = 18 = 2 * 3^2
    (1, 2)(3, 4)(5, 6)
    (2, 4, 6)
    (1, 3, 5)(2, 4, 6)
    Id(q1).
```

We repeated the process of factoring by the largest abelian subgroup, here we are able to see that $q1$ is made up 3^3 and 2; and their relationship between these two subgroups is that of a semi-direct product. We now know that $q1$ is made up of $3^2 : 2$, and we will use MAGMA to help us prove this;

```
> nlq1[5];
Permutation group acting on a set of cardinality 6
Order = 9 = 3^2
    (2, 6, 4)
    (1, 5, 3)(2, 4, 6)
> A:=q1! (2, 6, 4);
> B:=q1!(1, 5, 3)(2, 4, 6);
> nlq1[5] eq sub<q1|A,B>;
true
> for C in q1 do if Order(C) eq 2 and sub<q1|A,B,C> eq q1 then C;
break; end if;
for|if> f; end for;
(1, 2)(3, 4)(5, 6)
```

```
> C:=q1!(1, 2)(3, 4)(5, 6);.
```

Again, to write the semi-direct product we have to raise each A and B to the power of C. As we will see in the code below, it might take some trial and error,

```
> A^C;
(1, 5, 3)
> A^C;
(1, 5, 3)
>  A^C eq A*B;
true
> B^C;
(1, 3, 5)(2, 6, 4)
> B^C eq A*B^2;
false
> B^C eq A^2*B;
false
>  B^C eq B^2;
true
```

and now all that will be left to do is to write a presentation for $q1$

```
>   H<a,b,c>:=Group<a,b,c|a^3,b^3,(a,b),c^2,a^c=a*b,b^c=b^2>;
>   #H;
18
>   f,h1,k2:=CosetAction(H,sub<H|Id(H)>);
>   s:=IsIsomorphic(q1,h1);
>   s;
true
>   H<a,b,c>:=Group<a,b,c|a^3,b^3,(a,b),c^2,a^c=a*b,b^c=b^2>;
>   #H;
18
>   f,h1,k2:=CosetAction(H,sub<H|Id(H)>);
>   s:=IsIsomorphic(q1,h1);
>   s;
true.
```

We have successfully written a semi-direct presentation for $q1$, and have proved that $q1$ is $3^2 : 2$ now we will incorporate the remainder of the subgroup q. What is left to explore is the remainder of q, consisting of $nlq[2]$ which is 2^4 and will also be a semi-direct product,

```
>nlq[2];
Permutation group acting on a set of cardinality 16
```

```
Order = 16 = 2^4
    (1, 15)(2, 11)(3, 16)(4, 9)(5, 7)(6, 13)(8, 14)(10, 12)
    (1, 4)(2, 7)(3, 10)(5, 11)(6, 14)(8, 13)(9, 15)(12, 16)
    (1, 6)(2, 3)(4, 14)(5, 12)(7, 10)(8, 9)(11, 16)(13, 15)
    (1, 7)(2, 4)(3, 14)(5, 15)(6, 10)(8, 16)(9, 11)(12, 13)

D:=q!(1, 15)(2, 11)(3, 16)(4, 9)(5, 7)(6, 13)(8, 14)(10, 12);
E:=q!(1, 4)(2, 7)(3, 10)(5, 11)(6, 14)(8, 13)(9, 15)(12, 16);
F:=q! (1, 6)(2, 3)(4, 14)(5, 12)(7, 10)(8, 9)(11, 16)(13, 15);
I:=q! (1, 7)(2, 4)(3, 14)(5, 15)(6, 10)(8, 16)(9, 11)(12, 13);


>K:=sub<q|D,E,F,I>;
>T:=Transversal(q,K);
>nlq[2] eq K;
true

>for i in [1..#T] do if ff1(T[i]) eq A then i; end if; end for;
2
>for i in [1..#T] do if ff1(T[i]) eq B then i; end if; end for;
3
>for i in [1..#T] do if ff1(T[i]) eq C then i; end if; end for;
4
>T[2];
>T2:=q!(2, 3)(4, 5)(7, 8)(9, 10)(12, 14)(13, 15);

>T[3];
>T3:=q!(2, 4, 7)(6, 10, 14)(9, 11, 15)(12, 13, 16);

>T[4];
>T4:=q!(2, 4, 7)(3, 5, 8)(6, 11, 16)(9, 13, 14)(10, 15, 12);

>ff1(T2) eq A;
true
>ff1(T3) eq B;
true
>ff1(T4) eq C;
true.
```

The Transversals allow us to write $A, B,$ and C and allow us to commute with D, E, F and I, the following loop will help us with this

```
>for i,j,k in [1..2] do if D^T2 eq D^i*E^j*F^k then i,j,k;
> end if; end for;
1 2 1
```

```
/*D*E^2*F
   D^A=D*E^2*F */

>for i,j,k in [1..2] do if E^T2 eq D^i*I^j*F^k then i,j,k;
>end if; end for;
1 1 2
/(D*I*F^2
E^A=D*I*F^2 */

>for i,j,k in [1..2] do if F^T2 eq D^i*E^j*F^k then i,j,k;
>end if; end for;
2 2 1
/*D^2*E^2*F
F^A=D^2*E^2*F */

>for i,j,k in [1..2] do if I^T2 eq D^i*E^j*F^k then i,j,k;
>end if; end for;
1 1 1
/*D*E*F
I^A=D*E*F  */




>for i,j,k in [1..2] do if D^T3 eq D^i*E^j*F^k then i,j,k;
>end if; end for;
1 1 2
/*D*E*F^2
D^B=D*E*F^2     */

>for i,j,k in [1..2] do if E^T3 eq D^i*E^j*I^k then i,j,k;
> end if; end for;
2 2 1
/*D^2*E^2*I
E^B=D^2*E^2*I  */

>for i,j,k in [1..2] do if F^T3 eq I^i*E^j*F^k then i,j,k;
>end if; end for;
1 2 1
/*I*E^2*F
F^B=I*E^2*F   */

>for i,j,k in [1..2] do if I^T3 eq I^i*E^j*F^k then i,j,k;
>end if; end for;
1 1 2
```

```
/*I*E*F^2
I^B=I*E*F^2     */
```

```
>for i,j,k in [1..2] do if D^T4 eq D^i*F^j*I^k then i,j,k;
> end if; end for;
1 1 1
/*D*F*I
D^C=D*F*I   */
```

```
>for i,j,k in [1..2] do if E^T4 eq D^i*F^j*I^k then i,j,k;
>end if; end for;
2 2 1
/*D^2*F^2*I
E^C=D^2*F^2*I    */
```

```
>for i,j,k,l in [1..2] do if F^T4 eq D^i*E^j*F^k*I^l then i,j,k,l;
>end if; end for;
1 1 2 1
/*D*E*F^2*I
F^C=D*E*F^2*I  */
```

```
>for i,j,k,l in [1..2] do if I^T4 eq D^i*E^j*F^k*I^l then i,j,k,l;
> end if; end for;
2 1 2 1
/*D^2*E*F^2*I
I^C=D^2*E*F^2*I  */.
```

Now we can write a presentation of q,

```
>H<a,b,c,d,e,f,i>:=Group<a,b,c,d,e,f,i| a^2, b^3, c^3, (b, c),
>a*c^-1*a*c, b^-1*a*b^-1*a*c, d^a=d*e^2*f, e^a=d*i*f^2,
>f^a=d^2*e^2*f, i^a=d*e*f, d^b=d*e*f^2,
>e^b=d^2*e^2*i, f^b=i*e^2*f, i^b=i*e*f^2, d^c=d*f*i,
>e^c=d^2*f^2*i, f^c=d*e*f^2*i, f^c=d*e*f^2*i,
>i^c=d^2*e*f^2*i>;
```

```
> #H;
288
> Order(q) eq #H;
true
>  f1,h1,k3:=CosetAction(H,sub<H|Id(H)>);
>  s:=IsIsomorphic(q,h1);
```

```
>  s;
true
> IsIsomorphic(q,h1);
true Mapping from: GrpPerm: q to GrpPerm: h1
Composition of Mapping from: GrpPerm: q to GrpPC and
Mapping from: GrpPC to GrpPC and
```

Therefore, we have proven $G \cong 2^2 :^\bullet (2^4 : (3^2 : 2))$.

Chapter 6

Tables of Homomorphic Images

6.1 Homomorphic Image of $2^{*15} : 3^{\bullet}(5 : 2)$

$$N = <x, y, z> \cong 2^{*15} : 3^{\bullet}(5 : 2)$$

where $x \sim (1, 2, 7, 12, 6, 5, 9, 15, 13, 10, 3, 8, 14, 11, 4)$,

$y \sim (1, 3, 5)(2, 8, 9)(4, 10, 6)(7, 14, 15)(11, 13, 12)$

$z \sim (2, 6)(4, 8)(7, 13)(9, 10)(11, 15)(12, 14)$

and the prgenitor is.

$$N < x, y, z, t >:= Group < x, y, z, t | y^3, z^2, (x, y),$$

$y^-1 * z * y * z, z * y^-1 * x^-1 * z * x^-1, x^5 * y, t^2, (t, x * z * y^-1) >$

also, we stabilize $t \sim t_{15}$.

The relations are:

```
(z*t)^r1,
(y*t)^r2,
(x*y^-1*t)^r3,
(x*z*t)^r4,
(x*t)^r5,
(x^2*t)^r6
```

Table 6.1: Some Finite Images of Progenitor $2^{*15} : 3^{\bullet}(5:2)$

#G	r1	r2	r3	r4	r5	r6	Isomorphism Type
1440	0	0	0	0	0	3	$2^3 :^{\bullet} (A_5 \times 3)$
720	0	0	0	0	5	4	$PGL_2(9)$
120	0	0	0	0	10	3	$A_5 \times 2$
201600	0	0	0	3	4	4	$5 : (A_8{}^{\bullet}2)$
240	0	0	2	0	0	4	$2^2 : (3:2)$
600	0	0	2	5	10	10	Alt_5
960	0	0	4	2	8	4	$2^4 : (5:2)$
14515200	0	0	4	3	6	4	
360	0	0	9	6	3	0	$A_5 \times 2$
5160960	0	4	3	0	0	7	
60	0	10	6	2	4	10	C_2
660	5	2	5	0	0	0	$L_2(11)$

6.2 Homomorphic Image of $2^{*15} : D_{15}$

$$N = <x,y,z> \cong 2^{*15} : D_15$$

where $x \sim (1,2,6,10,4)(3,7,12,14,9)(5,8,13,15,11)$,

$y \sim (2,4)(3,5)(6,10)(7,11)(8,9)(12,15)(13,14)$

$z \sim (1,3,5)(2,7,8)(4,9,11)(6,12,13)(10,14,15)$

and the prgenitor is

$$N <x,y,z,t> := Group <x,y,z,t|y^2, z^3, (x^-1*y)^2,$$

$(x,z), (y*z^-1)^2, x^-5, t^2, (t,y*x*z)>,$

also, we stabilize $t \sim t_{15}$.

The relations used are:

```
(x *z*t)^r1,
(z * x^-1*t)^r2,
(z*t)^r3,
(x^2*t)^r4,
(x^2 * z^-1*t)^r5,
(x^2 * z*t)^r6,
(x*t)^r7,
(y*t )^r8,
(z*t)^r9
```

Table 6.2: Some Finite Images of Progenitor $2^{*15} : D_{15}$

#G	r1	r2	r3	r4	r5	r6	r7	r8	r9	Isomorphism Type
360	0	0	0	0	0	0	0	3	2	$A_5 \times 2$
14880	0	0	0	0	0	0	0	3	3	$L_2(31)$
120	0	0	0	0	0	0	2	0	5	$A_5 \times 2$
504	0	0	0	0	0	0	2	7	9	$L_2(8)$
20520	0	0	0	0	0	0	2	9	10	$L_2(19) \times (3:2)$
21600	0	0	0	0	0	0	2	10	8	$PGL_2(9):6$
40320	0	0	0	0	0	0	3	6	4	S_8
720	0	0	0	0	0	0	4	6	2	$S_5 \times 2$
1440	0	0	0	0	0	0	4	8	2	$PGL_2(9) \times 2$
29160	0	0	0	0	0	3	5	6	0	$3^5:(A_5 \times 2)$
24360	0	0	0	0	0	4	0	7	2	$PGL_2(29)$
4320	0	0	0	0	0	10	2	8	0	$PGL_2(9):(3:2)$
12180	0	0	0	0	3	0	7	5	7	$L_2(29)$
7920	0	0	0	0	4	4	0	6	2	$PGL_2(11) \times 2$
979200	0	0	0	0	4	4	6	4	0	$2^{\bullet}(A_5 : PGL_2(16))$
29760	0	0	0	0	10	3	0	0	4	$2^{\bullet}L_2(31)$
704880	0	0	0	3	0	4	0	0	0	$3^{\bullet}L_2(89)$

6.3 Some Finite Images of Progenitor $2^{*5} : A_5$

$$N = < x, y > \cong 2^{*5} : A_5$$

where $x \sim (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)$,

$y \sim (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)$,

and the prgenitor is

$N < x, y, t > := Group < x, y, t | x^5, y^3, (x * y * x)^2, t^2, (t, x^2 * y), (t, x * y^{-1} * x^{-1} * y * x) >$,

also, we stabilize $t \sim t_1$.

The relations use are:

```
((y*x)^2*t)^r1,
(y*x^-2*t)^r2,
(x*y*x^-1*t)^r3,
(y*x*y^-1*x^-1*y*t)^r4.
```

Table 6.3: Some Finite Images of Progenitor $2^{*5} : A_5$

#G	r1	r2	r3	r4	Isomorphism Type
75000	10	10	6	10	$5^4 : (A_5^{\bullet}2)$
1920	10	10	6	4	$2^5 :^{\bullet} A_5$
9720	10	10	6	6	$3^4 : (A_5 :^{\bullet} 2)$
30720	10	10	6	8	$2^8 : (A_5 \times 2)$
960	10	5	6	4	$2^4 : S_5$
175560	7	0	0	0	J1
117600	8	8	0	0	$PGL_2(249)$
720	6	6	8	6	S_6

6.4 Some Finite Images of Progenitor $2^{*6} : 3^2 : (2^{2^{\bullet}}2)$

$$N = < x, y > \cong 3^2 : (2^{2^{\bullet}}2)$$

where $x \sim (1, 2, 4)(5, 6)$,

$y \sim (1, 3)(2, 5)(4, 6)$,

and the prgenitor is

$< x, y, t > := Group < x, y, t | x^6, y^2, (x^-1 * y)^4, (x * y * x^-1 * y)^2, t^2, (x^3 * y * x^2 * y, t), (y * x^-2 * y, t), (y * x^3 * y, t), (x^3, t) >,$

also, we stabilize $t \sim t_1$.

The relations use are:

```
(y*t)^r1,
(y*x*t)^r2,
(x*y*x*t)^r3,
((y*x)^2*t)^r4,
(x^2*y*x^-2*y*t)^r5.
```

Table 6.4: Some Finite Images of Progenitor $2^{*6} : 3^2 : (2^{2^{\bullet}}2)$

#G	r1	r2	r3	r4	r5	Isomorphism Type
288	0	0	4	0	0	$4 :^{\bullet} 2$
432	0	0	6	2	0	$6^{\bullet}2$
80	0	0	10	0	2	$2^{\bullet}(5 : 2)$
20160	0	0	10	0	3	$4 :^{\bullet} S_7$
144	0	4	6	0	0	C_2
5040	0	7	0	0	3	S_7
10080	0	7	10	0	0	$2^{\bullet} S_7$
4608	4	0	0	4	0	$2^6 : 3^2 : (2^{2^{\bullet}}2)$
576	8	0	0	2	6	$2^3 :^{\bullet} 2$
5806080	8	0	9	4	6	$S(6,2)^{\bullet}2^2$
903168	6	0	8	0	0	$2^2 : (L_2(7) \times L_2(7) \times 2^2)^{\bullet}2$

6.5 Some Finite Images of Progenitor $2^{*5} : Alt_5$

$$N = < x, y > \cong Alt_5$$

where $x \sim (1, 2, 4, 5, 3)$,

$y \sim (3, 5, 4)$,

and the prgenitor is

$N < x, y, t >:= Group < x, y, t | x^5, y^3, (x * y^-1 * x)^2, t^2, (y, t), ((x^-1, y^-1), t) >$,

also, we stabilize $t \sim t_1$.

The relations use are:

```
(x^y*t)^r1,
(x*y*t)^r2,
(y*x^-1*t)^r3,
(x*y*x^-1*y^-1*x*y*t)^r4.
```

Table 6.5: Some Finite Images of Progenitor $2^{*5} : Alt_5$

#G	r1	r2	r3	r4	Isomorphism Type
120	0	0	0	2	C_2
1920	0	0	0	4	$2^5 :^{\bullet} A_5$
960	0	0	5	4	$A_5 : 2^4$
720	0	0	6	6	S_6
175560	0	0	7	0	J_1
117600	0	0	8	8	$L_2(49) \times 2$
9720	0	6	0	6	$3^4 : (A_5{}^{\bullet}2)$
75000	6	0	0	10	$5^3 : (A_5{}^{\bullet}2)$

6.6 Some Finite Images of Progenitor $2^{*12} : (2^{3^{\bullet}}2) : (S_3^2)$

$$N = < v, w, x, y, z > \cong (2^{3^{\bullet}}2) : (S_3^2)$$

where $v \sim (1, 2)(3, 6)$

$w \sim (3, 6)(9, 11)(10, 12)$

$x \sim (1, 3)(2, 6)(4, 9)(5, 10)(7, 11)(8, 12),$

$y \sim (1, 4, 5)(2, 7, 8)(3, 9, 10)(6, 11, 12),$

$z \sim (4, 5)(7, 8)(9, 10)(11, 12)$

and the prgenitor is

$N < v, w, x, y, z, t >:= Group < v, w, x, y, z, t | v^2, w^2, x^2, y^3, z^2, (v*w)^2, (v* x)^2, w*y^-1*w*y, x*y^-1*x*y, (v*z)^2, (w*z)^2, (x*z)^2, (y^-1*z)^2, (w*x)^4, (v*y*v* y^-1)^2, y^-1*x*w*x*w*v*y^-1*v*y^-1*v, t^2, (w, t), (z, t), (v*w*x*w*x, t), (y*v*y^-1, t),$

also, we stabilize $t \sim t_1$.

The relations use are:

```
(x*t)^r1,
(y * v * y * v * z*t)^r2,
(v * x*t)^r3,
(v * w*t)^r4,
(v * x * z * y*t)^r5,
(x * y*t)^r6,
(w * y*t)^r7,
(w * x * y*t)^r8.
```

Table 6.6: Some Finite Images of Progenitor $2^{*12} : (2^{3^\bullet}2) : (S_3{}^2)$

#G	r1	r2	r3	r4	r5	r6	r7	r8	$Isomorphism\ Type$
645120	0	0	0	0	6	7	4	7	$2^7 :^\bullet S_7$
1344	0	0	0	0	6	8	3	0	$2^\bullet PGL_2(7)$
864	0	0	0	0	6	6	0	9	$3^2 : (6 : 2)$
288	0	0	0	0	6	6	2	0	$(2 \times 3) :^\bullet 2$
16128	0	0	0	0	6	0	3	0	$6 : (PGL_2(7)$
1344	0	0	0	0	6	0	3	8	$2^\bullet PGL_2(7)$
103680	0	0	0	0	5	9	4	0	$S(4,3) : 2$
1358954496	0	0	0	0	4	9	6	0	
169869312	0	0	0	0	4	9	6	9	

6.7 Some Finite Images of Progenitor $2^{*12} : 2^4 : 3$

$$N = <v, w, x, y, z> \cong 2^4 : 3$$

where $v \sim (1,3)(4,5)(6,11)(10,12)$

$w \sim (1,3)(2,7)(4,5)(6,10)(8,9)(11,12)$

$x \sim (1,5)(2,8)(3,4)(6,10)(7,9)(11,12)$,

$y \sim (1,4)(2,7)(3,5)(6,11)(8,9)(10,12)$,

$z \sim (1,2,6)(3,8,11)(4,9,10)(5,7,12)$

and the prgenitor is

$N < v, w, x, y, z, t > := Group < v, w, x, y, z, t | v^2, w^2, x^2, y^2, z^3, (v*w)^2, (v* x)^2, (w*x)^2, (v*y)^2, (w*y)^2, (x*y)^2, y*z^-1*x*z, z*x*v*z^-1*w, z^-1*x*w*v* z*w, v*z*w*v*z^-1*y, t^2, (t, w*x), (t, v*y),,$

also, we stabilize $t \sim t_{12}$.

The relations use are:

```
(z^-1*t)^r1,
(z*t)^r2,
(v*w*t)^r3,
(v*t)^r4,
(w*t)^r5,
(x*t)^r6.
```

Table 6.7: Some Finite Images of Progenitor $2^{*12} : 2^4 : 3$

$\#G$	$r1$	$r2$	$r3$	$r4$	$r5$	$r6$	$Isomorphism\ Type$
96	0	2	0	0	0	0	C_2
768	0	3	0	0	0	4	$2^6 : (3^\bullet 2)$
384	0	3	0	0	0	6	$2^3 :^\bullet (2^2 : 3)$
192	0	3	0	2	0	2	$2^2 : 3$
768	0	3	0	4	0	0	$2^6 :^\bullet (2^2 : 3)$
660	0	5	0	2	6	3	$L_2(11)$
675840	0	5	0	3	4	0	$2^{10} : L_2(11)$
3420	0	5	2	5	10	10	$L_2(19)$
435600	0	5	3	6	0	6	$L_2(11) \times L_2(11)$
52416	0	6	0	3	2	8	$2^2 : L_2(13)$

Chapter 7

MAGMA Code: Progenitor Transitive(15,5), Companion To Chapter 2

```
N:=TransitiveGroup(15,5);
N;
S:=Sym(15);
xx:=S!(1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8);
yy:=S!(1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13);
N:=sub<S|xx,yy>;
#N;
/*60*/

FPGroup(N);
NN<x,y>:=Group<x,y|x^5,y^3,(x*y*x)^2>;


/*
Normal subgroup lattice
-----------------------

[2]  Order 60  Length 1  Maximal Subgroups: 1
---
[1]  Order 1   Length 1  Maximal Subgroups:
```

```
> IsIsomorphic(N,Alt(5));
true Homomorphism of GrpPerm: N, Degree 15,
Order 2^2 * 3 * 5 into GrpPerm: $,
Degree 5, Order 2^2 * 3 * 5 induced by
    (1, 9, 10, 3, 14)(2, 15, 7, 12, 6)(4, 5, 11, 13, 8)
|--> (1, 4, 5, 3, 2)
    (1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)(12, 14, 13)
|--> (3, 4, 5)
*/
```

```
N1:=Stabilizer(N,1);
N1;
/*Permutation group N1 acting on a set of cardinality 15
Order = 4 = 2^2
    (2, 11)(3, 4)(5, 12)(7, 9)(10, 13)(14, 15)
    (2, 5)(3, 13)(4, 10)(7, 14)(9, 15)(11, 12)*/
```

```
NN<x,y>:=Group<x,y|x^5,y^3,(x*y*x)^2>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..60]];
for i in [2..60] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1 then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2 then P[j]:=yy; end if;
if Eltseq(Sch[i])[j] eq -2 then P[j]:=yy^-1; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;

for i in [1..60] do if ArrayP[i] eq
N!(2, 11)(3, 4)
```

```
(5, 12)(7, 9)(10, 13)(14, 15) then
print Sch[i];  end if; end for;
for i in [1..60] do if ArrayP[i] eq
N!(2, 5)(3, 13)(4, 10)
(7, 14)(9, 15)(11, 12)then print
Sch[i];  end if; end for;
/*
x^2 * y
x * y^-1 * x^-1 * y * x
*/


NN<x,y,t>:=Group<x,y,t|x^5, y^3, (x*y*x)^2, t^2,
(t,x^2*y), (t,x*y^-1*x^-1*y*x)>;



C:=Classes(N);
Classes(N);
#C;

for i in [2..5] do
i,Orbits(Centralizer(N,C[i][3]));
end for;

for j in [2..5] do
C[j][3];
for i in [1..60] do
if ArrayP[i] eq C[j][3]
then Sch[i]; end if;
end for;
end for;



/*here we are just calculating the conjugations
 for all of the first order relations*/


/*
[2](1, 7)(3, 8)(5, 14)(6, 11)
(10, 12)(13, 15)
(x*y*x*t)^r
(x*y*x*t^(y^x*y)^r
```

```
(x*y*x*t^(y))^r
(x*y*x*t^(x))^r
(x*y*x*t^(x^3))^r
(x*y*x*t^((yy^xx)^2)^r


[3](1, 4, 10)(2, 5, 8)(3, 7, 11)
(6, 9, 15)(12, 14, 13)
(y*t)^r
(y*t^(y^x*y))^r
(y*t^(x^3))^r
(y*t^((y^x)^2))^r
(y*t^(x*y)^3)^r


[4](1, 9, 10, 3, 14)(2, 15, 7, 12, 6)
(4, 5, 11, 13, 8)
(x*t)^r
(x*t^(y^x*y))^r
(x*t^y)^r

[5](1, 10, 14, 9, 3)(2, 7, 6, 15, 12)
(4, 11, 8, 5, 13)
(x^2*t)^r
(x^2*t^(y^x*y))^r
(x^2*t^y)^r
*/


xx:=S!(1, 9, 10, 3, 14)
(2, 15, 7, 12, 6)
(4, 5, 11, 13, 8);
yy:=S!(1, 4, 10)(2, 5, 8)
(3, 7, 11)(6, 9, 15)
(12, 14, 13);



Orbits(Stabiliser(Centraliser(N,yy),1));

(y*t*t^(y))^r
(y*t*t^(x))^r
(y*t*t^(x^2))^r
```

```
(y*t*t^(x^3))^r



/*FAMOUS LEMMA*/
N12:=Stabiliser(N,[1,2]);
C12:=Centraliser(N,N12);
C12;
for i in [1..60] do if ArrayP[i] eq N!
(1, 9, 10, 3, 14)(2, 15, 7, 12, 6)
(4, 5, 11, 13, 8)
then print Sch[i];  end if; end for;

for i in [1..60] do if ArrayP[i] eq N!
(1, 4, 10)(2, 5, 8)(3, 7, 11)(6, 9, 15)
(12, 14, 13)
then print Sch[i];  end if; end for;

/* x, y are already relations*/



/*tran15,5 rr */



for r1, r2, r3, r4,r5, r6, r7, r8, r9, r10,
r11, r12 in [0..10] do
G<x,y,t>:=Group<x,y,t|x^5, y^3,
(x*y*x)^2, t^2, (t,x^2*y),
(t,x*y^-1*x^-1*y*x),
(x*y*x*t)^r1,
(y*t*t^(x))^r2,
(y*t*t^(x^2))^r3,
(x*y*x*t^(y))^r4,
(x*y*x*t^(x))^r5,
(y*t*t^(y))^r6,
(y*t*t^(x^3))^r7,
(x*y*x*t^(x^3))^r8,
(x^2*t)^r9,
(y*t^(x^3))^r10,
(x*t)^r11,
(y*t)^r12>;
if #G gt 60 then r1, r2, r3, r4, r5, r
6, r7, r8, r9, r10, r11, r12,
#G;
```

```
end if;
end for;
```

Chapter 8

MAGMA Code: Isomorphism Type of $L_2(31)^{\bullet}2$

```
G<x,y,z,t>:=Group<x,y,z,t|y^2,z^3,(x^-1 * y)^2,
(x, z),
(y * z^-1)^2, x^-5,t^2, (t,y * x * z),(z*t)^4,
(x^2 * z^-1*t)^10,(x^2 * z*t)^3>;
Index(G,sub<G|x,y,z>);

f,G1,k:=CosetAction(G,sub<G|x,y,z>);



CompositionFactors(G1);

Center(G1);

NL:=NormalLattice(G1);
NL;

q,ff:=quo<G1|NL[2]>;
q;


a:=q!(2, 3, 6, 14, 11)(4, 10, 19, 15, 7)
(5, 8, 16, 18, 13)(9, 21, 37, 73, 85)(12,
25, 49, 98, 72)(17, 30, 59, 112, 65)(20
, 34, 69, 26, 50)(22, 41, 84, 74,
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, 42)(27, 54, 105, 190, 125)(28,
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313)(103, 140, 245,
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, 220)(223, 354, 465,

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, 421, 484)(292, 385,
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d:=q!(2, 5, 10)(3, 8, 19)(4, 11, 13)(6, 16, 15)(7, 14, 18)(
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244)(101, 184,
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371)(318, 372, 439)(321, 373, 437)(322, 441, 323)(326,
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344)(340, 453,
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(380, 460,

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430, 431)(410,
422, 432)(419, 484, 420)(427, 457, 486)(429, 468, 481)
(438, 474,
496)(442, 444, 491)(443, 485, 490)(458, 482, 495)(459,
494, 483)(467,
489, 473)(469, 478, 492)(470, 493, 477);




IsIsomorphic(q,PSL(2,31));


FPGroup(q);

/* a^5,
b^2,
c^3,
d^2,
(a^-1 * b)^2,
(a, c),
(b * c^-1)^2,
d * c^-1 * d * a^-1 * b * d * c,
a * d * c^-1 * b * a * d * b * c,
(a^-1 * d * b * d * b)^2,
a^-1 * d * c^-1 * d * c^-1 * a^2 * d * a * b * d *
a^-1,
a^-1 * d * a^-1 * d * a * d * c^-1 * a * d * a * d *
a^-1 * d * c, */




T:=Transversal(G1,NL[2]);




e:=G1!(1, 93)(2, 37)(3, 91)(4, 46)(5, 15)(6, 45)
(7, 18)(8, 207)(9, 114)(10,
35)(11, 188)(12, 44)(13, 113)(14, 36)(16, 112)(17,
110)(19, 109)(20,
88)(21, 119)(22, 89)(23, 90)(24, 377)(25, 275)
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126)(52, 117)(53, 108)(54, 208)(55, 209)(56, 204)
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395)(60, 423)(61, 184)(62, 590)(63, 375)(64, 486)
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, 165)(75, 82)(76,
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(447, 911)(448, 849)(450,
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(518, 869)(519, 870)(520,
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(555, 839)(556, 666)(557,
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(562, 971)(563, 864)(567,
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(579, 778)(580, 779)(581,
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734)(603, 621)(604, 686)(605, 923)(607, 768)(
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17, 954)(618, 867)(620,
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653)(642, 702)(643, 703)(644, 704)(645, 705)(6
46, 649)(648, 651)(650,
735)(652, 737)(656, 739)(658, 891)(659, 892)(6
60, 797)(661, 838)(667,
978)(668, 980)(669, 812)(671, 836)(672, 953)
(673, 919)(679, 832)(685,
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(696, 970)(697, 940)(698,
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985)(729, 984)(730, 873)(731, 874)(732, 792)(733,
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990)(743, 808)(745, 915)(746, 881)(748, 798)(751,
793)(766, 787)(767,
788)(780, 841)(781, 942)(782, 957)(783, 822)(785,
890)(786, 889)(789,
958)(790, 805)(794, 992)(795, 907)(796, 818)(799,
 811)(800, 809)(806,
912)(820, 924)(821, 929)(823, 965)(824, 899)(825,
966)(826, 967)(828,
866)(829, 851)(833, 955)(834, 844)(837, 875)(840,
 843)(842, 943)(847,
949)(850, 960)(852, 969)(853, 934)(854, 857)(856,
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974)(877, 982)(880, 935)(882, 987)(887, 921)(894,
931)(895, 932)(896,
933)(897, 901)(898, 900)(902, 950)(905, 939)(906,
 941)(908, 977)(916,
979)(917, 975)(918, 946)(920, 981)(928, 947)(938,
952)(945, 983)(948,

976)(963, 972)(973, 991);

e:=G1!(2, 4)(3, 9)(5, 14)(6, 7)(8, 19)(10, 22)(11, 25)(12, 17)(13, 30)(15, 36)(16,
39)(18, 45)(21, 49)(23, 55)(24, 60)(26, 65)(27, 42)(28, 41)(29, 72)(31,
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166)(112, 239)(113, 232)(115, 119)(117, 241)(118, 128)(120, 246)(121,
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554)(293, 454)(294,
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419)(303, 315)(305,
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 588)(313, 543)(314,
404)(316, 597)(319, 321)(320, 322)(323, 501)(325,
610)(326, 612)(327,
431)(328, 520)(330, 621)(331, 622)(332, 625)(333,
470)(335, 629)(336,
633)(337, 478)(340, 530)(343, 345)(346, 420)(347,
643)(348, 498)(349,
495)(350, 496)(351, 581)(352, 380)(353, 655)(356,
661)(357, 412)(358,
411)(359, 663)(362, 668)(364, 672)(366, 387)(368,
585)(369, 438)(370,
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400)(383, 399)(384,
686)(386, 689)(388, 503)(389, 671)(391, 537)(393,
696)(394, 514)(395,
504)(398, 684)(401, 424)(403, 703)(406, 508)(407,
636)(408, 410)(413,
678)(416, 708)(417, 479)(421, 716)(425, 569)(426,
519)(428, 436)(429,
516)(430, 515)(432, 640)(434, 603)(435, 473)(437,
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609)(461, 691)(462,
755)(464, 472)(465, 756)(467, 492)(471, 761)(474,
627)(475, 763)(477,
619)(484, 664)(485, 509)(487, 564)(488, 550)(490,
776)(493, 780)(494,
758)(497, 784)(499, 656)(502, 525)(506, 544)(510,
688)(512, 695)(513,
560)(521, 799)(522, 753)(523, 623)(524, 804)(527,
608)(528, 807)(531,
809)(532, 638)(535, 654)(536, 718)(538, 694)(540,
574)(542, 592)(546,
823)(547, 662)(549, 740)(556, 747)(557, 611)(558, 8
06)(561, 833)(562,

795)(563, 639)(565, 573)(566, 572)(570, 595)(571, 637)(575, 842)(576, 766)(578, 790)(579, 826)(580, 825)(583, 679)(584, 851)(586, 845)(587, 860)(589, 764)(590, 827)(591, 774)(593, 660)(594, 707)(596, 869)(598, 677)(599, 676)(600, 605)(601, 675)(602, 721)(604, 717)(613, 738)(614, 724)(616, 782)(617, 781)(618, 798)(620, 884)(624, 801)(626, 819)(628, 887)(630, 890)(631, 739)(634, 838)(635, 719)(641, 7 05)(642, 895)(644, 898)(645, 653)(646, 652)(647, 760)(648, 857)(649, 737)(650, 767)(651, 854)(657, 713)(658, 659)(665, 762)(666, 692)(669, 918)(670, 800)(673, 920)(674, 915)(680, 725)(681, 757)(685, 700)(687, 791)(690, 881)(697, 794)(698, 927)(701, 803)(702, 932)(704, 900)(709, 911)(710, 743)(711, 921)(712, 785)(720, 937)(722, 796)(723, 797)(726, 727)(728, 929)(730, 947)(732, 938)(733, 948)(734, 923)(735, 788)(741, 916)(742, 910)(744, 953)(746, 813)(748, 867)(750, 816)(751, 908)(754, 876)(759, 943)(765, 841)(769, 965)(770, 831)(771, 832)(772, 829)(773, 830)(775, 787)(777, 805)(778, 967)(779, 966)(783, 894)(786, 905)(789, 974)(792, 952)(793, 977)(802, 883)(808, 848)(810, 980)(811, 872)(812, 946)(814, 836)(815, 970)(817, 868)(818, 903)(820, 975)(821, 985)(822, 931)(824, 853)(828, 858)(834, 875)(835, 936)(837, 844)(840, 987)(843, 882)(846, 849)(847, 963)(850, 856)(855, 951)(859, 964)(861, 962)(863, 871)(864, 959)(865, 958)(866, 893)(870, 885)(873, 928)(877, 945)(878, 957)(879, 986)(880, 941)(886, 984)(888, 976)(889, 939)(891, 892)(896, 973)(897, 902)(899, 934)(901, 950)(904, 922)(906, 935)(907, 971)(909, 912)(913, 925)(914,

```
926)(917, 924)(919, 981)(930, 944)(933, 991)(940,
992)(942, 954)(949,
972)(955, 989)(956, 988)(960, 968)(961, 978)(979,
990)(982, 983);
```

```
H<a,b,c,d,e>:=Group<a,b,c,d,e|a^5,b^2,c^3,d^2,
(a^-1 * b)^2,(a, c),
(b*c^-1)^2,d*c^-1*d*a^-1*b*d*c,a*d*c^-1*b*a*d*b*c,
(a^-1*d*b*d*b)^2,a^-1*d*c^-1*d*c^-1*a^2*d*a*b*d* a^-1,
a^-1*d*a^-1*d*a*d*c^-1*a*d*a*d*a^-1*d*c,
e^2,(a,e), (b,e), (c,e), (d,e)>;
```

```
f,H1,k:=CosetAction(H,sub<H|Id(H)>);
s,t:=IsIsomorphic(G1,H1);

s;
t;
```

Chapter 9

MAGMA Code: DCE $L_2(31)$

```
S:=Sym(15);
XX:=S!(1, 2, 6, 10, 4)(3, 7, 12, 14, 9)(5, 8, 13, 15, 11);
YY:=S!(2, 4)(3, 5)(6, 10)(7, 11)(8, 9)(12, 15)(13, 14);
ZZ:=S!(1, 3, 5)(2, 7, 8)(4, 9, 11)(6, 12, 13)(10, 14, 15);

/*
a:=S!(1, 13, 9, 2, 15, 3, 6, 11, 7, 10, 5, 12, 4, 8, 14);
b:=S!(1, 12, 11, 2, 14, 5, 6, 9, 8, 10, 3, 13, 4, 7, 15);
*/




N:=sub<S|XX,YY,ZZ>;
#N;

G<x,y,z,t>:=Group<x,y,z,t|y^2,z^3,(x^-1 * y)^2,
(x, z),(y * z^-1)^2, x^-5,t^2
, (t,y * x * z),(z*t)^4,(x^2 * z^-1*t)^10,(x^2 * z*t)^3,
y * x * t * z * t * z^-1 * t>;

/*pi:=S!(1, 12, 11, 2, 14, 5, 6, 9, 8, 10, 3, 13, 4, 7, 15);*/

f,G1,k:=CosetAction(G,sub<G|x,y,z>);
CompositionFactors(G1);
IN:=sub<G1|f(x),f(y),f(z)>;
```

```
sub<N|YY*XX*ZZ> eq Stabiliser(N,15);

#DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
DoubleCosets(G,sub<G|x,y,z>,sub<G|x,y,z>);
/*
{ <GrpFP, Id(G), GrpFP>,
<GrpFP, t, GrpFP>,
 <GrpFP, t * x * t, GrpFP>,
<GrpFP, t * y * t, GrpFP>,
<GrpFP,t * x^2 * t, GrpFP>,
 <GrpFP, t * x * t * y * t, GrpFP>,
 <GrpFP, t * x * t * x * t, GrpFP>,
  <GrpFP, t * x * z^-1 * t, GrpFP>,
<GrpFP, t * y * t * x * t, GrpFP>,
 <GrpFP, t * x * t * x^-1 * t, GrpFP>,
<GrpFP, t * y * t * x^-1 * t, GrpFP>,
 <GrpFP, t * x^2 * t * y * t, GrpFP>,
<GrpFP, t * x * t * x * z^-1 * t, GrpFP>,
 <GrpFP, t * x * t * z * x^-1 *t, GrpFP>
<GrpFP, t * x^-1 * t * y * t * x^-1 * t, GrpFP>,
<GrpFP, t * y * t * x * t * x * t, GrpFP>,
<GrpFP, t * x * z^-1 * t * x * z^-1 *t, GrpFP>,
<GrpFP, t * x * t * x^-1 * t * x * t, GrpFP>,
 <GrpFP, t * x * t * y *t * y * t, GrpFP>,
<GrpFP, t * x * t * y * t * x * t, GrpFP>,
<GrpFP, t * x * t * x * t * y * t, GrpFP>,
 <GrpFP, t * x * t * x * t * x * t, GrpFP>,
 <GrpFP, t * x^2 * y * t * x^-1 * t, GrpFP>,
<GrpFP, t * y * t * x * z^-1 * t, GrpFP>,
<GrpFP, t * x * z^-1 * t * x^-1 * t, GrpFP>,
}
*/



prodim:=function(pt, Q, I)
  v := pt;
   for i in I do
       v := v^(Q[i]);
    end for;
return v;
end function;

ts := [Id(G1): i in [1 .. 15] ];
```

```
ts[15]:=f(t); ts[1]:=f(t^(z*y*x^-2));
ts[2]:=f(t^(z*x^-2)); ts[3]:=f(t^((x*z)^2));
ts[4]:=f(t^(x*z)); ts[5]:=f(t^(x^2));
ts[6]:=f(t^(z*y)); ts[7]:=f(t^(x*y));
ts[8]:=f(t^(x^3)); ts[9]:=f(t^(y*x^2));
ts[10]:=f(t^z); ts[11]:=f(t^x); ts[12]:=f(t^y);
ts[13]:=f(t^(x^4)); ts[14]:=f(t^(z^2));
cst := [null : i in [1 .. Index(G,sub<G|x,y,z>)]]
where null is [Integers() | ];

 for i := 1 to 15 do
      cst[prodim(1, ts, [i])] := [i];
    end for;

m:=0;

for i in [1..30] do if cst[i] ne [] then m:=m+1;
end if; end for; m;

cst;


/*the code below is to investigate new
double cosets.
Here we find the equal names it might
share with other double cosets,
the order of the coset stabilizing group,
and the orbits*/


N0:=Stabiliser (N,15);
Orbits(N0);
N0s:=N0;
T0:=Transversal(N,N0s);
T0;
for i in [1..#T0] do
ss:=[15]^T0[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..30] do if cst[i] ne []
then m:=m+1; end if; end for;  m;  /*15*/
Orbits(N0);
```

```
N02:=Stabiliser(N,[15,2]);  /*need to check equal names*/
 SSS:={[15,2]};
 SSS:=SSS^N;
SSS;
Seqq:=Setseq(SSS);
 Seqq;

 for i in [1..#SSS] do for n in IN do
 if ts[15]*ts[2] eq n*ts[Rep(Seqq[i])[1]]*
ts[Rep(Seqq[i])[2]] then print Rep(Seqq[i]);
 end if;
 end for;
 end for;
/*[ 15, 2 ]
[ 12, 4 ]*/

N02s:=N02;
for n in N do if 15^n eq 12 and 2^n eq 4 then
N02s:=sub<N|N02s,n>;
 end if; end for;
 #N02s;
N02s;
/*Permutation group N02s acting on a set of
cardinality 15
Order = 2
(2, 4)(3, 5)(6, 10)(7, 11)(8, 9)(12, 15)(13, 14)*/
[15,2]^N02s;

N02:=Stabiliser(N,[15,2]);
N02;
 N02:=sub<N|(2, 4)(3, 5)(6, 10)(7, 11)
(8, 9)(12, 15)(13, 14)>;
#N02;
[15,2]^N02;
T:=Transversal(N,N02);
for i in [1..#T] do {[15,2]^N02}^T[i];
 end for;
for n in IN do if ts[15]*ts[2] eq
n*ts[12]*ts[4] then n; end if; end for;
ts[15]*ts[2] eq f(x)*ts[12]*ts[4];
/*false, we need to find what x is in f(x)
and the next code will tell us*/
```

```
/* Add Relation */
for n in IN do if ts[15]*ts[2]
eq n*ts[12]*ts[4] then n; end if; end for;
for n in IN do if ts[15]*ts[2] eq
n*ts[12]*ts[4] then A:=n; end if; end for;
W,phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
/*function(W)
w5 := W.1^2; w3 := W.3^-1;
w6 := w5 * w3; return w6;
end function
*/
g:=function(W);
 w5 := W.1^2; w3 := W.3^-1; w6 := w5 * w3;
 return w6;
end function;
g(G);
/*x^2 * z^-1*/
ts[15]*ts[2] eq f(x^2 * z^-1)*ts[12]*ts[4];
/*true */

N02; #N02;
/*(2, 4)(3, 5)(6, 10)(7, 11)(8, 9)(12, 15)(13, 14)
*/
T02:=Transversal(N,N02);
 for i in [1..#T02] do ss:=[15,2]^T02[i];
 cst[prodim(1,ts,ss)]:=ss;
 end for;
m:=0; for i in [1..496] do if cst[i] ne []
then m:=m+1; end if; end for; m;
/*30
[15,2] ~ [12,4]*/
Orbits(N02);

/*we follow this process until we have reached
the total number of single cosets, this number is
(#G1/#N)-1
*/
```

```
/* This code below lets us know where each
double coset lives,
and we have labeled each double coset 1-24*/

f,G1,k:=CosetAction(G,sub<G|x,y,z>);
A:=[Id(G1): i in [1..24]];
A[1]:=f(t);   /* 1  */
A[2]:=f(t * x * t);  /*2*/
A[3]:=f(t * y * t);  /*2*/
A[4]:=f(t * x^2 * t);  /*2*/
A[5]:=f(t * x * t * y * t); /*3*/
A[6]:=f(t * x * t * x * t);  /*3*/
A[7]:=f(t * x * z^-1 * t);  /*2*/
A[8]:=f(t * y * t * x * t); /*3*/
A[9]:=f(t * x * t * x^-1 * t); /*3*/
A[10]:=f(t * y * t * x^-1 * t); /*3*/
A[11]:=f(t * x^2 * t * y * t); /*3*/
A[12]:=f(t * x * t * x * z^-1 * t); /*3*/
A[13]:=f(t * x * t * z * x^-1 *t); /*3*/
A[14]:=f(t * x^-1 * t * y * t * x^-1 * t); /*4*/
A[15]:=f(t * y * t * x * t * x * t);  /*4*/
A[16]:=f(t * x * z^-1 * t * x * z^-1 *t);  /*3*/
A[17]:=f(t * x * t * x^-1 * t * x * t); /*4*/
A[18]:=f(t * x * t * y *t * y * t); /*4*/
A[19]:=f(t * x * t * y * t * x * t); /*4*/
A[20]:=f(t * x * t * x * t * y * t); /*4*/
A[21]:=f(t * x * t * x * t * x * t); /*4*/
A[22]:=f(t * x^2 * y * t * x^-1 * t); /*3*/
A[23]:=f(t * y * t * x * z^-1 * t); /*3*/
A[24]:=f(t * x * z^-1 * t * x^-1 * t); /*3*/

/*In the code below we find our where the potential
new double coset lives after we right multiply.
Listed are a few example*/

/* [15] */
for i in [1..24] do for m, n in IN do
if ts[15]*ts[1]eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[1] eq m*(A[i])^n then i; end if;
end for; end for;
```

```
for i in [1..24] do for m, n in IN do
if ts[15]*ts[2] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[2] eq m*(A[i])^n then i; end if;
 end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[4] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4] eq m*(A[i])^n then i; end if;
end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[5] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[5] eq m*(A[i])^n then i; end if;
end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[6] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[6] eq m*(A[i])^n then i; end if;
end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[10] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[10] eq m*(A[i])^n then i; end if;
end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[11] eq m*(A[i])^n then "true";
 end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[11] eq m*(A[i])^n then i; end if;
end for; end for;
```

```
/* [ 15,4]*/
for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[1] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[1] eq m*(A[i])^n then i;
end if; end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[2] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[2] eq m*(A[i])^n then i; end if;
end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[3] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[3] eq m*(A[i])^n then i; end if;
end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[4] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[4] eq m*(A[i])^n then i;
end if; end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[6] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[6] eq m*(A[i])^n then i;
end if; end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[9] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[9] eq m*(A[i])^n then i;
 end if; end for; end for;
```

```
for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[10] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[10] eq m*(A[i])^n then i;
end if; end for; end for;

for i in [1..24] do for m, n in IN do
if ts[15]*ts[4]*ts[14] eq m*(A[i])^n then "true";
end if; end for; end for;
 for i in [1..24] do for m,n in IN do
if ts[15]*ts[4]*ts[14] eq m*(A[i])^n then i; end if;
end for; end for;


/*the following code will help us produce
the relations,
listed are a few exaples*/
[15,1] done
for m, n in IN do if ts[15]*ts[1] eq
m*(ts[15])^n then A:=m; B:=n; end if;
end for;

W, phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
B@rho;
g:=function(W);
    return W.2;
end function;
g(G);
ts[15]*ts[1] eq f(x)*ts[15]^f(y);

/*for example, here we are showing that
the double coset [15,2,4] goes to [15]*/
[15,2,4]
for m, n in IN do if ts[15]*ts[2]*ts[4] eq
m*(ts[15])^n then A:=m; B:=n; end if; end for;

W, phi:=WordGroup(G1);
rho:=InverseWordMap(G1);
A@rho;
B@rho;
```

```
g:=function(W);
    return W.2;
end function;
g(G);
ts[15]*ts[2]*ts[4] eq f(x^2 * z^-1)*(ts[15])^f(y);
```

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